Towards Stability Criteria For Multidimensional Distributed Systems: Buffered Aloha Case

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Abstract

A distributed system can be viewed as a multidimensional, not necessarily Markovian, stochastic process over a large (typically infinite) state space. Assessing stability of such multidimensional systems is notoriously difficult. In this paper we consider the standard discrete-time slotted ALOHA system with a finite number of buffered users. The stability region for this system is known only for two users and for the symmetric system. We propose a new method of studying the stability of distributed systems - including ALOHA - by means of a simple concept of isolating single users, applying Loynes' stability criteria for an isolated queue, and using stochastic dominance to verify required stationarity assumptions in the Loynes' criterion. As a result, we derive sufficient conditions and necessary conditions for stability of the ALOHA system, and we also indicate that these conditions are sufficient and necessary. In fact, our method allows to assess stability of a subset of users in the ALOHA system. Such a stability we name partial stability, and it is of considerable interest to engineers. Finally, we generalize our approach to assess stability of a class of distributed systems using a more sophisticated extension of the ALOHA system. This generalized approach is next illustrated on coupled-processors systems.

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1. INTRODUCTION

A fundamental issue in the design of any distributed system is its stability, loosely defined as its ability to possess required properties in the presence of some disturbances. Hereafter, by stability we understand an ability of a system to keep a quantity of interest (e.g., queue length, waiting time, etc.) in a bounded region, or more restrictly the existence of the limiting distribution for the quantity of interest. Important examples of distributed systems are local area networks (e.g., ALOHA system, Ethernet, FDDI ring, token ring), multiprocessor systems (e.g., concurrent execution of tasks on multiprocessors), distributed computations (cooperative problem solving by sets of distributed processors), etc. More general and thus more important examples are multidimensional queuing systems with applications which include the ALOHA system (Fayolle et al. [FGL77], Tsybakov and Mikhailov [TsM79], Szpankowski [SZP88], Rao and Ephremides [RaE89], Borovkov [BOR89], etc.), backoff protocol for multiaccess channels (Aldous [ALD87]), Kelly [KEL85], Hastad et al. [HLRS7], Goodman et al. [GGM88]), data base systems with concurrent processing (Courcoubetis et al. [CRS87], Tsitsiklis et al. [TPH86], Baccelli and Liu [BaL89]), and so forth. From these examples it is clear that the stability problem is of considerable importance to the engineering community.

In this paper we concentrate on the buffered ALOHA system, propose a new method of evaluating its stability, and show that this new approach can be extended to a larger class of distributed systems. We consider a standard discrete-time slotted ALOHA system with $M$ users, each having a buffer of infinite capacity for incoming fixed-length packets (cf. [TsM79, SaE81, SZP88]). Assessing stability of the ALOHA system is notoriously difficult, and satisfactory solution to this problem is customary viewed as a good gauge how far we can advance stability analysis for a class of multidimensional distributed systems. This problem has been investigated in the past by several people. Stability analysis of the buffered ALOHA system was initiated by Tsybakov and Mikhailov [TsM79] who obtained a simple upper bound for the stability region, and exact sufficient and necessary conditions for the ergodicity of the symmetric system (e.g., all input rates and probability of transmissions are the same). These authors used the stochastic dominance technique to derive their bound. This was simplified and generalized in Szpankowski [SZP88] who derived some improved upper bounds for the stability region, and some new lower bounds. The same technique was used by Falin [FAL88] to obtain similar stability criteria for a more general arrival process. The Lyapunov test function approach was first adopted by Falin [FAL81] who derived better upper bounds for the stability region in the case of very asym-
metric system (e.g., very different input rates and probability of transmissions). This was further improved in Szpankowski [SZP88]. Recently Rao and Ephremides [RaE89] using the stochastic dominance approach constructed the best up-to-date upper bound for not-too-asymmetric buffered ALOHA system. Finally, Anantharam [ANA88] for very simple model of the arrival process – computed the ergodicity region for another formulation of the stability problem. Namely, the stability region considered therein contains every input rate vector for which there exists such a vector of transmission probabilities resulting in the stable ALOHA system. This is a different stability problem, and it was first investigated by Tsybakov and Mikhailov [TsM79] (see also Rao and Ephremides [RaE89]). It is easy to notice that stability region of this kind is an envelope of the stability region that we plan to investigate, and while the latter does not have a closed-form solution for stability condition, the former one enjoys such an explicit solution. At last, explicit solution to $M = 2$ user ALOHA system is known since the paper of Tsybakov and Mikhailov due to pioneering work of Malyshev [MAL72] (cf. also Rosenkrantz [ROS89], Payolle [PAY89], and Vaninskii and Lazareva [VaL88]). In addition, Mensikov [MEN74], and Malyshev and Mensikov [MaMS81] constructed stability criteria for a special class of three-dimensional Markov chains, but they are very difficult to verify in practice. Despite the fact that these criteria are known for almost twenty years, very few real systems have been analysed through this approach (see also [RaE89]).

Our approach to the stability problem of the ALOHA (and some other distributed) systems is new, and we do not use the Lyapunov test function approach. We based our idea on three simple techniques. Namely, at first we show that stability of an $M$-dimensional ALOHA system can be reduced to stability of an isolated single queue. Secondly, we apply an old result of Loynes [LOY62] that allows to assess stability of a general $G|G|1$ queue with any dependent arrival and service processes. Finally, to verify a technical stationarity requirement in Loynes' criteria we will apply the stochastic dominance technique. Using this approach we shall derive in this paper sufficient conditions and necessary conditions for the stability of an $M$ users ALOHA system. We also indicate that these conditions are sufficient and necessary. As expected, such a criterion depends on the probability of whether users are empty or not, and therefore no closed-form solution exists. This is an inherent characteristic of stability for the ALOHA system (cf. [SZP90]), and many other multiqueue distributed systems. Our method can be used to derive all other stability bounds found so far. We re-derive some old bounds, and present some new bounds. In fact, our criterion can be used to construct stability region of the $M$ users ALOHA system from the stability region of the ALOHA system with $K < M$ users (i.e., by induction). In particular,
for $M = 2$ and $M = 3$ we present explicit formulas for the stability region. Finally, our technique allows to assess stability of a subset of users in the ALOHA system, and we coin a term partial stability for this type of stability. Some preliminary results of this study have been presented in [SzR87] and [SZP90].

Finally, we discuss a generalization of our idea to a larger class of multiqueue distributed systems (e.g., token passing ring, coupled-processors, etc.). This generalization is based on three concepts discussed above with some modifications. Namely, we view a single user as an isolated queue with dependent service times. This service time is not the physical service time of a customer, but a new construction called the effective service time. Such an effective service time includes the physical service time, but in the presence of other queues the actual (i.e., effective) service time stretches out to comprehend the effect of the other queues. For such a modified queue we apply Loynes' criteria to assess stability of the queue in terms of the effective service time. This criterion involves a technical stationarity requirement. We present some results which circumvent this problem. We illustrate our new approach on a coupled-processors example.

2. MAIN RESULTS

This section present our main results concerning stability of the buffered ALOHA system. In fact, the ALOHA system serves as a motivating example for a more general stability analysis of some multiqueue distributed systems discussed in the next section.

We start with a short description of the buffered ALOHA system. The system consists of $M$ distributed users, each having an infinite buffer for storing fixed-length packets. The packets are transmitted through a broadcast channel. The channel is slotted, and a slot duration is equal to a packet transmission time. Each nonempty user transmits a packet with a probability $p_i$ in a slot, where $i \in M$ and $M = \{1, 2, \ldots, M\}$ is the set of users. If two or more users transmit simultaneously, then a collision occurs and the packets must be retransmitted in the future. When exactly one packet is transmitted in a slot, then a successful transmission takes place, the packet is removed from its queue, and another packet, if the queue is nonempty, gets its chance to be served. The arrival process is i.i.d. with respect to slots, and arrival processes are independent from a user to a user. Let $N_j^t$ represent the queue length in the $j$th user at the beginning of the $t$th slot, where $i$ is a nonnegative integer that indexes slots. Under the above assumptions, the $M$-dimensional process $N^t = (N_1^t, N_2^t, \ldots, N_M^t)$ is a Markov chain [SeE81, SZP86]. To see this, we note that the $j$th queue evolves according to the following stochastic equation

$$N_j^{t+1} = [N_j^t - Y_j^t]^+ + X_j^t$$  \hspace{1cm} (2.1a)

3
where $X^j_t$ represents the number of new customers arriving during the $t$th slot to the $j$th user, and $Y^j_t$ takes only two values, namely $Y^j_t = 1$ when a transmission is successful, and $Y^j_t = 0$ otherwise. In the above, $x^+ = \max\{0, x\}$. According to our assumptions, $X^j_t$ is an i.i.d. sequence of random variables with respect to $t = 0, 1, \ldots$ and $j \in \mathcal{M}$. We also assume that $X^j_t$ has finite first moment $\lambda_j = EX^j_t < \infty$. On the other hand, $Y^j_t$ depends on the $M$-dimensional vector $N^t = (N^t_1, \ldots, N^t_M)$, and as easy to see (cf. [SZP86]) for every $j \in \mathcal{M}$ we have

$$Y^j_t = R^j_t \left[ 1 - \sum_{k \in \mathcal{M} - \{j\}} R^t_k \chi(N^t_k) \right]^+, \quad (2.1b)$$

provided the event $\{N^t_j > 0\}$ holds. In the above, the transmission decision variable $R^t_k$ is equal to one when the $k$th user attempts to transmit in the $t$th slot and zero otherwise. Also, by definition $\chi(x) = 1$ for $x > 0$ and $\chi(0) = 0$. In words, (2.1a) and (2.1b) imply directly that $N^t$ is a Markov chain. Our task is to find conditions under which this Markov chain is ergodic (stable). In passing, we note that $Y^j_t$ is defined only for nonempty $j$th queue.

2.1 Preliminary results

In this subsection, we set up a general methodology to deal with stability problems for a class of multiqueue systems. Therefore, for the purpose of this subsection, we assume that a system is described by a multidimensional process $N^t = (N^t_1, N^t_2, \ldots, N^t_M)$ where the $j$th component $N^t_j$ of $N^t$ satisfies the stochastic equation (2.1), however, we do not require that $N^t$ is a Markov chain (e.g., we allow $X^j_t$ and $Y^j_t$ in (2.1) to be general processes representing the arrival stream and the output traffic, respectively). By stability of such processes we mean that the distribution of $N^t$ as $t \to \infty$ exists and the distribution is honest. In other words, $N^t$ is stable if for $x \in \mathcal{Z}^M$, where $\mathcal{Z}$ is a set of nonnegative integers, the following holds

$$\lim_{t \to \infty} \Pr\{N^t < x\} = F(x) \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1 \quad (2.2a)$$

where $F(x)$ is the limiting distribution function, and by $x \to \infty$ we understand that $x_j \to \infty$ for all $j \in \mathcal{M} = \{1, \ldots, M\}$. If a weaker condition holds, namely,

$$\lim_{x \to \infty} \lim_{t \to \infty} \inf \Pr\{N^t < x\} = 1, \quad (2.2b)$$

then the process is called substable [LOY62] or tight [BIL86] or bounded in probability sense. Otherwise, the system is unstable (for more details see [LOY62, BOR76, WAL88, BFL90]). The relationship between stability and substability, is of course, that a stable sequence is necessary substable, and a substable sequence is stable if the distribution function tends to
a limit. For example, if $N'$ is an aperiodic and irreducible Markov chain, then substability is equivalent to stability, since a limiting distribution exists (it may be degenerate) for any such Markov chain. In addition to this standard definitions, we also introduce a new notion of partial (sub)stability. Let $\mathcal{K}$ be a subset of $\mathcal{M}$. By $N^t_{\mathcal{K}}$ we denote a $|\mathcal{K}|$-dimensional process consisting of those coordinates of the original process $N^t$ that indices fall into the set $\mathcal{K}$. In other words, let $i_j \in \mathcal{K} \subset \mathcal{M}$ for $j = 1, \ldots, |\mathcal{K}|$. Then, $N^t_{\mathcal{K}} = (N^t_{i_1}, \ldots, N^t_{i_{|\mathcal{K}|}})$. We say that a system is partially (sub)stable with respect to $\mathcal{K}$ if the process $N^t_{\mathcal{K}}$ is (sub)stable according to (2.1a) or (2.1b).

Now we are ready to present our first preliminary result that is further referred as isolation lemma. Below, we prove that for substability of the process $N^t$ one requires stability (substability) of all its components.

**Lemma 1.** (i) If for all $j \in \mathcal{M}$ the one dimensional processes $N^t_j$ are stable (substable), then the $M$-dimensional process $N^t = (N^t_1, N^t_2, \ldots, N^t_M)$ is substable.

(ii) If for some $j$, say $j^*$, $N^t_{j^*}$ is unstable, then $N^t$ is also unstable.

**Proof.** We first prove part (i). Since each component of the process $N^t$ is at least substable, then by definition (2.2) for all $j \in \mathcal{M}$

\[
\lim_{x_j \to \infty} \lim_{t \to \infty} \Pr\{N^t_j > x_j\} = 0
\]

But

\[
1 \geq \lim_{x_j \to \infty} \lim_{t \to \infty} \Pr\{N^t_j \leq x_j, \text{ for } j = 1, 2, \ldots, M\} \geq 1 - \sum_{j=1}^{M} \lim_{x_j \to \infty} \lim_{t \to \infty} \Pr\{N^t_j > x_j\} = 1
\]

Thus

\[
\lim_{x_j \to \infty} \lim_{t \to \infty} \Pr\{N^t < x\} = 1
\]

and $N^t$ is substable by (2.2b). If $N^t$ is a Markov chain, then substability implies stability.

For part (ii) we notice that instability of $N^t_{j^*}$ implies

\[
\lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf \Pr\{N^t_{j^*} < x_{j^*}\} < 1
\]

Then

\[
\lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf \Pr\{N^t < x\} \leq \lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf \Pr\{N^t_{j^*} < x_{j^*}\} < 1
\]

which proves Lemma 1. 

In summary, the above isolation lemma moves the burden of the stability analysis from a multidimensional process $N^t$ itself to an individual, isolated components $N^t_j$. In our case,
\( N_j \) represents the queue length satisfying (2.1). We note that an isolated queue length \( N_j \) is not Markovian even when the whole process \( N_t \) is a Markov chain. Fortunately, stability for a general \( G|G|1 \) queue was investigated in depth by Loynes [LOY62] (see also [WAL88], [BFL90]), who proved the following important result. Below, a slight variation of Loynes' result, adopted to equation (2.1), is presented.

**Theorem 2.** (Loynes 1962). Let the pair \( \{X_j, Y_j\} \) be a strictly stationary and ergodic (metrically transitive) process. We denote by \( EX_j = \lambda_j \) and \( EY_j \) the corresponding mean values of \( X_j \) and \( Y_j \) respectively. Then the following holds

(i) if \( \lambda_j < EY_j \), then the queue \( G|G|1 \) described by (2.1) is stable in the sense of definition (2.2a),

(ii) if \( \lambda_j > EY_j \), then the \( G|G|1 \) queue is unstable, and \( \lim_{t \to \infty} N_j = \infty \) (a.s.),

(iii) if \( \lambda_j = EY_j \), then the queue may be stable, substable or unstable. If \( Y_j \) and \( X_j \) are independent of each other, and one of them is formed of non-constant mutually independent random variables, then the queue is unstable.

**Proof.** A simple proof of the above is already given by Loynes (cf. also [WAL88]). For the second part of (ii), we proceed as follows. Let \( U_j = X_j - Y_j \), and note that \( U_j \) is strictly stationary under the assumption of the theorem. Then, (2.1a) implies immediately

\[
N_j \geq N_j^0 + \sum_{k=0}^t U_j^k .
\]

But, by Strong Law of Large Numbers (SLLN) \( \sum_{k=0}^t U_j^k \to \infty \) (a.s.) provided \( EU_j > 0 \) [BIL86]. This implies that with a positive probability there exists a sample path of \( N_j \) such that it never returns to zero. This simple fact is of prime importance for our stability analysis.

**Remark 2.1.** We note that (2.1a) can be immediately reduced to Loynes’ standard formula (cf. [LOY62]) by setting \( W_j^{t+1} = N_j^{t+1} - X_j \) and \( U_j^t = X_j^{t-1} - Y_j \), so then

\[
W_j^{t+1} = (W_j^t + U_j^t)^+ .
\]  

**Remark 2.2.** From the above discussion, it should be clear that \( Y_j \) is defined conditionally on \( N_j \) > 0. Indeed, to measure \( EY_j \) we observe \( Y_j \) during \( m \) slots when the \( j \)-th queue is
nonempty (i.e., only in busy periods of the \( j \)th queue). Then, by stationarity of \( Y_j \) and by SLLN [BIL86] one obtains the following

\[
EY_j = \lim_{m \to \infty} \frac{\sum_{t=1}^{m} Y_j^t}{m} \quad \text{a.s.}
\]

(2.4)

In passing, we note that \( EY_j \) represents the average successful service rate, and hence \( 1/EY_j \) stands for the average effective service time (see (3.3) in Section 3). Having this in mind, we see that \( Y_j \) can be only defined during a busy period (see also Section 3).

2.2 Stability criteria for the buffered ALOHA system

Hereafter, \( N_t \) is an \( M \)-dimensional Markov chain representing the queue lengths in the ALOHA system at the beginning of the \( t \)th slot. By Theorem 2, for stability of \( N_t \) we need to evaluate \( EY_j \) which is equal to the probability of a successful transmission from the \( j \)th user in the \( t \)th slot. We define \( EY_j = P_{\text{suc}}^{(j)} \) provided \( Y_j \) is a stationary sequence. It is evident that this probability is a function of the whole process \( N_t \), however, ultimately \( P_{\text{suc}}^{(j)} \) is a function a non-Markovian process \( Z^t = (Z^t_1, Z^t_2, \ldots, Z^t_M) \) where \( Z^t_j = \chi(N^t_j) \) and \( \chi(\cdot) \) is defined after equation (2.1b). In words, \( Z^t_j \) indicates whether the \( j \)th queue is empty or not. To describe all possible states of this process, we introduce an \( M \)-dimensional zero-one vector \( z = (z_1, z_2, \ldots, z_M) \) such that for every \( j \in M \) one has \( z_j \in \{0, 1\} \). The set of all zero-one \( M \)-tuples is denoted by \( \Theta_M \), that is,

\[
\Theta_M = \{ z : z = (z_1, \ldots, z_M), \ z_j \in \{0, 1\}, \ j \in M \}.
\]

In addition, \( z^{(j)} \in \Theta_{M-1} \) denotes \((M-1)\)-tuple with the \( j \)th coordinate deleted, i.e., \( z^{(j)} = (z_1, z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_M) \in \Theta_{M-1} \). In the same manner, we define an \((M-1)\)-dimensional vector \( Z^{(j)}_t \) as a part of \( Z_t \) vector with the \( j \)th component deleted. For a stationary \( Z^{(j)}_t \) we set \( P(z^{(j)} \mid N^t_j > 0) = \Pr[Z^{(j)}_t = z^{(j)} \mid N^t_j > 0] \).

According to Theorem 2, for stability of the Markov chain \( N_t \) we need to evaluate the probability of success \( P_{\text{suc}}^{(j)} = EY_j \) for all \( j \in M \). We recall that \( Y_j \) is defined only for non-empty \( j \)th queue. For ergodic and stationary \( N_t \), we define the probability of success \( P_{\text{suc}}^{(j)} \) as follows

\[
P_{\text{suc}}^{(j)} = r_j \sum_{\pi^{(j)} \in \Theta_{M-1}} P(z^{(j)} \mid N^t_j > 0) \prod_{k \in M-\{j\}} (1 - r_k)^{x_k}
\]

(2.5)

Then, the following fact is direct a consequence of Lemma 1 and Theorem 2.

**Lemma 3. Necessity conditions.** (i) If the \( M \)-dimensional Markov chain \( N_t \) is ergodic (stable), then

\[
\lambda_j \leq P_{\text{suc}}^{(j)} \quad \text{for all} \ j \in M
\]

(2.6a)
where $P_{i,j}$ is given in (2.5).

(ii) Sufficiency condition. If $Y^t = (Y_1^t, \ldots, Y_M^t)$ is strictly stationary (ergodic) sequence, and if

$$\lambda_j < EY_j \quad \text{for all } j \in \mathcal{M},$$

then the Markov chain $N^t$ is stable.

Proof. To prove necessity we apply Lemma 1, Theorem 2, and the fact that $N^t$ is a Markov chain. By Lemma 1 we know that stability of $N^t$ requires substability of every component $N_j^t$, $j \in \mathcal{M}$. By Theorem 2 we need only to show that $Y^t$ is a strictly stationary and ergodic sequence. Since $N^t$ is ergodic, we can select an initial probability for $N^0$ in such a manner that it is equal to the limiting distribution. Then, $N^t$ is stationary and ergodic, and naturally $Y^t$ given by (2.1b) is stationary and ergodic since $Y^t$ is a measurable function (cf. (2.1b)) of $N^t$ (cf. [BRE68, Chap. 6]). Therefore, we can apply Loynes' criterion. Finally, note that for an irreducible and aperiodic Markov chains defined on a countable state space, the limiting distribution does not depend on the initial distribution. This completes the proof of part (i). For part (ii), the stationarity assumption of $Y^t$ is explicitly involved in the formulation of the theorem, so it follows directly from Lemma 1 and Theorem 2.

The main purpose of this paper is to prove that slightly modified condition (2.6a) is sufficient and necessary for stability of the ALOHA system (see Remark 2.5 after Theorem 5). Moreover, we would like to reformulate (2.6a) (i.e., re-compute (2.5)) in such a manner that one can design a simple numerical algorithm to evaluate the stability region. To accomplish this, we first deal with sufficient condition, and then return to the necessary condition.

Sufficient Conditions

The trouble with Lemma 3(ii) is that one needs to verify stationarity of $Y^t$ in order to prove sufficient condition for stability. In general this is difficult, and another approach has to be adopted. The idea is to apply a stochastic dominance technique [STO83, SZP88, RA89] together with Theorem 2. This relies on a very simple observation that for the ALOHA system the increase of the queue length at any user, say the $\ell$th one, leads to the increase of queue lengths in the whole system, and decrease of probabilities of success. In other words, this can be summarized as follows

$$N^t_{\ell} \leq_{st} \bar{N}^t_{\ell} \quad \text{and} \quad \forall j \in \mathcal{M} \setminus \{\ell\}, \quad N^t_j = \bar{N}^t_j \quad \Rightarrow \quad \forall j \in \mathcal{M}, \forall \ell \geq \ell, \quad N^t_j \leq_{st} \bar{N}^t_j \quad \text{and} \quad Y^t_j \geq_{st} Y^t_{\ell},$$

(2.7)
where \( \leq_{st} \) means stochastically smaller [STO83]. If the stationarity of \( Y^i_j \) is easier to establish than for the original system (and it is usually the case), then by Lemma 3(ii) \( \lambda_j < EY_j \) for all \( j \in M \) will imply stability of \( N^i \). In passing, we note that (2.7) can be rigorously established by the Sample Path Theorem [STO83], and the reader is referred to [SZP88] or [RaE89] for such a proof. In this paper we adopt (2.7) without further discussions.

We now construct two dominant systems \( \bar{A} \) and \( \bar{A} \) of the original ALOHA system which hereafter we also denote as \( A \). We concentrate on the stability of one queue, say the \( j \)th one, and define \( M_j = M - \{ j \} \). For given \( j \), the first dominant system \( \bar{A}_j \) is characterized by the following properties:

- arrival processes \( \{ X_i \}_{i=1}^M \) in the original system \( A \) and the dominant \( \bar{A}_j \) are identical,
- transmission decisions \( \{ R_i \}_{i=1}^M \) in both systems are identical,
- whenever the \( j \)th queue is empty, it continues to transmit "dummy packets" with transmission probability \( r_j \), and the other queues behave as in the original system \( A \), that is, the \( j \)th queue is viewed by other queues as never empty.

By (2.7), \( \bar{A}_j \) dominates \( A \), and we write it as \( A \preceq \bar{A}_j \). This implies that \( N_j^i \leq \bar{N}_j^i \) and \( N_{(j)}^i \leq \bar{N}_{(j)}^i \), where \( N_{(j)}^i = (N_1^i, ..., N_{j-1}^i, N_{j+1}^i, ..., N_M^i) \). We note also that the dominant system \( \bar{A}_j \) is really the ALOHA system with \( M - 1 \) users and slightly modified probability of transmissions (e.g., for the \( l \)th user the probability of transmission changes to \( r_l(1 - r_j) \), but probability of being silent is still equal to \( 1 - r_l \)). Finally, we note that the \( j \)th queue may be stable even if some of the remaining queues are unstable. Therefore, we partition the set of other queues \( M_j \) in \( \bar{A}_j \) into a set \( S \) of stable queues and the set \( U \) of unstable queues, that is, \( M_j = S \cup U \). For a given partition \( P = (S, U) \) we write \( \bar{A}_j^{(S,U)} = \bar{A}_j^P \) for the first dominant system.

For a given partition \( P = (S, U) \) of \( M_j \), we define the second dominant system \( \bar{A}_j^{(S,U)} = \bar{A}_j^P \) as the one satisfying the above three properties of the first dominant system \( \bar{A}_j^P \), and the following additional postulate:

- all stable queues in the new dominant system \( \bar{A}_j^{(S,U)} \) work as in \( \bar{A}_j^{(S,U)} \), but unstable queues in \( \bar{A}_j^{(S,U)} \) send dummy packets whenever empty, that is, unstable queues – from the perspective of all other queues – are never empty.

By (2.7), for a given partition \( P = (S, U) \) we have \( A \preceq \bar{A}_j^P \preceq \bar{A}_j^P \). In \( \bar{A}_j^P \) we define
the probability of success as

\[ P_{\text{succ}}^{(i)}(\mathcal{P}) = r_i \prod_{k \in \mathcal{M}} (1 - r_k) \sum_{z^{(j)}_s \in \Theta_{|S|}} P(z^{(j)}_s) \prod_{k \in S} (1 - r_k)^{x_k} \quad (2.8) \]

where \( z^{(i)}_s \in \Theta_{|S|} \), that is, \( z^{(i)}_s = (z_i, z_{i+1}, \ldots, z_{|S|}) \) and \( i_k \in S \) with \( i_k \neq j \) for all \( k \in \{1, 2, \ldots, |S|\} \). The probability \( P(z^{(j)}_s) \) is evaluated as follows. Let \( \overline{N} = (\overline{N}_j, \overline{N}^{(j)}) \), and we partition \( \overline{N}^{(j)} \) as \( \overline{N}^{(j)} = (\overline{N}_S, \overline{N}_U) \). Note that \( \overline{N}_S \) is an \( |S| \) -dimensional stable Markov chain. For a stationary version of \( \overline{N}_S \) we compute \( P(z^{(j)}_S) = \Pr(\chi(\overline{N}_S) = \overline{z}_S = z^{(j)}_S) \) where \( \overline{Z}_S \) is an \( |S| \) -dimensional zero-one process representing empty-nonempty queues in the set \( S \). In passing, we stress that \( P(z^{(j)}_S) \) in contrast to \( P(z^{(j)}_S | N_i > 0) \) is unconditional probability, with the \( j \)th queue never empty.

The above construction suggests the following computable sufficient conditions for partial stability of the ALOHA system.

**Theorem 4. Sufficient Conditions.** For a particular \( j \) and for a particular partition \( \mathcal{P} = (S, \mathcal{U}) \), let \( C^{(j)}_P \) be the region of the vectors \( \lambda^{(j)} = (\lambda_1, \ldots, \lambda_j-1, \lambda_j+1, \ldots, \lambda_M) \) such that \( \mathcal{P} \) is actually the partition of the users of \( M_j = M - \{j\} \) into stable and unstable ones. Let \( K \subset M \) be a subset of users. Then, the system is partially stable with respect to \( K \) in the region \( R_K = \cap_j R_j \) with

\[ R_J = \bigcup_{\mathcal{P}} \{ (\lambda_1, \ldots, \lambda_M) : \lambda^{(j)} \in C^{(j)}_\mathcal{P} \text{ and } \lambda_j < P_{\text{succ}}^{(j)}(\mathcal{P}) \} , \]

where \( P_{\text{succ}}^{(j)}(\mathcal{P}) \) is given in (2.8), and the above summation is taken over all partitions \( \mathcal{P} \) of \( M_j \).

**Proof.** It follows directly from our dominance relationship \( A \preceq \overline{A}_j^\mathcal{P} \preceq \overline{A}_j^\mathcal{P} \), and Lemma 3. Indeed, we only need to show that there exists a stationary version of \( \overline{Y}_j^\mathcal{P} \), or equivalently stationary version of \( \overline{N}_S \). But this is easy since \( \overline{N}_S \) is a stable (ergodic) \( |S| \) -dimensional Markov chain, and hence it has a stationary and ergodic version as required. This completes the proof by the same arguments as in the proof of Lemma 3(i). •

**Remark 2.3.** The probability of success given in (2.8) is almost of the same form as the one given in (2.5). Indeed, for a given partition \( \mathcal{P} = (S, \mathcal{U}) \) we can write (2.8) as

\[ P_{\text{succ}}^{(j)}(\mathcal{P}) = r_j \sum_{z^{(j)}_S \in \Theta_{M-1}} P_j(z^{(j)}_S) \prod_{k \in M - \{j\}} (1 - r_k)^{x_k} , \quad (2.8a) \]
where \( P_j(z^{(j)}) = P(z^{(j)}) \) in \( \overline{A}_j \), and this probability is unconditional, that is, the \( j \)th queue is never empty (or sends dummy packets whenever empty). Moreover, in (2.8a) we implicitly assume that \( P_j(z^{(j)}) = P(z^{(j)}_1)P(z^{(j)}_2) \) where \( P(z^{(j)}_1) = 1 \) if and only if \( z^{(j)}_1 = 1_U \) (\( 1_U \) is a \( |U| \)-dimensional vector of ones) and otherwise \( P(z^{(j)}_1) = 0 \).

Remark 2.4. Theorem 4 presents computable stability conditions even if (2.8) seems to be cumbersome to evaluate. This is evident since the probability of success \( P^{(j)}_{\text{succ}}(\mathcal{P}) \) can be computed by induction. Indeed, if one knows how to evaluate \( P(z^{(j)}) \) for the ALOHA system with \( K < M \) users, then these estimates can be used to estimate \( P(z^{(j)}) \) in \( M \) users ALOHA, hence to assess stability conditions for \( M \) users ALOHA system in the dominant system \( \overline{A}_j \). Since a single user ALOHA is simple, hence by induction we can estimate stability regions for multi-users ALOHA system. We illustrate this approach for \( M = 2 \) and \( M = 3 \) ALOHA system below.

**Necessary Conditions**

Theorem 4 provides sufficient condition for partial stability. If one is only interested in the stability of the whole system, then necessary condition can be formulated. In such a case, it is convenient to reformulate the stability conditions of Theorem 4. In particular, we obtain another – more compact – representation for the stability region \( \mathcal{R}_M \). Let \( \mathcal{P}^{(j)}(\mathcal{M}_j,\emptyset) \), that is, \( \mathcal{P}^{(j)} \) is a partition with \( \mathcal{M}_j \) stable queues and none unstable queues. Note that there are \( M \) such partitions of \( \mathcal{M} \), one for each queue \( j \). In the ALOHA system, as well as many other distributed systems (e.g., token passing rings, coupled-processors, etc.) the following representation of the whole stability region \( \mathcal{R}_M \) defined in Theorem 4 holds

\[
\mathcal{R}_M = \bigcup_{\mathcal{P}^{(j)}} \{ (\lambda_1, \ldots, \lambda_M) : \lambda^{(j)} \in C^{(j)}_{\mathcal{P}^{(j)}}, \text{ and } \lambda_j < P^{(j)}_{\text{succ}} \}, \quad (2.9a)
\]

where

\[
P^{(j)}_{\text{succ}} = P^{(j)}_{\text{succ}}(\mathcal{P}^{(j)}) = r_j \sum_{x^{(j)} \in \Theta_{M-1}} P(z^{(j)}) \prod_{k \in M-\{j\}} (1 - \tau_k)^{x_k}. \quad (2.9b)
\]

where \( P(z^{(j)}) \) is computed in \( \overline{A}_j^{\mathcal{P}^{(j)}} = \overline{A}_j \mathcal{P}^{(j)} \), and the summation in (2.9b) is taken only over the partitions \( \mathcal{P}^{(j)} \). Roughly speaking, the representation (2.9a) is true since for the stability of the whole system one requires stability of all queues, that is, stability of \( \mathcal{M}_j \) queues when investigating stability of the \( j \)th queue. Finally, note that for (2.9) one must consider only \( M \) partitions instead of \( 2^M \) required for the criteria of Theorem 4.

Now we are ready to discuss necessary stability condition for the whole system. Using representation (2.9), we will show that for the partition \( \mathcal{P}^{(j)} = (\mathcal{M}_j,\emptyset) \) the original system
$A$ and the dominant system $\mathcal{A}_j^{(\mathcal{M}_j,\emptyset)}$ are equivalent from the instability viewpoint. More precisely, let for a given partition $\mathcal{P}_j$ of $\mathcal{M}_j$ the original system $A$ be also denoted as $A^{P(j)}$. Naturally, $A^{P(j)} \leq \mathcal{A}_j^{P(j)} = \mathcal{A}_j^{P(j)}$. We shall prove that with a nonzero probability $A^{P(i)} = \mathcal{A}_j^{P(j)}$ whenever the $j$th queue is unstable. The equivalence between $A^{(\mathcal{M}_j,\emptyset)}$ and $\mathcal{A}_j^{(\mathcal{M}_j,\emptyset)}$ for unstable $j$th queue relies on the observation that these two systems are indistinguishable as long as the $j$th queue in $A^{P(j)}$ does not empty out.

**Theorem 5. Necessary Condition.** (i) Let hypotheses and notations of Theorem 4 hold together with (2.9), that is, the stability region $\mathcal{R}_\mathcal{M}$ obtained in Theorem 4 has the representation given in (2.9a). Define for every $j \in \mathcal{M}$ and its partition $\mathcal{P}_j$, a new set $\mathcal{R}_j$ as follows

$$\mathcal{R}_j = \{(\lambda_1, \ldots, \lambda_M) : \lambda_j \in \mathcal{C}^{(j)}_{\mathcal{M}_j,\emptyset} \text{ and } \lambda_j > \mu^{(j)}_{\text{succ}}\}, \quad (2.10a)$$

where $\mu^{(j)}_{\text{succ}}$ is defined in (2.9b). Then, the $\text{ALOHA}$ system is unstable in the region $\mathcal{R}_\mathcal{M} = \bigcup_{j \in \mathcal{M}} \mathcal{R}_j$.

(ii) In addition, if $\mathcal{R}_\mathcal{M} = \bigcap_{j=1}^M \mathcal{R}_j$ where

$$\mathcal{R}_j = \{(\lambda_1, \ldots, \lambda_M) : \lambda_j > r_j \prod_{i=1,i\neq j}^M (1-r_i)\}, \quad (2.10b)$$

then the system is unstable in $\mathcal{R}_\mathcal{M}$ too.

**Proof.** Part (ii) of the theorem was already proved for the $\text{ALOHA}$ system by Tsybakov and Mikhailov [TsM79], and in a more general form, by Szpankowski [SZP88]. Therefore, we shall concentrate on part (i). Let $\lambda \in \mathcal{R}_j$. From our construction of Theorem 4, we know that for this $\lambda$ the $j$th queue is unstable in the dominant system $\mathcal{A}_j^{P(j)}$. We now prove that for the same $\lambda$ the $j$th queue is unstable in the original system $A^{P(j)}$. This is done by showing that for $\lambda \in \mathcal{R}_j$ with a nonzero probability both systems $A^{P(j)}$ and $\mathcal{A}_j^{P(j)}$ are indistinguishable, that is, the $j$th queue never empties out in $A^{P(j)}$. We use anti-coupling arguments to prove this. Indeed, let $N_j^0 = \overline{N}_j = m > 0$ and $N_j^0 = \overline{N}_j$ for all $\ell \in \mathcal{M}_j$. Define $T = \min\{\tau : N_j^\tau = \overline{N}_j = 0\}$. Note that for $t < T$ both systems are identical. It suffices to show that $\Pr\{T = \infty\} > 0$ for $\lambda \in \mathcal{R}_j$. Consider $0 \leq t < T$. We argue in terms of the dominant system $\mathcal{A}_j^{P(j)}$. To establish the claim we refer to Theorem 2(ii). For this we need stationarity and ergodicity of $Y_j^j$ in $\mathcal{A}_j^{P(j)}$. Since all queues in $\mathcal{M}_j$ are stable, then by Lemma 1 also the Markov chain $\overline{N}_{\mathcal{M}_j}$ is stable. Choosing appropriate initial distribution, we consider only stationary and ergodic version of the Markov chain $\overline{N}_{\mathcal{M}_j}$. But then $Y_j^j$ is stationary and ergodic, since it is a measurable function of $\overline{N}_{\mathcal{M}_j}$ (cf. [BER68, Chap. 6]).
To complete the proof note that for $0 < t < T$ we have $Y^j_t = Y^j_0$, hence by Theorem 2(i) $N^j_t \to \infty$ (a.s.), and $\Pr \{ T = \infty \} > 0$. That is, the queue is unstable in $\mathcal{A}^P(j)$ whenever it is unstable in $\mathcal{A}^P_\lambda(j)$ for $\lambda \in \mathcal{R}_j$. This completes the proof. •

Remark 2.5. Informally, we shall argue that Theorem 4, together with Theorem 5, establish sufficient and necessary stability conditions for the ALOHA system. First of all, note that the complementary set $\mathcal{R}_M = \cap_{j=1}^M \mathcal{R}_j$ to the stability region $\mathcal{R}_M$ is not the sum of $\mathcal{R}_M$ and $\mathcal{R}_M$. In fact, it is easy to show that $\mathcal{R}_M \cup \mathcal{R}_M \subset \mathcal{R}_M$, and the inclusion is proper except in the case of $M = 2$ users. To see this, note that

$$\mathcal{R}_j = \{ (\lambda_1, \ldots, \lambda_M) : \lambda^{(j)} \notin C^{(j)}_{(\lambda_j, \emptyset)} \text{ or } \lambda_j > P^{(j)}_{\text{succ}} \} . \quad (3.10c)$$

(In this informal discussion we ignore boundary points of $\mathcal{R}_M$.) To prove that indeed $\mathcal{R}_M$ is the instability region, we argue as follows. From Theorem 5 we know that $\lambda \in \mathcal{R}_j$ belongs to instability of both $\mathcal{A}^P(j)$ and $\mathcal{A}^P_\lambda(j)$. We also know that with a nonzero probability the above two systems are identical as long as the instability is concerned, and in particular the $j$th queue in $\mathcal{A}^P(j)$ never returns to zero (which is necessary for the equivalence of the systems).

Now let us consider a partition of $\mathcal{M}_j$ that has one more unstable queue, say the $k$th one, that is, $\mathcal{P}(j) = (M_j - \{k\}, k)$. Such a partition can be reached from $\mathcal{P}(j)$ by increasing the input rate $\lambda_k$ to the $k$th queue. But by (2.7) this can only increase all other queue lengths, in particular, for the $j$th queue. Note, that in the region $\mathcal{R}_j$ the $j$th queue with nonzero probability never returns to zero for $\mathcal{P}(j)$, hence even more likely the queue has a sample path never hitting zero for the partition $\mathcal{P}(jk) = (M_j - \{k\}, k)$. Therefore, $\mathcal{A}^P(jk) = \mathcal{A}^P_\lambda(jk)$ as long as the instability is concerned. This extends the instability region to the partition $\mathcal{P}(jk)$. Repeating this process, we will finally exhaust all possible partitions, and prove instability in the region $\mathcal{R}_j$. These arguments can be formalized, but unfortunately this requires very heavy notations. Therefore, we postpone it to a forthcoming paper. In the next subsection, however, we present more detailed discussion for the case of $M = 3$ users.

Remark 2.6. In the proof of Theorem 5 we have used the fact that the stability region $\mathcal{R}_M$ defined in Theorem 4 can be represented in the form of (2.9a). Although intuitively (2.9a) is obvious, a formal proof of this fact is required for the ALOHA system. Cumbersome algebraic manipulations are needed for such a proof, and they do not contribute too much to a real understanding of the stability region of the ALOHA (as long as the probability $P(z^{(j)})$ is unknown). To circumvent this, we brought (2.9a) into the hypotheses of Theorem 5. We do prove below (2.9a) for $M = 2$ and $M = 3$ since in these cases we can compute explicitly the probabilities $P(z^{(j)})$, and hence the stability region. Finally, the representation (2.9) is
not required for validity of Theorem 5. We adopt it mainly to simplify the description of the stability region.

2.3 Special cases and bounds

In this subsection we apply Theorems 4 and 5 to derive explicit formulas for the stability regions for the case of $M = 2$ and $M = 3$ users. In addition, we construct some bounds for the stability region in the case $M > 3$. In fact, we indicate that any bound obtained so far (cf. [TsM79, SZP88, RaE89]) can be derived from our results.

Let us start with $M = 2$ users ALOHA system. Using (2.8a) we have

$$P^{(1)}_{\text{suc}}(P) = \tau_1\{P_1(0) + \tilde{r}_1P_1(1)\},$$

(2.11a)

$$P^{(2)}_{\text{suc}}(P) = \tau_2\{P_2(0) + \tilde{r}_2P_2(1)\}.$$  

(2.11b)

To recall, $P_1(0) = 1 - P_1(1) = \Pr\{N_1^t = 0 \mid N_1^t > 0, \forall \tau \leq t\}$ and the same for $P_2(0)$ and $P_2(1)$, that is, these probabilities are computed in the dominant systems $\overline{A}_1$ and $\overline{A}_2$. Naturally, in $\overline{A}_1$ we have $P_1(0) = \max\{0, 1 - \lambda_2/\tau_2\tilde{r}_1\}$, and similar in $\overline{A}_2$.

Now, we are ready to apply Theorem 4 and Theorem 5 to establish sufficient and necessary condition for stability. Using the construction from Theorem 4, we identify two partitions in $\overline{A}_1$, namely $\mathcal{P}_1 = (S, U) = (\{2\}, \emptyset)$ and partition $\mathcal{P}_2 = (S, U) = (\emptyset, \{2\})$. Moreover, using the above formula on $P_1(0)$ we note that for $\lambda_2 < \tau_2\tilde{r}_1$ the partition $\mathcal{P}_1$ takes place, while for $\lambda_2 > \tau_2\tilde{r}_1$ the partition $\mathcal{P}_2$ should be used. Therefore, for $\lambda_2 < \tau_2\tilde{r}_1$, (2.11a) implies

$$P^{(1)}_{\text{suc}}(\mathcal{P}_1) = \tau_1(1 - \lambda_2/\tilde{r}_1)$$

while for $\lambda_2 > \tau_2\tilde{r}_1$

$$P^{(1)}_{\text{suc}}(\mathcal{P}_2) = \tau_1\tilde{r}_2.$$ 

In a similar manner we obtain

$$P^{(2)}_{\text{suc}}(\mathcal{P}_1) = \tau_2(1 - \lambda_1/\tilde{r}_2) \quad \text{for} \quad \lambda_1 < \tau_1\tilde{r}_2,$$

$$P^{(2)}_{\text{suc}}(\mathcal{P}_2) = \tau_2/\tilde{r}_1 \quad \text{for} \quad \lambda_1 \geq \tau_1\tilde{r}_2,$$

and the first equality is obtained for the partition $\mathcal{P}_1' = (S, U) = (\{1\}, \emptyset)$ and the second for $\mathcal{P}_2' = (S, U) = (\emptyset, \{1\})$ of $M_2$. In Figure 1 for every queue and every partition we show stability regions $\mathcal{R}_1(\mathcal{P}_1), \mathcal{R}_2(\mathcal{P}_2), \mathcal{R}_1(\mathcal{P}_1'), \text{and} \mathcal{R}_2(\mathcal{P}_2')$. The total stability region for the whole system becomes $\mathcal{R} = \{\mathcal{R}_1(\mathcal{P}_1) \cup \mathcal{R}_2(\mathcal{P}_2)\} \cap \{\mathcal{R}_1(\mathcal{P}_1') \cup \mathcal{R}_2(\mathcal{P}_2')\}$. We prove below that the system is stable if and only if the input rates $(\lambda_1, \lambda_2)$ lie inside $\mathcal{R}$. 

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In order to apply Theorem 5, we need to check whether the representation (2.9a) holds. But this is easy, since in this case according to Figure 1 $\mathcal{R} = \mathcal{R}_1(P_1) \cup \mathcal{R}_2(P_1')$, as needed for (2.9a). Note that both partitions $P_1$ and $P_1'$ are of the form $(M_j, \emptyset)$. Moreover, $\mathcal{R} = \mathcal{R} \cup \mathcal{R}'$ where $\mathcal{R}, \mathcal{R}'$ and $\mathcal{R}$ are defined in (3.10a), (3.10b) and (3.10c) respectively. In fact, $\mathcal{R}$ is a complementary set to the stability region $\mathcal{R}$ except boundary points (which are excluded from both $\mathcal{R}$ and $\mathcal{R}'$). In conclusion, Theorem 4 and Theorem 5 imply the following corollary.

**Corollary 6.** For $M = 2$ users buffered ALOHA system, the Markov chain $N^t = (N^t_1, N^t_2)$ is stable for all $(\lambda_1, \lambda_2) \in \mathcal{R}$ such that

$$\mathcal{R} = \{\lambda_1 < r_1(1 - \lambda_2/\bar{r}_1) \text{ and } \lambda_2 < \bar{r}_1 r_2 \} \cup \{\lambda_1 < \bar{r}_1 \bar{r}_2 \text{ and } \lambda_2 < r_2(1 - \lambda_1/\bar{r}_2)\},$$

(2.12a)

and the system is unstable for $(\lambda_1, \lambda_2) \in \mathcal{R}'$ where

$$\mathcal{R}' = \{\lambda_1 > r_1(1 - \lambda_2/\bar{r}_1) \text{ or } \lambda_2 > \bar{r}_1 r_2 \} \cap \{\lambda_1 > \bar{r}_1 \bar{r}_2 \text{ or } \lambda_2 > r_2(1 - \lambda_1/\bar{r}_2)\},$$

(2.12b)

where $\bar{r}_j = 1 - r_j$. ■

Now we consider the case of $M = 3$ queues which is by far more difficult. Using the construction from the Theorem 4, as in (2.8a) we have in $\mathcal{A}_1$

$$\underline{P}^{(1)}\text{succ}(P) = r_1P_1(0, 0) + \bar{r}_2P_1(1, 0) + \bar{r}_3P_1(0, 1) + \bar{r}_2\bar{r}_3P_1(1, 1),$$

(2.13)

---

1 A similar statement is established in Rao and Ephremides [RaE89]. In fact, the proof in [RaE89] is not complete since the instability region $\mathcal{R}$ is ignored. In addition, no formal proof for instability of $\mathcal{R}$ is given in [RaE89]. Nevertheless, the idea of the instability in Corollary 6 is similar to the one in [RaE89].
and similarly in $\overline{A}_2$ and $\overline{A}_3$. Consider now only $\overline{A}_1$. The remaining queues $\overline{M}_1 = \{2,3\}$ are divided into four partitions, namely: (i) $\overline{P}_1 = \{(S,\mathcal{U})=\emptyset,\{2,3\}\}$, (ii) $\overline{P}_2 = \{\{2\},\{3\}\}$, $\overline{P}_3 = \{\{3\},\{2\}\}$, and (iii) $\overline{P}_4 = \{\{2,3\}\}$. Case (i) is easy since both queues are unstable in $\overline{A}_1$, hence $P_1(0,0) = P_1(1,0) = P_1(0,1) = 0$ and $P_1(1,1) = 1$. This implies that $E^{(1)}_{\text{succ}}(\overline{P}_1) = r_1\tilde{r}_2\tilde{r}_3$. Therefore, for this partition the stability region $\mathcal{R}_1(\overline{P}_1)$ is defined as

$$\mathcal{R}_1(\overline{P}_1) = \{\lambda_1 < r_1\tilde{r}_2\tilde{r}_3, \lambda_2 > r_2\tilde{r}_1\tilde{r}_3, \lambda_3 > r_3\tilde{r}_2\tilde{r}_1\}.$$  

(2.14)

In the second case, one queue is stable and one is unstable. Consider first $\overline{P}_2$. Then, in $\overline{A}_1$ we have $P_1(0,0) = P_1(0,1) = 0$, and $P_1(1,1) = 1 - P(1,0) = \lambda_3/r_3\tilde{r}_1\tilde{r}_2$. Elementary computations show that the first queue is stable in the region $\mathcal{R}_1(\overline{P}_2)$ defined as

$$\mathcal{R}_1(\overline{P}_2) = \{\lambda_1 < r_1\tilde{r}_3[1 - \lambda_2/(\tilde{r}_1\tilde{r}_3)], \lambda_2 < \tilde{r}_1r_2\tilde{r}_3, \lambda_3 > \tilde{r}_1r_3(1 - \lambda_2/(\tilde{r}_1\tilde{r}_3))\},$$  

(2.15a)

and in a similar manner for $\overline{P}_3$ we have

$$\mathcal{R}_1(\overline{P}_3) = \{\lambda_1 < r_1\tilde{r}_2[1 - \lambda_3/(\tilde{r}_1\tilde{r}_2)], \lambda_2 > \tilde{r}_1r_2\tilde{r}_3, \lambda_3 < \tilde{r}_1\tilde{r}_2r_3\}.$$  

(2.15b)

In the third case, two remaining queues are stable, hence we must compute the joint probabilities $P_1(0,0), P_1(0,1), P_1(0,1)$ and $P_1(1,1)$. Note that these probabilities are evaluated in $\overline{A}_1^{P_4} = \overline{A}_1^{P_4}$, that is, when the first queue is never empty and the other two queues are stable. In other words, one needs to solve two-users ALOHA model with slightly modified probabilities of transmission. Such an analysis was done by Nain in [NAI85]. We briefly summarize some of Nain’s results adopted to our setting. Let $F_i(x, y)$ denote the generating function of $(N_1, N_2)$ under the condition $N_i > 0$ for all $t$. Then, with a minor modification, it is proved in [NAI85] (see also [SZP86]) that

$$\begin{align*}
\lambda_2 &= \tilde{r}_1r_2r_3[1 - F_1(0,1)] + \tilde{r}_1r_2r_3[F_1(1,0) - F_1(0,0)] \\
\lambda_3 &= \tilde{r}_1r_2r_3[1 - F_1(1,0)] + \tilde{r}_1r_2r_3[F_1(0,1) - F_1(0,0)]
\end{align*}$$

Noting that $P_1(1,0) = F_1(1,0) - F_1(0,0), P_1(0,1) = F_1(0,1) - F_1(0,0), P_1(1,1) = 1 - F_1(0,1) - F_1(1,0) + F_1(0,0)$, and taking into account the above we easily compute the probability of success in $\overline{A}_1$, namely

$$E^{(1)}_{\text{succ}}(\overline{P}_3) = r_1 \left\{ 1 - \frac{\lambda_2\tilde{r}_2/\tilde{r}_1 + \lambda_3\tilde{r}_3/\tilde{r}_1 + r_2r_3[F_1(0,0) - 1]}{1 - r_2 - r_3} \right\}.$$  

(2.16a)

The probability $P_1(0,0)$ shown above is computed in [NAI85] using the method of the Riemann-Hilbert boundary problem (see (4.10) in [NAI85]), where either $(r_2 + r_3 \neq 1)$

$$P_1(0,0) = \left[ 1 - \frac{\lambda_2}{\tilde{r}_3\tilde{r}_1} - \frac{\lambda_3}{r_3\tilde{r}_1} \right] \exp\left[ \frac{\gamma(1)}{2\pi i} \int_{|t|=1} \frac{\log g(t)}{t[t - \gamma(1)]} dt \right]$$  

(2.17a)
or

\[ P_1(0,0) = \left[ 1 - \frac{\lambda_2}{\tau_2} - \frac{\lambda_3}{\tau_3} \right] \exp \left[ \frac{\gamma(t)}{2i\pi} \right] \int_{t=1}^{\infty} \frac{\log g_1(t)}{t(t - \gamma(t))} \, dt \]  

(2.17b)

depending on whether \( P_1(0,0) \) is computed from \( F_1(0,y) \) or \( F_1(x,0) \). But \( F_1(x,0) |_{x=0} = F(0,y) |_{y=0} = P_1(0,0) \), and note that the first term in (2.17a) can be expressed in two different ways as shown in (2.17). The region of validity of (2.17a) and (2.17b) is defined in [NAI85]. In (2.17), \( \gamma(x) |_{x=1} \) is the inverse of a conformal mapping of a unit circle onto a curve \( L_x \) defined in [NAI85] (see [NAI85], p.54 and Lemma 4.1). The functions \( g(t) \) and \( g_1(t) \) are defined in [NAI85], too (see [NAI85], p. 58).

Now, we can compute the stability region of the first queue under the partition \( P_4 \). After some algebra we obtain

\[ R_1(P_4) = \{ \lambda_1 < P_{\text{succ}}^1(P_4), \lambda_2 < \bar{r}_1 \tau_2(1 - \lambda_3/(\bar{r}_1 \tau_3)), \lambda_3 < \bar{r}_1 \tau_3(1 - \lambda_2/(\bar{r}_1 \tau_3)) \} \]  

(2.18)

where \( P_{\text{succ}}^1(P_4) \) is computed in (2.16a). This completes the stability analysis of the first queue. For the system stability we must repeat the above for the second and third queues. In particular, we define in the same manner as above the partitions \( P'_i \) and \( P''_i \), \( i = 1,2,3,4 \), of \( M_2 \) and \( M_3 \) respectively, and we construct stability regions \( R_2(P'_i) \) and \( R_2(P''_i) \) for \( 1 \leq i \leq 4 \). For the fourth partitions \( P'_4 \) and \( P''_4 \) we need to replace (2.16a) with

\[ P_{\text{succ}}^2(P'_4) = r_2 \left\{ 1 - \frac{\lambda_1 \bar{r}_1 / \bar{r}_2 + \lambda_3 \bar{r}_3 / \bar{r}_2 + \bar{r}_1 \tau_3 P_3(0,0) - 1}{1 - \tau_1 - \tau_3} \right\} \]  

(2.16b)

\[ P_{\text{succ}}^3(P''_4) = r_3 \left\{ 1 - \frac{\lambda_1 \bar{r}_1 / \bar{r}_3 + \lambda_2 \bar{r}_2 / \bar{r}_3 + \bar{r}_1 \tau_2 P_3(0,0) - 1}{1 - \tau_1 - \tau_2} \right\} \]  

(2.16c)

respectively, where \( P_3(0,0) \) and \( P_3(0,0) \) have the same pattern as (2.17).

Finally, the stability region of the whole system is \( R = R_1 \cap R_2 \cap R_3 \cap R_4 \), where the first queue is stable in \( R_1 = \bigcup_{j=1}^{4} R_1(P_j) \), the second queue in \( R_2 = \bigcup_{j=1}^{4} R_2(P'_j) \), and finally the last queue in \( R_3 = \bigcup_{j=1}^{4} R_3(P''_j) \). Stability region \( R \) is shown in our Figure 2.

In particular, note that the following three points belong to the boundary of the stability region: \( \omega = (\lambda_1, \lambda_2, \lambda_3) = (r_1 \bar{r}_2 \bar{r}_3, \bar{r}_1 \tau_2 \bar{r}_3, \bar{r}_1 \tau_1 \bar{r}_3) \), \( A = (\bar{r}_1 \bar{r}_2, \bar{r}_1 \tau_2, 0) \), \( B = (r_1, 0, 0) \). With (2.17b), one proves that \( \omega, B \) and \( C = (\bar{r}_1 \tau_3, 0, \bar{r}_1 \tau_3) \) belong to the boundary region, too. Using (2.16b), (2.16c) and an appropriate formula on \( P_3(0,0) \) and \( P_3(0,0) \) we can also show that \( \omega \) belongs to the boundary region, together with \( D = (0, 0, r_3), E(0, r_2 \bar{r}_3, \bar{r}_2 \bar{r}_3) \) and \( F = (0, r_2, 0) \). In passing, we stress the fact that the probability of success for the fourth partitions (cf. (2.16)) does depend explicitly on the probability \( P(0,0) \), which is a function of the input rates. This implies that the stability region for the ALOHA system for \( M \geq 3 \) is not linear function of \((\lambda_1, \lambda_2, \lambda_3)\). Moreover, (2.17) indicates that there are no simple explicit formulas for the stability region for \( M > 3 \).
Figure 2: Stability region for $M = 3$ users in the slotted ALOHA system.
To match the sufficient conditions with the necessary ones, we appeal to Theorem 5. It is easy to verify that (2.9a) holds, that is,

$$\mathcal{R} = \mathcal{R}_1(P_4) \cup \mathcal{R}_2(P'_4) \cup \mathcal{R}_3(P''_4),$$

(2.19)

where $\mathcal{R}_1(P_4)$ is computed in (2.18), while $\mathcal{R}_2(P'_4)$ and $\mathcal{R}_3(P''_4)$ stability regions follow the same pattern as in (2.18) with (2.16a) replaced by (2.16b) and (2.16c) respectively (see Fig. 2). We shall prove that the complementary set $\bar{\mathcal{R}}$ of the stability region $\mathcal{R}$ (with boundary points excluded) is the region for instability of the ALOHA system. Unfortunately, this time $\mathcal{R} \cup \bar{\mathcal{R}} \subset \mathcal{R}$, and the inclusion is proper. We first identify the points not covered by Theorem 5, and then prove — as it was indicated in Remark 2.5 — that the idea of the Theorem 5 can be extended to these points. For this we need a good understanding of the sets $\mathcal{R}$ and $\bar{\mathcal{R}}$. The set $\mathcal{R}$ is easy to visualize in Figure 2, however, some difficulties may arise with $\bar{\mathcal{R}}$. Therefore, we note that $\bar{\mathcal{R}} = \bar{\mathcal{R}}_1(P_4) \cup \bar{\mathcal{R}}_2(P'_4) \cup \bar{\mathcal{R}}_3(P''_4)$ where, for example,

$$\bar{\mathcal{R}}_1(P_4) = \{ \lambda_1 > P^{(1)}_{\text{succ}}(P_4), \lambda_2 < \bar{r}_1 r_2 (1 - \lambda_3 / (\bar{r}_1 \bar{r}_2)), \lambda_3 < \bar{r}_1 r_3 (1 - \lambda_2 / (\bar{r}_1 \bar{r}_3)) \} .$$

In Figure 3, the stability region $\mathcal{R}_1(P_4)$ is represented by the set of points inside $\omega A A' \omega' C C B$, and the instability region $\bar{\mathcal{R}}_1(P_4)$ is the "corridor" $A'' \Lambda \omega' \omega C'' C$.

Consider now a set of points not covered by $\bar{\mathcal{R}}$. For example,

$$\bar{\mathcal{R}}_4 = \{ \lambda_1 > P^{(1)}_{\text{succ}}(P_4), \lambda_2 < \bar{r}_1 r_2 \bar{r}_3, \lambda_3 < \bar{r}_1 r_3 (1 - \lambda_2 / (\bar{r}_1 \bar{r}_3)) \} ,$$

which is a direct neighbor of $\bar{\mathcal{R}}_4(P_4)$, but with the second queue also unstable. The set $\bar{\mathcal{R}}_4$ is represented in Figure 3 by points lying inside $C'' C B$. We prove now that the first queue is also unstable in $\bar{\mathcal{R}}_4$. Let us first consider a point $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ inside $\bar{\mathcal{R}}_1(P_4)$. We have already proved in Theorem 5 that in $\bar{\mathcal{R}}_1(P_4)$ the first queue is unstable, and in the original system $A^{(P_4)}$ with nonzero probability this queue grows to infinity without returning to zero. Increase now $\lambda_2$ such that the second queue becomes unstable, that is, the point $\lambda$ moves from $\bar{\mathcal{R}}_1(P_4)$ to the neighbor set $\bar{\mathcal{R}}_4$. Naturally, the increase of $\lambda_2$ leads only to the increase of the queue length $N^2_1$, and by (2.7) this finally may only imply increase of the queue length in the first queue $N^1_1$. Therefore, in this scenario the queue length $N^1_1$ even with heigher nonzero probability will not return to zero. As, in Theorem 5 we conclude that the first queue is unstable in $\bar{\mathcal{R}}_4$. In a similar manner, we prove that $\bar{\mathcal{R}}_5$ and $\bar{\mathcal{R}}_6$ defined as above (with obvious changes) are regions of instability.

In summary, we can characterize stability region of the ALOHA system with $M = 3$ users as follows.
Figure 3: Illustration to instability region for the ALOHA system with $M = 3$ users.
Corollary 7. The buffered ALOHA system with \( M = 3 \) is stable in the region \( R \) as defined in (2.19), and the system is unstable in the complementary set \( \bar{R} \) but the boundary points (where our methodology cannot give a definite answer to the stability problem).

A generalization for \( M > 3 \) is more intricate, since we need to estimate \( P(z^{(j)}) \), and so far only two-users ALOHA system has been analyzed. Nevertheless for \( M > 3 \), some bounds are easy to obtain from Theorem 4. In some cases an alternative formula for \( P(z^{(j)}) \) may be useful. Namely, noting that \( \sum_{z^{(j)} \in \Theta_{M-1}} P(z^{(j)}) = 1 \) we can easily transfer (2.8a) into

\[
P_{\text{succ}}^{(j)} = r_j \left[ 1 - \sum_{k=1}^{M-1} \sum_{(i_1, \ldots, i_k) \in \mathcal{M}_k} (-1)^k r_{i_1} \cdots r_{i_k} \Pr\{N_{i_1}^j \geq 1, \ldots, N_{i_k}^j \geq 1 \mid N_j^\tau \geq 1, \forall \tau \leq t \} \right]
\]

(2.20)

For example, the bound obtained by Tsybakov and Mikhailov [TsM79] (see also [SZP88]), directly follows from (2.8a). Indeed, since \( \prod_{k \in \mathcal{M}_j} (1 - r_k)^{z_k} \geq \prod_{k \in \mathcal{M}_j} (1 - r_k) \), one immediately proves that \( \lambda_j < r_j \prod_{k \in \mathcal{M}_j} (1 - r_k) \) for \( j \in \mathcal{M} \) is sufficient for stability. On the other hand, since \( \prod_{k \in \mathcal{M}_j} (1 - r_k)^{z_k} \leq 1 \) we prove that \( \lambda \geq r_j \) for some \( j \in \mathcal{M} \), is sufficient for instability of the ALOHA system (see also [SZP88]).

To obtain more sophisticated bounds, we need a better estimate for the probability \( P(z^{(j)}) \). Let us mention here one possibility (for a more sophisticated approach see [SZP88, RaE89]). We use (2.20) instead of (2.8a) and let the probability in (2.20) be denoted as

\[
\Pr\{N_{i_1}^j \geq 1, \ldots, N_{i_k}^j \geq 1 \mid N_j^\tau \geq 1, \forall \tau \leq t \} = P_j(1^{i_1}, \ldots, 1^{i_k})
\]

Let also for \( i_{\ell} \in \mathcal{M} - \{j\} \)

\[
P_j(1^{i_{\ell}}) = 1 - P_j(0^{i_{\ell}}) = \Pr\{N_{i_{\ell}}^j \geq 1 \mid N_j^\tau \geq 1, \forall \tau \leq t \}
\]

Note that

\[
P_j(1^n) - \sum_{i_{\ell} \neq n} P_j(0^{i_{\ell}}) = 1 - \sum_{\ell=1}^k P_j(0^{i_{\ell}}) \leq P_j(1^{i_1}, \ldots, 1^{i_k}) \leq P_j(1^{i_{\ell}})
\]

for some \( n, \ell \in \{1, 2, \ldots, k\} \). The probabilities \( P_j(1^{i_{\ell}}) \) can be estimated using the dominance arguments presented before. For example

\[
\frac{\lambda_{i_{\ell}}}{r_{i_{\ell}}} \leq P_j(1^{i_{\ell}}) \leq \frac{\lambda_{i_{\ell}}}{r_{i_{\ell}} \prod_{k=1, k \neq \ell} (1 - r_k)}
\]

(2.21)

To obtain the LHS of (2.21), it was assumed that all buffers except the \( j \)th are always empty, while in the RHS of (2.21) we have postulated that all buffers except the \( j \)th are
always nonempty. Using (2.20) and the above, upper and lower bounds on \( P_{\text{succ}}^{(j)} \) can be obtained, whence via Theorem 4 also bounds on the stability region of the ALOHA system.

Finally, the most sophisticated bound suggested by Rao and Ephremides [RaE89] (and the best up-to-date for not-very-asymmetric ALOHA system) follows from our Theorem 4. In this case, however, the estimate of \( P(z^{(j)}) \) is more careful, and therefore more lengthy. In fact, this bound extends the idea of Tsybakov and Mikhailov [TsM79] by analyzing more terms in (2.8a). More precisely, all probabilities \( P_j(z^{(j)}) \) in (2.8a) are skillfully bounded by one dimensional probability \( \Pr\{N_k > 0\} \). The interested reader is referred to the original paper [RaE89]. Other bounds suggested in [SZP88] that have been derived from the Lyapunov function approach, also follow from our Theorem 4 after some algebraic manipulations.

3. SOME GENERALIZATIONS

The construction of the stability condition from the last section seems to be a general one, and it can be extended to some other multiqueue systems that share common features with the ALOHA model. In fact, one may notice that Lemma 3, Theorem 4 and Theorem 5 are based on very general Lemma 1 and Theorem 2. This section presents some preliminary results along these lines (see also [SzR87, SZP90]). We illustrate our methodology on a coupled-processors system. In a forthcoming paper (cf. [GaS91]) we shall show how one can apply our methodology to prove stability result for the token passing ring, another long standing open problem in stability analysis of distributed systems.

In order to extend our Lemma 3, we need a generalization of a successful transmission, that is, the random variable \( Y_j \). Figure 4 illustrates a typical behavior of the \( j \)th queue isolated from a multiqueue distributed system. Note that the packet from the front of the queue will attempt a transmission, and will continue it until a successful transmission.

---

Figure 4: Illustration of the modified service time \( C_j \) for the ALOHA system (dashed area represents collision, and white boxes represent successful transmissions).
occurs. Therefore, the next packet in the queue has to wait more than one slot – which is the physical service time for a packet – to get an access to the server. In fact, it has to wait for a period of time that elapses between two successful transmissions. We call it the effective service time, and the sequence of such time intervals for the \( j \)th queue is denoted as \( C_j^t \) for \( t = 0,1, \ldots \). (when the queue is empty and a new customer arrives, then the effective service time starts at the arrival epoch.) What is the relationship between \( Y_j^t \) and \( C_j^t \)? To answer this question we adopt the following assumption:

\[(A)\] The sequence of effective service times \( C_j^t \) is a strictly stationary and ergodic random sequence with the average \( EC_j \).

To measure \( EC_j \) one observes \( n \) effective service times (only when the queue is nonempty!) and then by SLLN [BIL86]

\[
EC_j = \lim_{n \to \infty} \frac{\sum_{k=0}^{n} C_j^k}{n} \quad \text{a.s.}
\] (3.1)

To derive a relationship between \( EC_j \) and \( EY_j \) we note that \( 1 = \sum_{t=0}^{G_j} Y_j^t \) and

\[
\sum_{k=0}^{n} C_j^k = m \quad \Leftrightarrow \quad \sum_{t=0}^{m} Y_j^t = n ,
\] (3.2)

hence by (2.4a) and (3.1)

\[
EC_j = \frac{1}{EY_j} \quad \text{a.s.}
\] (3.3)

This equality suggests to call \( EY_j \) the effective service rate. Therefore, under assumption \((A)\), Lemma 3 can be rephrased as follows. The ALOHA system is stable when \( \lambda_j EC_j < 1 \) for all \( j \in M \), and unstable if \( \lambda_j EC_j > 1 \) for at least one \( j \in M \).

Now, we are a position to generalize the ALOHA example. Naturally, the notion of the effective service time is not only restricted to the ALOHA system. A class of multiqueue systems, such as coupled-processors, buffered exponential back-off, token passing rings, etc., can be studied in a similar manner. The main idea behind such a stability analysis is to use the isolation lemma to reduce the stability problem of a distributed system to assessing stability of a single isolated queue by the Loynes’ criteria (cf. Theorem 2). To proceed along these lines, one needs a precise definition of the effective service time. A mathematical construction of \( C_j^t \) can be found in [SzR87, SZP90]. For the purpose of this paper, we adopt the following informal definition. First of all, we assume that every customer has its own (physical) service time, which is distributed according to some distribution function \( B(\cdot) \). Then, the effective service time is the time that elapses from the first instance a customer
(from the front of the queue) receives access to the server until the next customer (may be a virtual one if there is no customers behind the one we consider) will have next access to the server. This definition explicitly assumes that the effective service time is defined for nonempty queues. In addition, we assume that for a newly arrived customer the effective service time starts immediately after the arrival, so vacation is not allowed. Such a general definition has the advantage of being applicable to a larger class of systems. However, every case needs some minor adjustments in the spirit of the above definition.

Now we are ready to formulate our general stability result.

**Theorem 8.** Let us consider a multiqueue system in which the effective service times $C_j$ is defined as above. The arrival process to every queue is an independent renewal process with the mean arrival rate $\lambda_j$ for $j \in M$. Let $N_t$ represent the queue length in all $M$ queues. Then, under assumption (A) the process $N_t$ is substable in the sense of definition (2.2b) if

$$\lambda_j C_j < 1 \quad \text{for all } j \in M,$$

and unstable if

$$\lambda_j C_j > 1 \quad \text{for at least one } j, \text{ say } j^* \in M.$$

**Proof.** This theorem is a direct consequence of Lemma 1 and Theorem 2. Indeed, Lemma 1 implies that for substability of $N_t$ one requires that every isolated queue is substable, which can be deduced from Theorem 2 in the presence of (A).

Applications of Theorem 8 depends upon a verification of the assumption (A). In the ALOHA case, Markovian property of $N_t$ helped to build a stationary version of $Y_j$ (cf. Theorem 4). In general, verification of (A) is difficult. It may be as difficult as establishing stability itself, but using the construction suggested in the ALOHA case, we can often circumvent this problem. Before we discuss such a construction, we present one general result concerning necessary conditions for stability. A nice feature of the lemma below is that it does not require the assumption (A) to verify necessary conditions for stability. The following theorem is a direct consequence of Little's formula.\(^3\)

\(^3\)In the token passing ring, an isolated queue can be modeled by a single queue with vacation. This is one source of troubles in the token passing ring. Another one comes from the fact that the stochastic dominance property (see (2.7) or (B) below) does not hold, since - in general - the increase in the vacation time does not lead to the increase in the queue length. Therefore, the token passing ring cannot be unified into the framework of this paper, and some additional generalizations are necessary (cf. a forthcoming paper [GeS91]).

\(^3\)The idea of applying Little's formula to this problem was suggested by Michael Kaplan, INRS, Canada.
Lemma 9. We adopt hypotheses of Theorem 8 without assumption (A). If $N^t$ is stable and assumptions of Little’s theorem [STI74, STI85] hold for every queue, then $\lambda_j E C_j \leq 1$ for all $j \in M$.

Proof. As in Theorem 8, after applying Lemma 1, we consider stability of an isolated queue, say the $j$th one. For such a queue, we apply Little’s formula to the server, that is, the waiting time has to be understood as the effective service time. But, the queue length for a single server, of course, cannot exceed one. Therefore, Little’s formula implies $\lambda_j E C_j \leq 1$, as needed.

To obtain more constructive sufficient stability conditions, we proceed as in Theorems 4 and 5. First of all, we adopt an assumption similar to (2.7), namely

(B) If $N^t \leq \overline{N}^t$ and $\forall j \in M - \{j\}$ $\overline{N}^t_j = \overline{N}^t_j$, then $\forall j \in M \forall t \geq \tau$ $\overline{N}^t_j \leq \overline{N}^t_j$ and $C^t_j \leq C^t_j$. Having (B) in mind, we construct, as in the ALOHA case, two dominant systems $\overline{A}_j^P$ and $\overline{A}_j^P$ for every partition $P$ of $M$. To recall, in $\overline{A}_j^P$ the $j$th queue sends dummy packets when empty, and in $\overline{A}_j^P$ additionally all unstable queues send dummy packets when empty. Then, direct extension of Theorem 4 and Theorem 5 lead to the following result.

Theorem 10. Assume (B) holds, and for given $j \in M$ and given partition $P = (S, U)$ the effective service time $C^t_j(P)$ in $\overline{A}_j^P$ is a stationary ergodic process. Let also for a particular $j$ and for a particular partition $P = (S, U)$, the set $C^{(j)}_P$ denotes the region of the vectors $\lambda^{(j)} = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_M)$ such that $P$ is actually the partition of the users of $M_j = M - \{j\}$ into stable and unstable ones.

(i) Sufficient Conditions. A system is partially stable with respect to $K \subset M$ in the region $R_K = \cap_{j \in K} R_j$, where

$$R_j = \bigcup_{P} \{ (\lambda_1, \ldots, \lambda_M) : \lambda^{(j)} \in C^{(j)}_P \text{ and } \lambda_j E C^t_j(P) < 1 \} . \quad (3.6)$$

(ii) Necessary Conditions. Let $P_{(j)} = (M_j, \emptyset)$, and $R_j(P_{(j)})$ denote the stability region defined in (3.6) for the partition $P_{(j)}$. Let also the whole stability region $R_M$ defined in (i) have the following representation

$$R_M = \bigcup_{j=1}^{M} R_j(P_{(j)}) . \quad (3.7)$$

Then, the system is unstable in $\hat{R}_M = \bigcup_{j=1}^{M} \hat{R}_j(P_{(j)})$ where

$$\hat{R}_j(P_{(j)}) = \{ (\lambda_1, \ldots, \lambda_M) : \lambda^{(j)} \in C^{(j)}_{P_{(j)}} \text{ and } \lambda_j E C^t_j(P_{(j)}) > 1 \} . \quad (3.8)$$
Proof. Part (i) is basically our Theorem 4 with obvious modifications. Part (ii) can be proved in a similar manner as Theorem 5. Details are left for the reader.

In passing, we mention one more result useful in verifying instability conditions. From the proof of Theorem 5 we conclude that for stability the following property is of prime importance:

(C) Let $k \in \mathcal{U}$ belongs to the set of unstable queues. Then the queue length $N^t_k$ grows to infinity almost surely. In other words, the queue returns to zero only finitely many times.

We know that (C) holds for a single queue with stationary arrival process and effective service time process. In a multidimensional environment, one can prove (cf. Theorem 5) that (C) holds if all other queues are stable. However, in general (C) may not hold, and verification of it can be sometimes very difficult. Nevertheless, we can provide one positive result. We note that if (C) is true, then the probability of returning to zero for this queue must be equal to zero, that is, for $k \in \mathcal{U}$ property (C) implies $\lim_{t \to \infty} \Pr\{N^t_k = 0\} = 0$, but in general the converse is not true. This is however hard to prove. We are able only to show the following lemma which sheds some lights on this problem.

Lemma 11. Let $N^t$ be an $M$-dimensional Markov chain whose $\mathcal{U}$ components are unstable in the sense of definition (2.2), that is, the $N^t$ can be partition as $N^t = (N^t_{\mathcal{S}}, N^t_{\mathcal{U}})$, where $N^t_{\mathcal{S}}$ and $N^t_{\mathcal{U}}$ are substable and unstable processes. If $\mathcal{U} \neq \emptyset$, then

$$
\lim_{t \to \infty} \Pr\{N^t_{\mathcal{U}} = \mathbf{0}_{\mathcal{U}}\} = 0,
$$

where $\mathbf{0}_{\mathcal{U}}$ denotes all-zeros vector of dimension $|\mathcal{U}|$.

Proof. For the simplicity of notation, we assume that $\mathcal{U} = \{1, 2, \ldots, L\}$ and $1 \leq L \leq M$. Since $N^t$ is an $M$-dimensional Markov chain, hence $\lim_{t \to \infty} \Pr\{N^t = k = (k_1, k_2, \ldots, k_M)\} = 0$. Now, using this fact and the stability of the $M$th component of $N^t$ we prove that $\lim_{t \to \infty} \Pr\{N^t_1 = k_1, \ldots, N^t_{M-1} = k_{M-1}\} = 0$ (note that the $M - 1$ dimensional process is not Markovian). Indeed, for any positive $\kappa$ and for $M \in \mathcal{S}$ we obtain

$$
\lim_{t \to \infty} \Pr\{N^t_1 = k_1, \ldots, N^t_{M-1} = k_{M-1}\} = \lim_{t \to \infty} \sum_{j=1}^{\kappa} \Pr\{N^t_1 = k_1, \ldots, N^t_M = j\} = 
$$

$$
\sum_{j=1}^{\kappa} \lim_{t \to \infty} \Pr\{N^t_1 = k_1, \ldots, N^t_M = j\} + \lim_{t \to \infty} \Pr\{N^t_1 = k_1, \ldots, N^t_M > \kappa\} \leq \lim_{t \to \infty} \Pr\{N^t_M > \kappa\}
$$
But, \( \lim_{\kappa \to \infty} \lim_{t \to \infty} \Pr\{N^t_M > \kappa\} = 0 \) since the \( M \)th component of \( N^t \) is stable (see (2.2a)). This implies \( \lim_{t \to \infty} \Pr\{N^t_1 = k_1, \ldots, N^t_{M-1} = k_{M-1}\} = 0 \) as desired. Repeating \((M-L)\) times the same type of "tightness" argument applied to stable components, we finally prove (3.9). \( \blacksquare \)

Finally, we illustrate some applications of Theorem 10. We first concentrate on two coupled-processor systems.

**Example 3.1. Two Coupled-Processors Systems**

In [Fal79] Fayolle and Iasnogrodski described a coupled-processor system. A queueing model for this consists of two \( M|M|1 \) queues with infinite capacities. The service rate of each server is \( \mu_1 \) and \( \mu_2 \) respectively, if the queues are nonempty. If the second queue is empty, then the service rate for the first queue is \( \mu_1^* \) (\( \mu_1^* \geq \mu_1 \)) and reverse, the second queue serves with rate \( \mu_2^* \) (\( \mu_2^* \geq \mu_2 \)) if the first queue is empty.

In order to apply Theorem 10 we need to define the effective service time, and in particular we must compute the average \( EC_j \) effective service time. Under \( M|M|1 \) assumption the two-dimensional process \( N^t = (N^t_1, N^t_2) \) representing queue lengths is a Markov chain. Then, in a stationary regime (i.e., when necessary stability conditions are discussed), the effective service rates \( 1/EC_j, j = 1,2 \) may be computed as follows

\[
\frac{1}{EC_1} = \mu_1 \Pr\{N^t_2 > 0 | N^t_1 > 0\} + \mu_2^* \Pr\{N^t_2 = 0 | N^t_1 > 0\} \tag{3.10a}
\]

\[
\frac{1}{EC_2} = \mu_2 \Pr\{N^t_1 > 0 | N^t_2 > 0\} + \mu_1^* \Pr\{N^t_1 = 0 | N^t_2 > 0\} \tag{3.10b}
\]

Then \( \lambda_j EC_j \leq 1, j = 1,2 \) are necessary for stability, according to Theorem 8.

As in the ALOHA case the difficulty with (3.10) is that the conditional probabilities in (3.10) are not easy to compute. To circumvent this, we appeal to Theorem 10. First of all, we note that assumption (B) (for \( \mu_i^* > \mu_i \) for \( i = 1,2 \)) is easy to verify. But, then in the dominant systems \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) we immediately prove the following

\[
\Pr\{\overline{N}^t_2 = 0\} = \Pr\{N^t_2 = 0 | N^t_1 > 0, \forall s \leq t\} = \max\{0, 1 - \lambda_2/\mu_2\} \tag{3.11a}
\]

\[
\Pr\{\overline{N}^t_1 = 0\} = \Pr\{N^t_1 = 0 | N^t_2 > 0, \forall s \leq t\} = \max\{0, 1 - \lambda_1/\mu_1\} \tag{3.11b}
\]

Furthermore, after some trivial algebra we verify that the representation (3.7) holds. This leads to the following corollary.
Corollary 11. Two couple-processors system is stable for \((\lambda_1, \lambda_2) \in \mathcal{R}\) where
\[
\mathcal{R} = \{ \lambda_1 < \mu_1^* + \lambda_2(\mu_1 - \mu_1^*)/\mu_2 \text{ and } \lambda_2 < \mu_2 \} \cup \{ \lambda_2 < \mu_1 \text{ and } \lambda_2 < \mu_2^* + \lambda_1(\mu_2 - \mu_2^*)/\mu_1 \},
\]
and the system is unstable in the complementary set \(\overline{\mathcal{R}}\) but boundary points where our methodology does not give an ultimate answer.

Stability conditions (3.12) have been derived for the first time by Fayolle and Iasnogrodkski [Fal79] using the complex analysis approach through the reduction to the Reimann-Hilbert problem. These conditions are also simple consequences of general stability criteria for two dimensional Markov chains, due to Rosenkranz [ROS89] and Fayolle [FAY89].

Finally, we note that although our method does not offer new results for the two coupled-processors system, a simple generalization of this does produce new stability results which have not been obtained by other methodologies.

Example 3.2 M Coupled-Processors

Let us consider \(M\) coupled-processors system as defined in [SZP88]. In such a system, the service rate for the \(k\)th queue depends on the state of \((k + 1)\)st \((\text{mod } M)\) queue. More precisely, the \(k\)th processor serves with rate \(\mu_k\) when the \((k + 1)\)st queue is non-empty, and with rate \(\mu_k^* \geq \mu_k\) when the \((k + 1)\)st queue is empty. Under \(M\) assumption regarding all the queues, the \(M\)-dimensional process \(N^t = (N^t_1, ..., N^t_M)\) is a Markov chain. In order to present some sufficient conditions, we apply Theorem 10 for very particular partition. For a given queue \(j\), we consider the partition \(P_j = (S, U) = (\mathcal{M}_j - \{j - 1\}, \{j - 1\} \mod M)\). Then, direct application of Theorem 10 implies that the system is stable in the region defined as below
\[
\lambda_j < \mu_j^* + \frac{\lambda_{j+1}}{\mu_{j+1}}(\mu_j - \mu_j^*) \text{ for all } j \in \mathcal{M} \mod M.
\]

It is also not difficult to notice that (3.14) is not necessary for stability. In fact, to compute sufficient and necessary stability condition for the \(M\) coupled-processors system we must know stability condition for \(K < M\) coupled-processors systems, as in the case of the ALOHA system. Using results of Fayolle and Iasnogrodkski [Fal79] we can, however, establish exact stability region for three coupled-processors system. Indeed, consider three dominant systems \(\overline{A}_1, \overline{A}_2\) and \(\overline{A}_3\). In particular, in \(\overline{A}_1\) the first queue is never empty, therefore, we reduce the problem to a two coupled-processors system with \(\mu_2 = \mu_2^*\) (since the first queue is always busy) and \(\mu_3, \mu_3^*\) as in the original three coupled-processors system.

For the stability analysis of \(\overline{A}_1\) we need the information about \(\Pr\{N^t_3 = 0\}\) (more precisely
Pr{N_3 = 0 | N_T > 0, \forall t \leq t}). But, using the results of Fayolle and Iasnogrodski [FaI79] for two coupled-processors system with \( \mu_2 = \mu_3^2 \) we obtain

\[
Pr\{N_3 = 0\} = F_1(1,0) = \frac{\mu_3^3}{\mu_3} \left( \mu_3^2 - \mu_3^3 - \frac{G(1)}{G(0)} \right) F(0,0)
\]

where

\[
F(0,0) = \frac{\mu_2 \lambda_2 - \mu_2 \lambda_3 - \mu_3^2 (\mu_2 - \lambda_2) G(0)}{\mu_2 \mu_3^3 G(1)}.
\]

In the above \( G(z) \) is a complex function that can be computed as a solution to the Riemann-Hilbert boundary problem (cf. [FaI79, pp. 341, Lemma 7.1]). Having this in mind, an application of Theorem 10 together with some additional analysis as in the case of ALOHA with \( M = 3 \) users, lead to the following corollary.

**Corollary 12.** Three coupled-processors system is stable inside the region \( \mathcal{R} \) and unstable outside \( \mathcal{R} \) (excluding boundary points), where the region \( \mathcal{R} = \bigcup_{i=1}^{3} \mathcal{R}_i(P_i) \) and

\[
\mathcal{R}_i(P_i) = \{ \lambda_i < \mu_i + F_i(1,0)(\mu_i^* - \mu_i) , \lambda_{i+1} < \mu_{i+1} \lambda_{i+2} + \lambda_{i+1}(\mu_{i+2}^* - \mu_{i+2}^*)/\mu_{i+1} \}
\]

where all indices are taken modulo 3. In the above \( F_i(1,0) \) represents the probability that \( i - 1 \mod 3 \) queue is empty in a two couple-processors systems with \( \mu_{i+1} = \mu_{i+1}^* \mod 3 \).

This probability was estimated in [FaI79], and the following formula can be used

\[
F_i(1,0) = \frac{\mu_{i+2}^*}{\mu_{i+2}} \left( \mu_{i+2}^* - \mu_{i+2}^* - \frac{G_i(1)}{G_i(0)} \right) F_i(0,0) \mod 3,
\]

where

\[
F_i(0,0) = \frac{\mu_{i+2}^* \lambda_{i+1} - \mu_{i+1}^* \lambda_{i+2} - \mu_{i+2}^* (\mu_{i+1}^* - \lambda_{i+1}) G_i(0)}{\mu_{i+1}^* \mu_{i+2}^*} \mod 3.
\]

The generating functions \( G_i(z) \) are given in [FaI79, Lemma 7.1].

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