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MECHANICAL SPRING RELIABILITY ASSESSMENTS BASED ON FEA GENERATED FATIGUE STRESSES AND MONTE CARLO SIMULATED STRESS / STRENGTH DISTRIBUTIONS

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ABSTRACT

The reliability assessment of a mechanical spring is required to evaluate different spring designs for various spring deflection applications. For the fatigue failure mode, the spring reliability is predominantly dependent on the service fatigue stress and the fatigue strength of the spring. In reality, significant statistical scatters of both fatigue stress and fatigue strength are often inherent and inevitable. Therefore, the spring reliability assessment for high-volume production is a statistical interference analysis of fatigue stress and fatigue strength. In many engineering cases, neither the fatigue stress nor the statistical interference can be analytically solved; finite element analysis (FEA) and Monte Carlo simulations are thus employed. Finite element analysis is used to generate the fatigue stresses as functions of spring deflections for displacement-controlled boundary conditions. The spring reliabilities as functions of fatigue life are predicted based on the Monte Carlo simulations of spring deflection and spring fatigue strength distributions.

The methodology described in this paper is equally good for force-controlled boundary conditions; and can be applied as a general probabilistic approach to evaluate the reliability of many other types of mechanical components. Application of this methodology has the potential for reducing the product developmental cost and improving product robustness.

1. INTRODUCTION

Finite element analysis (FEA) is a very powerful and versatile technique to analyze complicated structural problems. A finite element model is developed to calculate the fatigue stresses from the spring static preload to the spring dynamic deflections. For displacement-controlled boundary conditions, the fatigue stresses are explicitly expressed as functions of spring deflection using curve fitting of FEA results to quadratic polynomials. The Goodman relationship is used to include the mean stress effect on fatigue life in the analysis.

Two input random variables, spring deflection and spring fatigue strength, are modeled by truncated normal distributions. The spring fatigue strength is characterized by the S-N approach. The spring fatigue stress is also a random variable and is calculated from the spring deflection using the quadratic polynomials of fatigue stress versus spring deflection.

High-speed computers that quickly and economically estimate the statistics of complex functions have led to the wide usage of Monte Carlo simulation methods in solving difficult engineering problems. In this analysis, since the fatigue stress distribution is not a standard probability density function (i.e., Normal, Log normal, or Weibull distributions), Monte Carlo simulation becomes a suitable technique to solve this statistical interference problem. Monte Carlo simulations virtually sample 1,000,000 spring assemblies; and each has different deflections and fatigue strengths in a random manner according to their respective probability distributions. The output random variable of the Monte Carlo simulations is the spring fatigue life. Finally, the probability of failure (or the reliability) is calculated from Monte Carlo simulation results as functions of fatigue life.
2. FEA GENERATED FATIGUE STRESSES

The FEA generated fatigue stresses are given in Figure 1 as functions of spring deflections for two different spring designs. For this displacement-controlled boundary condition problem, the FEA results can be curve fitted to yield explicit expressions of fatigue stress versus spring deflection:

\[ S(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 \]  

(1)

where \( a_1 \) through \( a_3 \) are coefficients; and \( x \) is the spring deflection. Quadratic polynomials are sufficiently accurate in representing the FEA results. Coefficients \( a_1 \) through \( a_3 \) are evaluated by a regression analysis.

The stress at \( x = 0 \) is due to spring static preload. The stress at \( x > 0 \) is the dynamic fatigue stress under dynamic service displacements. Therefore, the minimum stress \( \sigma_{\text{min}} \) is the stress at zero deflection \( (x = 0) \), and the maximum stress \( \sigma_{\text{max}} \) is the stress at the maximum spring deflection. Both minimum and maximum stresses can be determined from Equation (1) as:

\[ \sigma_{\text{min}} = S(x = 0) \]

\[ \sigma_{\text{max}} = S(x = \text{max}) \]  

(2)

The mean stress \( \sigma_m \) and the stress amplitude \( \sigma_a \) are calculated by following equations:

\[ \sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2} \]

\[ \sigma_a = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \]  

(3)

The mean stress effect on fatigue life is included using Goodman relationship:

\[ \frac{\sigma_a + \sigma_m}{S_e} = \frac{1}{S_u} \]  

(4)

where \( S_u \) is the ultimate strength; and \( S_e \) is the fully reversed stress amplitude corresponding to the same life as that obtained with the stress condition \( \sigma_a \) and \( \sigma_m \) in Equation (3). The fatigue stress of a spring assembly is of constant amplitude; however, the amplitude varies randomly among assemblies. Combining Equations (1) to (4), the fully reversed stress amplitude can be finally determined by:

\[ S_e = \frac{\left[ S(x = \text{max}) - S(x = 0) \right] S_u}{2S_u - [S(x = \text{max}) + S(x = 0)]} \]  

(5)

3. STATISTICAL CHARACTERIZATIONS OF RANDOM VARIABLES

The spring dynamic deflection is a linear tolerance stack-up of several components involved in the assembly. Each assembly has a distinct value of maximum spring deflection; different assemblies have different deflections. Therefore, the spring deflection is treated as a random variable. In general, the tolerances of these components involved in an assembly can be considered as random and statistically independent to each other. It becomes reasonable to approximate the spring deflection distribution by a normal distribution according to the central limit theorem for sums. Furthermore, it is assumed that the manufacturing process is capable of maintaining the tolerance ranges to finite values for high volume production. Hence, the spring deflection distribution is truncated with a range of \( \mu_x \pm 3\sigma_x \). Note that \( \mu_x \) and \( \sigma_x \) are mean and standard deviation of the normal distribution for spring
deflection respectively. The Monte Carlo simulated histograms of spring deflection are shown in Figure 2 for both large and small deflection ranges.

The fatigue stress is calculated from the spring deflection using Equation (1). The fatigue stress is also a random variable; however, its statistical characteristics are not explicitly expressed and are realized through Monte Carlo simulations.

A truncated normal distribution is also assumed for the spring fatigue strength distribution with the range of \( \mu_s \pm 3\sigma_s \), where \( \mu_s \) and \( \sigma_s \) are mean and standard deviation of the fatigue strength normal distribution. \( \mu_s \) is also the fatigue strength corresponding to 50% probability of failure and a fully reversed loading situation (\( S_{E, p=50\%} \)). In this analysis, the standard deviation \( \sigma_s \) is set to 9% of the fatigue strength at 50% probability of failure, or

\[
\sigma_s = 0.09 \mu_s
\]  

The Monte Carlo simulated histogram of spring fatigue strength is given in Figure 3. It is also assumed that the spring deflection and the fatigue strength are statistically independent.

The probability density function of a truncated normal distribution is modified from a non-truncated normal distribution. The general mathematical expression of a truncated normal distribution of random variable \( y \) has four distribution parameters (ANSYS Release 8.0 Documentation), namely a mean \( \mu \) and a standard deviation \( \sigma \) of the non-truncated normal distribution, and the lower limit \( y_{min} \) and upper limit \( y_{max} \). The probability density function of a truncated normal distribution is:

\[
f_y(y) = \begin{cases} 
0 & \text{for } y < y_{min} \text{ or } y > y_{max} \\
\frac{1}{\left[\Phi\left(\frac{y_{max} - \mu}{\sigma}\right) - \Phi\left(\frac{y_{min} - \mu}{\sigma}\right)\right]} \phi\left(\frac{y - \mu}{\sigma}\right) & \text{for } y_{min} \leq y \leq y_{max}
\end{cases}
\]  

The cumulative distribution function of the truncated normal distribution is:

\[
F_y(y) = \frac{\Phi\left(\frac{y - \mu}{\sigma}\right) - \Phi\left(\frac{y_{min} - \mu}{\sigma}\right)}{\Phi\left(\frac{y_{max} - \mu}{\sigma}\right) - \Phi\left(\frac{y_{min} - \mu}{\sigma}\right)}
\]  

where \( \phi(z) \) and \( \Phi(z) \) are probability density function and cumulative distribution function of the standard normal distribution, respectively. \( \phi(z) \) and \( \Phi(z) \) are expressed as:

\[
\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \quad \text{for } -\infty \leq z \leq +\infty
\]

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{z^2}{2}\right) dz
\]  

The S-N curve in Figure 4 corresponds to 50% probability of failure and a fully reversed loading situation. It is closely modeled within the available data range by a third-order exponential decay function as follows:
$S_{e, p=50\%} = c_0 + c_1 \exp \left( -\frac{N}{t_1} \right) + c_2 \exp \left( -\frac{N}{t_2} \right) + c_3 \exp \left( -\frac{N}{t_3} \right)$

where $N$ is the fatigue life; and $c_1$ to $c_3$ and $t_1$ to $t_3$ are coefficients determined by a regression analysis. The method presented in this paper for determining the fatigue stress can be similarly applied for force-controlled boundary conditions.

4. RELIABILITY ASSESSMENTS USING MONTE CARLO SIMULATIONS

The limit state function is defined as:

$$g(r, s) = N(r, s) - N_S$$

where $r$ is a vector of the design parameters affecting the design strength, and $s$ is a vector of the design parameters affecting the applied stress. In Equation (11), $N_s$ is the service fatigue life for which an estimate of reliability or probability of failure is required. $N=N(r, s)$ is the random variable denoting fatigue life (the number of cycles to failure). Apparently $N$ is a function of design parameters. The limit state function is used to identify both the safe state and the failure state:

- **Safe state:** $g(r, s) = N(r, s) - N_S > 0$
- **Failure state:** $g(r, s) = N(r, s) - N_S < 0$

If $P(.)$ denotes probability, the probability of failure can be calculated using the limit state function as:

$$F(N_S) = P\left[g(r, s) < 0\right]$$

Alternatively, the probability of failure can be written as:

$$F(N_S) = \int \cdots \int f_x(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots \, dx_n$$

where $x_i (i = 1, \ldots, n$) are the generalized design parameters of interest. $f_x$ is the joint density function of $x_i (i = 1, \ldots, n)$. The integration in Equation (14) is performed over the region of the failure state, and can not be evaluated analytically in this analysis. The direct Monte Carlo simulations technique is thus used to evaluate Equations (11) to (13).

In Monte Carlo simulations, 1,000,000 assemblies are virtually sampled; all of them differ in a random manner from each other. Each assembly of a Monte Carlo simulation cycle has the spring deflection and the spring fatigue strength randomly generated according to their respective truncated normal distributions. The fatigue stress is then calculated from the spring deflection using Equation (1), and the fully reversed fatigue stress amplitude is determined by Equation (5). Knowing spring fatigue stress and fatigue strength, the spring fatigue life is then evaluated. Because of the limited range of the $S$-$N$ curve available, however, in some cases the spring fatigue stresses fall outside the stress range of the $S$-$N$ curve. The corresponding fatigue lives can not be exactly determined, but their upper or lower limit values are known: either above or below the life range of the $S$-$N$ curve. Even though a complete spring fatigue life distribution analysis is usually not feasible, this does not create any difficulties to estimate the probabilities of failure in the range of fatigue life defined by the $S$-$N$ curve. Mathematically, the probability of failure defined in Equations (11) and (13) can be estimated in Monte Carlo simulations as a function of $N_s$ by the following equation:
Similarly, the reliability, as a function of $N_s$, can be estimated by:

$$R(N_s) = \frac{n_s(N_s)}{n_f(N_s) + n_s(N_s)}$$  \hspace{1cm} (16)

where $n_f(N_s)$ is the number of simulation cycles in which the spring fatigue lives are less than $N_s$ ($g(r, s) < 0$); and $n_s(N_s)$ is the number of simulation cycles in which the spring fatigue lives are greater than or equal to $N_s$ ($g(r, s) \geq 0$). The total number of simulation cycles is $n_f(N_s) + n_s(N_s) = 1,000,000$. It is apparent that $F(N_s) + R(N_s) = 1$ from Equations (15) and (16). The evaluations of $F(N_s)$ and $R(N_s)$ are carried out over the whole range of fatigue life defined by the $S-N$ curve. The estimated probabilities of failure are expressed as functions of fatigue life in Figure 5.

A sufficiently large number of Monte Carlo simulation cycles is needed in order to achieve converged results. Monte Carlo simulations with different total numbers of simulation cycles are tested, revealing that the total number of 1,000,000 simulation cycles is enough to obtain converged results for this analysis. The accuracy of converged results is basically not a function of the total number of simulation cycles, but rather the uncertainties associated with the distributions of input random variables, including the uncertainties in the FEA modeling. The confidence levels are not specified for the statistics of input random variables, usually implying they are approximately average values. Accordingly, the confidence levels of the estimated probabilities of failure and the reliabilities are around 50%.

### 5. CONCLUSIONS

The methodology described in this paper has been successfully used to assess the reliabilities of different spring designs for various spring deflection range applications. A robust design has been identified and proposed at an early stage of the product development.

The combination of FEA and Monte Carlo simulations in this methodology grants this methodology to solve a wide range of complicated engineering problems. In fact, the methodology can be used as a general probabilistic approach to analyze many other kinds of mechanical components. This methodology can also play an important role in designing robust products while minimizing the product developing cost.

The accuracy of the assigned distributions for the random variables can significantly affect the precision of the analysis results. Therefore, the integration of the experimental data into this methodology is often necessary to more accurately define the distributions. Bench fatigue tests can also be performed to calibrate the analysis results for at least one of the cases investigated. However, the end analysis results are always more meaningful when used as comparisons among different cases investigated.

### REFERENCES

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Figure 1  FEA generated spring fatigue stresses as functions of spring deflections for two different spring designs
(1 inch = 0.0254 meter, 1 ksi = 6,894,757 Pascal)

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Figure 2 Monte Carlo simulated histograms of spring deflection for large spring deflection range (upper one) and small spring deflection range (lower one). 1 inch = 0.0254 meter.

Figure 3 Monte Carlo simulated histograms of spring fatigue strength
Figure 4  Spring fatigue property. The S-N curve represents the fatigue strength corresponding to 50% probability of failure and a fully reversed loading situation (1 ksi = 6,894,757 Pascal)

Figure 5  The estimated probabilities of spring failure from Monte Carlo simulations