Properties of Ambiguity Functions

John Mulcahy-Stanislawczyk

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GRADUATE SCHOOL
Thesis/Dissertation Acceptance

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By  Johnathan Mulcahy-Stanislawczyk

Entitled
Properties of Ambiguity Functions

For the degree of  Master of Science in Electrical and Computer Engineering

Is approved by the final examining committee:

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PROPERTIES OF AMBIGUITY FUNCTIONS

A Thesis

Submitted to the Faculty

of

Purdue University

by

John Mulcahy-Stanislawczyk

In Partial Fulfillment of the

Requirements for the Degree

of

Master of Science in Electrical and Computer Engineering

May 2014

Purdue University

West Lafayette, Indiana
ACKNOWLEDGMENTS

First, I would like to thank my parents for their constant support through my long, long academic journey. Second, I would like to thank Professor Bell for his great help and insight guiding me through my graduate school experience, and for the always stimulating conversations. I would also like to thank the other members of my committee, Professor Krogmeier and Professor Ersoy, for their assistance. Finally, I would like to thank the long list of people at Sandia National Laboratories who both provided me with this opportunity here at Purdue and also some incredible summers in New Mexico.

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-AC04-94AL85000. (SAND2014-3376)
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ABSTRACT

Mulcahy-Stanislawczyk, John M.S.E.C.E., Purdue University, May 2014. Properties of Ambiguity Functions. Major Professor: Mark R. Bell.

The use of ambiguity functions in radar signal design and analysis is very common. Understanding the various properties and meanings of ambiguity functions allow a signal designer to understand the time delay and doppler shift properties of a given signal. Through the years, several different versions of the ambiguity function have been used.

Each of these functions essentially have the same physical meaning; however, the use of different functions makes it difficult to be sure that certain properties hold for different formulations. This work proves several parallel properties of two versions of the asymmetric ambiguity function, the symmetric ambiguity function, and Woodwords original ambiguity function. It also provides some visualizations comparing the ambiguity functions for some common signals.
1. INTRODUCTION

Radar systems have become widely used in both civilian and military applications. The military may use them for tracking targets, while civilian applications include things such as weather radar. All of these systems work by essentially transmitting an electromagnetic pulse, and then listening to the reflection. Critically important to this is that pulses reflected by moving targets become Doppler shifted in frequency. Typically, the receivers in these systems are implemented using matched filters. Ambiguity functions describe the behavior of these Doppler shifted reflected signals after they go through a filter matched to the transmitted signal. Knowing and understanding these functions aids signal design and knowledge of radar performance.

1.1 Radar Systems

Radar systems work by transmitting an electromagnetic signal into a space and then listening for the reflected echo. Analysis of the returned signal can give information on targets in the space, such as their position, velocity, and size. The transmitted signal is usually taken to be some baseband signal $s(t)$ modulated with a radio or microwave carrier signal. The returned signal $Z(t)$ contains the reflection of $s(t)$ and noise from the channel. The reflected version of $s(t)$ will be time delayed based on the distance of from the transmitter to the target, Doppler shifted based on the relative velocity of the target, and its amplitude modified based on the radar cross section and distance to the target.
1.2 The Matched Filter Receiver

As stated above, radar systems are usually implemented using a matched filter receiver. These receivers in their simplest form essentially have three stages. First, the microwave or radio frequency received signal $Z(t)$ is demodulated, leaving only the baseband message signal $s(t)$ and noise from the channel $N(t)$. In the case of communication systems, $s(t)$ corresponds to one of the symbols of the system. For instance when transmitting binary, the signal $s_1(t)$ could correspond to a 1, while $s_0(t)$ could correspond to a 0. In radar, $s_1(t)$ could be taken to be the reflection from the transmitted signal $s(t)$ while $s_0(t)$ could be taken to be the absence of reflection. Second, the baseband signal is filtered with some sort of analysis filter, in this case, the matched filter. Finally, the result of the filtering process is sampled at some time and sent through a block to determine which case it was.

This type of receiver was originally introduced by North for radar detection of stationary targets, and is considered to be optimal under certain conditions [1]. The matched filter is simply the following:

$$h(t) = s^*(t).$$

(1.1)

In other words, the matched filter $h(t)$ for a signal $s(t)$ is $s(t)$ time reversed and complex conjugated [2].

The key assumptions here are that the received signal is identical in shape to the transmitted signal other than some change in its amplitude. Also, the channel only adds additive white Gaussian noise to the received signal with variance $N_0/2$. Under these assumptions, the matched filter receives this following signal as its input:
\[ X(t) = as(t) + N(t). \] (1.2)

Here \( s(t) \) is the transmitted message signal, \( N(t) \) is the noise term added by the transmission medium, and \( a \) is a constant value that represents the propagation loss in the amplitude of the transmitted signal. This signal \( X(t) \) then goes through the analysis filter \( h(t) \), which has the output signal \( Y(t) \).

\[ Y(t) = h(t) \ast X(t) = h(t) \ast (as(t) + N(t)) = h(t) \ast as(t) + h(t) \ast N(t) \] (1.3)

This signal is then sampled at some time, \( t_0 \), yielding \( Y(t_0) \). This value is then sent through the final block, which decides whether a reflection is present or not.

### 1.2.1 Optimal Matched Filter Decision Block

Optimal in this case means minimizing the probability of error. Because \( N(t) \) is additive white Gaussian noise, that makes \( Y(t_0) \) a Gaussian random variable under both hypotheses. In the case of a detected reflection, the variable has some mean \( y_0 \), but in the case of no reflection it has mean zero. In both cases it has the same variance \( \sigma_y \).

\[
E[Y(t_0)|\text{no reflection}] = 0 \\
Var[Y(t_0)^2|\text{no reflection}] = \sigma_y \\
E[Y(t_0)|\text{reflection}] = y_0 \\
Var[Y(t_0)|\text{reflection}] = \sigma_y
\] (1.4)

Since both cases are Gaussian random variables, the probably of error is minimized by maximizing the distance between the means of each case, and minimizing the variance. This is equivalent to maximizing the signal-to-noise ratio (SNR).

\[
\text{SNR} = \frac{(h(t) \ast s(t))^2}{E[(h(t) \ast N(t))^2]} \] (1.5)
1.2.2 Optimality of the Matched Filter

The optimum filter $h(t)$ is then the filter that maximizes the SNR. The denominator is equal to the following:

$$E[(h(t) * N(t))^2] = E \left[ \left( \int_{-\infty}^{\infty} N(\tau) h(t-\tau) \, d\tau \right) \left( \int_{-\infty}^{\infty} N(t') h(t'-t') \, dt' \right) \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[N(\tau)N(t')]h(t-\tau)h(t'-t') \, d\tau \, dt'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N_0}{2} \delta(\tau - t')h(t-\tau)h(t'-t') \, d\tau \, dt' \quad (1.6)$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t-t') \, dt'$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t'') \, dt''.$$

Using the Cauchy-Schwarz inequality, the numerator reduces to the following:

$$(h(t) * s(t))^2 = \left( \int_{-\infty}^{\infty} h(\tau) s(t-\tau) \, d\tau \right)^2$$

$$\leq \int_{-\infty}^{\infty} h^2(t) \, dt \int_{-\infty}^{\infty} s^2(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} h^2(t) \, dt \int_{-\infty}^{\infty} s^2(-t') \, dt'. \quad (1.7)$$

By a property of the Cauchy-Schwarz inequality, equality is only met on the inequality when $\lambda s^*(-t) = h(t)$. Note that $\lambda$ here also absorbs the propagation loss term $a$.

Putting this together, the expression for the SNR becomes the following:

$$\text{SNR} = \frac{\lambda \int_{-\infty}^{\infty} s^2(t) \, dt}{\frac{N_0}{2} \lambda^2 \int_{-\infty}^{\infty} s^2(t) \, dt}$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} s^2(t) \, dt. \quad (1.8)$$

This result shows that the matched filter $h(t) = s^*(-t)$ is optimal.
1.3 Ambiguity Functions

In radar systems, however, the transmitted signal is not necessarily equal to the received signal. In particular, the signal may be Doppler shifted. The Doppler shifted signal has the following complex envelope in the narrowband case:

\[ s_r(t) = s(t)e^{i2\pi \nu t}. \]  

(1.9)

The narrowband assumption works here for radar, since in general the speed of the target Doppler shifting the reflected signal is a small fraction of the speed of light, and the bandwidth of the signal is a small fraction of the carrier frequency. However, in sonar applications this may not be true where the speed of wave propagation in the medium is much slower. It should also be noted here that the optimality of the matched filter no longer holds since the received signal is no longer \( s(t) \).

At any rate, the output of the matched filter then becomes the following:

\[ \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{i2\pi \nu t} \, dt = \chi_s(\tau, \nu). \]  

(1.10)

Here \( \chi_s(\tau, \nu) \) is an ambiguity function [3]. Conceptually though, without loss of generality the complex envelope of the returned signal could also be taken to have a negative in the exponential. This produces another ambiguity function:

\[ \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi \nu t} \, dt = \Psi_s(\tau, \nu). \]  

(1.11)

Another variation can be formed by changing the convolution to the symmetric case:

\[ \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi \nu t} \, dt = \Gamma_s(\tau, \nu). \]  

(1.12)

In the original formulation by Woodward, he reverses the sign on \( \tau \), creating his version of the ambiguity function [4]:

\[ \int_{-\infty}^{\infty} s(t)s^*(t+\tau)e^{-i2\pi \nu t} \, dt = W_s(\tau, \nu). \]  

(1.13)
All of these formulations are essentially equivalent, and all of them are used within the literature. While their meanings are equivalent, there are differences in how the fundamental properties of ambiguity functions manifest themselves depending on the formulation that is being used. The rest of this work is concerned with proving various properties of several different forms of ambiguity functions and visualizing common signals for each of these formulations.
2. PROPERTIES OF AMBIGUITY FUNCTIONS

2.1 Positive Asymmetric Ambiguity Function

**Definition 1** For some signal $s(t)$, the positive asymmetric ambiguity function is given by

$$\chi_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^*(t - \tau)e^{2\pi i \nu t} \, dt.$$ 

**Theorem 1 (Value at Origin)**

$$|\chi_s(0, 0)| = E_s.$$ 

**Proof**

$$|\chi_s(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t - \tau)e^{2\pi i \nu t} \, dt \right|$$

$$|\chi_s(0, 0)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t - 0)e^{2\pi i (0) t} \, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} s(t)s^*(t) \, dt \right|$$

$$|\chi_s(0, 0)| = E_s.$$ 

**Theorem 2 (Maximum Property)**

$$|\chi_s(\tau, \nu)|^2 \leq E_s^2 = |\chi_s(0, 0)|^2.$$
Proof

\[
|\chi_s(\tau, \nu)|^2 = \left| \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{i2\pi\nu t} dt \right|^2
\]

\[
= \left| \int_{-\infty}^{\infty} [s(t)e^{i\nu t}] [s(t-\tau)e^{-i\nu t}]^* dt \right|^2
\]

\[
\leq \int_{-\infty}^{\infty} |s(t)e^{i\nu t}|^2 dt \int_{-\infty}^{\infty} |s(t-\tau)e^{-i\nu t}|^2 dt
\]

\[
= \int_{-\infty}^{\infty} |s(t)|^2 dt \int_{-\infty}^{\infty} |s(t-\tau)|^2 dt
\]

\[
= E_s^2
\]

\[
|\chi_s(\tau, \nu)|^2 \leq E_s^2 = |\chi_s(0, 0)|^2.
\]

\[\square\]

**Theorem 3 (Volume Property)**

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 d\tau d\nu = E_s^2.
\]

Proof

\[
\chi_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{i2\pi\nu t} dt.
\]

Let \(F^{-1}\) be the inverse Fourier Transform operator.

\[
\chi_s(\tau, \nu) = F^{-1}\{s(t)s^*(t-\tau)\} = F^{-1}\{\kappa(\tau, t)\}.
\]

By Parseval’s Theorem

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |F^{-1}\{x(t)\}(\nu)|^2 d\nu.
\]

\[
\int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 d\nu = \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 dt
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 d\nu d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 dt d\tau
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t)s^*(t-\tau)s^*(t)s(\tau) d\tau dt
\]

\[
= \int_{-\infty}^{\infty} |s(t)|^2 \int_{-\infty}^{\infty} |s(t-\tau)|^2 d\tau dt
\]

\[
= \int_{-\infty}^{\infty} |s(t)|^2 E_s dt
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 d\tau d\nu = E_s^2.
\]
Theorem 4 (Symmetry Property)

\[ |\chi_s(-\tau, -\nu)| = |\chi_s(\tau, \nu)|. \]

Proof

\[
\chi_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t) s^*(t + \tau) e^{-i2\pi\nu t} \, dt.
\]

Let \( t' = t + \tau. \)

\[
\chi_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t' - \tau) s^*(t') e^{-i2\pi\nu(t' - \tau)} \, dt'
\]

\[
= e^{i2\pi\nu\tau} \int_{-\infty}^{\infty} s(t' - \tau) s^*(t') e^{-i2\pi\nu t'} \, dt'
\]

\[
= e^{i2\pi\nu\tau} \left[ \int_{-\infty}^{\infty} s(t') s^*(t' - \tau) e^{i2\pi\nu t'} \right]^* \, dt'
\]

\[
= e^{i2\pi\nu\tau} \chi_s^*(\tau, \nu)
\]

\[ |\chi_s(-\tau, -\nu)| = |\chi_s(\tau, \nu)|. \]

Theorem 5 (Fourier Transform of Modulus Squared)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 e^{-i2\pi ft} e^{i2\pi\nu \tau} \, d\tau d\nu = |\chi_s(t, f)|^2.
\]
Proof

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 e^{-i2\pi f_1} e^{i2\pi t \nu} d\tau d\nu = \int_{\mathbb{R}^2} \chi_s(\tau, \nu) \chi_s^*(\tau, \nu) e^{-i2\pi f_2} e^{i2\pi t \nu} d\tau d\nu
\]

\[
= \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} s(t_1) s^*(t_1 - \tau) e^{i2\pi t_1 \nu_1} dt_1 \right) \left( \int_{-\infty}^{\infty} s^*(t_2) s(t_2 - \tau) e^{-i2\pi t_2 \nu_2} dt_2 \right) 
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} s(t_1) s^*(t_1 - \tau) s^*(t_2) s(t_2 - \tau) e^{i2\pi \nu(t_1 - t_2 + t)} e^{-i2\pi f_\tau} d\tau dt_1 dt_2 d\nu 
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(t_1) s^*(t_1 - \tau) s^*(t_2) s(t_2 - \tau) e^{-i2\pi f_\tau} \delta(t_1 - t_2 + t) dt_1 dt_2 d\tau 
\]

\[
= \int_{\mathbb{R}^2} s(t_2 - t) s^*(t_2 - t - \tau) s^*(t_2) s(t_2 - \tau) e^{-i2\pi f_\tau} dt_2 d\tau.
\]

Let \( \tau = t_2 - z \Rightarrow d\tau = -dz. \)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_2 - t) s^*(t_2 - t - (t_2 - z)) s^*(t_2) s(t_2 - (t_2 - z)) e^{-i2\pi f_\tau(t_2 - z)} dt_2 (-dz)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(z) s^*(z - t) e^{i2\pi f_\tau} s^*(t_2) s(t_2 - t) e^{-i2\pi f_\tau} dt_2 d\tau d\tau
\]

\[
= \left( \int_{-\infty}^{\infty} s(z) s^*(z - t) e^{i2\pi f_\tau} d\tau \right) \left( \int_{-\infty}^{\infty} s^*(t_2) s(t_2 - t) e^{-i2\pi f_\tau} dt_2 \right)
\]

\[
= \left( \int_{-\infty}^{\infty} s(z) s^*(z - t) e^{i2\pi f_\tau} d\tau \right) \left( \int_{-\infty}^{\infty} s^*(t_2) s(t_2 - t) e^{i2\pi f_\tau} dt_2 \right)^*
\]

\[
= \chi_s(t, f) \chi_s^*(t, f)
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_s(\tau, \nu)|^2 e^{-i2\pi f_\tau} e^{i2\pi t \nu} d\tau d\nu = |\chi_s(t, f)|^2.
\]

\[\square\]

**Theorem 6 (Fourier Transform)**

\[
\int_{\mathbb{R}^2} \chi_s(\tau, \nu) e^{-i2\pi f_\tau} e^{i2\pi t \nu} d\tau d\nu = s(-t) S^*(t) e^{i2\pi f t} = R_s(-t, -f),
\]

where \( R_s(t, f) \) is the Rihaczek distribution.
Proof

\[
\int \int_{\mathbb{R}^2} \chi_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi \nu \tau} d\tau d\nu = \int \int_{\mathbb{R}^3} s(t') s^*(t' - \tau) e^{i2\pi \nu t'} e^{-i2\pi f \tau} e^{i2\pi \nu \tau} dt' d\tau d\nu
\]
\[
= \int \int_{\mathbb{R}^2} s(t') s^*(t' - \tau) e^{-i2\pi f \tau} \int_{-\infty}^{\infty} e^{i2\pi \nu (t' + t)} dv dt' d\tau
\]
\[
= \int \int_{\mathbb{R}^2} s(t') s^*(t' - \tau) e^{-i2\pi f \tau} \delta(t' + t) dt' d\tau
\]
\[
= \int_{-\infty}^{\infty} s(-t) s^*(-t - \tau) e^{-i2\pi f \tau} d\tau
\]
\[
= s(-t) \int_{-\infty}^{\infty} s^*(-(\tau + t)) e^{-i2\pi f \tau} d\tau.
\]

Notice that this is the Fourier transform. Apply properties to find that
\[
= s(-t) S^*(f) e^{i2\pi f t} = R_s(-t, -f).
\]

Theorem 7 (Time Shift Property)

Given \( v(t) = s(t - \Delta) \), then
\[
\chi_v(\tau, \nu) = e^{i2\pi \nu \Delta} \chi_s(\tau, \nu).
\]

Proof

\[
\chi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t) v^*(t - \tau) e^{i2\pi \nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t - \Delta) s^*(t - \Delta - \tau) e^{i2\pi \nu t} dt.
\]

Let \( u = t - \Delta \Rightarrow du = dt. \)
\[
= \int_{-\infty}^{\infty} s(u) s^*(u - \tau) e^{i2\pi \nu (u + \Delta)} du
\]
\[
= e^{i2\pi \nu \Delta} \int_{-\infty}^{\infty} s(u) s^*(u - \tau) e^{i2\pi \nu u} du
\]
\[
\chi_v(\tau, \nu) = e^{i2\pi \nu \Delta} \chi_s(\tau, \nu).
\]
Theorem 8 (Time Scaling Property)

Given \( v(t) = s(at) \), then

\[
\chi_v(\tau, \nu) = \frac{1}{|a|} \chi_s(a\tau, \nu/a).
\]

Proof

\[
\chi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t - \tau)e^{i2\pi\nu t} dt
= \int_{-\infty}^{\infty} s(at)s^*(a(t - \tau))e^{i2\pi\nu t} dt.
\]

Let \( u = at \Rightarrow du = adt \).

\[
= \int_{-\infty}^{\infty} s(u)s^*(u - a\tau)e^{i2\pi\nu u/|a|} du/|a|
\]

\[
\chi_v(\tau, \nu) = \frac{1}{|a|} \chi_s(a\tau, \nu/a).
\]

Theorem 9 (Modulation Property)

Given \( v(t) = s(t)e^{i2\pi f t} \), then

\[
\chi_v(\tau, \nu) = e^{i2\pi f \tau} \chi_s(\tau, \nu).
\]

Proof

\[
\chi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t - \tau)e^{i2\pi\nu t} dt
= \int_{-\infty}^{\infty} s(t)e^{i2\pi f t} \left( s(t - \tau)e^{i2\pi f(t-\tau)} \right)^* e^{i2\pi\nu t} dt
= e^{i2\pi f \tau} \int_{-\infty}^{\infty} s(t)s^*(t - \tau)e^{i2\pi\nu t} dt
= e^{i2\pi f \tau} \chi_s(\tau, \nu).
\]
Theorem 10 (Quadratic Phase Shift Property)

Given \( v(t) = s(t)e^{i2\pi \alpha t^2} \), then
\[
|\chi_v(\tau, \nu)| = |\chi_s(\tau, \nu + \alpha \tau)|.
\]

Proof

\[
\chi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{i2\nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t)e^{i\pi \alpha t^2} \left(s(t-\tau)e^{i\pi \alpha(t-\tau)^2}\right)^* e^{i2\nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t)e^{i\pi \alpha t^2} s^*(t-\tau)e^{-i\pi \alpha(\tau^2-2\tau t+\tau^2)} e^{i2\nu t} dt
\]
\[
= e^{-i\pi \alpha \tau^2} \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{i2\pi(\nu+\alpha \tau)t} dt
\]
\[
= e^{-i\pi \alpha \tau^2} \chi_s(\tau, \nu + \alpha \tau)
\]
\[
|\chi_v(\tau, \nu)| = |\chi_s(\tau, \nu + \alpha \tau)|.
\]

\[\blacksquare\]

2.2 Negative Asymmetric Ambiguity Function

Definition 2 For some signal \( s(t) \), the negative asymmetric ambiguity function is given by
\[
\Psi_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\nu t} dt.
\]

Theorem 11 (Value at Origin)
\[
|\Psi_s(0, 0)| = E_s.
\]
Proof

\[ |\Psi_s(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi \nu t} dt \right| \]

\[ |\Psi_s(0,0)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t-0)e^{-i2\pi (0)t} dt \right| = \left| \int_{-\infty}^{\infty} s(t)s^*(t) dt \right| \]

\[ |\Psi_s(0,0)| = E_s. \]

\[ |\Psi_s(\tau, \nu)|^2 \leq E_s^2 = |\Psi_s(0,0)|^2. \]

Theorem 12 (Maximum Property)

Proof

\[ |\Psi_s(\tau, \nu)|^2 = \left| \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi \nu t} dt \right|^2 \]

\[ = \left| \int_{-\infty}^{\infty} [s(t)e^{-i\pi \nu t}] [s(t-\tau)e^{i\pi \nu t}]^* dt \right|^2 \]

\[ \leq \int_{-\infty}^{\infty} |s(t)e^{-i\pi \nu t}|^2 dt \int_{-\infty}^{\infty} |s(t-\tau)e^{i\pi \nu t}|^2 dt \]

\[ = \int_{-\infty}^{\infty} |s(t)|^2 dt \int_{-\infty}^{\infty} |s(t-\tau)|^2 dt \]

\[ |\Psi_s(\tau, \nu)|^2 \leq E_s^2 = |\Psi_s(0,0)|^2. \]

Theorem 13 (Volume Property)

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 d\tau d\nu = E_s^2. \]
Proof

\[ \Psi_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^*(t - \tau)e^{-i2\pi \nu t} \, dt. \]

Let \( \mathcal{F} \) be the Fourier Transform operator.

\[ \Psi_s(\tau, \nu) = \mathcal{F}\{s(t)s^*(t - \tau)\} = \mathcal{F}\{\kappa(\tau, t)\}. \]

By Parseval's Theorem

\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathcal{F}\{x(t)\}(\nu)|^2 \, d\nu. \]

\[ \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 \, d\nu = \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 \, d\nu d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt d\tau \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t)s^*(t - \tau)s^*(t)s(t - \tau) \, d\tau dt \]

\[ = \int_{-\infty}^{\infty} |s(t)|^2 \int_{-\infty}^{\infty} |s(t - \tau)|^2 \, d\tau dt \]

\[ = \int_{-\infty}^{\infty} |s(t)|^2 \, E_s \, dt \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 \, d\tau d\nu = E_s^2. \]

---

Theorem 14 (Symmetry Property)

\[ |\Psi_s(-\tau, -\nu)| = |\Psi_s(\tau, \nu)|. \]

Proof

\[ \Psi_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{i2\pi \nu t} \, dt. \]

Let \( t' = t + \tau \).

\[ \Psi_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t' - \tau)s^*(t')e^{i2\pi \nu (t' - \tau)} \, dt' \]

\[ = e^{-i2\pi \nu \tau} \int_{-\infty}^{\infty} s(t' - \tau)s^*(t')e^{i2\pi \nu t'} \, dt' \]

\[ = e^{-i2\pi \nu \tau} \left[ \int_{-\infty}^{\infty} s(t')s^*(t' - \tau)e^{-i2\pi \nu t'} \right] ^* \, dt' \]

\[ = e^{-i2\pi \nu \tau} \Psi_s^*(\tau, \nu) \]

\[ |\Psi_s(-\tau, -\nu)| = |\Psi_s(\tau, \nu)|. \]
Theorem 15 (Fourier Transform of Modulus Squared)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 e^{-i2\pi ft} e^{i2\pi t\nu} d\tau d\nu = |\Psi_s(t, f)|^2.
\]

Proof

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 e^{-i2\pi ft} e^{i2\pi t\nu} d\tau d\nu = \iint_{\mathbb{R}^2} \Psi_s(\tau, \nu) \Psi_s^*(\tau, \nu) e^{-i2\pi ft} e^{i2\pi t\nu} d\tau d\nu \\
= \iint_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} s(t_1)s^*(t_1 - \tau) e^{-i2\pi \nu t_1} dt_1 \right) \left( \int_{-\infty}^{\infty} s^*(t_2)s(t_2 - \tau) e^{i2\pi \nu t_2} dt_2 \right) \\
\cdots e^{-i2\pi ft} e^{i2\pi t\nu} d\tau d\nu \\
= \iint_{\mathbb{R}^4} s(t_1)s^*(t_1 - \tau)s^*(t_2)s(t_2 - \tau) e^{i2\pi \nu (t_2 - t_1 + t)} e^{-i2\pi ft} e^{i2\pi \nu (t_2 - t_1 + t)} dt_1 dt_2 dt_2 dt_1 d\tau \\
= \iint_{\mathbb{R}^3} s(t_1)s^*(t_1 - \tau)s^*(t_2)s(t_2 - \tau) e^{-i2\pi ft} \delta(t_2 - t_1 + t) dt_2 dt_1 d\tau \\
= \iint_{\mathbb{R}^2} s(t_1)s^*(t_1 - \tau)s^*(t_1 - t)s(t_1 - t - \tau) e^{-i2\pi ft} dt_1 d\tau.
\]

Let $\tau = t_1 - z \Rightarrow d\tau = -dz$.

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_1)s^*(t_1 - (t_1 - z))s^*(t_1 - t)s(t_1 - t - (t_1 - z)) e^{-i2\pi f(t_1 - z)} (-dz) dt_1 \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_1)s^*(t_1 - t) e^{-i2\pi ft_1} s^*(z) s(z - t) e^{i2\pi fz} dt_1 dz \\
= \left( \int_{-\infty}^{\infty} s(t_1)s^*(t_1 - t) e^{-i2\pi ft_1} dt_1 \right) \left( \int_{-\infty}^{\infty} s^*(z) s(z - t) e^{i2\pi fz} dz \right) \\
= \left( \int_{-\infty}^{\infty} s(t_1)s^*(t_1 - t) e^{-i2\pi ft_1} dt_1 \right) \left( \int_{-\infty}^{\infty} s(z)s^*(z - t) e^{-i2\pi fz} dz \right)^* \\
= \Psi_s(t, f) \Psi_s^*(t, f) \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_s(\tau, \nu)|^2 e^{-i2\pi ft} e^{i2\pi t\nu} d\tau d\nu = |\Psi_s(t, f)|^2.
\]
Theorem 16 (Fourier Transform)
\[
\iiint_{\mathbb{R}^2} \Psi_s(\tau, \nu) e^{-i2\pi ft} e^{i2\pi tv} \, d\tau d\nu = s(t)S^*(f)e^{-i2\pi ft} = R_s(t, f),
\]
where \( R_s(t, f) \) is the Rihaczek distribution.

Proof
\[
\iiint_{\mathbb{R}^2} \Psi_s(\tau, \nu) e^{-i2\pi ft} e^{i2\pi tv} \, d\tau d\nu = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} s(t')s^*(t' - \tau) e^{-i2\pi vt'} e^{-i2\pi ft} \, dt' d\tau d\nu \\
= \int_{\mathbb{R}^2} s(t')s^*(t' - \tau) \int_{-\infty}^{\infty} e^{-i2\pi vt'} \, dv dt' d\tau \\
= \int_{\mathbb{R}^2} s(t')s^*(t' - \tau) \int_{-\infty}^{\infty} \delta(t' - t) \, dv dt' d\tau \\
= \int_{-\infty}^{\infty} s(t)s^*(t - \tau) e^{-i2\pi ft} \, dt \\
= s(t) \int_{-\infty}^{\infty} s^*(-(t - \tau)) e^{-i2\pi ft} \, dt.
\]

Notice that this is the Fourier transform. Apply properties to find that
\[
\iiint_{\mathbb{R}^2} \Psi_s(\tau, \nu) e^{-i2\pi ft} e^{i2\pi tv} \, d\tau d\nu = s(t)S^*(f)e^{-i2\pi ft} = R_s(t, f).
\]

Theorem 17 (Time Shift Property)
\[
\text{Given } v(t) = s(t - \Delta), \text{ then} \\
\Psi_v(\tau, \nu) = e^{-i2\pi \nu \Delta} \Psi_s(\tau, \nu).
\]

Proof
\[
\Psi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t - \tau) e^{-i2\pi vt} \, dt \\
= \int_{-\infty}^{\infty} s(t - \Delta)s^*(t - \Delta - \tau) e^{-i2\pi vt} \, dt \\
= \int_{-\infty}^{\infty} s(u)s^*(u - \tau) e^{-i2\pi v(u + \Delta)} \, du \\
= e^{-i2\pi \nu \Delta} \int_{-\infty}^{\infty} s(u)s^*(u - \tau) e^{-i2\pi \nu u} \, du \\
\Psi_v(\tau, \nu) = e^{-i2\pi \nu \Delta} \Psi_s(\tau, \nu).
Theorem 18 (Time Scaling Property)

Given \( v(t) = s(at) \), then

\[
\Psi_v(\tau, \nu) = \frac{1}{|a|} \Psi_s(a\tau, \nu/a).
\]

Proof

\[
\Psi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t - \tau)e^{-i2\pi\nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(at)s^*(a(t - \tau))e^{-i2\pi\nu t} dt.
\]

Let \( u = at \Rightarrow du = adt \).

\[
\Psi_v(\tau, \nu) = \frac{1}{|a|} \Psi_s(a\tau, \nu/a).
\]

Theorem 19 (Modulation Property)

Given \( v(t) = s(t) e^{i2\pi ft} \), then

\[
\Psi_v(\tau, \nu) = e^{i2\pi f\tau} \Psi_s(\tau, \nu).
\]

Proof

\[
\Psi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t - \tau)e^{-i2\pi\nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t)e^{i2\pi ft} (s(t - \tau)e^{i2\pi f(t-\tau)})^* e^{-i2\pi\nu t} dt
\]
\[
= e^{i2\pi f\tau} \int_{-\infty}^{\infty} s(t)s^*(t - \tau)e^{-i2\pi\nu t} dt
\]

\[
\Psi_v(\tau, \nu) = e^{i2\pi f\tau} \Psi_s(\tau, \nu).
\]
Theorem 20 (Quadratic Phase Shift Property)

Given \( v(t) = s(t)e^{i2\pi \alpha t^2} \), then

\[ |\Psi_v(\tau, \nu)| = |\Psi_s(\tau, \nu - \alpha \tau)|. \]

Proof

\[
\Psi_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi \nu t} dt
\]

\[
= \int_{-\infty}^{\infty} s(t)e^{i\pi \alpha t^2} \left( s(t-\tau)e^{i\pi \alpha (t-\tau)^2} \right)^* e^{-i2\pi \nu t} dt
\]

\[
= \int_{-\infty}^{\infty} s(t)e^{i\pi \alpha t^2} s^*(t-\tau)e^{-i\pi \alpha (t^2-2t\tau+\tau^2)}e^{-i2\pi \nu t} dt
\]

\[
= e^{-i\pi \alpha \tau^2} \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi (\nu-\alpha \tau)t} dt
\]

\[
= e^{-i\pi \alpha \tau^2} \Psi_s(\tau, \nu - \alpha \tau)
\]

\[ |\Psi_v(\tau, \nu)| = |\Psi_s(\tau, \nu - \alpha \tau)|. \]

2.3 Symmetric Ambiguity Function

Definition 3 For some signal \( s(t) \), the symmetric ambiguity function is given by

\[ \Gamma_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi \nu t} dt. \]

Theorem 21 (Value at Origin)

\[ |\Gamma_s(0, 0)| = E_s. \]

Proof

\[ |\Gamma_s(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi \nu t} dt \right| \]

\[ |\Gamma_s(0, 0)| = \left| \int_{-\infty}^{\infty} s(t+0/2)s^*(t-0/2)e^{-i2\pi (0) t} dt \right| \]

\[ = \left| \int_{-\infty}^{\infty} s(t)s^*(t) dt \right| \]

\[ |\Gamma_s(0, 0)| = E_s. \]
Theorem 22 (Maximum Property)

\[ |\Gamma_s(\tau, \nu)|^2 \leq E_s^2 = |\Gamma_s(0, 0)|^2. \]

Proof

\[
|\Gamma_s(\tau, \nu)|^2 = \left| \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi\nu t} \, dt \right|^2
\]

\[
= \left| \int_{-\infty}^{\infty} [s(t)e^{-i\pi\nu t}][s(t - \tau)e^{i\pi\nu t}]^* \, dt \right|^2
\]

\[
\leq \int_{-\infty}^{\infty} |s(t + \tau/2)e^{-i\pi\nu t}|^2 \, dt \int_{-\infty}^{\infty} |s(t - \tau/2)e^{i\pi\nu t}|^2 \, dt
\]

\[
= \int_{-\infty}^{\infty} |s(t + \tau/2)|^2 \, dt \int_{-\infty}^{\infty} |s(t - \tau/2)|^2 \, dt
\]

\[ |\Gamma_s(\tau, \nu)|^2 \leq E_s^2 = |\Gamma_s(0, 0)|^2. \]

Theorem 23 (Volume Property)

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 \, d\tau d\nu = E_s^2. \]
Proof

\[ \Gamma_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi \nu t} \, dt. \]

Let \( \mathcal{F} \) be the Fourier Transform operator.

\[ \Gamma_s(\tau, \nu) = \mathcal{F}\{s(t + \tau/2)s^*(t - \tau/2)\} = \mathcal{F}\{\kappa(\tau, t)\}. \]

By Parseval’s Theorem \( \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathcal{F}\{x(t)\}(\nu)|^2 \, d\nu. \)

\[ \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 \, d\nu = \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 \, d\nu d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt d\tau \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)s^*(t + \tau/2)s(t - \tau/2) \, d\tau dt. \]

Let \( u = t + \tau/2 \Rightarrow du = dt. \)

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(u)s^*(u - \tau)s(u)\, d\tau du \]

\[ = \int_{-\infty}^{\infty} |s(u)|^2 \int_{-\infty}^{\infty} |s(u - \tau)|^2 \, d\tau du \]

\[ = \int_{-\infty}^{\infty} |s(u)|^2 E_s \, du \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 \, d\tau d\nu = E_s^2. \]

\[ \blacksquare \]

\[ \text{Theorem 24 (Symmetry Property)} \]

\[ \Gamma_s(-\tau, -\nu) = \Gamma_s(\tau, \nu). \]

Proof

\[ \Gamma_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t - \tau/2)s^*(t + \tau/2)e^{i2\pi \nu t} \, dt \]

\[ \Gamma_s(-\tau, -\nu) = \left(\int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi \nu t} \, dt\right)^* \]

\[ \Gamma_s(-\tau, -\nu) = \Gamma_s(\tau, \nu). \]

\[ \blacksquare \]
Theorem 25 (Fourier Transform of Modulus Squared)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = |\Gamma_s(t, f)|^2.
\]

Proof

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = \iint_{\mathbb{R}^2} \Gamma_s(\tau, \nu)\Gamma_s^*(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu \\
= \iint_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} s(t_1 + \tau/2) s^*(t_1 - \tau/2) e^{-i2\pi \nu t_1} dt_1 \right) \\
\cdots \left( \int_{-\infty}^{\infty} s(t_2 + \tau/2) s^*(t_2 - \tau/2) e^{i2\pi \nu t_2} dt_2 \right) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu \\
= \iiint_{\mathbb{R}^4} s(t_1 + \tau/2) s^*(t_1 - \tau/2) s^*(t_2 + \tau/2) s(t_2 - \tau/2) e^{-i2\pi f \tau} \\
\cdots e^{i2\pi \nu (t_2 - t_1 + t)} dt_1 dt_2 dt_2 dt_1 \\
= \iiint_{\mathbb{R}^3} s(t_1 + \tau/2) s^*(t_1 - \tau/2) s^*(t_2 + \tau/2) s(t_2 - \tau/2) e^{-i2\pi f \tau} \\
\cdots \int_{-\infty}^{\infty} e^{i2\pi \nu (t_2 - t_1 + t)} dt_1 dt_2 dt_2 dt_1 \\
= \iiint_{\mathbb{R}^2} s(y + t/2) s^*(x + t/2) s^*(y - t/2) s(x - t/2) e^{-i2\pi f (x-y)} J dx dy.
\]

Let \( t_1 - \tau/2 = x + t/2 \) and \( t_1 + \tau/2 = y + t/2 \) \( \Rightarrow \tau = y - x \).

\[
\int_{\mathbb{R}^2} s(y + t/2) s^*(x + t/2) s^*(y - t/2) s(x - t/2) e^{-i2\pi f (x-y)} J dx dy.
\]

Where the Jacobian \( J = \frac{\partial y}{\partial \tau} \frac{\partial y}{\partial t_1} - \frac{\partial x}{\partial \tau} \frac{\partial x}{\partial t_1} = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 1. \)

\[
= \left( \int_{-\infty}^{\infty} s(y + t/2) s^*(y - t/2) e^{-i2\pi f y} dy \right) \left( \int_{-\infty}^{\infty} s^*(x + t/2) s(x - t/2) e^{i2\pi f x} dx \right) \\
= \left( \int_{-\infty}^{\infty} s(y + t/2) s^*(y - t/2) e^{-i2\pi f y} dy \right) \left( \int_{-\infty}^{\infty} s(x + t/2) s^*(x - t/2) e^{-i2\pi f x} dx \right)^* \\
= \Gamma_s(t, f)\Gamma_s^*(t, f) \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma_s(\tau, \nu)|^2 e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = |\Gamma_s(t, f)|^2.
\]
Theorem 26 (Fourier Transform)

\[ \int \int_{\mathbb{R}^2} \Gamma_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d
u = \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi f t} d\tau \]

\[ = \mathcal{W}_s(t, f), \]

where \( \mathcal{W}_s(t, f) \) is the Wigner distribution.

Proof

\[ \int \int_{\mathbb{R}^2} \Gamma_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = \int \int_{\mathbb{R}^3} s(t' + \tau/2)s^*(t' - \tau/2)e^{-i2\pi v t'} e^{-i2\pi f \tau} \]

\[ \ldots e^{i2\pi t \nu} dt' d\tau d\nu \]

\[ = \int \int_{\mathbb{R}^2} s(t' + \tau/2)s^*(t' - \tau/2)e^{-i2\pi f \tau} \int_{-\infty}^{\infty} e^{-i2\pi \nu(t' - t)} d\nu dt' d\tau \]

\[ = \int \int_{\mathbb{R}^2} s(t' + \tau/2)s^*(t' - \tau/2)e^{-i2\pi f \tau} \delta(t' - t) dt' d\tau \]

\[ = \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi f \tau} d\tau \]

\[ \int \int_{\mathbb{R}^2} \Gamma_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = \int_{-\infty}^{\infty} s(t + \tau/2)s^*(t - \tau/2)e^{-i2\pi f \tau} d\tau = \mathcal{W}_s(t, f). \]

\[ \square \]

Theorem 27 (Time Shift Property)

Given \( v(t) = s(t - \Delta) \), then

\[ \Gamma_v(\tau, \nu) = e^{-i2\pi \nu \Delta} \Gamma_s(\tau, \nu). \]
Proof

\[
\Gamma_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t + \tau/2)v^*(t - \tau/2)e^{-i2\pi vt} dt
\]
\[
= \int_{-\infty}^{\infty} s(t - \Delta + \tau/2)s^*(t - \Delta - \tau/2)e^{-i2\pi vt} dt.
\]

Let \( u = t - \Delta \Rightarrow du = dt \).

\[
= \int_{-\infty}^{\infty} s(u + \tau/2)s^*(u - \tau/2)e^{-i2\pi(vu + \Delta)} du
\]
\[
= e^{-i2\pi\nu\Delta} \int_{-\infty}^{\infty} s(u + \tau/2)s^*(u - \tau/2)e^{-i2\pi
\nu u} du
\]
\[
\Gamma_v(\tau, \nu) = e^{-i2\pi\nu\Delta} \Gamma_s(\tau, \nu).
\]

\[
\text{Theorem 28 (Time Scaling Property)}
\]

Given \( v(t) = s(at) \), then

\[
\Gamma_v(\tau, \nu) = \frac{1}{|a|} \Gamma_s(a\tau, \nu/a).
\]

Proof

\[
\Gamma_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t + \tau/2)v^*(t - \tau/2)e^{-i2\pi vt} dt
\]
\[
= \int_{-\infty}^{\infty} s(a(t + \tau/2))s^*(a(t - \tau/2))e^{-i2\pi vt} dt.
\]

Let \( u = at \Rightarrow du = adt \).

\[
= \int_{-\infty}^{\infty} s(u + a\tau/2)s^*(u - a\tau/2)e^{-i2\pi vu/a} du/|a|
\]
\[
\Gamma_v(\tau, \nu) = \frac{1}{|a|} \Gamma_s(a\tau, \nu/a).
\]
Theorem 29 (Modulation Property)

Given \( v(t) = s(t)e^{i2\pi ft} \), then
\[
\Gamma_v(\tau, \nu) = e^{i2\pi f\nu} \Gamma_s(\tau, \nu).
\]

Proof

\[
\Gamma_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t + \tau/2) v^*(t - \tau/2) e^{-i2\pi \nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t + \tau/2) e^{i2\pi f(t + \tau/2)} \left( s(t - \tau/2) e^{i2\pi f(t - \tau/2)} \right)^* e^{-i2\pi \nu t} dt
\]
\[
= e^{i2\pi f\tau} \int_{-\infty}^{\infty} s(t + \tau/2) s^*(t - \tau/2) e^{-i2\pi \nu t} dt
\]
\[
\Gamma_v(\tau, \nu) = e^{i2\pi f\tau} \Gamma_s(\tau, \nu).
\]

Theorem 30 (Quadratic Phase Shift Property)

Given \( v(t) = s(t)e^{i2\pi \alpha t^2} \), then
\[
\Gamma_v(\tau, \nu) = \Gamma_s(\tau, \nu - \alpha \tau).
\]

Proof

\[
\Gamma_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t + \tau/2) v^*(t - \tau/2) e^{-i2\pi \nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t + \tau/2) e^{i\pi \alpha (t + \tau/2)^2} \left( s(t - \tau/2) e^{i\pi \alpha (t - \tau/2)^2} \right)^* e^{-i2\pi \nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t + \tau/2) e^{i\pi \alpha (t^2 + t\tau + \tau^2/4)} s^*(t - \tau) e^{-i\pi \alpha (t^2 - t\tau + \tau^2/4)} e^{-i2\pi \nu t} dt
\]
\[
= \int_{-\infty}^{\infty} s(t + \tau/2) s^*(t - \tau/2) e^{-i2\pi (\nu - \alpha \tau)t} dt
\]
\[
\Gamma_v(\tau, \nu) = \Gamma_s(\tau, \nu - \alpha \tau).
\]
2.4 Woodward’s Ambiguity Function

Definition 4 For some signal \( s(t) \), Woodward’s ambiguity function is given by

\[
W_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{-j2\pi\nu t} dt.
\]

Theorem 31 (Value at Origin)

\[|W_s(0, 0)| = E_s.\]

Proof

\[
|W_s(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{-j2\pi\nu t} dt \right|
\]

\[|W_s(0, 0)| = \left| \int_{-\infty}^{\infty} s(t)s^*(t + 0)e^{-j2\pi(0) t} dt \right|
\]

\[= \left| \int_{-\infty}^{\infty} s(t)s^*(t) dt \right|
\]

\[|W_s(0, 0)| = E_s.\]

\[\square\]

Theorem 32 (Maximum Property)

\[|W_s(\tau, \nu)|^2 \leq E_s^2 = |W_s(0, 0)|^2.\]

Proof

\[
|W_s(\tau, \nu)|^2 = \left| \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{-j2\pi\nu t} dt \right|^2
\]

\[= \left| \int_{-\infty}^{\infty} [s(t)e^{-j\pi\nu t}][s(t + \tau)e^{j\pi\nu t}]^* dt \right|^2
\]

\[\leq \int_{-\infty}^{\infty} |s(t)e^{-j\pi\nu t}|^2 dt \int_{-\infty}^{\infty} |s(t + \tau)e^{j\pi\nu t}|^2 dt
\]

\[= \int_{-\infty}^{\infty} |s(t)|^2 dt \int_{-\infty}^{\infty} |s(t + \tau)|^2 dt
\]

\[|W_s(\tau, \nu)|^2 \leq E_s^2 = |W_s(0, 0)|^2.\]

\[\square\]
Theorem 33 (Volume Property)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_s(\tau, \nu)|^2 \, d\tau \, d\nu = E_s^2.
\]

Proof

\[
W_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t)s^* (t + \tau) e^{-i2\pi \nu t} \, dt.
\]

Let \( \mathcal{F} \) be the Fourier Transform operator.

\[
W_s(\tau, \nu) = \mathcal{F}\{s(t)s^*(t + \tau)\} = \mathcal{F}\{\kappa(\tau, t)\}.
\]

By Parseval’s Theorem

\[
\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathcal{F}\{x(t)\}(\nu)|^2 \, d\nu.
\]

\[
\int_{-\infty}^{\infty} |W_s(\tau, \nu)|^2 \, d\nu = \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_s(\tau, \nu)|^2 \, d\nu d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\kappa(\tau, t)|^2 \, dt d\tau
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t)s^*(t + \tau) s^*(t) s(t + \tau) \, d\tau \, dt
\]

\[
= \int_{-\infty}^{\infty} |s(t)|^2 \int_{-\infty}^{\infty} |s(t + \tau)|^2 \, d\tau \, dt
\]

\[
= \int_{-\infty}^{\infty} |s(t)|^2 E_s \, dt
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_s(\tau, \nu)|^2 \, d\tau \, d\nu = E_s^2.
\]

Theorem 34 (Symmetry Property)

\[
|W_s(-\tau, -\nu)| = |W_s(\tau, \nu)|.
\]
Proof

\[ W_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{i2\pi \nu t} \, dt. \]

Let \( t' = t - \tau \).

\[ W_s(-\tau, -\nu) = \int_{-\infty}^{\infty} s(t' + \tau)s^*(t')e^{i2\pi \nu (t' + \tau)} \, dt' \]
\[ = e^{i2\pi \nu \tau} \int_{-\infty}^{\infty} s(t')s^*(t')e^{i2\pi \nu t'} \, dt' \]
\[ = e^{i2\pi \nu \tau} \left[ \int_{-\infty}^{\infty} s(t')s^*(t' + \tau)e^{-i2\pi \nu t'} \right]^* \, dt' \]
\[ = e^{i2\pi \nu \tau} W_s^*(\tau, \nu) \]

\[ |W_s(-\tau, -\nu)| = |W_s(\tau, \nu)|. \]

\[ \square \]

Theorem 35 (Fourier Transform of Modulus Squared)

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_s(\tau, \nu)|^2 e^{-i2\pi f \tau} e^{i2\pi \nu \tau} \, d\tau d\nu = |W_s(t, f)|^2. \]
Proof

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| W_s(\tau, \nu) \right|^2 e^{-i2\pi f\tau} e^{i2\pi t\nu} \, d\tau \, d\nu = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_s(\tau, \nu) W_s^*(\tau, \nu) e^{-i2\pi f\tau} e^{i2\pi t\nu} \, d\tau \, d\nu
\]

\[
= \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} s(t_1) s^*(t_1 + \tau) e^{-i2\pi \nu t_1} \, dt_1 \right) \left( \int_{-\infty}^{\infty} s^*(t_2) s(t_2 + \tau) e^{i2\pi \nu t_2} \, dt_2 \right)
\]

\[
\quad \cdot e^{-i2\pi f\tau} e^{i2\pi t\nu} \, d\tau \, d\nu
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_s(\tau, \nu) W_s^*(\tau, \nu) \, d\tau \, d\nu
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(t_1) s^*(t_1 + \tau) s^*(t_2) s(t_2 + \tau) e^{-i2\pi f\tau} \int_{-\infty}^{\infty} e^{-i2\pi \nu(t_1-t_2-t)} \, d\nu \, d\tau \, dt_1 \, dt_2
\]

\[
= \int_{\mathbb{R}^2} s(t_2 + t) s^*(t_2 + t + \tau) s^*(t_2) s(t_2 + \tau) e^{-i2\pi f\tau} \, dt_2 \, d\tau
\]

Let \( \tau = z - t_2 \Rightarrow d\tau = dz. \)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_2 + t) s^*(t_2 + t + (z - t_2)) s^*(t_2) s(t_2 + (z - t_2)) e^{-i2\pi f(z-t_2)} \, dt_2 \, dz
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(z) s^*(z + t) e^{-i2\pi f z} s^*(t_2) s(t_2 + t) e^{i2\pi f t_2} \, dt_2 \, dz
\]

\[
= \left( \int_{-\infty}^{\infty} s(z) s^*(z + t) e^{-i2\pi f z} \, dz \right) \left( \int_{-\infty}^{\infty} s^*(t_2) s(t_2 + t) e^{i2\pi f t_2} \, dt_2 \right)
\]

\[
= \left( \int_{-\infty}^{\infty} s(z) s^*(z + t) e^{-i2\pi f z} \, dz \right) \left( \int_{-\infty}^{\infty} s(t_2) s^*(t_2 + t) e^{-i2\pi f t_1} \, dt_2 \right)
\]

\[
= W_s(t, f) W_s^*(t, f)
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| W_s(\tau, \nu) \right|^2 e^{-i2\pi f\tau} e^{i2\pi t\nu} \, d\tau \, d\nu = \left| W_s(t, f) \right|^2.
\]

Theorem 36 (Fourier Transform)

\[
\int_{\mathbb{R}^2} W_s(\tau, \nu) e^{-i2\pi f\tau} e^{i2\pi t\nu} \, d\tau \, d\nu = s(t) S^*(-f) e^{i2\pi f t} = R_s(t, -f),
\]

where \( R_s(t, f) \) is the Rihaczek distribution.
Proof

\[
\begin{align*}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} s(t') s^*(t' + \tau) e^{-i2\pi \nu t'} e^{-i2\pi f \tau} e^{i2\pi t \nu} dt' d\tau d\nu \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} s(t') s^*(t' + \tau) e^{-i2\pi f \tau} \int_{-\infty}^{\infty} e^{-i2\pi \nu (t' - t)} d\nu dt' d\tau \\
&= \int_{\mathbb{R}^2} s(t') s^*(t' + \tau) e^{-i2\pi f \tau} \delta(t' - t) dt' d\tau \\
&= \int_{-\infty}^{\infty} s(t) s^* (t + \tau) e^{-i2\pi f \tau} d\tau \\
&= s(t) \int_{-\infty}^{\infty} s^* (t + \tau) e^{-i2\pi f \tau} d\tau.
\end{align*}
\]

Notice that this is the Fourier transform. Apply properties to find that

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_s(\tau, \nu) e^{-i2\pi f \tau} e^{i2\pi t \nu} d\tau d\nu = s(t) S^* (-f) e^{i2\pi ft} = R_s (t, -f).
\]

Theorem 37 (Time Shift Property)

Given \( v(t) = s(t - \Delta) \), then

\[ W_v(\tau, \nu) = e^{-i2\pi \nu \Delta} W_s(\tau, \nu). \]

Proof

\[
W_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t) v^*(t + \tau) e^{-i2\pi \nu t} dt
\]

\[
= \int_{-\infty}^{\infty} s(t - \Delta) s^*(t - \Delta + \tau) e^{-i2\pi \nu t} dt.
\]

Let \( u = t - \Delta \Rightarrow du = dt \).

\[
= \int_{-\infty}^{\infty} s(u) s^*(u + \tau) e^{-i2\pi \nu (u + \Delta)} du
\]

\[
= e^{-i2\pi \nu \Delta} \int_{-\infty}^{\infty} s(u) s^*(u + \tau) e^{-i2\pi \nu u} du
\]

\[ W_v(\tau, \nu) = e^{-i2\pi \nu \Delta} W_s(\tau, \nu). \]
Theorem 38 (Time Scaling Property)

Given \(v(t) = v(t) = s(at)\), then

\[
W_v(\tau, \nu) = \frac{1}{|a|} W_s(a\tau, \nu/a).
\]

Proof

\[
W_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t) v^*(t + \tau) e^{-i2\pi \nu t} dt
= \int_{-\infty}^{\infty} s(at) s^*(a(t + \tau)) e^{-i2\pi \nu t} dt.
\]

Let \(u = at \Rightarrow du = adt\).

\[
= \int_{-\infty}^{\infty} s(u) s^*(u + a\tau) e^{-i2\pi \nu u/a} du/|a|
W_v(\tau, \nu) = \frac{1}{|a|} W_s(a\tau, \nu/a).
\]

Theorem 39 (Modulation Property)

Given \(v(t) = s(t)e^{i2\pi ft}\), then

\[
W_v(\tau, \nu) = e^{-i2\pi f \tau} W_s(\tau, \nu).
\]

Proof

\[
W_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t) v^*(t + \tau) e^{-i2\pi \nu t} dt
= \int_{-\infty}^{\infty} s(t)e^{i2\pi ft} \left(s(t + \tau)e^{i2\pi f(t+\tau)}\right)^* e^{-i2\pi \nu t} dt
= e^{-i2\pi f \tau} \int_{-\infty}^{\infty} s(t) s^*(t + \tau) e^{-i2\pi \nu t} dt
W_v(\tau, \nu) = e^{-i2\pi f \tau} W_s(\tau, \nu).
\]
Theorem 40 (Quadratic Phase Shift Property)

Given \( v(t) = s(t)e^{i\pi \tau^2} \), then

\[ |W_v(\tau, \nu)| = |W_s(\tau, \nu + \alpha \tau)|. \]

Proof

\[
W_v(\tau, \nu) = \int_{-\infty}^{\infty} v(t)v^*(t + \tau)e^{-i2\pi \nu t} dt
= \int_{-\infty}^{\infty} s(t)e^{i\pi \tau^2} \left( s(t + \tau)e^{i\pi \alpha (t+\tau)^2} \right)^* e^{-i2\pi \nu t} dt
= \int_{-\infty}^{\infty} s(t)e^{i\pi \tau^2} s^*(t + \tau)e^{-i\pi \alpha (t^2 + 2t\tau + \tau^2)} e^{-i2\pi \nu t} dt
= e^{-i\pi \alpha \tau^2} \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{-i2\pi (\nu + \alpha \tau)t} dt
= e^{-i\pi \alpha \tau^2} W_s(\tau, \nu + \alpha \tau)
\]

\[ |W_v(\tau, \nu)| = |W_s(\tau, \nu + \alpha \tau)|. \]
3. SUMMARY OF PROPERTIES

Tables 3.1 and 3.2 summarize the properties of these four different ambiguity functions. Of particular note are the first four shared properties and the first unshared property. These properties restrict what are in general valid ambiguity functions. The volume property essentially states that the total volume of an ambiguity function is constant and bounded. This means that no matter how the signal \( s(t) \) is modified the energy of the signal must end up somewhere. Removing side lobes somewhere requires that they appear elsewhere. The Fourier Transform of the Modulus squared being equal to itself is also an interesting and exotic property that greatly restricts the possible choices of ambiguity functions. Another way to view these restrictions is that picking any particular function as an ambiguity function does not necessarily have a valid signal that produces it.

The remaining properties give more insight into what happens when certain changes are made to a signal, like shifting it in time. For the most part these changes are similar across the different formulations of ambiguity functions, but can introduce some phase shifts or sign changes in the result. Adding a quadratic phase term to a signal is also sometimes called applying a chirp. Put together, all of these properties help inform a signal designer of what ambiguity functions are possible and what some simple modifications to a signal will do to its ambiguity function.
Table 3.1.
Shared Properties of Ambiguity Functions

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Value at Origin</td>
<td>$</td>
</tr>
<tr>
<td>Maximum Property</td>
<td>$</td>
</tr>
<tr>
<td>Volume Property</td>
<td>$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}</td>
</tr>
<tr>
<td>Symmetry Property</td>
<td>$</td>
</tr>
<tr>
<td>Fourier Transform of</td>
<td>$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}</td>
</tr>
<tr>
<td>Modulus Squared</td>
<td></td>
</tr>
<tr>
<td>Time Scaling</td>
<td>$f_v(\tau, \nu) = \frac{1}{</td>
</tr>
</tbody>
</table>

Table 3.2.
Unshared Properties of Ambiguity Functions

<table>
<thead>
<tr>
<th></th>
<th>$\chi_v(\tau, \nu)$</th>
<th>$\Psi_v(\tau, \nu)$</th>
<th>$\Gamma_v(\tau, \nu)$</th>
<th>$W_v(\tau, \nu)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourier Transform</td>
<td>$R_v(-t, -f)$</td>
<td>$R_v(t, f)$</td>
<td>$W_v(t, f)$</td>
<td>$R_v(t, -f)$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$e^{i2\pi \nu \Delta} \chi_v(\tau, \nu)$</td>
<td>$e^{-i2\pi \nu \Delta} \Psi_v(\tau, \nu)$</td>
<td>$e^{-i2\pi \nu \Delta} \Gamma_v(\tau, \nu)$</td>
<td>$e^{-i2\pi \nu \Delta} W_v(\tau, \nu)$</td>
</tr>
<tr>
<td>$v(t) = s(t - \Delta)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modulation</td>
<td>$e^{i2\pi f \tau} \chi_v(\tau, \nu)$</td>
<td>$e^{i2\pi f \tau} \Psi_v(\tau, \nu)$</td>
<td>$e^{i2\pi f \tau} \Gamma_v(\tau, \nu)$</td>
<td>$e^{-i2\pi f \tau} W_v(\tau, \nu)$</td>
</tr>
<tr>
<td>$v(t) = s(t)e^{i2\pi ft}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Magnitude of Quadratic</td>
<td>$</td>
<td>\chi_v(\tau, \nu + \alpha \tau)</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>Phase Shift</td>
<td>$v(t) = s(t)e^{i2\pi \alpha t^2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. PLOTS OF AMBIGUITY FUNCTIONS

4.1 50% Duty Cycle Pulse

A more concrete way to understand ambiguity functions is by looking at their plots. Probably the most basic signal possible is the square pulse, as seen in the figures in this section. Note that all of these plots use normalized time and frequency units. Another way to think of that is that each unit of time is proportional to one second, while each unit of frequency is proportional to one Hertz.

![Square Pulse with 50% Duty Cycle](image)

**Fig. 4.1.** Square Pulse with 50% Duty Cycle
Each ambiguity function produces a plot for the square pulse that is essentially the same as the others in magnitude, providing for some variance due to plotting a numerical approximation to them. The phase for each of them is slightly more interesting. The asymmetric cases at least produce phases that appear oriented in the same way. Woodward’s ambiguity function and the symmetric one produce nearly identical phase.

The magnitude of the ambiguity function of this square pulse essentially looks like an impulse that linearly falls down as it expands out along the delay axis. In frequency, it is essentially a series of sinc functions that become wider and wider as the delay leaves zero. This makes sense considering that as the absolute value of
the time delay increases, the product of $s(t)s^*(t - \tau)$ becomes a smaller and smaller square pulse.
Fig. 4.4. Negative Asymmetric Ambiguity Function of 50% Duty Cycle Pulse
Fig. 4.5. Symmetric Ambiguity Function of 50% Duty Cycle Pulse
Fig. 4.6. Woodward's Ambiguity Function of 50% Duty Cycle Pulse
4.2 75% Duty Cycle Pulse

In this same vein, it would be good to understand what happens when the duty cycle of the pulse is increased.

![Square Pulse with 75% Duty Cycle](image.png)

Fig. 4.7. Square Pulse with 75% Duty Cycle

With a longer pulse, the ambiguity function is stretched out along the delay axis. The maximum Doppler shift remains more or less the same since it corresponds to the delay with the minimum amount of overlapping pulse. The characteristics of each type of ambiguity function is essentially the same as the shorter pulse.
Fig. 4.8. 3D plot of Magnitude of Ambiguity Function of 75% Duty Cycle Pulse
Fig. 4.9. Positive Asymmetric Ambiguity Function of 75% Duty Cycle Pulse
Fig. 4.10. Negative Asymmetric Ambiguity Function of 75% Duty Cycle Pulse
Fig. 4.11. Symmetric Ambiguity Function of 75% Duty Cycle Pulse
Fig. 4.12. Woodward’s Ambiguity Function of 75% Duty Cycle Pulse
4.3 Pulse Train

In a real radar system, typically pulses are repeated in a pulse train. This section contains plots for the ambiguity functions of a train of square pulses with a 25% duty cycle.

![Real Part of Signal](image)

Fig. 4.13. Pulse Train with 25% Duty Cycle

This is fairly different than the two single square pulse cases. In the magnitude, there are small versions of the square pulse’s ambiguity function repeated over and over, with the highest peak being in the center. The pulses are interfering with each other. Because the center pulse is on average closer to all of the other pulses than any of the others, it has the highest peak. The structure seen in the phase of the single square pulses is now almost totally gone. This highlights a trade off in signal design. The sooner a pulse can be repeated, the sooner information can be updated about targets in the beam path, but at the same time, the sooner you repeat, the
more prior pulses interfere with present pulses. The pulse train is also sometimes said to produce an ambiguity function that looks like a bed of nails due to all of the interference.
Fig. 4.15. Positive Asymmetric Ambiguity Function of 25% Duty Cycle Pulse Train
Fig. 4.16. Negative Asymmetric Ambiguity Function of 25% Duty Cycle Pulse Train
Fig. 4.17. Symmetric Ambiguity Function of 25% Duty Cycle Pulse Train
Fig. 4.18. Woodward’s Ambiguity Function of 25% Duty Cycle Pulse Train
4.4 Up Chirp

A common way to avoid the interference from neighboring pulses in the pulse train is to use a chirped signal that better separates the transmitted frequencies in time.

![Real Part of Signal](image)

Fig. 4.19. Up Chirp

The big feature of the chirped pulse is that the ambiguity function is no longer symmetric about zero delay. The different ambiguity functions are now not as similar to each other. In particular, the magnitude is flipped between several of the cases. The fact that the magnitude has this slant now would make it so that a pulse train of these chirped signals could be bit together to reduce interference.
Fig. 4.20. 3D plot of Magnitude of Ambiguity Function of Up Chirp
Fig. 4.21. Positive Asymmetric Ambiguity Function of Up Chirp
Fig. 4.22. Negative Asymmetric Ambiguity Function of Up Chirp
Fig. 4.23. Symmetric Ambiguity Function of Up Chirp
Fig. 4.24. Woodward's Ambiguity Function of Up Chirp
5. CONCLUSION

Ambiguity functions are extremely useful tools for understanding radar system behavior. They characterize the mismatch between the matched filter and the reflected radar signal that may be Doppler shifted. The ambiguity function itself essentially gives the output of the matched filter in both magnitude and phase as a function of time delay and potential Doppler shift. Using this information, it is possible to create better signals for use in radar.

There are a large number of properties that ambiguity functions have, but they can vary in their manifestations between formulations. This can cause a significant amount of confusion and waste the time of an engineer who might repeatedly be reproving these properties in the course of their work. With this large set of proofs and properties, it is hoped that some of that can be avoided.
LIST OF REFERENCES
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VITA
John Mulcahy-Stanislawczyk was born in Rahway, New Jersey. A few years after his birth, he moved to Missouri, where in 2008 he graduated from Lafayette High School in Wildwood, Missouri. Later that year he began attending Missouri University of Science and Technology in Rolla, Missouri. In 2011, John was hired by Sandia National Laboratories as an intern, which continued until 2012 when he was hired as a Critical Skills Masters Program student. Midway through 2012 he graduated from Missouri S&T with degrees in Electrical Engineering, Computer Engineering, and Physics. Later in 2012, John began attending Purdue University in West Lafayette, Indiana.