Oscillation of quenched slowdown asymptotics of random walks in random environment in Z

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By Sung Won Ahn

Entitled
OSCILLATION OF QUENCHED SLOWDOWN ASYMPTOTICS OF RANDOM WALKS IN RANDOM ENVIRONMENT IN Z

For the degree of Doctor of Philosophy

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Approved by: Goldberg, David 7/21/2016

Head of the Departmental Graduate Program Date
OSCILLATION OF QUENCHED SLOWDOWN ASYMPTOTICS
OF RANDOM WALKS IN RANDOM ENVIRONMENT IN Z

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of
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of
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ABSTRACT

Ahn, Sung Won PhD, Purdue University, August 2016. Oscillation of Quenched Slowdown Asymptotics of Random Walks in Random Environment in \( \mathbb{Z} \). Major Professor: Jonathon Peterson.

We consider a one dimensional random walk in a random environment (RWRE) with a positive speed \( \lim_{n \to \infty} \frac{X_n}{n} = v_\alpha > 0 \). Gantert and Zeitouni [1] showed that if the environment has both positive and negative local drifts then the quenched slowdown probabilities \( P_\omega(X_n < xn) \) with \( x \in (0, v_\alpha) \) decay approximately like \( \exp\{-n^{1-1/s}\} \) for a deterministic \( s > 1 \). More precisely, they showed that \( n^{-\gamma} \log P_\omega(X_n < xn) \) converges to 0 or \( -\infty \) depending on whether \( \gamma > 1 - 1/s \) or \( \gamma < 1 - 1/s \). In this paper, we improve on this by showing that \( n^{-1+1/s} \log P_\omega(X_n < xn) \) oscillates between 0 and \( -\infty \), almost surely. This had previously been shown only in a very special case of random environments [2].
1. Introduction

1.1 Definition and Background of RWRE

The random walk is a well-known probability field that mathematically formulates random paths on $\mathbb{Z}^d$, trees, or etc. The process of the random walk is determined by a series of steps in which the direction of each step is chosen from the transition probability. The transition probability is the set of probability the walk moves from the current location(site) to the other approachable location. A classical random walk typically has a spatially homogeneous transition probability. In other words, the transition probability of each location is identical in direction. In practice, however, it is natural to observe that the transition probability of each location may not be fixed due to defects or impurities as time progresses. Random walk in random environment (RWRE) is the mathematical model that catches such spatially inhomogeneity by randomizing the transition probabilities on a certain measure. The initial set up of the RWRE is consisted of the set of locations(sites) and the set of edges between locations, denoting them by $V$ and $E = \{(x, y)\}$ for some $x, y \in V$ respectively. For each $v \in V$, we define $N_x$ to be the set of locations connected to $x$ through the edges. That is,

$$N_x = \{y \in V : (x, y) \in E\}.$$

Also, for each edge connecting $x$ to $y \in N_v$, we assign a weight $\omega_x(y) \in [0, 1]$ so that

1. $\omega_x(y) \geq 0 \quad \forall y \in V$
2. $\omega_x(y) = 0 \quad \forall y \notin N_x$
3. $\sum_{y \in N_x} \omega_x(y) = 1$
We call the set of weights \{\omega_x(y), \ y \in N_x\} a transition probability at the location(site) \(x\). Due to the randomness of the transition probability, there are two stages of randomness that occur from this process. Let us denote \(M_1(N_x)\) as a collection of transition probabilities at \(x\) and denote \(\Omega = \prod_{x \in V} M_1(N_x)\) as the product collection of the \(M_1(N_x), \ x \in V\). By setting \(\alpha\) as the probability measure on \(\Omega\), we can call the element of \(\Omega\) the environment denoted by \(\omega\). At the first stage, an environment is chosen randomly from the environment space \(\alpha\). The next stage is to generate the walk \(X_n, \ n \in \mathbb{N}\) under the chosen environment where \(X_n\) is the location of random walk at time \(n\). Due to the two stages of the randomness, there are two different laws: quenched and annealed laws. If the path \(X_n\) starting at \(x\) is generated under one particular \(\omega\), the corresponding law is called quenched law denoted by \(P^x_{\omega}(\cdot)\), and its expectation is denoted by \(E^x_{\omega}[\cdot]\). Without conditioning on the environment \(\omega\), the law of \(X_n\) is called the annealed law denoted by

\[
P_{\alpha}^x(\cdot) = \int P_{\omega}^x(\cdot) \alpha(d\omega) = E_{\alpha}[P_{\omega}^x(\cdot)],
\]
and its corresponding expectation is denoted by \(E_{\alpha}^x[\cdot]\). For simplicity we will write \(P_{\omega}(\cdot), \ E_{\omega}[\cdot], \ P_{\alpha}(\cdot), \ E_{\alpha}[\cdot]\) when the walk is started at \(x = 0\). Our research is devoted to the nearest neighbor RWRE with i.i.d. environment on \(\mathbb{Z}\). Here, i.i.d. environment is the collection of transition probabilities of \(x \in V\) which are independent and identically distributed under the measure \(\alpha\). Also, the nearest neighbor RWRE means that the walk can only jump up to locations adjacent to the current location. Precisely, the location set \(V\) our model is the one dimensional integer set \(\mathbb{Z}\), and the set of locations connected to \(x\), denoted by \(N_x\), is \(x - 1\) and \(x + 1\) since the walk can only jump to its neighbor. Hence, the environment set is \(\Omega = \{\omega = \{\omega_x\} \in [0,1]^Z\}\) where the associated transition probability of the walk is

\[
P_\omega(X_{n+1} = x + 1|X_n = x) = \omega_x
\]
\[
P_\omega(X_{n+1} = x - 1|X_n = x) = 1 - \omega_x.
\]
We end this section with a further definition which means to be an additional assumption throughout our study. The RWRE has an elliptic environment if \( \alpha(\omega_0 = 0 \text{ or } 1) = 0 \).

1.2 Structure of Thesis

The structure of this Thesis will be as follows.

In Chapter 2, we will enter with the preliminary results of RWRE on \( \mathbb{Z} \) as demonstrated by the recurrence/transience and the speed of \( X_n \) in Section 2.1. The results are computed using notations and methods introduced by Peterson in [3]. His methods simplify the formulation of probabilities and speed. This methods allow us to identify local behaviors of the walk such as a local crossing time in our later chapters. And then, we will briefly summarize the annealed and quenched limit laws of the random walk \( X_n \) and the hitting time \( T_n \) with a precise centering and scaling factor characterized by a key parameter.

Section 2.2 is a discussion of the annealed and quenched large deviation principle of RWRE on an exponential scale. In particular, Comets, Gantert, and Zeitouni derived a quenched and annealed LDP for \( X_n/n \) in Theorem 2.2.1 from the result of LDP for \( T_n/n \). We give a brief overview of their approach used to prove Theorem 2.2.1. Their result shows that both of quenched and annealed rate function vanishes on the interval \([0, r_\alpha]\) on an exponential scale when the environment has “positive speed with zero drift” (i.e. \( \alpha(\omega_0 \geq 1/2) = 1 \)) or “positive speed with mixed drift” (i.e. \( \alpha(\omega_0 < 1/2) \in (0, 1) \)). This result concludes that the order of decay rate has to be sub-exponential scale in those regime.

Section 2.3 summarizes Gantert and Zeitouni’s work on the precise decay rates of the quenched and annealed large deviation for \( X_n/n \) under the sub-exponential scale [1]. When the environment has “positive speed with negative drift”; however, they partly revealed the rough estimation of the decay rate of the quenched case, and they conjectured that \( \exp(-n^{1-1/s}) \) should be the precise rate. Our main result
confirms that their conjecture is true. Section 2.4 provides a brief overview of our main results.

Chapter 3 will provide a result of the large deviation of $T_n/n$ under a special random walk which is purposefully always stepped to the right with an exponential waiting period with i.i.d. random means for each site. This study serves as a preliminary exercise because the large deviation of $T_n/n$ under the special environment results similarly to the large deviation of $T_n/n$ under the original RWRE. Also, this study is a helpful resource in that some of techniques used in the proof are applied to our main result.

In Chapter 4, we will provide the full proof of our main result which solves the sharp decay rate of a roughly estimated quenched large deviation in a “positive speed with negative drift” condition. Section 4.2 shows the explicit upper estimate of the exponential moment given that one way node is placed to the far left of the current location. Then in Section 4.3, we prove that the quenched expectation of crossing time over vast region is negligible compared to the total amount of time it takes to cross the whole interval. In Section 4.4, we prove our main theorem from the result obtained from Section 4.2 and 4.3.
2. Large Deviations of RWRE on $\mathbb{Z}$

2.1 Review of the Preliminary Results of RWRE on $\mathbb{Z}$

This section consists of the preliminary results of RWRE on $\mathbb{Z}$, including the recurrence/transience and speed. It also shows the results about limit laws and the large deviation principle of the walk $X_n$. Before presenting the results, it is necessary to introduce some notations that are used throughout the section. Let $T_n$ be the hitting time for the walk reaching at site $n$ at the first time given by $T_n := \inf\{i \geq 0 : X_i = n\}$. It turns out that the limit results of $X_n$ are connected to the limit results of the hitting time $T_n$ by the following relation with $\lim_{n \to \infty} T_n/n < \infty$. That is,

$$\lim_{n \to \infty} \frac{X_n}{n} = \lim_{n \to \infty} \frac{n}{T_n}.$$

The key notation throughout this section is the random variable $\rho_x$ which is defined by

$$\rho_x = \frac{(1 - \omega_x)}{\omega_x}.$$

2.1.1 Recurrence, Transience, and the Effect of the Speed of $X_n$ on $\mathbb{Z}$

The first mathematical result for RWRE was the limit behavior of $X_n$ discovered by Solomon in [4] that proved the recurrence and transience of the RWRE on $\mathbb{Z}$. Solomon was able to show the limit behavior with the following assumptions that are listed below.

**Assumption 1** $\alpha$ is an i.i.d. product measure of the environment $\omega$ on $\mathbb{Z}$.

**Assumption 2** $E_\alpha[\log \rho_0]$ is well defined (with $+\infty$ or $-\infty$ as possible values).
Assumption 1 means that $\{\omega_x, x \in \mathbb{Z}\}$ is an i.i.d. sequence of random variables in $(0, 1)$. Then, Solomon’s work shows that the recurrence and transience of the RWRE is characterized by the sign of $E_\alpha[\log \rho_0]$.

**Theorem 2.1.1 (Solomon, [4])** Under Assumption 1 and 2,

$$E_\alpha[\log \rho_0] < 0 \Rightarrow \lim_{n \to \infty} X_n = +\infty \quad \mathbb{P}_\alpha \text{ a.s}$$

$$E_\alpha[\log \rho_0] > 0 \Rightarrow \lim_{n \to \infty} X_n = -\infty \quad \mathbb{P}_\alpha \text{ a.s}$$

$$E_\alpha[\log \rho_0] = 0 \Rightarrow -\infty = \liminf_{n \to \infty} X_n < \liminf_{n \to \infty} X_n = +\infty \quad \mathbb{P}_\alpha \text{ a.s}$$

Later, Alili generalized Solomon’s result of the recurrence and transience of RWRE in stationary ergodic environments in [5]. Solomon also proved that $X_n$ satisfies a law of large numbers, developing an explicit formula of the speed of RWRE, $v_\alpha = \lim_{n \to \infty} \frac{X_n}{n}$, under the i.i.d. environments. In this dissertation, we simplify our computations using notations introduced in [3]. Recall that for an environment $\omega = (\omega_x)_{x \in \mathbb{Z}}$, we have defined $\rho_x = \frac{1 - \omega_x}{\omega_x}$.

Then, for any integers $i \leq x \leq j$ where $i, x, j \in \mathbb{Z}$, we notate that $P_x^\omega(T_i < T_j)$ is harmonic in that it satisfies

$$P_x^\omega(T_i < T_j) = \omega_x P_x^{\omega+1}(T_i < T_j) + (1 - \omega_x) P_x^{\omega-1}(T_i < T_j)$$

with the boundary condition of

$$P_i^\omega(T_i < T_j) = 1, \text{ and } P_j^\omega(T_i < T_j) = 0$$

Then using notations on (2.1) and solving the recursive equation, we derive the following probabilities:

$$P_x^\omega(T_j < T_i) = \frac{R_{i,x-1}}{R_{i,j-1}} \text{ and } P_x^\omega(T_j > T_i) = \frac{\Pi_{i,x-1}R_{x,j-1}}{R_{i,j-1}}.$$
Then, we have the following recursive formula:

\[ E_\omega[T_1] = \omega_0 + (1 - \omega_0)(1 + E_\omega^{-1}[T_0] + E_\omega[T_1]), \] (2.2)

which becomes

\[ E_\omega[T_1] = 1 + \rho_0 + \rho_0 E_\omega^{-1}[T_0]. \]

By iterating the previous steps to \( E_\omega^x[T_{x+1}] \) for all \( x \leq -1 \), we get

\[ E_\omega[T_0] = 1 + 2 \sum_{i=0}^{\infty} \Pi_{-i,0} = 1 + 2W_0. \]

Then, the following theorem is the explicit representation of the speed under stationary ergodic environments by Alili in [5]. The formula is modified using notations in (2.1) for the simplification.

**Theorem 2.1.2 (Alili, [5])** We define the following notations:

\[ \bar{S} = 1 + 2 \sum_{i=0}^{\infty} \Pi_{0,i}, \quad \bar{F} = 1 + 2 \sum_{i=0}^{\infty} \Pi_{-1,0}^{-1} \]

Assume the environment space is stationary and ergodic and then \( \mathbb{P}_\alpha \)-a.s.:

\[ v_\alpha = \begin{cases} \frac{1}{E_\alpha(\bar{S})} & \text{if } E_\alpha(\bar{S}) < \infty \\ -\frac{1}{E_\alpha(\bar{F})} & \text{if } E_\alpha(\bar{F}) < \infty \\ 0 & \text{if } E_\alpha(\bar{S}) = \infty \text{ and } E_\alpha(\bar{F}) = \infty \end{cases} \]

Suppose \( \alpha \) is i.i.d. product measure, then

\[ E_\alpha[\bar{S}] = 1 + 2E_\alpha[W_2] = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^{i} E_\alpha[\rho_j] = 1 + 2 \sum_{i=1}^{\infty} E_\alpha[\rho_0]^i. \] (2.3)

Using (2.3), the explicit form of speed in Theorem 2.1.2 is simplified to the following forms which is also given by Solomon in [4].
Corollary 2.1.3 (Solomon, [4]) Under Assumption 1,

$$v_\alpha = \begin{cases} \frac{1 - E_\alpha[\rho_0]}{1 + E_\alpha[\rho_0]} & \text{if } E_\alpha[\rho_0] < 1 \\ \frac{1 - E_\alpha[\rho_0^{-1}]}{1 + E_\alpha[\rho_0^{-1}]} & \text{if } E_\alpha[\rho_0] < 1 \\ 0 & \text{if } \frac{1}{E_\alpha[\rho_0]} \leq 1 \leq E_\alpha[\rho_0^{-1}]. \end{cases}$$

Notice from the zero speed case where $1 \leq E_\alpha[\rho_0]$ that, by Jensen’s inequality, it is possible for $E_\alpha[\log \rho_0] \leq 0 \leq \log E_\alpha[\rho_0]$. This means us that the walk can be transient to the right (from Theorem 2.1.1) even though the speed is 0.

2.1.2 Limiting Distributions of RWRE on $\mathbb{Z}$

An extension to Solomon’s work, Kesten, Kozlov, and Spitzer studied the limiting distributions of transient RWRE under the annealed law in [6]. In addition to the Assumption 1 and 2, the following assumptions on environmental space $\Omega$ is used in their paper.

Assumption 3 There exists unique $s > 1$ satisfying $E_\alpha[\rho_0^s] = 1$ and $E_\alpha[\rho_0^s \log \rho_0] < \infty$.

It turns out that parameter $s > 0$, defined by the equation

$$E_\alpha[\rho_0^s] = 1, \quad s > 0,$$

proved to be a key factor determining both the scaling factor and the limit law of the random walk. In part,

- If $s \in (1, 2)$, then under annealed law, $\frac{X_n - mn_\alpha}{n^{1/s}} \Rightarrow$ a stable law of index $s$.

- If $s > 2$, then under annealed law with a constant $\sigma > 0$, $\frac{X_n - mn_\alpha}{\sigma n^{1/2}} \Rightarrow$ a standard normal law.

Limiting distributions for $s \in (0, 1]$ and $s = 2$ are also shown in [6]. The first result of the quenched limiting distribution was given by Alili in [5]. Like the annealed limiting
law, the scaling factor is determined by the parameter $s > 0$ for $E_0^s[\rho^s_0] = 1$. Alili proved the quenched central limit theorem of hitting time $T_n$ when $s > 2$. The limiting distributions of the walk $X_n$ in transient and recurrent RWRE under the quenched law was later studied by Peterson, Goldsheid, and Samorodnitsky in [3], [7], [8]. Peterson and Goldsheid independently proved the quenched central limit theorem of walk for $s > 2$ which requires the environment dependent centering. That is, if $s > 2$ with some constant $\sigma' > 0$ and $\lim_{n \to \infty} \frac{X_n}{n} = v_\alpha$, 

$$P_\omega \left( \frac{X_n - nv_\alpha + Z_n(\omega)}{\sigma' \sqrt{n}} \leq x \right) \to \Phi(x)$$

where $Z_n(\omega)$ is an explicit environment dependent centering. The case of $s \in (0, 2)$ is quite different. Peterson proved in [9] that the quenched limiting distribution of $P_\omega \left( \frac{X_n - nv_\alpha}{n^{1/s}} \leq x \right)$ does not exists and oscillates by finding two different subsequences which converge to two different types of limiting distribution. Then, Peterson and Samorodnitsky proved that the quenched distribution of the hitting times and the walk converges in distribution to a random variable rather than a deterministic value in [8]. More precisely, if $s \in (0, 2)$ and $\lim_{n \to \infty} \frac{X_n}{n} = v_\alpha$, 

$$\alpha \left( P_\omega \left( \frac{X_n - nv_\alpha}{n^{1/s}} \leq x \right) \leq z \right) \to F_x(z)$$

with deterministic function of $F_x(\cdot)$. Moreover, Peterson and Samorodnitsky discovered the explicit form of the random variable for $s \in (0, 2)$ which consists of a non-homogeneous poisson point process [8].

### 2.2 Review of Annealed/Quenched Large Deviation Principle of RWRE on $\mathbb{Z}$

This section consists of a review on known results for the annealed and quenched Large Deviation Principle (LDP) of RWRE on $\mathbb{Z}$. As a basic set up for the LDP of RWRE on $\mathbb{Z}$, we introduce the following definitions found from [10]:
Definition 2.2.1 A function \( I : \mathbb{R} \to [0, \infty] \) is a rate function if it is lower semicontinuous. It is a good rate function if its level sets \( \{ x \in \mathbb{R} | I(x) \leq a \} \) for any \( a \geq 0 \) are compact.

Definition 2.2.2 Let \( \alpha \) be the environment measure on \( \Omega \). A sequence of \( \mathbb{R} \) valued random variables \( \{ Z_n \} \) satisfies the quenched Large Deviation Principle with speed \( n \) and deterministic rate function \( I_q^\alpha(\cdot) \) if for any Borel set \( A \),

\[
-I^\alpha_q(A^c) \leq \liminf_{n \to \infty} \frac{1}{n} \log P_\omega(Z_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_\omega(Z_n \in A) \leq -I^\alpha_q(\bar{A}) \quad \alpha - \alpha.s.
\]

where \( A^c \) denotes the interior of \( A \), \( \bar{A} \) the closure of \( A \), and for any Borel set \( F \),

\[
I^\alpha_q(F) = \inf_{x \in F} I^\alpha_q(x).
\]

Definition 2.2.3 Let \( \alpha \) be the environment measure of \( \Omega \). A sequence of \( \mathbb{R} \) valued random variables \( \{ Z_n \} \) satisfies the annealed Large Deviation Principle with speed \( n \) and deterministic rate function \( I^\alpha_a(\cdot) \) if for any Borel set \( A \),

\[
-I^\alpha_a(A^c) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_\alpha(Z_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_\alpha(Z_n \in A) \leq -I^\alpha_a(\bar{A}) \quad \tag{2.5}
\]

Definition 2.2.4 A LDP is called weak if the upper bound in (2.4) or (2.7), holds only with \( \bar{A} \) compact.

In particular, if the Borel set \( A \) satisfies continuity on \( I^\alpha_q \) and \( I^\alpha_a \) such that \( I^\alpha_q(A^c) = I^\alpha_q(\bar{A}) \) and \( I^\alpha_a(A^c) = I^\alpha_a(\bar{A}) \) respectively, we have the following equality.

\[
\lim_{n \to \infty} \frac{1}{n} \log P_\omega(Z_n \in A) = I^\alpha_q(A) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_\alpha(Z_n \in A) = I^\alpha_a(A) \quad \tag{2.6}
\]

It turns out that rate functions of both the quenched and annealed LDPs of RWRE on \( \mathbb{Z} \) are deterministic and continuous on \( \mathbb{R} \) in the most cases. Therefore, we will state large deviation results in the form of (2.6) instead of Definition 2.2.2 or Definition 2.2.3.

The first result of LDP for \( X_n/n \) under the quenched measure was studied by Greven and den Hollander [11]. They approach the problem by looking at the RWRE
as a Markov chain in the environment space, and find the rate function explicitly using some variational technique. Later, Comets, Gantert, and Zeitouni [12] used a different approach, obtaining an LDP for $X_n/n$ as a byproduct of an LDP for $T_n/n$. This approach had the advantage of supplying LDPs under both the quenched and annealed measures. Moreover, the approach in [12] led to a good qualitative description of the quenched and annealed large deviation rate functions. In this approach, a stronger assumption needs to be first imposed so that $\mathbb{E}_{\alpha}[\log \rho_0]$ is bounded below and above by a finite constant.

**Assumption 4** RWRE is uniformly elliptic. That is there exist a small $\epsilon > 0$ such that $\omega_x \in (\epsilon, 1 - \epsilon)$ for all $x \in \mathbb{Z}$ with probability 1.

In addition to Assumption 4, we introduce the following notations and functions

$$
\tau_x = T_{x+1} - T_x, \quad x \geq 0
$$

$$
\phi(\lambda, \omega) := \mathbb{E}_\omega[e^{\lambda \tau_0} \mathbf{1}_{\tau_0 < \infty}], \quad G(\lambda, \alpha, u) := \lambda u - \int \log \phi(\lambda, \omega) \alpha(d\omega).
$$

We will denote by $I^{\tau,q}_{\alpha}$ and $I^q_{\alpha}$ the rate functions for the quenched LDP associated with $T_n/n$ and $X_n/n$ respectively. In the end, we have the following theorem:

**Theorem 2.2.1 (Comets, Gantert, Zeitouni, [12])** Under Assumption 1 and 4,

1. The random variables $\{T_n/n\}$ satisfy the weak quenched LDP with speed $n$ and convex rate function

$$
I^{\tau,q}_{\alpha}(u) = \sup_{\lambda \in \mathbb{R}} G(\lambda, \alpha, u).
$$

2. Assume further that $\mathbb{E}_{\alpha}[\log \rho_0] \leq 0$. Then, the random variables $X_n/n$ satisfy the quenched LDP with speed $n$ and good convex rate function

$$
I^q_{\alpha}(v) = \begin{cases} v I^{\tau,q}_{\alpha} \left( \frac{1}{v} \right) & \text{if } 0 < v \leq 1 \\ |v| \left( I^{\tau,q}_{\alpha} \left( \frac{1}{|v|} \right) - \mathbb{E}_{\alpha}[\log \rho_0] \right) & \text{if } -1 \leq v < 0 \end{cases}
$$

and

$$
I^q_{\alpha}(0) = \lim_{v \to 0^+} v I^{\tau,q}_{\alpha} \left( \frac{1}{v} \right)
$$
For $\alpha$–a.e. $\omega$ the distributions of $X_n/n$ under $P_\omega$ satisfy a large deviation principle with convex, good rate function $I^q_\alpha$.

The relationship between the rate functions of $X_n/n$ and $T_n/n$ in Theorem 2.2.1 is obtained by renewal duality. That is, for $v \in (0, 1)$

$$P_\omega \left( \frac{X_n}{n} \sim v \right) \approx P_\omega(T_{nv} \sim n) = P_\omega \left( \frac{T_{nv}}{nv} \sim \frac{1}{v} \right).$$

Hence, we get

$$\lim_{n} \frac{1}{n} \log P_\omega \left( \frac{X_n}{n} \sim v \right) \approx v \lim_{n \to \infty} \frac{1}{nv} \log P_\omega \left( \frac{T_{nv}}{nv} \sim \frac{1}{v} \right),$$

which leads to the following equality: $I^q_v(v) = vI^r_{\alpha}(\frac{1}{v})$.

For $v \in [-1, 0)$, suppose $I^{-\tau,q}_\alpha$ is defined to be the rate function of quenched LDP associated with $T_{-n}/n$. Then, we claim that the LDP rate function for $T_{-n}/n$ is related to the LDP rate function for $T_{n}/n$ by

$$I^{-\tau,q}_\alpha(u) := I^{r,q}_\alpha(u) - E[\log \rho_0], \quad 1 \leq u < \infty. \quad (2.7)$$

In order to show the sketch of the proof of (2.7), we define new notations for an exponential moment of hitting time to the left direction such that,

$$\bar{\tau}_k = T_{k-1} - T_k, \quad k \leq 0.$$

$$\bar{\phi}(\lambda, \omega) := E_\omega[e^{\lambda \bar{\tau}_0} \mathbf{I}_{\bar{\tau}_0 < \infty}], \quad \bar{G}(\lambda, \alpha, u) := \lambda u - \int \log \bar{\phi}(\lambda, \omega) \alpha(d\omega).$$

Then, note that it follows from the first part of Theorem 2.2.1 that the rate function for $T_{-n}/n$ is given by $I^{-\tau,q}_\alpha = \sup_{\lambda \in \mathbb{R}} \bar{G}(\lambda, \alpha, u)$. The proof begins with the path decompositions of $\tau_0$ and $\bar{\tau}_0$ which characterize $\phi(\lambda, \omega)$ and $\bar{\phi}(\lambda, \omega)$ as recursive formulas like (2.2). That is,

$$\phi(\lambda, \omega) = \omega_0 e^\lambda + (1 - \omega_0) e^\lambda \phi(\lambda, \theta^{-1} \omega) \phi(\lambda, \omega)$$

and

$$\bar{\phi}(\lambda, \omega) = (1 - \omega_0) e^\lambda + \omega_0 e^\lambda \bar{\phi}(\lambda, \theta \omega) \bar{\phi}(\lambda, \omega),$$
where $\theta$ is a left-shift operator given by $(\theta \omega)_x = \omega_{x+1}$. Combining the recursive formulas of $\phi(\lambda, \omega)$ and $\bar{\phi}(\lambda, \omega)$, we have

$$
\frac{\phi(\lambda, \omega)}{\bar{\phi}(\lambda, \omega)} = \frac{(1 - \bar{\phi}(\lambda, \theta \omega) \phi(\lambda, \omega))}{(1 - \phi(\lambda, \omega) \phi(\lambda, \theta^{-1} \omega))}.
$$

Taking the expectation over $\alpha$ after the logarithm on both sides,

$$
E_\alpha[\log \rho_0] + E_\alpha[\log \phi(\lambda, \omega)] - E_\alpha[\log \bar{\phi}(\lambda, \omega)] = E_\alpha[\log(1 - \bar{\phi}(\lambda, \theta \omega) \phi(\lambda, \omega))]
- E_\alpha[\log(1 - \phi(\lambda, \omega) \phi(\lambda, \theta^{-1} \omega))].
$$

By the stationarity of $\alpha$, the right hand side of the previous equation is 0, and so

$$
E_\alpha[\log \bar{\phi}(\lambda, \omega)] = E_\alpha[\log \phi(\lambda, \omega)] + E_\alpha[\log \rho_0].
$$

Then for any fixed $\lambda \in \mathbb{R}$, we have

$$
\lambda u - E_\alpha[\log \bar{\phi}(\lambda, \omega)] = \lambda u - E_\alpha[\log \phi(\lambda, \omega)] - E_\alpha[\log \rho_0]
$$
or $\bar{G}(\lambda, \alpha, u) = G(\lambda, \alpha, u) - E_\alpha[\log \rho_0]$. Taking supremum of the both sides over all $\lambda \in \mathbb{R}$ and recalling the definitions of $I^{\tau,q}_\alpha(u)$ and $I^{-\tau,q}_\alpha(u)$, we prove the claim of (2.7). For $E_\alpha[\log \rho_0] > 0$ where the walk is transient to the left, we derive the rate function from Theorem 2.2.1 with a modified environment measure $\alpha^{\text{Inv}} := \alpha \circ \text{Inv}^{-1}$ where $\text{Inv} : \Omega \to \Omega$ reflects the mirror image of the transition probability such that $(\text{Inv} \omega)_i = 1 - \omega_{-i}$. Under $\alpha^{\text{Inv}}$, the walk becomes transient to the right and, by Theorem 2.2.1, the LDP for $X_n/n$ holds with good convex rate function $I_{a}^q(v) = I_{a}^q(\text{Inv}(-v))$.

We refer the result of the annealed LDP of $X_n/n$ and $T_n/n$ from [12]. Their approach requires an extra condition that the environment measure $\alpha$ needs to satisfy an empirical process $R_n = n^{-1} \sum_{j=0}^{n-1} \delta_{\theta^j \omega}$ where $\theta$ is a left-shift operator given by $(\theta \omega)_x = \omega_{x+1}$. Then, $\alpha$ satisfy an annealed LDP with good rate function

$$
I_{a}^q(v) = \inf_{\eta \in M} \left[ I_{a}^q(v) + |v|h(\eta|\alpha) \right]
$$
where $M$ denotes the space of environment measure satisfying Assumption 1, and $h(\cdot|\alpha)$ is a relative entropy with respect to $\alpha$. 
We are interested in certain large deviation asymptotics when the RWRE is a positive speed accompanied by mixed local drifts; that is, $v_\alpha > 0$ and $\alpha(\omega_0 \leq 1/2) > 0$. In this case, the results in [12] show that both the quenched and averaged large deviation rate functions vanish on the interval $[0, v_\alpha]$. That is,

$$\{ x : I^q_\alpha(x) = 0 \} = \{ x : I^\varnothing_\alpha(x) = 0 \} = [0, v_\alpha].$$

Thus, in the case of a positive speed with mixed local drifts, the probability of the random walk moving at a positive but slower than typical speed decays sub-exponentially in $n$. It was shown in several papers that the precise rate of decay of these large deviation slowdown probabilities is different under the quenched and annealed measures and that the sub-exponential rate depends on the specifics of the distribution $\alpha$ on environments [1], [13], [14]. We explain this with more detail in the next section.

### 2.3 Review of Annealed/Quenched Sup-exponential Tail of RWRE on $\mathbb{Z}$

This section summarizes the annealed and quenched sup-exponential regime in which the rate function vanishes under exponential scale when the speed of the random walk is positive. First, we note from the previous section that the speed of the walk is positive if and only if $\mathbb{E}_\alpha[\rho_0] < 1$. Due to the convexity of $\mathbb{E}_\alpha[\rho_s^0]$ in $s$, given that $\mathbb{E}_\alpha[\rho_0] < 1$, there exists a unique $s > 1$ satisfying $\mathbb{E}_\alpha[\rho_s^0] = 1$. Therefore, we have the following assumption throughout this section.

**Assumption 5** There exists unique $s > 1$ satisfying $\mathbb{E}_\alpha[\rho_s^0] = 1$.

Recall that the same $s$ showed up in the limiting distributions. The precise sub-exponential quenched and annealed rates of decay of the slowdown probabilities have also been studied under the assumption that the environment has “positive or zero drift”; that is, $\alpha(\omega_0 \geq 1/2) = 1$ and $\alpha_0 := \alpha(\omega_0 = 1/2) \in (0, 1)$. In this case, the precise quenched and annealed asymptotics of the slowdown probabilities are given in [15] and [14], respectively. In particular,
Theorem 2.3.1 (Pisztora, Povel, Zeitouni [15], [14]) Suppose that $E[\rho_0] < 1$, $\alpha(\omega_0 \geq 1/2) = 1$ and $\alpha_0 := \alpha(\omega_0 = 1/2) \in (0, 1)$. Then,
\[
\lim_{n \to \infty} \frac{(\log n)^2}{n} \log P_{\omega}(X_n < nv) = -\frac{(\pi \log \alpha_0)^2}{8} \left(1 - \frac{v}{v_\alpha}\right), \quad \forall v \in (0, v_\alpha),
\]
and
\[
\lim_{n \to \infty} \frac{1}{n^{1/3}} \log \mathbb{P}_\alpha(X_n < nv) = -\left\{\frac{27(\pi \log \alpha_0)^2}{32} \left(1 - \frac{v}{v_\alpha}\right)\right\}^{1/3}, \quad \forall v \in (0, v_\alpha).
\]

Note that they quantify the fact that the annealed probabilities of large deviations are larger than their quenched counterparts, due to the possibility of rare fluctuations in the environment which may slow down the RWRE. Under the assumption that the environment has “positive or negative drift”; that is, $\alpha(\omega_0 \geq 1/2) = 1$ and $\alpha(\omega_0 < 1/2) \in (0, 1)$, $s$ becomes a key parameter which determines the sub-exponential quenched and annealed decay rate. The precise sub-exponential annealed rates of decay of the slowdown probabilities have also been discovered in [1]. In particular,

Theorem 2.3.2 (Dembo, Perez, Zeitouni, [13]) Under Assumption 5, $\alpha(\omega_0 < 1/2) \in (0, 1)$, and let $\nu \in (0, \nu_\alpha)$. Then, for $\mathbb{P}_\alpha - a.a.\omega$, the following statements hold:
\[
\lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{P}_\alpha(X_n < nv) = 1 - s, \quad \forall v \in (0, \nu_\alpha).
\]

On the other hand, the quenched rates of decay are only roughly estimated by the following theorem.

Theorem 2.3.3 (Gantert, Zeitouni, [1]) Under Assumption 5, $\alpha(\omega_0 < 1/2) \in (0, 1)$, and let $\nu \in (0, \nu_\alpha)$. Then, for $\mathbb{P}_\alpha - a.a.\omega$, the following statements hold:

1. For any $\delta > 0$,
\[
\limsup_{n \to \infty} \frac{1}{n^{1-1/s-\delta}} \log P_{\omega}(X_n < nv) = -\infty
\]

2. For any $\delta > 0$,
\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s+\delta}} \log P_{\omega}(X_n < nv) = 0
\]
One should compare the rate of decay obtained in Theorem 2.3.3 with the annealed polynomial rate of decay \( P_\alpha(X_n < n^{1-s}) \sim n^{1-s} \). Again, the rate in Theorem 2.3.2 should be compared with the annealed rate \( (\text{cf. Theorem 2.3.1}) \ P_\alpha(X_n < n^{1-s}) \sim n^{1-s} \). One might suspect from this that \( P_\omega(X_n/n \leq v) \) decays on a stretched exponential scale like \( \exp(-Cn^{1-1/s}) \) for some deterministic constant \( C > 0 \) depending on \( v \in (0, v_\alpha) \). However, in [1] Gantert and Zeitouni showed that for any \( v \in (0, v_\alpha) \),

\[
\limsup_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega \left( \frac{X_n}{n} < v \right) = 0, \quad \alpha\text{-a.s.,} \quad (2.8)
\]

and conjectured that the corresponding liminf is equal to \(-\infty\). The lower bound was proved in a special case by Gantert in [2] in which \( \alpha(\omega_0 \in \{p, 1\}) = 1 \) for some fixed \( p < 1/2 \). In this case, the environment \( \omega \) consists of scattered “one-way nodes” (i.e., sites \( x \) with \( \omega_x = 1 \)) and all remaining sites have a fixed drift to the left. We note that the results in [2] also include cases in which the distribution \( \alpha \) is such that the environment \( \omega = \{\omega_x\}_{x \in \mathbb{Z}} \) is ergodic rather than i.i.d. In our result, we restrict ourselves to the i.i.d. environments, but we remove the requirement that the support of \( \omega_0 \) is \( \{p, 1\} \), and we generalize the distribution of the \( \omega_x \) for \( x \in \mathbb{Z} \).

### 2.4 Outline of the Main Result

This section briefly summarizes the full proof of our main result in Chapter 4 which solves the conjecture that the lim inf of (2.8) is \(-\infty\). For the main result, we make the following assumptions:

**Assumption 6** The distribution \( \alpha \) on environments is such that \( E_\alpha[\log \rho_0] < 0 \) and \( E_\alpha[\rho_0^s] = 1 \) for some \( s > 1 \).

**Remark 1** It follows from Hölder’s inequality that \( \gamma \mapsto E_\alpha[\rho_0^\gamma] \) is a convex function. Moreover, the slope of this function at \( \gamma = 0 \) is \( E_\alpha[\log \rho_0] < 0 \). Thus, it follows from Assumption 6 that \( E_\alpha[\rho_0] < 1 \) and therefore the RWRE is transient to the right with a positive speed of \( v_\alpha > 0 \). Moreover, since \( \rho_0 < 1 \iff \omega_0 > 1/2 \) it follows that
\( \alpha(\omega_0 > 1/2) > 0 \) and \( \alpha(\omega_0 < 1/2) > 0 \). Since the environment is assumed to be i.i.d. this implies that \( \alpha \)-a.e. environment has sites with local drifts to the right and to the left.

**Assumption 7** The distribution of \( \log \rho_0 \) is non-lattice under \( \alpha \) and that \( E_{\alpha}[\rho_0 \log \rho_0] < \infty \).

**Remark 2** The conditions in Assumption 7 are needed for certain precise tail asymptotics that we will use throughout our chapters. The main result may be true without these additional technical assumptions, but this would require dealing with rougher tail asymptotics throughout the proof of our main result. The conditions in Assumption 7 have also been used in many previous papers in one-dimensional RWRE [6], [16], [17], [18], [8], [19], [20].

Thus, the main result is the following:

**Theorem 2.4.1** If Assumptions 6 and 7 hold, then for any \( v \in (0, v_\alpha) \),

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega \left( \frac{X_n}{n} < v \right) = -\infty, \quad \alpha - a.s.
\]

(2.9)

Together with (2.8), we conclude that \( \frac{1}{n^{1-1/s}} \log P_\omega(X_n/x < v) \) fluctuates between 0 and \(-\infty\), \( \alpha \)-a.s. Our goal is to obtain the main result from the sub-exponential deviation results of a hitting time \( T_n \). More precisely, the main part of the proof of Theorem 2.4.1 is devoted to showing the following theorem.

**Theorem 2.4.2** Under the same assumption as Theorem 2.4.1, for any \( u \in (\frac{1}{v_\alpha}, \infty) \),

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_n > un) = -\infty, \quad \alpha - a.s.
\]

(2.10)

Then, the renewal duality of \( T_n/n \) and \( X_n/n \), with minor corrections, completes the proof of Theorem 2.4.1 (See the proof of Theorem 4.1.1 for a more precise statement.) In order to find the appropriate subsequence of \( n \) satisfying (2.10), we briefly discuss
several techniques and notations used throughout the proof of (2.10). The first technique involves visualizing an environment by constructing potential $V(x)$ as it was introduced by Sinai in the analysis of recurrent RWRE [21]. That is,

$$V(x) = \begin{cases} \sum_{i=0}^{x-1} \log \rho_i & \text{if } x \geq 1 \\ 0 & \text{if } x = 0 \\ -\sum_{i=x}^{-1} \log \rho_i & \text{if } x \leq -1. \end{cases}$$

The potential $V(x)$ enables us to cut an environment into blocks by using “ladder points”, $\{\nu_i, i \in \mathbb{Z}\}$, defined by

$$\nu_0 = \sup \{y \leq 0 : V(y) < V(k), \forall k < y\}, \quad (2.11)$$
and for $i \geq 1$,

$$\nu_i = \inf \{x > \nu_{i-1} : V(x) < V(\nu_{i-1})\}, \text{ and } \nu_{-i} = \sup \{y < \nu_{i+1} : V(y) < V(k), \forall k < y\}.$$ 

Between each consecutive ladder points, we measure the height of a peak, called the

![Figure 2.1. The locations of ladder points $\{\nu_i\}_{i \in \mathbb{Z}}$ on $\mathbb{Z}$.

‘exponential height’, denoted by

$$M_i := \max \{\Pi_{\nu_i,j} : \nu_i \leq j \leq \nu_{i+1}\} = \max \{e^{V(j) - V(\nu_i)} : \nu_i < j \leq \nu_{i+1}\}, \quad i \in \mathbb{Z}.$$
This exponential height has a crucial role as it estimates the rough amount of the crossing time between the ladder points [17]. Since the environment is i.i.d. under the measure $\alpha$, it follows that the blocks of the environment between adjacent ladder points $\mathbb{B}_i = \{\omega_x : x \in [\nu_i, \nu_{i+1})\}$ are i.i.d. for $i \neq 0$. In particular, $\{M_i\}_{i \neq 0}$ are both i.i.d. sequences of random variables. However, the interval of environment between the ladder points on either side of the origin have a different distribution. For this reason, it is convenient to, at times, work with a related measure on environment $Q$ given by

$$Q(\cdot) = \alpha(\cdot | \nu_0 = 0).$$

The distribution $Q$ was first introduced in [17]. Under the measure $Q$ the blocks between adjacent ladder points $\mathbb{B}_i$ are i.i.d. for all $i \in \mathbb{Z}$ with each having the same distribution as $\mathbb{B}_1$ under the original measure $\alpha$ on environments. In particular, this implies that $\{M_i\}_{i \in \mathbb{Z}}$ is an i.i.d. sequence under the measure $Q$. Figure 2.2 is a figure of the potential under measure $Q$ with ladder points and their corresponding exponential heights. Our analysis shows that any events under the measure $Q$ occur almost surely under the measure $\alpha$ (see Lemma 4.4.2). Therefore, as $\{\nu_i\}_{i \geq 0}$ is the subsequence of $n$, the following proposition concludes (2.10).
Proposition 2.4.3 Under the same assumptions as Theorem 2.4.1, for any \( u \in (\frac{1}{v_0}, \infty) \),
\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{\nu_n} > u\nu_n) = -\infty, \quad Q\text{-a.s.} \tag{2.12}
\]

**Proof** [Sketch of the proof of Proposition 2.4.3] The idea of the proof for Proposition 2.4.3 is to divide the environment into large blocks with \( \omega \)-dependent size and analyze the crossing times of these blocks. First, we define subsequence \( n_k \) for an integer \( m > s \),
\[
n_k = m^{n_k} \text{ for } k \geq 0.
\]
Then let \( a_k = n_k^{1/s}/D \) for some fixed \( D > 1 \), and we set the blocks of the environment as the intervals between ladder locations \( \nu_{ja_k} \) and \( \nu_{(j+1)a_k} \) for \( j \in \mathbb{Z} \). To simplify this notation, let us denote the ladder locations at the edges of the blocks by
\[
\nu(j,k) := \nu_{ja_k}, \quad j \in \mathbb{Z}, k \geq 1.
\]
Then, the path of the random walk \( X_n \) on \( \mathbb{Z} \) naturally defines a birth-death chain by observing how the random walk moves from one \( \nu(j,k) \) to either \( \nu(j-1,k) \) or \( \nu(j+1,k) \). To be precise, let \( \{t_i\}_{i \geq 0} \) be the sequence of times when the random walk reaches a ladder point \( \nu(j,k) \) different from the last such ladder point visited. That is, \( t_0 = 0 \) and
\[
t_i = \inf \{n > t_{i-1} : X_n \in \{\nu(j,k)\}_{j \in \mathbb{Z}} \text{ and } X_n \neq X_{t_{i-1}}\}, \quad i \geq 1.
\]
We then obtain a birth-death process \( \{Z_i\}_{i \geq 0} \) on \( \mathbb{Z} \) by letting \( X_{t_i} = \nu(Z_i, k) \). If we let \( \Theta_i = t_i - t_{i-1} \), then it follows that
\[
T_{\nu_{n_k}} \leq \sum_{i=1}^{N_k} \Theta_i,
\]
where \( N_k = \inf\{i \geq 1 : Z_i \geq n_k/a_k\} \) is the time needed for the induced birth-death process to move at least \( n_k/a_k \) to the right. If we also define \( \tilde{N}_k = \inf\{i \geq 1 : |Z_i| \geq \}

to be the time for the birth-death process to exit \((-n_k/a_k, n_k/a_k)\), then it follows that
\[
P_\omega(T_{n_k} > u\nu_{n_k})
\leq P_\omega(N_k \neq \tilde{N}_k) + P_\omega(\tilde{N}_k > L, N_k = \tilde{N}_k) + P_\omega\left(\sum_{i=1}^{N_k} \Theta_i > u\nu_{n_k}, \tilde{N}_k \leq L\right).
\]

(2.13)

The first term on the right of (2.13) is the probability that the birth-death process ever backtracks to \(-n_k/a_k\). The second term on the right of (2.13) is the probability that the birth-death process never backtracks to \(-n_k/a_k\) but crosses blocks within \((-n_k/a_k, n_k/a_k)\) more than \(L\) times before hitting \(n_k/a_k\). Now, for some fixed constant \(D_0 > 0\) and any small constant \(\delta > 0\), let us define
\[
L_k = \frac{D}{1 - \delta} n_k^{1-1/s}, \quad \text{and} \quad \lambda_k = \frac{D_0}{n_k^{1/s}},
\]
where the constant \(D > 0\) comes from \(a_k = n_k^{1/s}/D\). The first two probabilities of the right hand side in (2.13) decays exponentially almost certain under \(Q\) such that
\[
\lim_{n \to \infty} \frac{1}{n^{1-1/s}} \log \left\{ P_\omega(N_k \neq \tilde{N}_k) + P_\omega(\tilde{N}_k > L, N_k = \tilde{N}_k) \right\} = -\infty.
\]

This is because the Birth-Death process typically steps to right with very high probability. (See Lemma 4.4.4 and (4.45) for more precise statements.) Regarding the third term, \(N_k = \tilde{N}_k\) implies that the distribution of the crossing time \(\Theta_i\) is determined by the location \(Z_{i-1} \in J_{n_k}\) for each \(i \leq \tilde{N}_k\), where
\[
J_{n_k} = [-n_k/a_k, n_k/a_k] \cap \mathbb{Z} = \left\{ -\left\lfloor \frac{n_k}{a_k} \right\rfloor, -\left\lfloor \frac{n_k}{a_k} \right\rfloor + 1, \ldots, \left\lfloor \frac{n_k}{a_k} \right\rfloor \right\}.
\]

Furthermore, Chebyshev’s inequality and \(\tilde{N}_k \leq L_k\) imply that
\[
P_\omega\left(\sum_{i=1}^{N_k} \Theta_i > u\nu_{n_k}, \tilde{N}_k \leq L_k\right) \leq P_\omega\left(\sum_{i=1}^{L_k} \Theta_i \mathbb{I}(Z_{i-1} \in J_{n_k}) > u\nu_{n_k}\right)
\leq E_\omega \left[\prod_{i=1}^{L_k} e^{\lambda_k \Theta_i \mathbb{I}(Z_{i-1} \in J_{n_k})}\right] e^{-\lambda_k u\nu_{n_k}},
\]

(2.14)
Let us define $E^x_\omega(m)[\cdot]$ for $x \leq m$ to be the expected value with a walk, starting at site $x$ given a reflection at site $m$ (i.e., $\omega_m = 1$). Recall that, each $\Theta_i$ is a crossing time from $\nu(Z_{i-1}, k)$ to either $\nu(Z_{i-1} - 1, k)$ or $\nu(Z_{i-1} + 1, k)$ whichever the walk visits first. Then, each $\Theta_i$ is less than the crossing time from $\nu(Z_{i-1}, k)$ to $\nu(Z_{i-1} + 1, k)$ with a reflection at $\nu(Z_{i-1} - 1, k)$ for $Z_{i-1} \in \mathbb{Z}$. Therefore, we have

$$E_\omega \left[ e^{\lambda_k \Theta_i} \mathbf{1}_{\{Z_{i-1} \in J_{n_k}\}} \right] = \sum_{j \in J_{n_k}} P(Z_{i-1} = j) \times E_\omega^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] + P(Z_{i-1} \notin J_{n_k})$$

$$\leq \max_{j \in J_{n_k}} E_\omega^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] \quad \text{for any } 1 \leq i \leq L_k.$$  

Thus, conditioning $L_k$ steps of induced birth-death process backward inductively from $L_k$ to 1 results in the following.

$$E_\omega \left[ \prod_{i=1}^{L_k} e^{\lambda_k \Theta_i} \mathbf{1}_{\{Z_{i-1} \in J_{n_k}\}} \right] \leq \left( \max_{j \in J_{n_k}} E_\omega^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] \right)^{L_k}, \quad \text{for any } L_k \geq 1.$$  

(2.15)

In order to control the right site of (2.15), we use the following corollary. (See Section 4.2 for the full proof of Corollary 2.4.4.)

**Corollary 2.4.4** Suppose $m < k_0 < k_1$ for any $m, k_0, k_1 \in \mathbb{Z}$. If $\lambda > 0$ is sufficiently small enough such that

$$e^{-\lambda} - \sinh(\lambda) E^m_{\omega(m)}[T_{k_1}] > 0,$$

where $\sinh(\lambda) = \frac{e^\lambda - e^{-\lambda}}{2}$, then

$$E_{\omega(k_0)}^{k_0}[e^{\lambda T_{k_1}}] \leq \exp \left( \frac{\sinh(\lambda) E^m_{\omega(m)}[T_{k_1}]}{e^{-\lambda} - \sinh(\lambda) E^m_{\omega(m)}[T_{k_1}]} \right).$$

Then in choosing $m = \nu(j - 1, k), k_0 = \nu(j, k)$ and $k_1 = \nu(j + 1, k)$, the right side of inequality in (2.15) has an upper bound in an explicit form. That is, with $\lambda_k > 0$ sufficiently small enough,

$$\max_{j \in J_{n_k}} E_{\omega(\nu(j-1,k))}^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] \leq \max_{j \in J_{n_k}} \exp \left( \frac{\sinh(\lambda_k) (E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}])}{e^{-\lambda_k} - \sinh(\lambda_k) (E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}])} \right),$$

(2.16)
By (2.14) and (2.16), we get
\[
\liminf_{k \to \infty} \frac{1}{n_k^{1-1/s}} \log P_{\omega} \left( \sum_{i=1}^{\hat{N}_k} \Theta_i > u \nu_{n_k}, \hat{N}_k \leq L_k \right)
\leq \liminf_{k \to \infty} \frac{1}{n_k^{1-1/s}} \log \left\{ \max_{j \in J_{\hat{N}_k}} E_{\omega(\nu(j-1,k))}^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] \right\} e^{-\lambda_k u \nu_{n_k}}
\leq \liminf_{k \to \infty} \frac{L_k}{n_k^{1-1/s}} \left( \log \max_{j \in J_{\hat{N}_k}} E_{\omega(\nu(j-1,k))}^{\nu(j,k)} \left[ e^{\lambda_k T_{\nu(j+1,k)}} \right] \right) - \frac{\lambda_k u \nu_{n_k}}{n_k^{1-1/s}}
\leq \liminf_{k \to \infty} \frac{D}{1 - \delta} \left( \max_{j \in J_{\hat{N}_k}} \frac{\sinh \lambda_k (E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}])}{e^{-\lambda_k} - \sinh \lambda_k (E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}])} \right) - \frac{D_0 u \nu_{n_k}}{n_k},
\]
(2.17)

The next step is to find a deterministic bound of \(E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}]\) and \(E_{\omega(\nu(j-1,k))}^{\nu(j-1,k)}[T_{\nu(j+1,k)}]\) in (2.17). Our method is to decompose each previously-mentioned quenched expectation with the sums of crossing times between ladder points. That is
\[
E_{\omega(\nu(j-1,k))}^{\nu(j,k)}[T_{\nu(j+1,k)}] = \sum_{i=j a_k}^{(j+1)a_k-1} E_{\omega(\nu(j-1,k))}^{\nu(i)}[T_{\nu(i+1)}],
\]
and,
\[
E_{\omega(\nu(j-1,k))}^{\nu(j-1,k)}[T_{\nu(j+1,k)}] = \sum_{i=(j-1)a_k}^{(j+1)a_k-1} E_{\omega(\nu(j-1,k))}^{\nu(i)}[T_{\nu(i+1)}].
\]

To simplify the above notation, for any integers \(i, j\) such that \(i \in [(j-1)a_k, (j+1)a_k-1]\) let \(\beta_i^\nu = E_{\omega(\nu(j-1,k))}^{\nu(i)}[T_{\nu(i+1)}]\) be the quenched expected crossing time from \(\nu_i\) to \(\nu_{i+1}\) with a reflection added at \(\nu(j-1, k)\). Then, we classify the sums of \(\beta_i^\nu\) into two groups by the size of \(M_i\) and determine an upper bound of the sums of each group separately. For a fixed \(\epsilon > 0\) we will refer to \(\{i : M_i > n_k^{(1-\epsilon)/s}\}\) and \(\{i : M_i \leq n_k^{(1-\epsilon)/s}\}\) as “big hills” and “small hills.” The following lemma shows that the maximums of sums of centered expected crossing times with a small hill, \(\{M_i \leq n_k^{(1-\epsilon)/s}\}\), are negligible in the limit. (See Section 4.3 and Lemma 4.4.6 for the full proof of Lemma 2.4.5.)

**Lemma 2.4.5** Let us define \(J'_{n_k} = J_{n_k} \cup \{-[n_k/a_k] - 1\}\). Then, for any \(\epsilon_1 > 0\),
\[
Q \left( \max_{j \in J'_{n_k}} \sum_{i=(j) a_k}^{(j+1)a_k-1} (\beta_i^\nu \mathbb{I}_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} - E_Q[\beta_0]) > \frac{\epsilon_1}{2} a_k \right. \text{ i.o.} \left. \right) = 0.
\]
An upper bound of \( \beta_i^j \) corresponding to big hills requires a more careful estimation because \( \beta_i^j \) with the biggest hill dominates all of the other \( \beta_i^j \)'s. The following corollary shows that some subsequence of \( n_k \) exists such that sums of \( \beta_i^j \) corresponding to big hills are bounded above by \( \epsilon' n_k^{1/s} \) for any \( \epsilon' > 0 \). (See Corollary 4.4.9 for the full proof of Corollary 2.4.6.)

**Corollary 2.4.6** Suppose \( 0 < \epsilon < \frac{s-1}{2s} \). Then, for any \( \epsilon' > 0 \),

\[
Q \left( \max_{j \in J_{n_k}} \sum_{i=(j-1)\alpha_k}^{(j+1)\alpha_k-1} \beta_i^j \mathbb{I}_{\{M_i > n_k^{1/s}\}} < \epsilon' n_k^{1/s} \right) = 1.
\]

Then, Lemma 2.4.5 implies that, for any \( \epsilon_1 > 0 \) and \( 0 < \epsilon < \frac{s-1}{2s} \), there is a \( K(\omega) \) such that for all \( k \geq K(\omega) \),

\[
\max_{j \in J_{n_k}} \sum_{i=(j-1)\alpha_k}^{(j+1)\alpha_k-1} \beta_i^j \mathbb{I}_{\{M_i \leq n_k^{1/s}\}} \leq \frac{E_Q[\beta_0] + \epsilon_1/2}{D} n_k^{1/s}, \tag{2.18}
\]

where \( E_Q[\beta_0] = E_Q[T_{\nu_0}] \). On the other hand, by Corollary 2.4.6 with \( \epsilon' = \epsilon_1/D \), we can find an environment dependent subsequence of \( n_k \) defined as \( n_{k'} \), such that

\[
\max_{j \in J_{n_{k'}}} \sum_{i=(j-1)\alpha_{k'}}^{(j+1)\alpha_{k'}-1} \beta_i^j \mathbb{I}_{\{M_i > n_{k'}^{1/s}\}} < \frac{\epsilon_1}{D} n_{k'}^{1/s}. \tag{2.19}
\]

Then, by a choice of \( D > 2(E_Q[\beta_0] + \epsilon_1)D_0 \), we can conclude that (2.17) along with subsequence \( n_{k'} \) is bounded above by

\[
\lim_{k' \to \infty} \frac{D}{1 - \delta} \frac{\sinh(\lambda_{k'}) E_Q[\beta_0] + 3\epsilon_1/2}{D} n_{k'}^{1/s} - \frac{D_0 \nu_{n_{k'}}}{n_{k'}} = \frac{D_0(E_Q[\beta_0] + 3\epsilon_1/2)}{(1 - \delta)(1 - 2D_0(E_Q[\beta_0] + \epsilon_1)/D)} - D_0 E_Q[\nu_1], \quad \text{Q-a.s.,}
\]

where in the last equality we used the fact that \( \sinh(\lambda_{n}) n^{1/s} \to D_0 \) and \( \nu_n/n \to E_Q[\nu_1] \) as \( n \to \infty \), Q-a.s. In summary, we have shown that for any \( D_0, \epsilon_1, \delta > 0 \) and for all sufficiently large \( D < \infty \),

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega}(T_{\nu_n} > u \nu_n) \leq \frac{D_0(E_Q[\beta_0] + 3\epsilon_1/2)}{(1 - \delta)(1 - 2D_0(E_Q[\beta_0] + \epsilon_1)/D)} - D_0 E_Q[\nu_1], \quad \text{Q-a.s.}
\]
By first taking $D \to \infty$ and then letting $\epsilon_1, \delta \to 0$, we can thus conclude that

$$\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{\nu_n} > u\nu_n) \leq D_0 E_Q[\nu_1] \left( \frac{E_Q[\beta_0]}{E_Q[\nu_1]} - u \right), \quad Q\text{-a.s.,} \quad (2.20)$$

for any $D_0 < \infty$. Finally, since

$$\frac{1}{\nu_a} = \lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{T_{\nu_n}}{\nu_n} = \lim_{n \to \infty} \frac{T_{\nu_n} n}{\nu_n} = \frac{E_Q[\beta_0]}{E_Q[\nu_1]},$$

it follows that the term in parenthesis in (2.20) is negative for $u > 1/\nu_a$, and thus the right side of (2.20) can be made smaller than any negative number by choosing sufficiently large $D_0$.  

\[ \blacksquare \]
3. Bounds of Slowdown Asymptotic Hitting Time with I.I.D. Random Mean Waiting Times

In this chapter, we will consider the asymptotic result of a random walk with the i.i.d sequence of mean waiting times on \( \mathbb{Z} \). The random walk is defined as follows. Starting from the origin, \( X_0 = 0 \), the walk always moves one step to the right and waits on each site for an exponentially distributed time with a random mean before moving to the next site. Note that the crossing time from one site to a neighbor site under this model is independent under the annealed measure as each crossing time only depends on the waiting time at the current visiting site. However, this is no longer true under the general RWRE on \( \mathbb{Z} \) because a walk may go to sites left of the current visiting site, and so \( \tau_i \) depends on \( \omega_x \) for all \( x \leq i \). However, this study is still a helpful resource because we will use some of proof’s techniques and apply them to our main result. Hence, we consider this study as a foundation for our main result under the original RWRE on \( \mathbb{Z} \).

3.1 Background and Statement of Results

In this section, we provide a list of notations and assumptions and then summarize our main results. Suppose \( T_i \) is the first time the walk reaches the site \( i \) such that \( T_i = \inf\{t \geq 0 : X_t = i\} \). Denote \( \tau_i \) as the waiting time on site \( i \). That is,

\[
\tau_i = T_{i+1} - T_i \quad \text{for } i \geq 0.
\]

Because there are two different randomnesses (the waiting time and its mean) accounted in the random walk, we construct two probability measures that are similar to the measures defined in the general RWRE. First, let \( \omega = \{\omega_i\} \in (0, \infty)^\mathbb{Z} \) be a sequence of independent, identically distributed random variables called the en-
vironment, and let $P$ be an i.i.d. product measure of environments on the space $(0, \infty)^\mathbb{Z}$. Also, under the measure $P$, we assume that $\omega_i$ is a non-degenerate heavy tailed random variable. Specifically, $s > 1$ and some constant $C > 0$ exist such that

$$ P(\omega_i > x) \sim \frac{C}{x^s} \text{ for large } x > 0 \text{ and } i \in \mathbb{N}. \quad (3.1) $$

If a walk $X_n$ is generated under a particular environment $\omega$, the corresponding law is called quenched law denoted by $P_\omega(\cdot)$, and its expectation is denoted by $E_\omega[\cdot]$. Without conditioning of the environments, the law of the walk is called an annealed law denoted by $P$ such that $P(\cdot) = E_P[P_\omega(\cdot)]$ and $E[\cdot] = E_P[E_\omega[\cdot]]$ where $E_P[\cdot]$ is the expectation with respect to $P$ on the environments. Also, we have another assumption that the waiting time at each site $i$, $\tau_i$, is exponentially distributed with a parameter $1/\omega_i$. The following list summarizes our assumptions which apply to many of the equations we use throughout this chapter.

**Assumption 8** \{\omega_i\} $\in (0, \infty)^\mathbb{Z}$ is a sequence of non-degenerate i.i.d. random variables with a continuous distribution under $P$ satisfying (3.1).

**Assumption 9** Given an environment $\omega$, \{\tau_i\}_{i \geq 0} is a sequence of the exponential random variable with parameter $1/\omega_i$ under $P_\omega$.

Assumption 9 implies that the quenched mean of waiting time denoted by $E_\omega[\tau_i]$ is $\omega_i$. Hence from (3.1), we have the asymptotic probability tail of $E_\omega[\tau_i]$ under $P$ such that

$$ P(E_\omega[\tau_0] > x) \sim \frac{C}{x^s} \text{ for large } x > 0. \quad (3.2) $$

Note that \{\tau_i\}_{i \geq 0} is independent under $P_\omega$ and $P$ as each $\tau_i$ is independent to the environment of any other sites $j \neq i$. Also, note that (3.2) implies the finite first moment of $\tau_i$ almost surely under $P$ since $E[\tau_i] = E_P[E_\omega[\tau_i]] = E_P[E_\omega[\tau_0]] = E_P[\omega_0] \in (0, \infty)$. As a result, \{\tau_i\}_{i \geq 0} is an i.i.d sequence under $P$ and, by the law of large number,

$$ \lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \tau_i}{n} = E[\tau_0] < \infty. \quad P - a.s. $$
Then, the following theorem regards an asymptotic relation between the hitting time $T_n$ and the walk $X_n$ which can be obtained easily by the renewal theory and the law of large number.

**Theorem 3.1.1** Under Assumption 8, suppose $\lim_{n \to \infty} T_n/n = 1/v_P$ for some constant $1/v_P \in (0, \infty)$. Then, the random walk $X_n$ has a positive speed of $v_P > 0$ such that

$$\lim_{n \to \infty} \frac{X_n}{n} = v_P.$$ 

Theorem 3.1.1 shows that the random walk is transient with positive speed. It turns out that there is a commonality between this model and the result of the transient walk with positive speed under the general RWRE. That is, the LDP of the hitting time $T_n$ under our toy model results in similar way to the LDP of $X_n$ and $T_n$ under the general transient RWRE with positive speed. The following result is our main goal for this section.

**Theorem 3.1.2** Under Assumption 8, suppose $\lim_{n \to \infty} \frac{T_n}{n} = \frac{1}{v_P}$. Then for any $u > 1/v_P$, we have

$$\limsup_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega \left( \frac{T_n}{n} > u \right) = 0,$$

$$(3.3)$$

and

$$\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega \left( \frac{T_n}{n} > u \right) = -\infty.$$

$$(3.4)$$

Note that we will use $c, c', C, C', \ldots$ as generic positive constants whose values are not important and may differ by one usage to another, and use $C_0, C_1, C_2, \ldots$ as constants constructed for a specific usage.
3.2 The Quenched Subexponential Tail of Sums of I.I.D. Expected Waiting Time

This sub-section contains the full proof of Theorem 3.1.2. The proof begins by decomposing the hitting time \( T_n \) by the series of waiting times \( \tau_i \). That is,

\[
T_n = \sum_{i=0}^{n-1} \tau_i.
\]

So for any \( u > 1/v_p \),

\[
P_\omega(T_n > un) = P_\omega\left( \sum_{i=0}^{n-1} \tau_i > un \right). \tag{3.5}
\]

Next, we show that the lower and upper bounds of (3.5) in forms of \( \max_{0 \leq i \leq n-1} E_\omega[\tau_i] \) becomes the main factor that determines subsequences to get (3.3) and (3.4). Regarding the lower bound, recall that each \( \tau_i \) is exponentially distributed with a parameter \( 1/\omega_j = 1/E_\omega[\tau_i] \) and \( \tau_j < \sum_{i=0}^{n-1} \tau_i \) for any \( 0 \leq j \leq n-1 \). Hence,

\[
P_\omega\left( \sum_{i=0}^{n-1} \tau_i > un \right) \geq \max_{0 \leq i \leq n-1} P_\omega(\tau_i > un)
\]

\[
= \max_{0 \leq i \leq n-1} \exp\left( -\frac{un}{E_\omega[\tau_i]} \right)
\]

\[
= \exp\left( -\frac{un}{\max_{0 \leq i \leq n-1} E_\omega[\tau_i]} \right). \tag{3.6}
\]

Regarding the upper bound, we use the Chebyshev’s inequality and independence of \( \tau_i \) under \( P_\omega \) such that for any \( \lambda > 0 \),

\[
P_\omega\left( \sum_{i=0}^{n-1} \tau_i > un \right) = P_\omega\left( e^{\lambda \sum_{i=0}^{n-1} \tau_i} > e^{\lambda un} \right)
\]

\[
\leq E_\omega\left[ e^{\lambda \sum_{i=0}^{n-1} \tau_i} \right] e^{-\lambda un} = \prod_{i=0}^{n-1} E_\omega\left[ e^{\lambda \tau_i} \right] e^{-\lambda un}. \tag{3.7}
\]

Recall that each \( \tau_i \) is equivalent to the exponential random variable with a mean \( E_\omega[\tau_i] \). Hence, suppose that \( \lambda \max_{0 \leq i \leq n-1} E_\omega[\tau_i] < 1 \), and we have the following equality.

\[
\max_{0 \leq i \leq n-1} E_\omega\left[ e^{\lambda \tau_i} \right] = \max_{0 \leq i \leq n-1} \frac{1}{1 - \lambda E_\omega[\tau_i]} = \frac{1}{1 - \lambda \max_{0 \leq i \leq n-1} E_\omega[\tau_i]} \tag{3.8}
\]
Then, from (3.7) and (3.8),

$$P_\omega \left( \sum_{i=0}^{n-1} \tau_i > u n \right) \leq e^{-\lambda u n} \left( \frac{1}{1 - \lambda \max_{0 \leq i \leq n-1} E_\omega[\tau_i]} \right)^n.$$  \hfill (3.9)

By observing (3.6) and (3.9), the key to prove Theorem 3.1.2 is to find the size of $\max_{0 \leq i \leq n-1} E_\omega[\tau_i]$. The following lemma shows that $O(n^{1/s})$ is on the right size.

**Lemma 3.2.1** Under the same assumption as Theorem 3.1.2, for any constant $C_0, C_1 > 0$,

$$P \left( \max_{0 \leq i \leq n-1} E_\omega[\tau_i] > C_0 n^{1/s} \ i.o. \right) = 1,$$  \hfill (3.10)

and,

$$P \left( \max_{0 \leq i \leq n-1} E_\omega[\tau_i] < C_1 n^{1/s} \ i.o. \right) = 1.$$  \hfill (3.11)

We will postpone the proof of Lemma 3.2.1 until the end. First, we will complete the remaining proof of Theorem 3.1.2 given that Lemma 3.2.1 is true.

**Proof** [Proof of Theorem 3.1.2] For the proof of (3.3), we have (3.6) and (3.10), and so we can find a subsequence $n_k$ such that

$$P_\omega(T_{n_k} > u n_k) \geq \exp \left( -\frac{u n_k}{\max_{0 \leq i \leq n_k-1} E_\omega[\tau_i]} \right) \geq \exp \left( -\frac{u n_k}{C_0 n_k^{1/s}} \right) \geq \exp \left( -\frac{u n_k^{1-1/s}}{C_0} \right).$$

Then,

$$\limsup_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_n > u n) \geq \limsup_{k \to \infty} \frac{1}{n_k^{1-1/s}} \log P_\omega(T_{n_k} > u n_k) \geq -\frac{u}{C_0}.$$  \hfill \(3.12\)

Finally, the last term can be made greater than any negative number by choosing $C_0$ sufficiently large, thereby completing the proof of (3.3).

For the proof of (3.4), recall from our assumption that $u > \frac{1}{v_n}$. Accordingly, choose $\eta > 1$ such that $u > \eta \cdot \frac{1}{v_n}$ still holds. Then, we choose $0 < \epsilon < 1$ small enough such that for any $\epsilon'$ with $0 < \epsilon' < \epsilon$,

$$\frac{1}{1 - \epsilon'} \leq e^{\eta \epsilon'}.$$  \hfill (3.12)
Also, we define \( \lambda = \lambda_n = C_2 n^{-1/s} \) for the fixed constant \( C_2 > 0 \). By (3.11) in Lemma 3.2.1, a subsequence \( n_{k'} \) exists such that \( E_{\omega} [\tau_i] < \frac{\epsilon}{C_2 n_{k'}}^{1/s} \) for all \( 0 \leq i \leq n_{k'} - 1 \). In applying (3.12) to the product of exponential moments of the crossing time in (3.7) given that \( \lambda_k = \lambda_{n_k} \),

\[
\prod_{i=0}^{n_{k'}-1} E_{\omega} [e^{\lambda_k \tau_i}] = \prod_{i=0}^{n_{k'}-1} \frac{1}{1 - \lambda_k E_{\omega} [\tau_i]} \leq \prod_{i=0}^{n_{k'}-1} \exp (\eta \lambda_k E_{\omega} [\tau_i]) = \exp (\eta \lambda_k E_{\omega} [T_{n_{k'}}]) .
\]

Therefore,

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega} (T_n > un) \leq \liminf_{k' \to \infty} \frac{1}{n_{k'}^{1-1/s}} \log P_{\omega} (T_{n_{k'}} > un_{k'}) \\
\leq \liminf_{k' \to \infty} \frac{1}{n_{k'}^{1-1/s}} \log \exp (\eta \lambda_k E_{\omega} [T_{n_{k'}}] - \lambda_k un_{k'}) \\
= \liminf_{k' \to \infty} \frac{\lambda_k n_{k'}^{1/s}}{n_{k'}} (\eta E_{\omega} [T_{n_{k'}}] - un_{k'}) \\
= \liminf_{k' \to \infty} C_2 \left( \frac{\eta E_{\omega} [T_{n_{k'}}]}{n_{k'}} - u \right) .
\]

Since \( \lim_{n \to \infty} \frac{E_{\omega} [T_n]}{n} = \frac{1}{v_\alpha} \), we finally have

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega} (T_n > un) \leq C_2 \left( \frac{\eta}{v_\alpha} - u \right) . \tag{3.13}
\]

The term in parenthesis in (3.13) is negative, and thus the right side of (3.13) can be made smaller than any negative number by choose a sufficiently large \( C_2 \).

We finish this section with the proof of Lemma 3.2.1 by applying the second Borel-Cantelli Lemma. In order to do that, we need the pick a subsequence of events inside probability (3.10) and (3.11) that are independent, and each probability sum is not summable. Hence, we define the subsequence

\[ n_k := k^{2k}, \quad k \geq 0 \]

Recall that \( E_{\omega} [\tau_i] = \omega_i \), so we will use \( \omega_i \) instead to simplify our notation. Given our assumption that the sequence \( \{\omega_i\}_{0 \leq i \leq n_k - 1} \) is independent, a following lemma shows that the largest quenched expected waiting time \( \omega_i \) among \( 0 \leq i \leq n_k - 1 \) occurs from \( i \in [n_{k-1}, n_k - 1] \).
Lemma 3.2.2 Under the same assumption as Theorem 3.1.2

\[ P \left( \max_{0 \leq i \leq n_k - 1} \omega_i \neq \max_{n_{k-1} \leq i \leq n_k - 1} \omega_i \ i.o \right) = 0. \]

Proof Since \( \omega_i \) is independent and \( \{ \omega_i \}_{n_{k-1} \leq i \leq n_k - 1} \subset \{ \omega_i \}_{0 \leq i \leq n_k - 1} \),

\[
\limsup_{k \to \infty} k^2 P \left( \max_{0 \leq i \leq n_k - 1} \omega_i \neq \max_{n_{k-1} \leq i \leq n_k - 1} \omega_i \right)
\]

\[= \limsup_{k \to \infty} k^2 \cdot \frac{n_{k-1}}{n_k} = \limsup_{k \to \infty} k^2 \cdot \frac{(k - 1)^{2(k-1)}}{k^{2k}} = \frac{1}{e}. \]

Then the sum of \( P(\max_{0 \leq i \leq n_k - 1} \omega_i \neq \max_{n_{k-1} \leq i \leq n_k - 1} \omega_i) \) over \( k \geq 0 \) is finite and the first Borel-Cantelli Lemma concludes the proof.

Finally, we have the completed proof of Lemma 3.2.1.

Proof [Proof of Lemma 3.2.1] Observe that (3.10) and (3.11) are the probabilities of events occurred within some open sets on \((0, \infty)\). Also by Lemma 3.2.2, there exists \( \omega \)-dependent large \( k_0(\omega) \) such that for any \( k > k_0(\omega) \)

\[
\max_{0 \leq i \leq n_k - 1} \omega_i = \max_{n_{k-1} \leq i \leq n_k - 1} \omega_i. \tag{3.14}
\]

Therefore, it is enough to prove that for any open set \( A \in (0, \infty) \)

\[ P \left( \max_{n_{k-1} \leq i \leq n_k - 1} \frac{\omega_i}{n_k^{1/s}} \in A \ i.o. \right) = 1. \tag{3.15} \]

Notice that the event inside the probability in (3.15) is independent of the environment \([0, n_{k-1} - 1]\) for each \( k \in \mathbb{N} \). Therefore, the events (3.15) are an independent sequence. Hence, in order to prove (3.15) using the second Borel-Cantelli Lemma, it is enough to show that

\[
\sum_{k=1}^{\infty} P \left( \max_{n_{k-1} \leq i \leq n_k - 1} \frac{\omega_i}{n_k^{1/s}} \in A \right) = \infty. \tag{3.16}
\]

To prove (3.16), note that

\[ P \left( \max_{n_{k-1} \leq i \leq n_k - 1} \frac{\omega_i}{n_k^{1/s}} \in A \right) = P \left( \max_{n_{k-1} \leq i \leq n_k - 1} \left( \frac{n_k - n_{k-1}}{n_k} \right)^{1/s} \frac{\omega_i}{(n_k - n_{k-1})^{1/s}} \in A \right). \]
Notice that \( \frac{n_k - n_{k-1}}{n_k} = \frac{k^{2k-(k-1)2(k-1)}}{k^{2k}} \rightarrow 1 \) as \( k \rightarrow \infty \). Hence for a large \( k > 0 \), we have a non-empty open set \( A' \subset A \) such that

\[
P\left( \max_{n_{k-1} \leq i \leq n_k-1} \frac{\omega_i}{n_k^{1/s}} \in A \right) \geq P\left( \max_{n_{k-1} \leq i \leq n_k-1} \frac{\omega_i}{(n_k - n_{k-1})^{1/s}} \in A' \right). \quad (3.17)
\]

By the tail asymptotic in (3.1), it follows from [22, Proposition 1.15] that

\[
\max_{n_{k-1} \leq i \leq n_k-1} \frac{\omega_i}{(n_k - n_{k-1})^{1/s}} \text{ converges weakly with an extreme value distribution with cumulative density function } \Phi_s(x) \text{ such that }
\]

\[
\Phi_s(x) = \begin{cases} 
0 & x < 0 \\
\exp(-x^{-s}) & x \geq 0.
\end{cases}
\]

Hence, the probabilities in (3.17) are uniformly bounded away from 0 for all \( k \), and thus (3.16) follows. \( \blacksquare \)
4. Oscillation of Quenched Slowdown Asymptotics of
RWRE in \( \mathbb{Z} \)

This chapter consists of the article *Oscillation of Quenched Slowdown Asymptotic of RWRE in \( \mathbb{Z} \)*, by Sung Won Ahn and Jonathon Peterson, which was published by the Electronic Journal of Probability. Before introducing our main result, we revisit necessary assumptions introduced in the earlier chapters.

**Assumption 10** The distribution \( \alpha \) on environments is such that \( E_{\alpha}[\log \rho_0] < 0 \) and \( E_{\alpha}[\rho_0^s] = 1 \) for some \( s > 1 \).

Additionally, we will also need the following assumption.

**Assumption 11** The distribution of \( \log \rho_0 \) is non-lattice under \( \alpha \) and that \( E_{\alpha}[\rho_0^s \log \rho_0] < \infty \).

Under these assumptions, this article contains the full proof of Theorem 2.4.1. In order to keep this chapter self-contained, we show the full proof in the article with relatively no changes. Although major notations and backgrounds used in this chapter are already introduced in Chapter 1 and Chapter 2, we repeat the notations and backgrounds (with additional technical lemmas) in the following section for the reader.

4.1 Notations and Backgrounds

First, we restate notations introduced in Chapter 2 for readers. Some notation will be used throughout the remainder of the paper. Recall that for an environment \( \omega = (\omega_x)_{x \in \mathbb{Z}} \), we have defined \( \rho_x = \frac{1-\omega_x}{\omega_x} \), and for integers \( i \leq j \) we define

\[
\Pi_{i,j} := \prod_{k=i}^{j} \rho_k, \quad W_{i,j} := \sum_{k=i}^{j} \Pi_{k,j}, \quad R_{i,j} := \sum_{k=i}^{j} \Pi_{i,k}
\]
\[ W_j := \sum_{k \leq j} \Pi_{k,j}, \quad R_i := \sum_{k=1}^{\infty} \Pi_{i,k}. \]

(Note that \( W_i \) and \( R_i \) are finite for all \( i \in \mathbb{Z} \) with probability one if \( E_\alpha [\log \rho_0] < 0 \).)

And,

\[ P^x_\omega (T_j < T_i) = \frac{R_{i,x-1}}{R_{i,j-1}}, \quad \text{and} \quad P^x_\omega (T_j > T_i) = \frac{\Pi_{i,x-1} R_{x,j-1}}{R_{i,j-1}}. \]

We will use these notations frequently in the next sections in order to simplify various expressions under the quenched law. In particular, note that we can obtain a quenched expectation of \( \tau_i = T_{i+1} - T_i \) (the time to cross from \( i \) to \( i + 1 \)) by

\[ E_\omega [\tau_i] = 1 + 2W_i, \quad (4.1) \]

which is derived from [23, (2.1.7) and (2.1.8)]. Also, we note that throughout paper, we will use \( c, c', C, C', ... \) as generic positive constants whose values are not important and may differ by one usage to another, and use \( C_0, C_1, C_2, ... \) as constants constructed for a specific usage. Throughout this section, we will use the method introduced by Sinai of the “potential” of an environment which allows us to visualize the environment as a sequence of “valleys” [21]. This technique was originally developed by Sinai to study the limiting distributions of recurrent RWRE but has also shown to be useful for transient RWRE [17], [18], [8], [19]. For a fixed environment \( \omega \), let the potential \( V(x) \) be the function

\[ V(x) = \begin{cases} 
\sum_{i=0}^{x-1} \log \rho_i & \text{if } x \geq 1 \\
0 & \text{if } x = 0 \\
-\sum_{i=-x}^{-1} \log \rho_i & \text{if } x \leq -1.
\end{cases} \]

The potential \( V(x) \) enables us to cut an environment into blocks by “ladder points”, \( \{\nu_i, i \in \mathbb{Z}\} \), defined by

\[ \nu_0 = \sup\{y \leq 0 : V(y) < V(k), \forall k < y\}, \quad (4.2) \]

and for \( i \geq 1,

\[ \nu_i = \inf\{x > \nu_{i-1} : V(x) < V(\nu_{i-1})\}, \quad \text{and} \quad \nu_{-i} = \sup\{y < \nu_{-i+1} : V(y) < V(k), \forall k < y\}. \]
Equivalently,
\[ \nu_0 = \sup \{ y \leq 0 : \Pi_{k,y-1} < 1, \forall k < y \} \]
and, for \( i \geq 1 \),
\[ \nu_i = \inf \{ x > \nu_{i-1} : \Pi_{\nu_{i-1},x-1} < 1 \}, \quad \text{and} \quad \nu_{-i} = \sup \{ y < \nu_{-i+1} : \Pi_{k,y-1} < 1, \forall k < y \}. \]

Figure 4.1 is an example of the locations of ladder points on \( \mathbb{Z} \). Let us denote the length between consecutive ladder points by
\[ l_i = \nu_{i+1} - \nu_i, \quad i \in \mathbb{Z}, \]
and the exponential height of the potential between the ladder points by
\[ M_i := \max \{ \Pi_{\nu_i,j} : \nu_i \leq j \leq \nu_{i+1} \} = \max \{ e^{V(j)-V(\nu_i)} : \nu_i < j \leq \nu_{i+1} \}, \quad i \in \mathbb{Z}. \]

This exponential height has a crucial role in our analysis because our result shows that the quenched expectation of the crossing times on sections with “big” \( M_i \) determines which subsequence to take for Theorem 4.1.1 to be satisfied. Also, we will show that the sums of the quenched expectation of crossing times on sections with a “small” \( M_i \) is negligible in the limit.
The ladder points of the environment form a convenient structure for studying the hitting times of the random walk. Since the environment is i.i.d. under the measure \( \alpha \), it follows that the blocks of the environment between adjacent ladder points \( \mathcal{B}_i = \{ \omega_x : x \in [\nu_i, \nu_{i+1}) \} \) are i.i.d. for \( i \neq 0 \). In particular, \( \{ l_i \}_{i \neq 0} \) and \( \{ M_i \}_{i \neq 0} \) are both i.i.d. sequences of random variables. However, the interval of environment between the ladder points on either side of the origin has a different distribution. In particular, under the measure \( \alpha \), the random variables \( l_0 \) and \( M_0 \) have a different distribution that \( l_i \) and \( M_i \) with \( i \neq 0 \). For this reason, it is convenient to at times work with a related measure on environments \( Q \) given by

\[
Q(\cdot) = \alpha(\cdot | \nu_0 = 0).
\]

The sequence \( \{ \omega_x \}_{x \in \mathbb{Z}} \) is no longer i.i.d. under the measure \( Q \), but this distribution has the convenient property that the environment is stationary under shifts of the ladder points of the environment. More precisely, if \( \theta \) is the natural left-shift operator on environments given by \( (\theta \omega)_x = \omega_{x+1} \), then for any \( k \in \mathbb{Z} \) the environments \( \omega \) and \( \theta^k \omega \) have the same distribution under \( Q \). Moreover, under the measure \( Q \) the blocks between adjacent ladder points \( \mathcal{B}_i \) are i.i.d. for all \( i \in \mathbb{Z} \) with each having the same distribution as \( \mathcal{B}_1 \) under the original measure \( \alpha \) on environments. In particular, this implies that \( \{ l_i \}_{i \in \mathbb{Z}} \) and \( \{ M_i \}_{i \in \mathbb{Z}} \) are both i.i.d. sequences under the measure \( Q \).

The distribution \( Q \) was first introduced in [17], and we will frequently refer to estimates under the measure \( Q \) that were proved in this section. We mention here a few of these that we will use throughout the remainder of the paper. First of all, under the measure \( Q \) the distances \( l_i \) between ladder points have exponential tails. That is, there exist constants \( C, C' > 0 \) such that

\[
Q(l_i > x) \leq Ce^{-C'x}. \tag{4.3}
\]

Secondly, it follows from a result of Iglehart [24, Theorem 1] that there exists a constant \( C_0 > 0 \) such that

\[
Q(M_i > x) \sim C_0 x^{-s}, \quad \text{as } x \to \infty. \tag{4.4}
\]
(Note that it follows from this asymptotic statement that \( Q(M_i > x) \leq Cx^{-s} \) for all \( x > 0 \) and some \( C > 0 \). At times we will use this upper bound rather than the asymptotics in (4.4).) One of the main ideas that will be used throughout the paper is that the expected time for the random walk to cross between adjacent ladder points \( E^{\nu_i}[T_{\nu_{i+1}}] \) is roughly comparable to the exponential height of the potential \( M_i \) between the ladder points. Thus, we expect that \( E_\omega[T_{\nu_1}] \) also has polynomial tails similar to (4.4). Indeed, it was shown in [17] that

\[
Q(E_\omega[T_{\nu_1}] > x) \sim K_{\infty}x^{-s}, \quad \forall x \geq 0,
\]

for some \( K_{\infty} > 0 \).

Under the previously mentioned assumptions, our main result is the following.

**Theorem 4.1.1** If Assumptions 10 and 11 hold, then for any \( v \in (0, v_\alpha) \),

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega \left( \frac{X_n}{n} < v \right) = -\infty, \quad \alpha - a.s. \tag{4.6}
\]

We conclude the introduction with an overview of the proof of Theorem 4.1.1. The proof of the quenched slowdown asymptotics for the hitting times is structured as follows. In Section 4.2, we give an explicit upper bound of the quenched moment generating function of the hitting times with a one way node placed on a site to the left of the starting point. This explicit form shows that the sums of quenched expected time between ladder locations control the quenched subexponential tail of hitting times. In Section 4.3, we will show the sums of the quenched expected crossing time between ladder locations with “small” \( M_i \) are negligible in the limit under a measure \( Q \). Finally, in Section 4.4 we will prove the needed quenched asymptotics of slowdown probabilities for hitting times to complete the proof of Theorem 4.1.1.

**4.2 The Moment Generating Function of Hitting Times with an added Reflection Point**

In this section, we show an upper bound of the quenched moment generating function of hitting time with a reflection point. We say a site \( x \) is a reflection point
if $\omega_x = 1$. Under our assumptions, if $\alpha(\omega_0 = 1) = 0$ (that is, there are no reflection points in the environment) then for $\alpha$-a.e. environment $\omega$ the moment generating function $E_\omega[e^{\lambda \tau}] = \infty$ for all $\lambda > 0$ [12]. However, if we place a reflection point to the left of the starting point of the random walk then the moment generating function is finite for small enough $\lambda > 0$ and we will give an upper bound for this modified moment generating function. For any environment $\omega$ and any $m \in \mathbb{Z}$, let $\omega(m)$ be the environment $\omega$ modified by adding a reflection point at $m$. That is,

$$\omega(m)_x = \begin{cases} 
\omega_x & x \neq m \\
1 & x = m.
\end{cases}$$

The main result in this section is the following lemma which gives an upper bound on quenched moment generating functions of hitting times with a reflection point added to the left of the starting point.

**Lemma 4.2.1** Let $m \leq n$. If $\lambda$ is small enough such that

$$e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{n+1}] - (n + 1 - m) \right) > 0$$

(4.7)

where $\sinh(\lambda) = \frac{e^{\lambda} - e^{-\lambda}}{2}$, then for all $m \leq k \leq n$,

$$E^{m}_{\omega(m)}[e^{\lambda \tau_k}] \leq e^\lambda \frac{e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k}] - (k - m) \right)}{e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k+1}] - (k + 1 - m) \right)}.$$  

(4.8)

**Remark 3** Since $E^{m}_{\omega(m)}[T_{n+1}] - (n + 1 - m) = \sum_{k=m}^{n} (E^{m}_{\omega(m)}[\tau_k] - 1)$ is non-decreasing in $n$, if $\lambda > 0$ is such that (4.7) holds then it follows that

$$e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k+1}] - (k + 1 - m) \right) > 0 \text{ for all } m \leq k \leq n,$$

and this is the condition that will be used in the proof below to obtain the upper bound (4.8).

**Proof** Clearly, it is enough to prove the statement of the lemma when $m = 0$. Therefore, for convenience of notation, let $g(k) = E^{m}_{\omega(0)}[e^{\lambda \tau_k}]$ for $k \geq 0$. We need to show that

$$g(k) \leq e^\lambda \frac{e^{-\lambda} - \sinh(\lambda) (E^{m}_{\omega(0)}[T_{k}] - k)}{e^{-\lambda} - \sinh(\lambda) (E^{m}_{\omega(0)}[T_{k+1}] - (k + 1))},$$

for $0 \leq k \leq n$,  

(4.9)
whenever $\lambda$ is small enough so that
\[ e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{n+1}] - n - 1) > 0. \] (4.10)

For $n = k = 0$, $g(0) = e^\lambda$ because a reflection point to the right is placed at a site 0. Thus, (4.9) clearly holds when $n = 0$ and so we need only to consider $n \geq 1$. For any $1 \leq k \leq n$, let us decompose $\tau_k$ into a series of crossing times from $k - 1$ to $k$ before reaching $k + 1$. Let $N$ be a number of times a walk steps from $k$ to $k - 1$ before stepping from $k$ to $k + 1$. Then, $N$ is a geometric random variable with a success probability of $\omega_k$ and
\[ \tau_k = N + 1 + \sum_{i=1}^{N} \tau_{k-1}^{(i)} \] in distribution,
where $\tau_{k-1}^{(i)}$ is an independent copy of $\tau_{k-1}$ for each $i$. Therefore, we have that
\[
g(k) = \sum_{n=0}^\infty E_{\omega(0)} \left[ e^{(N+1+\sum_{i=1}^{N} \tau_{k-1}^{(i)})} \right] P(N = n)
= \sum_{n=0}^\infty e^{(n+1)} g(k-1)^n (1 - \omega_k)^n \omega_k
= \omega_k e^\lambda \sum_{n=0}^\infty ((1 - \omega_k) e^\lambda g(k-1))^n.
\]
Here, we claim the following statement and postpone its proof until the end that (4.10) is a sufficient condition for
\[
(1 - \omega_k) e^\lambda g(k-1) < 1, \quad 1 \leq k \leq n. \quad (4.11)
\]
Then with a sufficiently small $\lambda$, we obtain a representation of the moment generating function introduced in terms of continued fraction or
\[
g(k) = \frac{\omega_k e^\lambda}{1 - (1 - \omega_k) e^\lambda g(k-1)}, \quad 1 \leq k \leq n. \quad (4.12)
\]
Using (4.12), we will give a proof of (4.9) by induction in $k$. If $k = 1$, then
\[
g(1) = \frac{\omega_1 e^\lambda}{1 - (1 - \omega_1) e^\lambda g(0)} = \frac{\omega_1 e^\lambda}{1 - (1 - \omega_1) e^{2\lambda}}
= \frac{\omega_1 e^\lambda}{e^{-\lambda} + \rho_1 e^{-\lambda} - \rho_1 e^\lambda}
= \frac{1}{e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_2] - 2)},
\]
where the last equality is obtained by noting that (4.1) implies \( E_{\omega(0)}[T_2] = 2 + 2\rho_1 \). Suppose that the inequality in (4.9) holds for \( g(k-1) \) such that

\[
g(k-1) \leq \frac{e^\lambda - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))}{e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k)}
\]

Then,

\[
g(k) = \frac{\omega_k e^\lambda}{1 - (1 - \omega_k) e^\lambda g(k-1)}
\]

\[
= \frac{e^\lambda}{1 + \rho_k - \rho_k e^\lambda g(k-1)}
\]

\[
< \frac{e^\lambda}{1 + \rho_k - \rho_k e^\lambda \cdot e^\lambda \left( \frac{e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))}{e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k)} \right)}
\]

\[
= \frac{e^\lambda}{1 + \rho_k - \rho_k \left( \frac{e^\lambda - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))}{e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k)} \right)}
\]

\[
\leq \frac{e^\lambda (e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k))}{(1 + \rho_k)(e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) - \rho_k(e^\lambda - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))))}
\]

(4.13)

The proof of (4.8) will then be complete if we can show the denominator in (4.13) is equal to the denominator in (4.9). To this end, note that (4.1) implies that

\[
E_{\omega(0)}[T_k] = k + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j} \Pi_{i,j}.
\]

(4.14)
the above proof shows that the inequality (4.9) holds for $g$. This verifies (4.11) for $k$. Finally, it remains to prove that (4.10) implies (4.11). The proof uses a mathematical induction in $k$ which is very similar to the proof of (4.9). If $k = 1$, (4.10) and Remark 3 implies

$$ e^{-\lambda} > \sinh(\lambda)(E_{\omega(0)}[T_2] - 2) = (e^\lambda - e^{-\lambda})\rho_1. $$

Since $\rho_1 = (1 - \omega_1)/\omega_1$ and $g(0) = e^\lambda$, this is equivalent to

$$ 1 > e^{2\lambda}(1 - \omega_1) = e^\lambda g(0)(1 - \omega_1). $$

This verifies (4.11) for $k = 1$. Suppose now that (4.11) holds up to $k - 1 < n$. Then, the above proof shows that the inequality (4.9) holds for $g(k - 1)$. Therefore,

$$ 1 - e^\lambda g(k - 1)(1 - \omega_k) $$

$$ \geq 1 - e^\lambda(1 - \omega_k)e^\lambda e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1)) $$

$$ e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) $$

$$ = e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1)) $$

$$ - e^\lambda(1 - \omega_k)(e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))) $$

$$ e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) $$

$$ \geq e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) - (1 - \omega_k)(e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))) $$

$$ e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) $$

$$ \geq \omega_k(1 + \rho_k)(e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k)) - \rho_k(e^\lambda - \sinh(\lambda)(E_{\omega(0)}[T_{k-1}] - (k - 1))) $$

$$ e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k) $$

$$ = \omega_k e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k+1}] - (k + 1)) $$

$$ = e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_k] - k), $$
where the last equality comes from (4.15). Since $E_\alpha[\log \rho_0] < 0$ implies $\omega_k > 0$ and Remark 3 implies
\[
e^{-\lambda} - \sinh(\lambda)(E_{\omega(0)}[T_{k+1}] - (k + 1)) > 0,
\]
we get that $1 > e^\lambda g(k - 1)(1 - \omega_k)$. □

As a corollary of Lemma 4.2.1 we obtain the following upper bound for the quenched moment generating function of the time to cross an interval with a reflection point at some point to the left of $X_0$.

**Corollary 4.2.2** Suppose $m < k_0 < k_1$ for any $m, k_0, k_1 \in \mathbb{Z}$. If $\lambda > 0$ is sufficiently small enough such that
\[
e^{-\lambda} - \sinh(\lambda) E_{\omega(m)}^m[T_{k_1}] > 0,
\]
then,
\[
E_{\omega(m)}^{k_0}[e^{\lambda T_{k_1}}] \leq \exp \left( \frac{\sinh(\lambda) E_{\omega(m)}^{k_0}[T_{k_1}]}{e^{-\lambda} - \sinh(\lambda) E_{\omega(m)}^m[T_{k_1}]} \right).
\]

**Proof** First of all, if $\lambda > 0$ is small enough so that (4.16) holds then Remark 3 implies that
\[
e^{-\lambda} - \sinh(\lambda) (E_{\omega(m)}^m[T_{i+1}] - (i + 1 - m)) > 0, \text{ for all } k_0 \leq i \leq k_1 - 1.
\]

By Lemma 4.2.1 and using the fact that the sequence $\{\tau_i\}_{k_0 \leq i \leq k_1 - 1}$ is independent under $P_{\omega(m)}$, we have
\[
E_{\omega(m)}^{k_0}[e^{\lambda T_{k_1}}] = E_{\omega(m)}[e^{\lambda \sum_{i=k_0}^{k_1-1} \tau_i}] = \prod_{i=k_0}^{k_1-1} E_{\omega(m)}[e^{\lambda \tau_i}]
\]
\[
\leq \prod_{i=k_0}^{k_1-1} \frac{e^{-\lambda} - \sinh(\lambda) (E_{\omega(m)}^m[T_{i+1}] - (i + 1 - m))}{e^{-\lambda} - \sinh(\lambda) (E_{\omega(m)}^m[T_{i+1}] - (i - m))}
\]
\[
= e^{\lambda(k_1 - k_0)} \left( 1 + \frac{\sinh(\lambda) (E_{\omega(m)}^{k_0}[T_{k_1}] - (k_1 - k_0))}{e^{-\lambda} - \sinh(\lambda) (E_{\omega(m)}^m[T_{k_1}] - (k_1 - m))} \right).
\]
Since $1 + x \leq e^x$ for any $x \in \mathbb{R}$ we can conclude that

$$e^{\lambda(k_1 - k_0)} \left( 1 + \frac{\sinh(\lambda) \left( E^{k_0}_{\omega(m)}[T_{k_1}] - (k_1 - k_0) \right)}{e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k_1}] - (k_1 - m) \right)} \right)$$

$$\leq \exp \left( \lambda(k_1 - k_0) + \frac{\sinh(\lambda) \left( E^{k_0}_{\omega(m)}[T_{k_1}] - (k_1 - k_0) \right)}{e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k_1}] - (k_1 - m) \right)} \right)$$

$$\leq \exp \left( \frac{\lambda(k_1 - k_0) + \sinh(\lambda) \left( E^{k_0}_{\omega(m)}[T_{k_1}] - (k_1 - k_0) \right)}{e^{-\lambda} - \sinh(\lambda) \left( E^{m}_{\omega(m)}[T_{k_1}] - (k_1 - m) \right)} \right)$$

$$\leq \exp \left( \frac{\sinh(\lambda) E^{k_0}_{\omega(m)}[T_{k_1}]}{e^{-\lambda} - \sinh(\lambda) E^{m}_{\omega(m)}[T_{k_1}]} \right),$$

where in the second inequality we used that the denominator inside the exponent is at most $e^{-\lambda} \leq 1$, and in the last inequality we used that $\lambda < \sinh(\lambda)$ for $\lambda > 0$. This completes the proof of the corollary.

### 4.3 Bounds for Quenched Expected Crossing Times

From the results of the previous section, we see that the quenched expected crossing times are key to obtaining bounds on the quenched moment generating functions of hitting times. In particular, it will be necessary to obtain control on how small $\lambda > 0$ must be for the bounds given by Corollary 4.2.2 to be valid. In order to consider this problem in more general setting, let us define a sequence $a_n = n^{\eta_1}$ for some $\eta_1 > 0$, and study $E_{\omega}[T_{\nu_n}]$ under the measure $Q$. First, we decompose $E_{\omega}[T_{\nu_n}]$ to the series of crossing time between consecutive ladder locations such that

$$E_{\omega}[T_{\nu_n}] = \sum_{i=0}^{\nu_{n-1}} E^{\nu_i}_{\omega} \left[ T_{\nu_{i+1}} \right].$$

For simplicity, we will introduce some notation.

$$\beta_i = E^{\nu_i}_{\omega} \left[ T_{\nu_{i+1}} \right], \quad i \in \mathbb{Z}.$$

Under the measure $Q$, recall that $\nu_0 = 0$ and that $\theta^{\nu_0} \omega$ has a same distribution for any $i \in \mathbb{Z}$. As a result, $\{\beta_i\}_{i \in \mathbb{Z}}$ is stationary under $Q$. Next, we determine i.i.d
components in \( i \) which mainly contribute to the size of each \( \beta_i \). It turns out that \( \beta_i \) is roughly comparable to \( M_i \). Suppose \( b_n = n^{\eta_2} \) for some \( \eta_1 > \eta_2 > 0 \). The main goal of this section is to show that the size of \( \beta_i \) with \( M_i \leq b_n \) is small enough that the sums of such \( \beta_i \)'s is unlikely to play a large role in the size of \( E_\omega [T_{\nu_n}] \). Therefore, the large deviation events are primarily dependent on the \( \beta_i \) for indices \( i \) with \( M_i > b_n \).

The following Proposition is the main result of this section.

**Proposition 4.3.1** Let \( a_n = n^{\eta_1} \) and \( b_n = n^{\eta_2} \) for some \( \eta_1 > \eta_2 > 0 \). Let Assumption 10 and 11 hold. Then, for any \( \epsilon > 0 \) there exist constants \( C, C' \) such that

\[
Q \left( \sum_{i=0}^{a_n-1} (\beta_i I_{\{M_i \leq b_n\}} - E_Q[\beta_0]) > a_n \epsilon \right) \leq C' a_n e^{-C(log n)^2}. \tag{4.18}
\]

The remainder of this section is devoted to the proof of Proposition 4.3.1. First of all, let \( c_n := \lfloor (log n)^2 \rfloor \) and define \( \beta^{(c_n)}_i \) to be a quenched expected crossing time from \( \nu_i \) to \( \nu_{i+1} \) with a reflection point located at \( \nu_i - (c_n - 1) \). That is,

\[
\beta^{(c_n)}_i := E_{\nu^{(c_n)}_{i-1}} \omega(\nu_{i-(c_n-1)}) \left[ T_{\nu_{i+1}} \right].
\]

The strategy of proof for (4.18) is first to show that the sums of differences of \( \beta_i I_{\{M_i \leq b_n\}} \) and \( \beta^{(c_n)}_i I_{\{M_i \leq b_n\}} \) are negligible in the limit, and then prove the inequality of (4.18) with \( \beta_i \) replaced by \( \beta^{(c_n)}_i \). More precisely, we have

\[
Q \left( \sum_{i=0}^{a_n-1} (\beta_i I_{\{M_i \leq b_n\}} - E_Q[\beta_0]) > a_n \epsilon \right) \leq Q \left( \sum_{i=0}^{a_n-1} (\beta_i - \beta^{(c_n)}_i) I_{\{M_i \leq b_n\}} > \frac{\epsilon}{2} a_n \right) + Q \left( \sum_{i=0}^{a_n-1} (\beta^{(c_n)}_i I_{\{M_i \leq b_n\}} - E_Q[\beta_0]) > \frac{\epsilon}{2} a_n \right), \tag{4.19}
\]

and we will show that each term in (4.19) is bounded above by \( C a_n e^{-C'(log n)^2} \). The following lemma does this for the first term of (4.19).

**Lemma 4.3.2** For any \( \epsilon > 0 \), there exist \( C, C' > 0 \) such that

\[
Q \left( \sum_{i=0}^{a_n-1} (\beta_i - \beta^{(c_n)}_i) I_{\{M_i \leq b_n\}} > \frac{\epsilon}{2} a_n \right) < C' a_n e^{-C'(log n)^2}.
\]
Proof. Using (4.1), we may write
\[ \beta_i = \sum_{j=\nu_i}^{\nu_{i+1}-1} (1 + 2W_j) \]
\[ = l_i + 2 \sum_{j=\nu_i}^{\nu_{i+1}-1} W_{\nu_i,j} + 2W_{\nu_i-1}R_{\nu_i,\nu_{i+1}-1}. \]  
(4.20)

Similarly, applying (4.1) with a reflection at \( \omega_{\nu_i-(n-1)} \) (so that \( \rho_{\nu_i-(n-1)} = 0 \)) gives
\[ \beta_i^{(c_n)} = l_i + 2 \sum_{j=\nu_i}^{\nu_{i+1}-1} W_{\nu_i,j} + 2R_{\nu_i,\nu_{i+1}-1}W_{\nu_i-(n-1),\nu_i-1}. \]  
(4.21)

Then by (4.20) and (4.21), we get
\[ \beta_i - \beta_i^{(c_n)} = 2(1 + W_{\nu_i-(n-1)-1})\Pi_{\nu_i-(n-1),\nu_i-1}R_{\nu_i,\nu_{i+1}-1} \]
Hence,
\[ Q \left( \sum_{i=0}^{a_n-1} \left( \beta_i - \beta_i^{(c_n)} \right) I_{\{M_i \leq b_n\}} > a_n \frac{\epsilon}{2} \right) \]
\[ = Q \left( \sum_{i=0}^{a_n-1} (1 + W_{\nu_i-(n-1)-1})\Pi_{\nu_i-(n-1),\nu_i-1}R_{\nu_i,\nu_{i+1}-1}I_{\{M_i \leq b_n\}} > a_n \frac{\epsilon}{4} \right) \]

Since \( \Pi_{i_1,i_2} \leq M_i \) for any \( i_1, i_2 \) such that \( \nu_i \leq i_1 \leq i_2 \leq \nu_{i+1} - 1 \),
\[ R_{\nu_i,\nu_{i+1}-1}I_{\{M_i \leq b_n\}} = \sum_{k=\nu_i}^{\nu_{i+1}-1} \Pi_{\nu_i,k}I_{\{M_i \leq b_n\}} \leq l_i M_i I_{\{M_i \leq b_n\}} \leq l_i b_n. \]

Also, from (4.3) and Lemma 2.2 in [17] there exist \( c, c' > 0 \) such that
\[ Q(l_0 > x) < ce^{-c'x}, \quad \text{and} \quad Q(1 + W_{-1} > x) < ce^{-c'x}. \]  
(4.22)

Applying (4.22) and Chebyshev Inequality,
\[ Q \left( \sum_{i=0}^{a_n-1} (1 + W_{\nu_i-(n-1)-1})\Pi_{\nu_i-(n-1),\nu_i-1}R_{\nu_i,\nu_{i+1}-1}I_{\{M_i \leq b_n\}} > a_n \frac{\epsilon}{4} \right) \]
\[ \leq Q \left( \exists i \in [0, a_n - 1] : l_i > (\log n)^2 \right) \]
\[ + Q \left( \exists i \in [-c_n + 1, a_n - c_n] : 1 + W_{\nu_i-1} > (\log n)^2 \right) \]
\[ + Q \left( \sum_{i=0}^{a_n-1} \Pi_{\nu_i-(n-1),\nu_i-1} > \frac{\epsilon a_n}{4(\log n)^4 b_n} \right) \]
\[ \leq 2ca_ne^{-c'(\log n)^2} + \frac{4(\log n)^4 b_n}{\epsilon} E Q[I_{\{\Pi_{0,\nu_1-1}\}}^{c_n-1}] \leq Ca_ne^{-C(\log n)^2} \text{ for some } C, C' > 0, \]
where the second to last inequality comes from the fact that $E_Q[\Pi_{\nu_i-k,i-1}] = E_Q[\Pi_{0,\nu_i-1}]$ (since blocks between ladder points are i.i.d. under $Q$), and the last inequality follows from $E_Q[\Pi_{0,\nu_i-1}] < 1$, $b_n (\log n)^4 \ll a_n$, and $c_n = \lceil (\log n)^2 \rceil$.

Regarding the second term of (4.19), we will begin by decomposing $\beta_i^{(c_n)}$ in a way that will help us to get control the dependence in the sequence. Recall the decomposition of $\beta_i^{(c_n)}$ in (4.21). Observe that the first two terms are i.i.d as sequences indexed by $i$, and the last term is stationary in $i$ but dependent under measure $Q$. Since (4.1) implies that

$$E_Q[\beta_0] = E_Q[l_0] + 2E_Q \left[ \sum_{j=0}^{\nu_i-1} W_{0,j} \right] + 2E_Q[W_{-1} R_{0,\nu_i-1}],$$

we can bound the second term of (4.19) by three different probabilities such that

$$Q\left( \sum_{i=0}^{a_n^{-1}} (\beta_i^{(c_n)} I_{\{M_i \leq b_n\}} - E_Q[\beta_0]) > \frac{\epsilon}{2} a_n \right)$$

$$\leq Q\left( \sum_{i=0}^{a_n^{-1}} (l_i - E_Q[l_0]) > a_n \frac{\epsilon}{6} \right)$$

$$+ Q\left( \sum_{i=0}^{a_n^{-1}} \left( \sum_{j=\nu_i}^{\nu_i+1-1} W_{\nu_i,j} I_{\{M_i < b_n\}} - E_Q \left[ \sum_{j=\nu_i}^{\nu_i-1} W_{\nu_0,j} \right] \right) > \frac{\epsilon}{12} a_n \right)$$

$$+ Q\left( \sum_{i=0}^{a_n^{-1}} \left( W_{\nu_i-(c_n-1),\nu_i-1} R_{\nu_i,\nu_i+1-1} I_{\{M_i \leq b_n\}} - E_Q[W_{-1} R_{0,\nu_i-1}] \right) > a_n \frac{\epsilon}{12} \right).$$

The proof of Proposition 4.3.1 then follows easily from the following three lemmas.

**Lemma 4.3.3** For any $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that

$$Q\left( \sum_{i=0}^{a_n^{-1}} (l_i - E_Q[l_0]) > a_n \epsilon \right) = O \left( e^{-c(\epsilon) a_n} \right).$$

**Lemma 4.3.4** For any $\epsilon > 0$, there exists $C, C' > 0$ such that

$$Q\left( \sum_{i=0}^{a_n^{-1}} \sum_{j=\nu_i}^{\nu_i+1-1} W_{\nu_i,j} I_{\{M_i < b_n\}} - E_Q \left[ \sum_{j=\nu_i}^{\nu_i-1} W_{\nu_0,j} \right] > a_n \epsilon \right) \leq C a_n e^{-C'(\log n)^2}. \quad (4.23)$$
Lemma 4.3.5 For any $\epsilon > 0$, there exists constants $C, C' > 0$ such that

$$Q \left( \sum_{i=0}^{n-1} \left\{ W_{\nu_i-(n-1)}R_{\nu_i,\nu_i+1-1}I_{M_i \leq b_n} - E_Q[W_{n-1}R_{0,n-1}] \right\} > a_n \epsilon \right) \leq C a n e^{-C'(\log n)^2}$$

(4.24)

The proof of Lemma 4.3.3 is a standard result in large deviation theory since the $l_i$ are i.i.d. with exponential tails. We will therefore only give the proofs of Lemmas 4.3.4 and 4.3.5. Although the summands in (4.23) are i.i.d, we cannot use the standard large deviation techniques involving exponential moments to obtain a bound like in Lemma 4.3.3 because the exponential moment is infinite as

$$\sum_{j=\nu_i}^{\nu_i+1-1} W_{\nu_i,j} > M_i$$

and

$$Q(M_i > x) \sim C''/x^s$$

for $1 < s$. Instead, we adapt a technique of Nagaev and Fuk used on estimating for large deviation probability of sums of heavy tailed independent random variables [25]. Let $X$ be a random variable on arbitrary probability space $\Omega$ and let $A$ be a measurable subset of $\Omega$. If $X \leq y$, we claim that for any $h > 0$,

$$E \left[ e^{hX} - 1 - hX \right] \leq \frac{e^{hy} - 1 - hy}{y^2} E \left[ X^2 \right].$$

(4.25)

It is easy to verify (4.25) by the fact that $(e^{hx} - 1 - hx)/x^2$ is non-decreasing in $x$.

Secondly, we state the following lemma which follows easily from the tail asymptotics (4.4) for $M_i$ under the measure $Q$.

Lemma 4.3.6 Let Assumptions 10 and 11 hold.

1. If $s < 2$, then $E_Q[M_0^2 I_{M_0 \leq x}] \sim \frac{C_0}{2-s} x^{2-s}$ as $x \to \infty$.

2. If $s = 2$, then $E_Q[M_0^2 I_{M_0 \leq x}] \sim 2C_0 \log x$ as $x \to \infty$.

3. If $s > 2$, then $E_Q[M_0^2] < \infty$.

Now we are ready to give the proof of Lemma 4.3.4.

Proof [Proof of Lemma 4.3.4] For the simplicity, let us first introduce a notation.

$$W := E_Q \left[ \sum_{j=\nu_0}^{\nu_1-1} W_{\nu_0,j} \right].$$
Also, define a positive function $\zeta(i,n)$ such that

$$\zeta(i,n) := \sum_{j=\nu_i}^{\nu_{i+1}-1} W_{\nu_i,j} I\{M_i < b_n\} \cap \{\nu_i, (\log n)^2\}.$$  

Since $\Pi_{i_1,i_2} \leq M_i$ for any $i_1, i_2$ such that $\nu_i \leq i_1 \leq i_2 \leq \nu_{i+1} - 1$,

$$\sum_{j=\nu_i}^{\nu_{i+1}-1} W_{\nu_i,j} = \sum_{j=\nu_i}^{\nu_{i+1}-1} \sum_{k=\nu_i}^{\nu_{i+1}-1} \Pi_{k,j} \leq \sum_{j=\nu_i}^{\nu_{i+1}-1} M_i \leq \sum_{j=\nu_i}^{\nu_{i+1}-1} l_i M_i \leq \ell_i^2 M_i.$$

As a result, we have a following bound of $\zeta(i,n)$.

$$\zeta(i,n) \leq (\log n)^4 M_i I\{M_i < b_n\} \leq (\log n)^4 b_n. \tag{4.26}$$

Replacing the notations in the problem by $\zeta(i,n)$ and $W$ the notation above, the problem is simplified to

$$Q\left( \sum_{i=0}^{a_n-1} \left( \sum_{j=\nu_i}^{\nu_{i+1}-1} W_{\nu_i,j} I\{M_i < b_n\} - W \right) > a_n \epsilon \right)$$

$$\leq Q\left( \exists i \in [0, a_n - 1] : l_i > (\log n)^2 \right) + Q\left( \sum_{i=0}^{a_n-1} \zeta(i,n) > a_n (\epsilon + W) \right). \tag{4.27}$$

By (4.22) and the stationarity of $l_i$ under $Q$, the first term is bounded by $ca_n e^{-c'(\log n)^2}$ for some $c, c' > 0$. So, it remains to prove a similar upper bound for the second term of (4.27). Recall that $\zeta(i,n)$ are i.i.d. sequences in $i$ under the measure $Q$. Then by Chebyshev Inequality, for any $\lambda \geq 0$

$$Q\left( \sum_{i=0}^{a_n-1} \zeta(i,n) > a_n (\epsilon + W) \right) \leq E_Q [e^{\lambda \sum_{i=0}^{a_n-1} \zeta(i,n)}] e^{-\lambda a_n(W + \epsilon)}$$

$$= e^{-\lambda a_n(W + \epsilon)} E_Q [e^{\lambda \zeta(0,n)}] [a_n]. \tag{4.28}$$

Note that $E_Q[\zeta(0,n)] \leq W$, and $\zeta(0,n) \leq (\log n)^4 b_n$. Then using (4.25) and (4.26),

$$E_Q [e^{\lambda \zeta(0,n)}] = 1 + \lambda E_Q [\zeta(0,n)] + E_Q [e^{\lambda \zeta(0,n)} - 1 - \lambda \zeta(0,n)]$$

$$\leq 1 + \lambda W + \frac{e^{\lambda (\log n)^2 b_n} - 1 - \lambda (\log n)^4 b_n E [\zeta(0,n)^2]}{(\log n)^4 b_n^2}$$

$$\leq 1 + \lambda W + \frac{e^{\lambda (\log n)^2 b_n} - 1 - \lambda (\log n)^4 b_n}{b_n^2} E [M_0^2 I\{M_0 < b_n\}]$$
With a choice of $\lambda = \frac{1}{(\log n)^4b_n}$, we get

$$E_Q \left[ e^{\lambda \zeta(0,n)} \right] \leq 1 + \frac{W}{(\log n)^4b_n} + \frac{2}{b_n^2} E \left[ M_0^2 I_{\{M_0 < b_n\}} \right] \leq \exp \left( \frac{W}{(\log n)^4b_n} + \frac{2}{b_n^2} E \left[ M_0^2 I_{\{M_0 < b_n\}} \right] \right).$$

Applying Lemma 4.3.6, we obtain that there exists some constant $C > 0$ such that

$$E_Q \left[ e^{\lambda \zeta(0,n)} \right] \leq \begin{cases} 
\exp \left( \frac{W}{(\log n)^4b_n} + \frac{C}{b_n^2} \right) & \text{if } 1 < s < 2 \\
\exp \left( \frac{W}{(\log n)^4b_n} + \frac{C \log n}{b_n} \right) & \text{if } s = 2 \\
\exp \left( \frac{W}{(\log n)^4b_n} + \frac{C}{b_n^2} \right) & \text{if } 2 < s. 
\end{cases} \quad (4.29)$$

Combining (4.28) and (4.29) we get that there exists a constant $c > 0$ such that

$$Q \left( \sum_{i=0}^{n-1} \zeta(i,n) > a_n(\epsilon + W) \right) \leq \begin{cases} 
c \times \exp \left( a_n \left( \frac{C}{b_n} - \frac{\epsilon}{(\log n)^4b_n} \right) \right) & \text{if } 1 < s < 2 \\
c \times \exp \left( a_n \left( \frac{C \log n}{b_n} - \frac{\epsilon}{(\log n)^4b_n} \right) \right) & \text{if } s = 2 \\
c \times \exp \left( a_n \left( \frac{C}{b_n^2} - \frac{\epsilon}{(\log n)^4b_n} \right) \right) & \text{if } 2 < s. 
\end{cases}$$

Note that all three cases are bounded above by $ce^{-c'a_n/(\log n)^4b_n}$ for some $c, c' > 0$ for $n$ large enough. Hence, the second term of (4.27) is bounded above by the right-hand side of (4.23) for large $n$. 

In preparation for the proof of Lemma 4.3.5, we introduce the following notation.

$$\tilde{W}_{i,n} := W_{\nu_i - (c_n - 1), \nu_i - 1} \text{ and } \tilde{R}_i := R_{\nu_i, \nu_i + 1 - 1},$$

and define

$$\psi(i,n) := \tilde{R}_i I_{\{M_i \leq b_n, l_i \leq (\log n)^2\}} \tilde{W}_{i,n} I_{\{\tilde{W}_{i,n} < (\log n)^2\}}.$$

Note that $\psi(i + c_n, n)$ is independent of $\psi(i,n)$ under the measure $Q$. Also, since $\tilde{R}_i = \sum_{k=\nu_i}^{\nu_i + 1} \Pi_{\nu_i,k} \leq l_i M_i$,

$$\psi(i,n) \leq (\log n)^4 M_i I_{\{M_i < b_n\}} \leq (\log n)^4 b_n. \quad (4.30)$$

Finally, we give the proof of Lemma 4.3.5.
Proof [Proof of Lemma 4.3.5] For the simplification to notation, denote \( W' := E_Q[W_{-1}] \) and \( R := E[R_{0,v_1-1}] \). Note that \( R_{0,v_1-1} \) and \( W_{-1} \) are independent because \( R_{0,v_1-1} \in \{ \omega_x : 0 \leq x \leq v_1 - 1 \} \) while \( W_{-1} \in \{ \omega_x : x \leq -1 \} \), so we get

\[
E_Q[W_{-1}R_{0,v_1-1}] = E_Q[W_{-1}]E_Q[R_{0,v_1-1}] = W'R.
\]

With new notations described above, the problem is simplified to

\[
Q \left( \sum_{i=0}^{a_n-1} \left\{ \tilde{W}_{i,n}\tilde{R}_i - W'R \right\} > a_n\epsilon \right) \leq Q \left( \exists i \in [0, a_n - 1] : l_i > (\log n)^2 \right)
\]

\[
+ Q \left( \exists i \in [0, a_n - 1] : W_{v_1-1} > (\log n)^2 \right) + Q \left( \sum_{i=0}^{a_n-1} (\psi(i, n) - W'R) > a_n\epsilon \right).
\]

(4.31)

By (4.22), the first and second terms of (4.31) are bounded by \( 2ca_ne^{-c'(\log n)^2} \) for some constant \( c, c' > 0 \). Hence, it is enough to show that there exist some constants \( C, C' > 0 \), such that

\[
Q \left( \sum_{i=0}^{a_n-1} \psi(i, n) > a_n(\epsilon + W'R) \right) \leq Ca_ne^{-C'(\log n)^2} \quad (4.32)
\]

A proof of (4.32) begins with grouping \( \{ \psi(i, n) \}_{0 \leq i \leq a_n-1} \) into \( c_n = \lfloor (\log n)^2 \rfloor \) smaller sums as follows. In particular, since

\[
\sum_{i=0}^{a_n-1} \psi(i, n) \leq \sum_{j=0}^{c_n-1} \left( \sum_{i=0}^{\lfloor a_n/c_n \rfloor} \psi(j + ic_n, n) \right),
\]

then the third term of (4.31) is bounded above by

\[
Q \left( \sum_{j=0}^{c_n-1} \left( \sum_{i=0}^{\lfloor a_n/c_n \rfloor} \psi(j + ic_n, n) \right) > a_n(\epsilon + W'R) \right) \quad (4.33)
\]

\[
\leq \sum_{j=0}^{c_n-1} Q \left( \sum_{i=0}^{\lfloor a_n/c_n \rfloor} \psi(j + ic_n, n) > \frac{a_n}{c_n}(\epsilon + W'R) \right)
\]

\[
= c_nQ \left( \sum_{i=0}^{\lfloor a_n/c_n \rfloor} \psi(ic_n, n) > \frac{a_n}{c_n}(\epsilon + W'R) \right),
\]

(4.34)
where the last equality follows by the stationarity of \( \psi(i, n) \) under \( Q \). Notice terms in the sum inside the probability in (4.34) are i.i.d. under \( Q \). Hence, applying Chebyshev Inequality to (4.34), for any \( \lambda > 0 \)

\[
Q \left( \sum_{i=0}^{\lceil n/c_n \rceil} \psi(i, n) > \frac{a_n}{c_n} (\epsilon + W'R) \right) \leq E_Q \left[ e^{\lambda \frac{a_n}{c_n} (\epsilon + W'R)} \right]
\]

where in the last line we used the first inequality in (4.30). With a choice of \( \lambda \)

\[
= E_Q \left[ e^{\lambda \psi(0,n)} \right]^{\lceil a_n/c_n \rceil + 1} e^{-\frac{\lambda an}{cn} (\epsilon + W'R)}.
\]

(4.35)

Note that \( E[\psi(0,n)] \leq W'R \) and recall that \( \psi(0,n) \leq (\log n)^4 b_n \). Therefore, using (4.25)

\[
E_Q \left[ e^{\lambda \psi(0,n)} \right] = 1 + \lambda E_Q [\psi(0,n)] + E_Q \left[ e^{\lambda \psi(0,n)} - 1 - \lambda \psi(0,n) \right]
\]

\[
\leq 1 + \lambda W'R + \frac{e^{\lambda (\log n)^4 b_n} - 1 - \lambda (\log n)^4 b_n}{((\log n)^4 b_n)^2} E[\psi(0,n)^2]
\]

\[
\leq 1 + \lambda W'R + \frac{e^{\lambda (\log n)^4 b_n} - 1 - \lambda (\log n)^4 b_n}{b_n^2} E\left[ M_0^2 I_{\{M_0 < b_n\}} \right],
\]

where in the last line we used the first inequality in (4.30). With a choice of \( \lambda = \frac{1}{(\log n)^4 b_n} \), we get

\[
E_Q \left[ e^{\lambda \psi(0,n)} \right] \leq 1 + \frac{W'R}{(\log n)^4 b_n} + \frac{2}{b_n^2} E\left[ M_0^2 I_{\{M_0 < b_n\}} \right]
\]

\[
\leq \exp \left( \frac{W'R}{(\log n)^4 b_n} + \frac{2}{b_n^2} E\left[ M_0^2 I_{\{M_0 < b_n\}} \right] \right),
\]

and thus, applying Lemma 4.3.6, there exists a constant \( C > 0 \) such that

\[
E_Q \left[ e^{\lambda \psi(0,n)} \right] \leq \begin{cases} 
\exp \left( \frac{W'R}{(\log n)^4 b_n} + \frac{C}{b_n^2} \right) & \text{if } 1 < s < 2 \\
\exp \left( \frac{W'R}{(\log n)^4 b_n} + \frac{C \log n}{b_n^2} \right) & \text{if } s = 2 \\
\exp \left( \frac{W'R}{(\log n)^4 b_n} + \frac{C}{b_n^2} \right) & \text{if } 2 < s.
\end{cases}
\]

(4.36)

Combining (4.34), (4.35) and (4.36), there exist constants \( c, C > 0 \) such that, for large \( n \),

\[
Q \left( \sum_{i=0}^{a_n-1} \left( \psi(i, n) - W'R \right) > a_n \epsilon \right) \leq \begin{cases} 
c \times c_n \times \exp \left( \frac{a_n}{c_n} \left( \frac{C}{b_n^2} - \frac{\epsilon}{(\log n)^4 b_n} \right) \right) & \text{if } 1 < s < 2 \\
c \times c_n \times \exp \left( \frac{a_n}{c_n} \left( \frac{C \log n}{b_n^2} - \frac{\epsilon}{(\log n)^4 b_n} \right) \right) & \text{if } s = 2 \\
c \times c_n \times \exp \left( \frac{a_n}{c_n} \left( \frac{C}{b_n^2} - \frac{\epsilon}{(\log n)^4 b_n} \right) \right) & \text{if } 2 < s.
\end{cases}
\]
Note that all three cases are bounded above by $c \times cn e^{-c' \frac{an}{n \log(n)^4} \mu_n} < Ca_n e^{-C'(\log n)^2}$ for some $c, c', C, C' > 0$ with $n$ large enough, which completes the proof of (4.32).

### 4.4 The Quenched Subexponential Tail of Hitting Time Large Deviations

The main goal of this section is to prove the following.

**Proposition 4.4.1** Under the same assumptions as Theorem 4.1.1, for any $u \in \left( \frac{1}{v_\alpha}, \infty \right)$,

$$
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{\nu_n} > uw_n) = -\infty, \quad \alpha\text{-a.s.} \tag{4.37}
$$

Before giving the proof of Proposition 4.4.1, we will first show how it can be used to complete the proof of Theorem 4.1.1.

**Proof** [Proof of Theorem 4.1.1] Let $v < v' < v_\alpha$, then

$$
P_\omega(X_n < nv) \leq P_\omega(T_{nv'} > n) + P^{nv'}_{\omega}(T_{nv} < \infty). \tag{4.38}
$$

First, we will show that Proposition 4.4.1 implies that

$$
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{nv'} > n) = -\infty, \quad \alpha\text{-a.s.} \tag{4.39}
$$

To this end, let $\mu$ and $\mu'$ be such that $v' < \mu < \mu' < v_\alpha$ and let $c_1 = \frac{\mu}{E_Q[\nu_1]}$. Since $\lim_{n \to \infty} \frac{\nu_n}{n} = E_Q[\nu_1], \alpha\text{-a.s.}$, it follows that

$$
\lim_{n \to \infty} \frac{\nu_{\lfloor c_1n \rfloor}}{n} = c_1 E_Q[\nu_1] = \mu.
$$

That is, $\frac{\nu_{\lfloor c_1n \rfloor}}{n} \in (v', \mu')$ for all $n$ sufficiently large (depending on $\omega$). Thus, for $\alpha$-a.e. environment and all $n$ large enough we have that

$$
P_\omega(T_{nv'} > n)
\leq P_\omega(T_{\lfloor c_1n \rfloor} > n) = P_\omega \left( T_{\lfloor c_1n \rfloor} > \frac{n}{\nu_{\lfloor c_1n \rfloor}} \nu_{\lfloor c_1n \rfloor} \right) \leq P_\omega \left( T_{\lfloor c_1n \rfloor} > \frac{1}{\mu'} \nu_{\lfloor c_1n \rfloor} \right),
$$

and since $1/\mu' > 1/v_\alpha$ it follows from Proposition 4.4.1 that (4.39) holds. Regarding the second term on the right of (4.38), it was shown in [16, Lemma 3.3] that there is
some constant $C > 0$ such that $\mathbb{P}_\alpha[T_m < \infty] \leq \exp(Cm)$ for any $m < 0$. Therefore, we have a following upper bound with a choice of small $\epsilon > 0$ such that

$$
\mathbb{P}_\alpha(P_{\omega}^{nv'}(T_{nv} < \infty) \geq e^{-\epsilon n}) \leq e^{\epsilon n} \mathbb{P}_\alpha(T_{nv} < \infty) \\
= e^{\epsilon n} \mathbb{P}_\alpha(T_n(v-v') < \infty) \leq e^{\epsilon n} e^{Cn(v-v')}.
$$

Since $v < v'$, if $\epsilon > 0$ is chosen sufficiently small then the upper bound given above is exponentially decreasing in $n$ and so the Borel-Cantelli Lemma implies that $P_{\omega}^{nv'}(T_{nv} < \infty)$ is almost surely eventually less than $e^{-C'n}$ for some constant $C' > 0$ for all $n$ large. In particular, this implies that

$$
\lim_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega}^{nv'}(T_{nv} < \infty) = -\infty, \quad \alpha\text{-a.s.},
$$

which concludes our proof.

To prove Proposition 4.4.1 let us first define a new measure $\tilde{\alpha}$ on environments by $\tilde{\alpha}(\omega \in \cdot) = \alpha(\theta^{-\nu_0} \omega \in \cdot)$. That is, $\alpha$ is the distribution of the environment shifted so that the ladder point $\nu_0 \leq 0$ is at the origin. Compare this with the distribution $Q$ which is obtained instead by conditioning $\nu_0$ to be at the origin. We show next that $\tilde{\alpha}$ is in fact absolutely continuous with respect to $Q$.

**Lemma 4.4.2** $\tilde{\alpha}$ is absolutely continuous with respect to $Q$.

**Proof** First of all, note that

$$
\{\nu_0 = -k\} = \{\Pi_{j,-k-1} < 1 \text{ for } j < -k, \Pi_{-k,j} \geq 1 \text{ for } -k \leq j \leq -1\}.
$$
Therefore, for any event $A \in \sigma(\{ \omega_z \}, z \in \mathbb{Z})$,

$$\tilde{\alpha}(\omega \in A) = \sum_{k=0}^{\infty} \alpha(\nu_0 = -k)\alpha(\theta^{-k}\omega \in A | \nu_0 = -k)$$

$$= \sum_{k=0}^{\infty} \alpha(\nu_0 = -k)\alpha(\theta^{-k}\omega \in A | \Pi_{j,-k-1} < 1 \text{ for } j < -k, \Pi_{-k,j} \geq 1 \text{ for } -k \leq j \leq -1)$$

$$= \sum_{k=0}^{\infty} \alpha(\nu_0 = -k)\alpha(\omega \in A | \Pi_{j,-1} < 1 \text{ for } j < 0, \Pi_{0,j} \geq 1 \text{ for } 0 \leq j \leq k - 1)$$

$$= \sum_{k=0}^{\infty} \alpha(\nu_0 = -k)\alpha(\omega \in A, \Pi_{0,j} \geq 1 \text{ for } 0 \leq j \leq k - 1 | \Pi_{j,-1} < 1 \text{ for } j < 0)$$

$$= \sum_{k=0}^{\infty} \alpha(\nu_0 = -k)\alpha(\omega \in A, \nu_1 > k)\frac{Q(\omega \in A, \nu_1 > k)}{Q(\nu_1 > k)}.$$ 

Therefore, if $Q(\omega \in A) = 0$ then $\tilde{\alpha}(\omega \in A) = 0$ also. That is, $\tilde{\alpha}$ is absolutely continuous with respect to $Q$. 

**Remark 4** In fact, the above proof shows that $\frac{d\tilde{\alpha}}{dQ}(\omega) = \sum_{k=0}^{\nu_1(\omega)-1} r_k$, where $r_k = \frac{\alpha(\nu_0 = -k)}{Q(\nu_1 > k)}$.

We now show how the measure $\tilde{\alpha}$ is helpful for proving Proposition 4.4.1. Since $\nu_0 \leq 0$ for any environment $\omega$, we have

$$P_{\omega}(T_{\nu_n} > u\nu_n) \leq P^{\nu_0}_{\omega}(T_{\nu_n} > u\nu_n),$$

and thus to prove Proposition (4.4.1) it will be enough to show that the conclusion holds with $\tilde{\alpha}$ in place of $\alpha$. That is, we need to show that

$$\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega}(T_{\nu_n} > u\nu_n) = -\infty, \quad \tilde{\alpha}\text{-a.s.}$$

However, since Lemma 4.4.2 shows that $\tilde{\alpha}$ is absolutely continuous with respect to $Q$, the above limit will follow if we can show the same almost sure limit under the measure $Q$. That is, we have reduced the proof of Proposition 4.4.1 to the following.
Proposition 4.4.3 Under the same assumptions as Theorem 4.1.1, for any \( u \in (\frac{1}{v_n}, \infty) \),

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_{\omega}(T_{\nu_n} > u \nu_n) = -\infty, \quad Q\text{-a.s.} \tag{4.40}
\]

The remainder of the paper is devoted to the proof of Proposition 4.4.3. We will follow the approach of [1] by dividing the environment into large blocks and then analyzing the crossing times of these large blocks. The main improvement we make is that we obtain better estimates on the quenched moment generating functions of these crossing times using the results from Section 4.2. To decompose the environment into blocks, fix an integer \( m > s \), let us define subsequence \( n_k \) such that

\[
n_k = m^{m^k} \text{ for } k \geq 0,
\]

and \( a_k = n_k^{1/s}/D \) for some fixed \( D > 1 \), which will later allow to be arbitrarily large. The blocks of the environment will be the intervals between ladder locations \( \nu_{ja_k} \) and \( \nu_{(j+1)a_k} \) for \( j \in \mathbb{Z} \). To simplify notation, let us denote the ladder locations at the edges of the blocks by

\[
\nu(j, k) := \nu_{ja_k}, \quad j \in \mathbb{Z}, \ k \geq 1.
\]

The path of the random walk \( X_n \) on \( \mathbb{Z} \) naturally defines a birth-death chain by observing how the random walk moves from one \( \nu(j, k) \) to either \( \nu(j - 1, k) \) or \( \nu(j + 1, k) \). To be precise, let \( \{t_i\}_{i \geq 0} \) be the sequence of times when the random walk reaches a ladder point \( \nu(j, k) \) different from the last such ladder point visited. That is, \( t_0 = 0 \) and

\[
t_i = \inf \{ n > t_{i-1} : X_n \in \{\nu(j, k)\}_{j \in \mathbb{Z}} \text{ and } X_n \neq X_{t_{i-1}} \}, \quad i \geq 1.
\]

We then obtain a birth-death process \( \{Z_i\}_{i \geq 0} \) on \( \mathbb{Z} \) by letting \( X_{t_i} = \nu(Z_i, k) \). If we let \( \Theta_i = t_i - t_{i-1} \), then it follows that

\[
T_{\nu_n} \leq \sum_{i=1}^{N_k} \Theta_i,
\]
where \( N_k = \inf\{i \geq 1 : Z_i \geq n_k/a_k\} \) is the time needed for the induced birth-death process to move at least \( n_k/a_k \) to the right. If we also define \( \tilde{N}_k = \inf\{i \geq 1 : |Z_i| \geq n_k/a_k\} \) to be the time for the birth-death process to exit \((-n_k/a_k, n_k/a_k)\) then it follows for any fixed \( L \) that

\[
P_\omega(T_{\nu n_k} > u \nu n_k) \\ \leq P_\omega(N_k \neq \tilde{N}_k) + P_\omega(\tilde{N}_k > L, N_k = \tilde{N}_k) + P_\omega \left( \sum_{i=1}^{\tilde{N}_k} \Theta_i > u \nu n_k, \tilde{N}_k \leq L \right).
\]  

(4.41)

We will show below that the environment is such that for \( k \) large enough the induced birth-death process has a very strong drift to the right so that by choosing \( L \) large enough we can make the first two probabilities on the right above very small. The last probability on the right is the key term, and we will obtain control on this by obtaining certain uniform upper bounds on the time it takes a random walk started at \( \nu(j,k) \) to reach either \( \nu(j-1,k) \) or \( \nu(j+1,k) \).

The following result shows that the first term in (4.41) has an exponential tail.

**Lemma 4.4.4** There exist \( \delta > 0 \) such that for \( Q \)-a.e. environment \( \omega \) there is an integer \( K(\omega) < \infty \) such that

\[
P_\omega(N_k \neq \tilde{N}_k) \leq e^{-\delta n_k}, \quad \forall k \geq K(\omega).
\]

**Proof** The event

\[
\{N_k \neq \tilde{N}_k\} \subset \{T_{\nu n_k} < \infty\} \subset \{T_{-n_k} < \infty\}.
\]

Therefore,

\[
Q \left( P_\omega(N_k \neq \tilde{N}_k) > e^{-\delta n_k} \right) \leq Q \left( P_\omega(T_{-n_k} < \infty) > e^{-\delta n_k} \right) \leq e^{\delta n_k} E_Q [P_\omega(T_{-n_k} < \infty)].
\]  

(4.42)

Since \( \alpha(\nu_0 = 0) > 0 \) and \( Q(\cdot) = \alpha(\cdot|\nu_0 = 0) \), we have

\[
E_Q [P_\omega(T_{-n_k} < \infty)] = \frac{E_\alpha [P_\omega(T_{-n_k} < \infty) 1_{\nu_0=0}]}{\alpha(\nu_0 = 0)} \leq \frac{P_\alpha(T_{-n_k} < \infty)}{\alpha(\nu_0 = 0)} \leq \frac{e^{-C n_k}}{\alpha(\nu_0 = 0)},
\]

(4.43)
where the last inequality holds by Lemma 3.3 in [16]. Finally, if $\delta > 0$ is chosen sufficiently small then (4.42) is summable in $k$ and so the Borel-Cantelli Lemma completes the proof.

In order to determine the decay rate of the second and third term in (4.41), we first define a set

$$J_{n_k} = [-n_k/a_k, n_k/a_k] \cap \mathbb{Z} = \left\{ -\left\lfloor \frac{n_k}{a_k} \right\rfloor, -\left\lceil \frac{n_k}{a_k} \right\rceil + 1, \ldots, \left\lfloor \frac{n_k}{a_k} \right\rfloor \right\}.$$  

Clearly, if $N_k = \tilde{N}_k$ then the birth-death process $Z_i \in J_{n_k}$ when $t_i < T_{\nu_{n_k}}$. So, we only need to observe paths of the birth-death process $\{Z_i\}_{i \geq 0}$ restricted to $J_{n_k}$ and analyze its associated probability. The following lemma gives a uniform upper bound (for all $k$ large enough) on the probability that the birth-death process steps to the left before time $\tilde{N}_k$.

**Lemma 4.4.5** There exist $\delta' > 0$ such that

$$Q\left( \max_{j \in J_{n_k}} P_{\omega}^{\nu(j,k)}(T_{\nu(j-1,k)} < T_{\nu(j+1,k)}) > e^{-\delta' a_k} \text{ i.o.} \right) = 0. \quad (4.44)$$

**Proof** First, note that

$$Q\left( \max_{j \in J_{n_k}} P_{\omega}^{\nu(j,k)}(T_{\nu(j-1,k)} < T_{\nu(j+1,k)}) > e^{-\delta' a_k} \right)$$

$$\leq \sum_{j \in J_{n_k}} Q\left( P_{\omega}^{\nu(j,k)}(T_{\nu(j-1,k)} < T_{\nu(j+1,k)}) > e^{-\delta' a_k} \right)$$

$$\leq 3 \frac{n_k}{a_k} Q\left( P_{\omega}(T_{\nu(-1,k)} < T_{\nu(1,k)}) > e^{-\delta' a_k} \right)$$

$$\leq 3 \frac{n_k}{a_k} Q\left( P_{\omega}(T_{-a_k} < \infty) > e^{-\delta' a_k} \right)$$

$$\leq 3 \frac{n_k}{a_k} E_Q [P_{\omega}(T_{-a_k} < \infty)] e^{\delta' a_k},$$

where the second inequality holds because $|J_{n_k}| \leq 3n_k/a_k$ and $Q$ is stationary under shifts of the ladder points of the environment, and the third inequality holds by $\{T_{\nu(-1,k)} < T_{\nu(1,k)}\} \subseteq \{T_{\nu-a_k} < \infty\} \subseteq \{T_{-a_k} < \infty\}$. Finally, it follows from (4.43) that the last line is bounded above by $C' \frac{n_k}{a_k} e^{-(C'-\delta')a_k}$. Since this is summable in $k$ for sufficiently small $\delta' > 0$, the Borel-Cantelli Lemma finishes the proof of (4.44).
Let \( \{S_i\}_{i \geq 0} \) be a simple random walk with
\[
P(S_{i+1} = S_i + 1 | S_i) = 1 - P(S_{i+1} = S_i - 1 | S_i) = 1 - e^{-\delta a_k}.
\]

Since this random walk steps to the right with very high probability, it is unlikely that the random walk takes too long to travel \( \lfloor n_k/a_k \rfloor \) steps to the right. In particular, if we fix \( \delta > 0 \) and let \( L_k = \frac{n_k}{a_k(1-\delta)} \), then it was shown in [1, Lemma 9] that
\[
P(\inf \{ i > 0 : S_i = \left\lceil \frac{n_k}{a_k} \right\rceil \} > L_k) \leq e^{-\delta n_k}.
\]

for some \( \delta_1 > 0 \). It follows from Lemma 4.4.5 that the probability of jumping to left under \( S_i \) dominates the probability of jumping to left under \( Z_i \) when \( Z_i = j \in J_{n_k} \). As a result, if the process \( Z_i \) stays within \( J_{n_k} \), then the random walk \( S_i \) will take longer than the process \( Z_i \) to reach \( \left\lceil \frac{n_k}{a_k} \right\rceil \). That is, for \( k \) sufficiently large (depending on \( \omega \)),
\[
P(\tilde{N}_k > L_k, N_k = \tilde{N}_k) \leq P(\inf \{ i > 0 : S_i = \left\lceil \frac{n_k}{a_k} \right\rceil \} > L_k) \leq e^{-\delta_1 n_k}. \tag{4.45}
\]

In order to estimate the decaying rate of the last term in (4.41), we first find an explicit upper bound of the exponential moment of \( \Theta_i \). Recall that, each \( \Theta_i \) is a crossing time from \( \nu(Z_{i-1}, k) \) to either \( \nu(Z_{i-1} - 1, k) \) or \( \nu(Z_{i-1} + 1, k) \) that the walk visits first. Then, each \( \Theta_i \) is less than the crossing time from \( \nu(Z_{i-1}, k) \) to \( \nu(Z_{i-1} + 1, k) \) with a reflection at \( \nu(Z_{i-1} - 1, k) \) for \( Z_{i-1} \in \mathbb{Z} \). Therefore, we have for \( \lambda > 0 \) that
\[
E_{\omega} \left[ e^{\lambda \Theta_i \mathbf{1}_{\{Z_{i-1} \in J_{n_k}\}}} \right] = \sum_{j \in J_{n_k}} P(Z_{i-1} = j) \times E_{\omega(\nu(j-1, k))} \left[ e^{\lambda T_{\nu(j+1, k)}} \right] + P(Z_{i-1} \notin J_{n_k}) \leq \max_{j \in J_{n_k}} E_{\omega(\nu(j-1, k))} \left[ e^{\lambda T_{\nu(j+1, k)}} \right]. \tag{4.46}
\]

By Corollary 4.2.2 with \( m = \nu(j - 1, k) \), \( k_0 = \nu(j, k) \) and \( k_1 = \nu(j + 1, k) \), the right side of inequality in (4.46) has an upper bound in an explicit form. That is, with \( \lambda > 0 \) sufficiently small enough such that
\[
\max_{j \in J_{n_k}} E_{\omega(\nu(j-1, k))} [T_{\nu(j+1, k)}] < \frac{e^{-\lambda}}{\sinh \lambda}. \tag{4.47}
\]
we have
\[ E_{\nu(j,k)}[e^{\lambda T_{\nu(j+1,k)}}] = \exp \left( \frac{\sinh \lambda (E_{\nu(j,k)}[T_{\nu(j+1,k)}])}{e^{-\lambda} - \sinh \lambda (E_{\nu(j,k)}[T_{\nu(j+1,k)}])} \right) \]
for each \( j \in J_{n_k} \). Therefore, we get
\[ E_{\omega}[\alpha_{i,j} I_{\{\omega \in J_{n_k}\}}] \leq \max_{j \in J_{n_k}} \exp \left( \frac{\sinh \lambda (E_{\omega(\nu(j-1,k))}[T_{\nu(j+1,k)}])}{e^{-\lambda} - \sinh \lambda (E_{\omega(\nu(j-1,k))}[T_{\nu(j+1,k)}])} \right). \tag{4.48} \]
Note that the requirement that \( \lambda > 0 \) is small enough so that (4.47) is satisfied is needed for (4.48) to ensure that certain moment generating functions are finite. Since \( e^{-\lambda}/\sinh \lambda \to \infty \) as \( \lambda \to 0^+ \), (4.47) is always satisfied for some small \( \lambda > 0 \). However, we will later want to apply the upper bound (4.48) with a deterministic choice of \( \lambda = \lambda_k = D_0 n_k^{-1/s} \) with some fixed \( D_0 > 0 \), and in this case the bound (4.47) may not necessarily be satisfied. However, we will prove a following claim and show that with this choice of \( \lambda_k \) there is an environment dependent subsequence of \( n_k \) where the condition (4.47) is met. For any fixed constant \( \epsilon_1 > 0 \), we will show that
\[ Q \left( \max_{j \in J_{n_k}} E_{\omega(\nu(j-1,k))}[T_{\nu(j+1,k)}] < 2(E_Q[\beta_0] + \epsilon_1)a_k \ i.o \right) = 1. \tag{4.49} \]
Recall that the sequence \( a_k = n_k^{1/s}/D \) for some \( D > 1 \). Since \( e^{-\lambda_k}/\sinh \lambda_k \sim \frac{1}{\lambda_k} = \frac{n_k^{1/s}}{D_0} \), it follows from (4.49) that if the constants \( D, D_0 \) and \( \epsilon_1 \) are chosen so that \( D > \frac{2(E_Q[\beta_0] + \epsilon_1)}{D_0} \) then
\[ \max_{j \in J_{n_k}} E_{\omega(\nu(j-1,k))}[T_{\nu(j+1,k)}] < \frac{2(E_Q[\beta_0] + \epsilon_1)}{D} n_k^{1/s} \leq \frac{e^{-\lambda_k}}{\sinh \lambda_k}, \text{ infinitely often.} \]
Therefore, it is enough to prove (4.49) to show that there is almost surely a subsequence of \( n_k \) for which (4.47) holds when \( \lambda = \lambda_k = D_0 n_k^{-1/s} \).

To simplify notation, for any integers \( i, j \) such that \( i \in [(j-1)a_k, (j+1)a_k-1] \) let \( \beta_i^j = E_{\omega(\nu(j-1,k))}[T_{\nu_i+1}] \) be the quenched expected crossing time from \( \nu_i \) to \( \nu_{i+1} \) with a reflection added at \( \nu(j-1,k) \). Then, we can restate (4.49) as
\[ Q \left( \max_{i=(j-1)a_k} \sum_{i=(j-1)a_k}^{(j+1)a_k-1} \beta_i^j < \frac{2(E_Q[\beta_0] + \epsilon_1)}{D} n_k^{1/s} \ i.o \right) = 1. \tag{4.50} \]
A strategy for proving (4.50) is to classify the sums of $\beta_i^j$ into two groups by the size of $M_i$ and determine an upper bound of the sums of each group separately. For a fixed $\epsilon > 0$ we will refer to $\{i: M_i > n_i^{(1-\epsilon)/s}\}$ and $\{i: M_i \leq n_i^{(1-\epsilon)/s}\}$ as “big hills” and “small hills,” respectively. Then, we begin by a lemma showing the upper bound of a group of $\beta_i^j$ corresponding to small hills using Proposition 4.3.1. An upper bound of $\beta_i^j$ corresponding to big hills requires a more careful estimation because $\beta_i^j$ with the biggest hill dominates all of the other $\beta_i^j$’s. The first step is to prove that $\beta_i^j$ corresponding to big hills are typically located outside of a small group of ladder blocks. Then, we show that at most one big hill is typically observed at each ladder block. Finally, we estimate a uniform bound of $\beta_i^j$ corresponding to big hills observed from each ladder block.

The following lemma shows that the maximums of sums of centered expected crossing time with a small hill, $\{M_i \leq n_k^{(1-\epsilon)/s}\}$, are negligible in the limit.

**Lemma 4.4.6** Let us define $J_{n_k}' = J_{n_k} \cup \{-\lfloor n_k/a_k \rfloor - 1\}$. Then, for any $\epsilon_1 > 0$,

$$Q\left(\max_{j \in J_{n_k}'} \sum_{i = (j)a_k}^{(j+1)a_k-1} (\beta_i^j I_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} - E_Q[\beta]) > \frac{\epsilon_1}{2} a_k \quad i.o.\right) = 0.$$

**Proof** Since $\beta_i^j < \beta_i$ for any $j \in J_{n_k}'$ and $i \in [(j)a_k, (j+1)a_k - 1]$, it suffices to prove

$$Q\left(\max_{j \in J_{n_k}'} \sum_{i = (j)a_k}^{(j+1)a_k-1} (\beta_i I_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} - E_Q[\beta]) > \frac{\epsilon_1}{2} a_k \quad i.o.\right) = 0.$$

Recall that $\beta_i, i \in \mathbb{Z}$ is stationary under $Q$. Hence,

$$Q\left(\max_{j \in J_{n_k}'} \sum_{i = (j)a_k}^{(j+1)a_k-1} (\beta_i I_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} - E_Q[\beta]) > \frac{\epsilon_1}{2} a_k\right) \leq \frac{3n_k}{a_k} Q\left(\sum_{i=0}^{a_k-1} (\beta_i I_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} - E_Q[\beta]) > \frac{\epsilon_1}{2} a_k\right) \leq Cn_k e^{-C' (\log n_k)^2},$$

for some $C, C' > 0$, where the last equality comes from Proposition 4.3.1. Then, the conclusion follows by the Borel-Cantelli Lemma.
Next, a following lemma shows that the maximum $\beta_i^j$ with big hill always occurs in $j \in J_{n_k} \setminus \{-1,0,1\}$ for $k$ large enough.

**Lemma 4.4.7** If $0 < \epsilon < 1 - 1/s$, then

\[
Q \left( \max_{j \in J_{n_k}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} \neq \max_{j \in J_{n_k} \setminus \{-1,0,1\}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} \text{ i.o.} \right) = 0.
\]

**Proof** We have the following inclusion,

\[
\left\{ \max_{j \in J_{n_k}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} \neq \max_{j \in J_{n_k} \setminus \{-1,0,1\}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} \right\} \\
\subseteq \left\{ \max_{-2a_k \leq i \leq 2a_k - 1} M_i > n_k^{(1-\epsilon)/s} \right\}.
\]

That is, in order for the two maximums to not be equal there must be at least one large hill corresponding to some $i \in [-2a_k, 2a_k - 1]$. Moreover, for large $n_k$,

\[
Q \left( \max_{-2a_k \leq i \leq 2a_k - 1} M_i > n_k^{(1-\epsilon)/s} \right) = (4a_k)Q(M_0 > n_k^{(1-\epsilon)/s}) = O \left( \frac{a_k}{n_k^{1-\epsilon/s}} \right) = O \left( \frac{1}{n_k^{1-\epsilon/s}} \right),
\]

where the second to last equality comes from the tail asymptotics of $M_0$ in (4.4) and the last equality comes from the definition of $a_k$. Then, the conclusion of the lemma follows from the Borel-Cantelli Lemma.

A following lemma shows that for $n_k$ large enough each interval $[(i-1)a_k, (i+1)a_k - 1]$ with $i \in J_{n_k}$ contains at most one big hill.

**Lemma 4.4.8** Suppose $0 < \epsilon < \frac{s-1}{2s}$, and define a set $A_k$ such that

\[
A_k := \left\{ \exists j \in J_{n_k} \text{ such that } \#\{i \in [(j-1)a_k, (j+1)a_k - 1] : M_i > n_k^{(1-\epsilon)/s}\} \geq 2 \right\}.
\]

Then,

\[
Q(A_k \text{ for infinitely many } k) = 0.
\]
Proof

Since \( \{M_i\}_{i \in \mathbb{Z}} \) is i.i.d. under \( Q \),

\[
Q \left( \exists j \in J_{n_k} \text{ such that } \sum_{i = (j-1)a_k, (j+1)a_k-1}^{(j+1)a_k-1} M_i > n_{k}^{(1-\epsilon)/s} \right) \\
\leq 3 \frac{n_k}{a_k} Q(\sum_{i = 0, 2a_k - 1}^{2a_k - 1} M_i > n_{k}^{(1-\epsilon)/s} ) \geq 2).
\]

For simplicity, let us denote \( N := \sum_{i = 0, 2a_k - 1}^{2a_k - 1} M_i > n_{k}^{(1-\epsilon)/s} \). Then, \( N \) is a binomial random variable with parameter \( n = 2a_k \) and \( p = Q(M_0 > n_{k}^{(1-\epsilon)/s}) \). Using the inequality \((1 - np) \leq (1 - p)^n\) for \( n \geq 0 \) and \( 0 \leq p \leq 1 \),

\[
Q(N \geq 2) = 1 - (1 - p)^n - np(1 - p)^{n-1} \leq n(n - 1)p^2 \leq (np)^2.
\]

Recall that \( a_k = n_{k}^{1/s}/D \) with some fixed constant \( D > 1 \) and \( Q(M_0 > n_{k}^{(1-\epsilon)/s}) \leq Cn^{\epsilon - 1} \) for some constant \( C > 0 \). Then, we have

\[
3 \frac{n_k}{a_k} P(N \geq 2) \leq 3 \frac{n_k}{a_k} \left( \frac{2Ca_k}{n_{k}^{1-\epsilon}} \right)^2 \leq \frac{C'}{n_{k}^{1-\epsilon - 2\epsilon}}, \text{ for some } C' > 0.
\]

Since \( 1 - 1/s - 2\epsilon > 0 \) by our assumption, the conclusion follows from the Borel-Cantelli Lemma.

Finally, we show that for some subsequence of \( n_k \) the sums of \( \beta_i^j \) corresponding to big hills are bounded above by \( \epsilon' n_{k}^{1/s} \) for any \( \epsilon' > 0 \).

**Corollary 4.4.9** Suppose \( 0 < \epsilon < \frac{s-1}{2s} \). Then, for any \( \epsilon' > 0 \),

\[
Q \left( \max_{j \in J_{n_k}} \sum_{i = (j-1)a_k}^{(j+1)a_k-1} \beta_i^j I_{\{M_i > n_{k}^{(1-\epsilon)/s}\}} < \epsilon' n_{k}^{1/s} \text{ i.o.} \right) = 1.
\]

**Proof** First, we prove that,

\[
Q \left( \max_{j \in J_{n_k} \setminus \{-1, 0, 1\}} \sum_{i = (j-1)a_k}^{(j+1)a_k-1} \beta_i^j I_{\{M_i > n_{k}^{(1-\epsilon)/s}\}} < \epsilon' n_{k}^{1/s} \text{ i.o.} \right) = 1. \tag{4.51}
\]

Since \( n_k = m^n \) for some \( m > s \) and \( a_k = n_{k}^{1/s}/D \) we have that \( n_{k-1} < a_k \) for all \( k \) large enough. Therefore, \( [\nu_{-n_{k-1}}, \nu_{n_{k-1}}] \subset [\nu_{-a_k}, \nu_{a_k}] = [\nu(-1, k), \nu(1, k)] \) and due to the reflections used in the definition of \( \beta_i^j \), the event inside the probability in (4.51) is
independent of the environment in the interval \([\nu_{-n_k-1}, \nu_{n_k-1}]\). Therefore, the events inside (4.51) are an independent sequence for \(k\) large enough and so to prove (4.51) by the second Borel-Cantelli Lemma it is enough to show that

\[
\sum_{k=1}^{\infty} Q \left( \max_{j \in J_{n_k} \setminus \{-1,0,1\}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} < \epsilon' n_k^{1/s} \right) = \infty. \tag{4.52}
\]

To prove (4.52), note that

\[
Q \left( \max_{j \in J_{n_k} \setminus \{-1,0,1\}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} < \epsilon' n_k^{1/s} \right) \geq Q \left( \max_{j \in J_{n_k}} \frac{\beta_i^j}{(2n_k)^{1/s}} < \epsilon' / 2 \right) \geq Q \left( \max_{i \in [-n_k, n_k-1]} \frac{\beta_i}{(2n_k)^{1/s}} < \epsilon' / 2 \right). \tag{4.53}
\]

It was shown in [8, Proposition 5.1] that \(\{\frac{\beta_i}{(2n)^{1/s}}, -n \leq i < n\}\) converges weakly to a nonhomogeneous Poisson point process with intensity measure \(\gamma x^{-s-1} dx\) for some \(\gamma > 0\). Hence, the probabilities in (4.53) are uniformly bounded away from 0 for all \(k\) and thus (4.52) follows.

By Lemma 4.4.7 and 4.4.8, we have, for \(k\) large enough,

\[
\max_{j \in J_{n_k} \setminus \{-1,0,1\}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} = \max_{j \in J_{n_k}} \beta_i^j I_{\{M_i > n_k^{(1-\epsilon)/s}\}} = \max_{(j-1)a_k 

\sum_{i = (j-1)a_k} I_{\{M_i > n_k^{(1-\epsilon)/s}\}}. \tag{4.54}
\]

Hence, the conclusion of the Corollary follows from (4.51) and (4.54).

We are now ready to give the proof of the main result of this section.
Proof [Proof of Proposition 4.4.1] Recall that \(a_k = \frac{n_k^{1/s}}{b_k}\) and \(L_k = \frac{n_k}{a_k(1-\delta)} = \frac{D_0}{1-\delta}n_k^{1-1/s}\) for some fixed \(\delta > 0\). And, choose \(\lambda = \lambda_k = \frac{D_0}{n_k}\) for any fixed \(D_0 > 0\). Taking \(L = L_k\) in (4.41),

\[
P_\omega(T_{\nu nk} > u\nu nk)
\]

\[
\leq P_\omega(N_k \neq \tilde{N}_k) + P_\omega(\tilde{N}_k > L_k, N_k = \tilde{N}_k) + P_\omega\left(\sum_{i=1}^{N_k} \Theta_i > u\nu nk, \tilde{N}_k \leq L_k\right).
\]

We have proved in Lemma 4.4.4 and (4.45) that the first two terms on the right side decay exponentially for \(Q\)-a.e. environment \(\omega\). Consequently,

\[
\lim_{n \to \infty} \frac{1}{n^{1-1/s}} \log \left\{ P_\omega(N_k \neq \tilde{N}_k) + P_\omega(\tilde{N}_k > L_k, N_k = \tilde{N}_k) \right\} = -\infty. \tag{4.55}
\]

Regarding the third term, for each \(i \leq \tilde{N}_k\) the distribution of the crossing time \(\Theta_i\) is determined by the location \(Z_{i-1} \in J_{nk}\). Also, since \(\tilde{N}_k \leq L_k\),

\[
P_\omega\left(\sum_{i=1}^{N_k} \Theta_i > u\nu nk, \tilde{N}_k \leq L_k\right) \leq P_\omega\left(\sum_{i=1}^{L_k} \Theta_i I_{\{Z_{i-1} \in J_{nk}\}} > u\nu nk\right)
\]

\[
\leq E_\omega \left[ \prod_{i=1}^{L_k} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} \right] e^{-\lambda_k u\nu nk}, \tag{4.56}
\]

where the second inequality comes from Chebyshev’s inequality. Now, we claim that

\[
E_\omega \left[ \prod_{i=1}^{L} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} \right] \leq \left( \max_{j \in J_{nk}} E_\omega^{\nu(j,k)} [e^{\lambda_k T_{\nu(j+1,k)}}] \right)^L, \quad \text{for any } L \geq 1. \tag{4.57}
\]

To see this, let \(\mathcal{G}_i := \sigma(X_n : n \leq \sum_{i=1}^i \Theta_i)\) be the \(\sigma\)-field generated by the walk up until the \(i\)-th step of the induced birth-death process on the blocks. Then,

\[
E_\omega \left[ \prod_{i=1}^{L} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} \right] = E_\omega \left[ E_\omega \left[ \prod_{i=1}^{L} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} | \mathcal{G}_{L-1} \right] \right]
\]

\[
= E_\omega \left[ \prod_{i=1}^{L-1} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} E_\omega \left[ e^{\lambda_k \Theta_L} I_{\{Z_{L-1} \in J_{nk}\}} | \mathcal{G}_{L-1} \right] \right]
\]

\[
\leq \max_{j \in J_{nk}} E_\omega^{\nu(j,k)} [e^{\lambda_k T_{\nu(j+1,k)}}] \times E_\omega \left[ \prod_{i=1}^{L-1} e^{\lambda_k \Theta_i} I_{\{Z_{i-1} \in J_{nk}\}} \right] \times E_\omega \left[ e^{\lambda_k \Theta_L} I_{\{Z_{L-1} \in J_{nk}\}} \right],
\]
where the last inequality comes from (4.46), and then (4.57) follows by induction. Applying (4.57) with $L = L_k$, we have

$$
\liminf_{k \to \infty} \frac{1}{n_k^{1-1/s}} \log P_{\omega} \left( \sum_{i=1}^{N_k} \Theta_i > u \nu_{n_k}, \tilde{N}_k \leq L_k \right)
\leq \liminf_{k \to \infty} \frac{1}{n_k^{1-1/s}} \log \left\{ \max_{j \in J_{n_k}} \left( e^{\nu_{(j-1,k)}} [e^{\lambda_k T_{(j+1,k)}}] \right)^{L_k} e^{-\lambda_k u \nu_{n_k}} \right\}
= \liminf_{k \to \infty} \frac{L_k}{n_k^{1-1/s}} \left( \log \max_{j \in J_{n_k}} e^{\nu_{(j-1,k)}} [e^{\lambda_k T_{(j+1,k)}}] \right) - \frac{\lambda_k u \nu_{n_k}}{n_k^{1-1/s}}
\leq \liminf_{k \to \infty} \frac{L_k}{n_k^{1-1/s}} \left( \max_{j \in J_{n_k}} \sinh(\lambda_k (E_{\omega,(j-1,k)}^\nu[T_{(j+1,k)}])) - \frac{\lambda_k u \nu_{n_k}}{n_k^{1-1/s}} \right),
$$

(4.58)

where the first inequality comes from (4.56) and (4.57), and the last inequality comes from (4.48). Recall from Lemma 4.4.6 that, for any $\epsilon_1 > 0$ and $0 < \epsilon < \frac{s-1}{2s}$, there is a $K(\omega)$ such that for all $k \geq K(\omega)$,

$$
\max_{j \in J_{n_k}} \sum_{i=(j-1)u_k}^{(j+1)u_k-1} \beta_i^j \mathbf{1}_{\{M_i \leq n_k^{(1-\epsilon)/s}\}} \leq \frac{E_Q[\beta_0] + \epsilon_1/2}{D} n_k^{1/s}.
$$

(4.59)

On the other hand, by Corollary 4.4.9 with $\epsilon' = \epsilon_1/D$, we can find an environment dependent subsequence of $n_k$ defined as $n_{k'}$ such that

$$
\max_{j \in J_{n_{k'}}} \sum_{i=(j-1)u_{k'}}^{(j+1)u_{k'}-1} \beta_i^j \mathbf{1}_{\{M_i > n_{k'}^{(1-\epsilon')/s}\}} \leq \frac{\epsilon_1}{D} n_{k'}^{1/s}.
$$

(4.60)

Then, by a choice of $D > 2(E_Q[\beta_0] + \epsilon_1)D_0$, (4.49) is satisfied for some subsequence $k'$. Therefore, we can conclude that (4.58) is bounded above by

$$
\lim_{k \to \infty} \frac{L_k}{n_k^{1-1/s}} e^{-\lambda_k} \left( e^{\nu_{(j-1,k)}} [e^{\lambda_k T_{(j+1,k)}}] \right)^{\frac{3\epsilon_1/2}{D} n_k^{1/s}} - \frac{\lambda_k u \nu_{n_k}}{n_k^{1-1/s}}
= \frac{D_0(E_Q[\beta_0] + 3\epsilon_1/2)}{(1-\delta)(1 - 2D_0(E_Q[\beta_0] + \epsilon_1)/C)} - D_0uE_Q[\nu_1], \quad Q\text{-a.s.},
$$

(4.61)
where in the last equality we used that \( \lambda_k = D_0 n_k^{-1/s} \), \( L_k = \frac{D_{k-1}}{1-s} n_k^{-1/s} \) and the fact that \( \nu_n/n \to E_Q[\nu_1], Q\text{-a.s.} \). In summary, we have shown that for any \( D_0, \epsilon_1, \delta > 0 \) and for all sufficiently large \( D < \infty \) that

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{\nu_n} > u\nu_n) \leq \frac{D_0(E_Q[\beta_0] + 3\epsilon_1/2)}{(1 - \delta)(1 - 2D_0(E_Q[\beta_0] + \epsilon_1)/D)} - D_0 u E_Q[\nu_1], \quad Q\text{-a.s.}
\]

By first taking \( D \to \infty \) and then letting \( \epsilon_1, \delta \to 0 \), we can thus conclude that

\[
\liminf_{n \to \infty} \frac{1}{n^{1-1/s}} \log P_\omega(T_{\nu_n} > u\nu_n) \leq D_0 E_Q[\nu_1] \left( \frac{E_Q[\beta_0]}{E_Q[\nu_1]} - u \right), \quad Q\text{-a.s.,} \quad (4.62)
\]

for any \( D_0 < \infty \). Finally, since

\[
\frac{1}{v_\alpha} = \lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{T_{\nu_n}}{\nu_n} = \lim_{n \to \infty} \frac{T_{\nu_n} n}{\nu_n} = \frac{E_Q[\beta_0]}{E_Q[\nu_1]},
\]

it follows that the term in parenthesis in (4.62) is negative for \( u > 1/v_\alpha \), and thus the right side of (4.62) can be made smaller than any negative number by choosing \( D_0 \) sufficiently large.

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REFERENCES
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VITA

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