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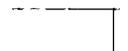
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MULTIPLE FLEXIBLE JOINT ROBOTS  
IN COOPERATIVE MOTION

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# Lyapunov Based Control of Multiple Flexible Joint Robots in Cooperative Motion

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## Abstract

In this paper, we address the problem of controlling multiple flexible joint robots manipulating a common load cooperatively. The load position and velocity tracking errors **are** shown to converge to zero. The internal forces exerted on the load are also shown to converge to their desired trajectories. Asymptotic stability of the system is insured regardless of the joint flexibility and despite the fact that force sensors mounted on the robot wrist are not employed.

## 1. Introduction

Recently considerable amount of research has focused on the problem of cooperative control and coordination of multiple robots. Interest in multi-robot systems has arisen because several tasks require the use of two or more robots. Examples of such tasks include the joining and securing of large pipes for the construction of space structures, picking up and carrying of heavy loads, and grasping odd shaped loads. Cooperative robots may be used in hazardous or unsafe environments such as in space, in deep waters and in radioactive environments. By using more than one robot the manipulation capability and the workspace of the system **are** further increased. However multi-robot systems are more difficult to control than single robots. Additional problems arise as the parameters of the robots and the manipulated load may not be known exactly.

Several control schemes, adaptive and non adaptive schemes have been proposed for cooperative multiple **robots** with rigid joints manipulating a common load. **Zheng** and Luh [17] considered the kinematic and dynamic model of the multi-robot

system and developed an inverse dynamics schemes for load position control. Hsu et al [3] developed a control algorithm for the coordinated manipulation of **multifingered** robot hand. **Their** control law guarantees the convergence of the load position and internal forces to the desired values respectively. Tarn, et al [13] developed a nonlinear control scheme which consisted of using nonlinear transformations to determine a nonlinear feedback which exactly linearized the robot dynamics. Linear optimal techniques were then used to develop a robust control of the linearized system. Yun, et al [14] also used exact linearization and output decoupling techniques to control multiple robots. Yoshikawa and **Zheng** [16] also developed tracking **control** laws for multiple robots, experiments were also reported.

Walker, et al [15] developed an algorithm to control the position of a load handled by multiple robots using Lyapunov-like functions. They also showed the internal forces could be controlled in an open-loop manner. A feature of their controller is that the computational requirements are low as their control law is amenable to Newton-Euler parallelization. The problem of manipulating a load using multiple robots when the load makes contact with an environment was addressed by Hyati [12], Cole [11] and, Hu and Goldenberg [10]. Several linearizing and direct adaptive control schemes which performed contact force control and load position control was developed by those **researchers**[10][11][12]. **Zribi** and **Ahmad** [18,19] proposed an adaptive controller for the multi-robot system manipulating a rigid object cooperatively. Their controller takes into account the dynamics of the manipulators and the load and does not require feedback of joint acceleration or the inversion of the inertia or the Jacobian matrices. They also considered the effects of bounded disturbances on the system and a control law which guarantees the convergence of the tracking error to a bounded set when a disturbance is present is also given.

There are very few papers which address the problem of controlling multiple robots with joint flexibility. **Ahmad** and Guo [1] addressed the problem of controlling multiple flexible-joint robots with linear dynamics. **Ahmad** [2] showed that the multiple flexible joint robot system is analogous to a single robot interacting with a **frictionless** surface, while in motion along the surface. He also developed an exactly linearizing control law that ensures the convergence to zero of the load position error and internal forces errors despite the fact that force sensors are not employed.

The complexity of controlling a **single** flexible joint robot is one of the reasons that the multiple flexible joint robots did not attract much attention. For an excellent survey on the past work on the control of flexible joint robots, the reader is referred to **Spong** [8].

In this paper we analyze the control of multiple flexible joint robots in cooperative motion while manipulating a common load, see Figure.1. We assume that the manipulated load is rigidly grasped. We first derive the dynamics of the manipulators and the load. We then combine the dynamics of the robots and that of the load by eliminating the contact **forces/moments**. As the internal grasping forces do not contribute to the motion of the load, they appear in the robot joint dynamical equations when the position, velocity and acceleration of the joints are expressed in terms of the load position. Next, we derive several properties of the combined system; these properties are used later to derive the controller. Finally we propose a control law that ensures the convergence to zero of the position and velocity **tracking** errors. We also show that the internal forces exerted on the load converge to their desired trajectories. The boundedness of all signals are also shown.

This paper is divided into four sections, in section two we derive the dynamic equations of the cooperative multi-robot system and their properties. In section three the load and the internal force control law is derived. In section four the conclusion and a summary of the paper is given.

## 2.1 Multi-Robot System Dynamics Model

The general dynamic model for a cooperative multi-robot system has been investigated thoroughly in the literature, and is also described in the below for completeness. We first state a few assumptions that will be used in the subsequent derivation.

### Assumptions

- (1) The manipulators are rigidly grasping the load, (**i.e.**, there is no motion between the contact point of the load and the robots end-effectors).
- (2) All the manipulators are **kinematically** non-singular along the desired trajectories. Further a finite set of joint angles exist which can position the end effector at any point on the workspace (of dimension  $\hat{m}$ ) subject to the manipulator joint motion limits.

### Dynamics Model

The dynamic equations of the  $i$ th flexible joint manipulator in cooperative motion with  $k$  robots can be written as,

$$D_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) + K_{si}(q_i - q_{im}) + J_i(q_i)^T F_e = 0 \quad (1)$$

$$\text{and, } D_{im}\ddot{q}_{im} - K_{si}(q_i - q_{im}) = u_i \quad \text{for } i=1, \dots, k$$

where,  $q_i \in \mathbf{R}^{n_i}$  is the vector of joint displacements,  $q_{im} \in \mathbf{R}^{n_i}$  is the vector of actuator displacements, and  $n_i$  is the number of joints of the  $i$ th robot. The inertia matrix of

the  $i$ th robot is  $D_i(q_i) \in \mathbf{R}^{n_i \times n_i}$ , this is a positive definite and **symmetric** matrix. The matrix of **centrifugal/Coriolis** forces is  $C_i(q_i, \dot{q}_i) \in \mathbf{R}^{n_i \times n_i}$ ; the gravitational forces acting on robot  $i$  is  $G_i(q_i) \in \mathbf{R}^{n_i}$ ; the manipulator Jacobian is  $J_i(q_i) \in \mathbf{R}^{\hat{m} \times n_i}$ . The actuator input torque for the  $i$ th robot is  $u_i \in \mathbf{R}^{n_i}$ ;  $D_{im} \in \mathbf{R}^{n_i \times n_m}$  is the constant, positive definite actuator inertia matrix;  $K_{si} \in \mathbf{R}^{n_i \times n_i}$  is the positive definite joint stiffness matrix. A non **redundant** robot operating in the workspace of dimension  $\hat{m}$  has  $n_i = \hat{m}$  degrees of joint motion. The vector of **forces/moments** applied by the  $i$ th manipulator on to the object at the point of contact is  $F_{e_i} \in \mathbf{R}^{\hat{m}}$  and it can be written in terms of the contact forces, given that  $\hat{m} = \hat{m}_1 + \dots + \hat{m}_k$ ,  $f_i \in \mathbf{R}^{\hat{m}_1}$  and contact moments  $\eta_i \in \mathbf{R}^{\hat{m}_2}$ , such that  $F_{e_i} = \begin{bmatrix} f_i^T & \eta_i^T \end{bmatrix}^T$  for  $i = 1, \dots, k$ .

Given  $n = \sum_{i=1}^k n_i = k\hat{m}$  now we can group the dynamics of the  $k$  robots to get,

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G_r(q) + K_s(q - q_m) + J(q)^T F_e = 0 \quad (2)$$

$$\text{and, } D_m \ddot{q}_m - K_s(q - q_m) = u \quad (3)$$

where  $D(q) \in \mathbf{R}^{n \times n}$ ,  $D_m \in \mathbf{R}^{n \times n}$ ,  $C(q, \dot{q}) \in \mathbf{R}^{n \times n}$ ,  $K_s \in \mathbf{R}^{n \times n}$  and  $J(q) \in \mathbf{R}^{n \times n}$  are block **diagonal** matrices whose diagonal elements are  $D_i(q_i)$ ,  $D_{im}$ ,  $C_i(q_i, \dot{q}_i)$ ,  $K_{si}$  and  $J_i(q_i)$  respectively. Note also the following definition of the vectors  $q, q_m, u, G_r \in \mathbf{R}^n$  and  $F_e \in \mathbf{R}^{k\hat{m}}$ ;

$$q = \begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_k \end{bmatrix}, \quad q_m = \begin{bmatrix} q_{1m} \\ \cdot \\ \cdot \\ \cdot \\ q_{km} \end{bmatrix}, \quad G_r = \begin{bmatrix} G_1 \\ \cdot \\ \cdot \\ \cdot \\ G_k \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_k \end{bmatrix} \quad \text{and, } F_e = \begin{bmatrix} F_{e_1} \\ \cdot \\ \cdot \\ \cdot \\ F_{e_k} \end{bmatrix}. \quad (4)$$

Note that in the above we have dropped functional dependencies whenever possible without ambiguity.

The equations of motion of the load are obtained from the Newton-Euler mechanics,

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{M}\mathbf{g}_1 = \sum_{i=1}^{i=k} \mathbf{f}_i, \quad (5)$$

$$\text{and, } \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \sum_{i=1}^{i=k} (\boldsymbol{\eta}_i + \mathbf{r}_i \times \mathbf{f}_i) \quad (5)$$

Here the position of the center of mass of the object expressed in the world coordinate frame is  $\mathbf{z} \in \mathbf{R}^{\hat{m}_1}$ . The rotational velocity of the object **expressed** in the world coordinate frame is  $\boldsymbol{\omega} \in \mathbf{R}^{\hat{m}_2}$ , and the vector of gravitational forces acting on the object is  $\mathbf{g}_1 \in \mathbf{R}^{\hat{m}_1}$ . The mass matrix  $\mathbf{M}_1 \in \mathbf{R}^{\hat{m}_1 \times \hat{m}_1}$  is a diagonal matrix whose diagonal elements are the mass of the load. The matrix  $\mathbf{I} \in \mathbf{R}^{\hat{m}_2 \times \hat{m}_2}$  is the load inertia matrix about the center of mass coordinate **frame**. The position of the end effector of the  $i$ th manipulator with respect to the object center of mass, expressed in the world coordinate frame, is  $\mathbf{r}_i \in \mathbf{R}^{\hat{m}_1}$ .

If we let  $\dot{\mathbf{x}} = [\dot{\mathbf{z}}^T, \boldsymbol{\omega}^T]^T$  then the motion of the load expressed by equations (4) and (5) can be re-written as,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{N}_2(\dot{\mathbf{x}})\dot{\mathbf{x}} = \mathbf{G}\mathbf{F}_e = \mathbf{F}_o, \quad (6)$$

where  $\mathbf{F}_o \in \mathbf{R}^{\hat{m}}$  is the net **force/torque** vector acting on the load center of mass, and  $\mathbf{G} \in \mathbf{R}^{\hat{m} \times n}$  is the grasp matrix, which is defined as,

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \cdots & \mathbf{G}_k \end{bmatrix}. \quad (7)$$

The matrix  $\mathbf{G}_i \in \mathbf{R}^{\hat{m} \times n_i}$  is obtained from equations (4) and (5) and (for  $\hat{m}=6$  with  $\hat{m}_1=3, \hat{m}_2=3$  and  $n_i=6$ ) it is given as,

$$\mathbf{G}_i = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \boldsymbol{\Omega}_i(\mathbf{r}_i) & \mathbf{I}_3 \\ \boldsymbol{\Omega} & 3 \end{bmatrix} \text{ and, } \boldsymbol{\Omega}_i(\mathbf{r}_i) = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix},$$

where  $\mathbf{I}_3$  is the 3 by 3 identity matrix, and  $\mathbf{r}_i = [r_{ix}, r_{iy}, r_{iz}]^T$ . Note that for other dimensions of  $\hat{m}_1, \hat{m}_2$  and  $n_i$  similar representation of the cross product and the summation in **equation(6)** can be easily determined. Note the **inertia/mass** matrix  $\mathbf{M} \in \mathbf{R}^{\hat{m} \times \hat{m}}$  and the vector  $\mathbf{N}_2(\dot{\mathbf{x}})\dot{\mathbf{x}} \in \mathbf{R}^{\hat{m}}$  are defined such that,

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \text{ and, } \mathbf{N}_2(\dot{\mathbf{x}})\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{M}\mathbf{g}_1 \\ \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) \\ \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) \end{bmatrix}. \quad (8)$$

The **position/orientation** vector of the center of mass of the object expressed in the

world coordinate frame is  $x \in \mathbf{R}^{\hat{m}}$  and the velocity vector is given by  $\dot{x} = \begin{bmatrix} \dot{z}^T & \omega^T \end{bmatrix}^T \in \mathbf{R}^{\hat{m}}$ .

## 22 Kinematic Model

We are interested in controlling the manipulators end effectors in some predefined Cartesian task space such that,

$$x_{ei} = K_i(q_i) \quad i = 1, \dots, k, \quad (9)$$

where  $K_i(\cdot) : \mathbf{R}^{n_i} \rightarrow \mathbf{R}^{\hat{m}}$ , with  $n_i = \hat{m}$ , is the transformation from the joint angle space of  $q_i$  to the task space containing  $x_{ei}$ , and  $x_{ei} \in \mathbf{R}^{\hat{m}}$  is the **position/orientation** of the point of contact of the  $i$ th manipulator with the load, expressed in the world coordinate frame. Notice that,  $\dot{x}_{ei} = \begin{bmatrix} \dot{z}_{ei}^T & \omega_{ei}^T \end{bmatrix}^T \in \mathbf{R}^{\hat{m}}$ . Recall that the robots are assumed to be non-redundant and are able to position the end-effectors at any point in the workspace subject to joint motion limits, thus  $n_i = \hat{m}$  for all  $i = 1, \dots, k$ .

If we differentiate equation (9) with respect to time, and if we define  $J_i(q_i)$  to be the differential map from the joint space to the Cartesian space (i.e.,  $J_i(q_i) = \frac{\partial K_i}{\partial q_i}$ ), then we can write,

$$\dot{x}_{ei} = J_i(q_i)\dot{q}_i \quad i = 1, \dots, k. \quad (10)$$

If these equations are stacked into a single vector by forming  $J_i$ 's into a block diagonal matrix, and concatenating the  $q_i$ 's into one vector  $q$ , we get,

$$v_e = Jq, \quad (11)$$

where  $v_e = \begin{bmatrix} \dot{x}_{e1}^T & \dot{x}_{e2}^T & \dots & \dot{x}_k^T \end{bmatrix}^T$ , and  $J(q) = \text{diag}(J_i(q_i)) \in \mathbf{R}^{n \times n}$ . If we write  $F_o$  as the total force acting on the center of mass of the object, then  $F_o$  correspond to the left hand side of equation (6), hence equation (6) can be written as,

$$F_o = GF_e. \quad (12)$$

Now from the duality between the forces and velocities, we can write,

$$G^T \dot{x} = v_e, \quad (13)$$

where  $\dot{x}$  is the velocity of the object. Thus for the  $k$  robots system, we can combine equations (11) and (13) to get,

$$\mathbf{G}^T \dot{\mathbf{x}} = \mathbf{J} \dot{\mathbf{q}}, \quad (14)$$

where  $\mathbf{G}$  is the grasp matrix for the multi-robot system, and  $\mathbf{J}$  is the Jacobian of the system. Now if we differentiate the above equation, we get,

$$\ddot{\mathbf{q}} = \mathbf{J}^{-1} \frac{d}{dt} (\mathbf{G}^T \dot{\mathbf{x}}) - \mathbf{J}^{-1} \dot{\mathbf{J}} \dot{\mathbf{q}}. \quad (15)$$

### 2.3 Definition of Internal Forces and Internal Force Errors

The end-effector force of the  $i$ th manipulator,  $\mathbf{F}_{e_i}$ , can be decomposed into two forces, the motion force and the internal grasping force. The internal grasping forces  $\mathbf{F}_I = [\mathbf{F}_{Ie1}, \dots, \mathbf{F}_{Iek}]^T \in \mathbf{R}^n$  do not cause any motion of the load. However we must control these end-effector internal forces,  $\mathbf{F}_{Ie_i} \in \mathbf{R}^{\hat{m}}$  with  $i=1, \dots, k$ , in order to prevent excessive compressive or expansive forces being applied to the load. We can calculate the internal force  $\mathbf{F}_I$  from equation (6) if  $\mathbf{F}_o$  is known and  $\text{rank}(\mathbf{G}) = \hat{m}$ , then

$$\mathbf{F}_e = \mathbf{G}^+ \mathbf{F}_o + \mathbf{F}_I. \quad (16)$$

Here,  $\mathbf{G}^+ = \mathbf{G}^T (\mathbf{G}\mathbf{G}^T)^{-1}$  and  $\mathbf{G}\mathbf{G}^+ = \mathbf{I}_{\hat{m}}$ , given  $\mathbf{I}_{\hat{m}}$  is an  $\hat{m} \times \hat{m}$  identity matrix. (For a discussion related to other choices of the inverse of the  $\mathbf{G}^+$  matrix see [21] and [22]. Notice that other choices of the inverse of  $\mathbf{G}$  does not effect the derivations presented in this paper.) Therefore we see that  $\mathbf{G}\mathbf{F}_I = \mathbf{0}$  and  $\mathbf{G}\mathbf{F}_e = \mathbf{F}_o$ , i.e., the internal forces do not contribute to the motion of the load. The desired internal forces  $\mathbf{F}_{I,d} \in \mathbf{R}^n$  also satisfy  $\mathbf{G}\mathbf{F}_{I,d} = \mathbf{0}$ . The internal force error  $\mathbf{e}_f = \mathbf{F}_{I,d} - \mathbf{F}_I$  also satisfies

$$\mathbf{G} \mathbf{e}_f = \mathbf{0}. \quad (17)$$

These internal force properties will be used to derive the control law.

### 2.4 Combined Model of the Multi-Robot System

From the dynamics of the robots, equation (2), we can solve for  $\mathbf{F}_e$  as,

$$\mathbf{F}_e = \mathbf{J}^{-T} [ -\mathbf{D}\ddot{\mathbf{q}} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{G}_r - \mathbf{K}_s(\mathbf{q} - \mathbf{q}_m) ]. \quad (18)$$

Now replace, in equation (7),  $\mathbf{F}_e$  by its value from equation (18), we get,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{N}_2\dot{\mathbf{x}} + \mathbf{G}_l = \mathbf{G}\mathbf{J}^{-T} [ -\mathbf{D}\ddot{\mathbf{q}} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{G}_r - \mathbf{K}_s(\mathbf{q} - \mathbf{q}_m) ]. \quad (19)$$

Replacing  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  by their values from equations (14) and (15), we get,

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{N}_2\dot{\mathbf{x}} + \mathbf{G}_1 = & -\mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}[\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\ddot{\mathbf{x}} + \frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}})\dot{\mathbf{x}}] - \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{C}\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\dot{\mathbf{x}} \\ & - \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{G}_r - \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s(\mathbf{q}-\mathbf{q}_m). \end{aligned} \quad (20)$$

Combining terms, we get,

$$\begin{aligned} (\mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}} + \mathbf{M})\ddot{\mathbf{x}} + (\mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}\frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}) + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{C}\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}} + \mathbf{N}_2)\dot{\mathbf{x}} \\ + (\mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{G}_r + \mathbf{G}_1) + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s(\mathbf{q}-\mathbf{q}_m) = 0. \end{aligned} \quad (21)$$

Letting,

$$\mathbf{D}^* = \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}} + \mathbf{M}, \quad (22)$$

$$\mathbf{C}^* = \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}\frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}) + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{C}\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}} + \mathbf{N}_2, \quad (23)$$

$$\mathbf{G}^* = \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{G}_r + \mathbf{G}_1, \quad (24)$$

we can rewrite equation (21) as,

$$\mathbf{D}^*\ddot{\mathbf{x}} + \mathbf{C}^*\dot{\mathbf{x}} + \mathbf{G}^* + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s(\mathbf{q}-\mathbf{q}_m) = 0. \quad (25)$$

Multiplying equation (3) by  $\mathbf{G}\mathbf{J}^{-\mathbf{T}}$ , we get,

$$\mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{D}_m\ddot{\mathbf{q}}_m - \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s(\mathbf{q}-\mathbf{q}_m) = \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{u} \quad (26)$$

Equations (25) and (26) represent the combined dynamics of the multi-robot system.

## 2.5 Properties of the Multi-Robot System Model

Several properties of the multi-robot system dynamic model will be shown as they will be used to derive the control law.

### Properties

(P1)  $\mathbf{D}$  and  $\mathbf{M}$  are symmetric positive definite matrices.

(P2)  $\dot{\mathbf{D}} - 2\mathbf{C}$  is skew symmetric matrix or  $\frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}(\dot{\mathbf{D}} - 2\mathbf{C})\dot{\mathbf{q}} = 0$ . The proof of property P2 can be found in [9] and [20].

(P3)  $\mathbf{M} - 2\mathbf{N}_2$  is skew symmetric matrix or,  $\frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}(\mathbf{M} - 2\mathbf{N}_2)\dot{\mathbf{x}} = 0$ . This property can be seen from energy considerations. In general if  $\mathbf{M}$  is not expressed in the object center of mass coordinate frame,  $\mathbf{M}=\mathbf{M}(\mathbf{x})$  and  $\mathbf{N}\dot{\mathbf{x}}=\mathbf{N}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{x}}$ . The total energy of the load is given by  $\mathbf{E}_L=\frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{x}} + \mathbf{h}(\mathbf{x})$  where  $\mathbf{h}(\mathbf{x})$  is the potential energy and  $\mathbf{G}_r=\frac{\partial}{\partial \mathbf{x}}\mathbf{h}(\mathbf{x})$ . As the power input to the load is given by,  $\frac{d}{dt}\mathbf{E}_L=\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{F}_o=\dot{\mathbf{x}}^{\mathbf{T}}(\mathbf{M}\ddot{\mathbf{x}} + \frac{1}{2}\dot{\mathbf{M}}\dot{\mathbf{x}} + \mathbf{G}_r)=\dot{\mathbf{x}}^{\mathbf{T}}(\mathbf{M}\ddot{\mathbf{x}} + \mathbf{N}\dot{\mathbf{x}} + \mathbf{G}_r)$ , thus we have the property,

$\frac{1}{2} \dot{\mathbf{x}}^T (\dot{\mathbf{M}} - 2\mathbf{N}) \dot{\mathbf{x}} = 0$ . These properties will now be used to show the properties of the combined multi-robot system model. Note that property P3 indicates that there is no energy being absorbed or being created by the **centrifugal/Coriolis** forces. If  $\mathbf{M}$  is expressed in the center of mass coordinate frame,  $\mathbf{M}$  is a constant and property P3 still holds.

Lemma 1:

Further multi-robot system properties can be shown, these are;

(P4)  $\mathbf{D}^*$  is symmetric positive definite matrix and.

(P5)  $\dot{\mathbf{D}}^* - 2\mathbf{C}^*$  is skew symmetric.

Proof: Note first that,

$$\mathbf{D}^* = \mathbf{G}\mathbf{J}^{-T}\mathbf{D}\mathbf{J}^{-1}\mathbf{G}^T + \mathbf{M} = (\mathbf{G}\mathbf{J}^{-T})\mathbf{D}(\mathbf{G}\mathbf{J}^{-T})^T + \mathbf{M}.$$

Thus  $\mathbf{D}^*$  is **symmetric** positive definite matrix because  $\mathbf{D}$  and  $\mathbf{M}$  are symmetric positive definite matrices (from the above properties P1 and P2). Next consider property P5, as,

$$\begin{aligned} \dot{\mathbf{D}}^* - 2\mathbf{C}^* &= \frac{d}{dt}(\mathbf{G}\mathbf{J}^{-T})\mathbf{D}(\mathbf{J}^{-1}\mathbf{G}^T) + \mathbf{G}\mathbf{J}^{-T}\dot{\mathbf{D}}\mathbf{J}^{-1}\mathbf{G}^T + \mathbf{G}\mathbf{J}^{-T}\mathbf{D}\frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^T) \\ &\quad + \dot{\mathbf{M}} - 2\mathbf{G}\mathbf{J}^{-T}\mathbf{D}\frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^T) - 2\mathbf{G}\mathbf{J}^{-T}\mathbf{C}\mathbf{J}^{-1}\mathbf{G}^T - 2\mathbf{N}_2 \\ &= \mathbf{G}\mathbf{J}^{-T}(\dot{\mathbf{D}} - 2\mathbf{C})\mathbf{J}^{-1}\mathbf{G}^T + (\dot{\mathbf{M}} - 2\mathbf{N}_2) + \frac{d}{dt}(\mathbf{G}\mathbf{J}^{-T})\mathbf{D}(\mathbf{J}^{-1}\mathbf{G}^T) \\ &\quad - \mathbf{G}\mathbf{J}^{-T}\mathbf{D}\frac{d}{dt}(\mathbf{J}^{-1}\mathbf{G}^T) \\ &= \mathbf{v}^T(\dot{\mathbf{D}} - 2\mathbf{C})\mathbf{v} + (\dot{\mathbf{M}} - 2\mathbf{N}_2) + \frac{d}{dt}(\mathbf{v}^T)\mathbf{D}\mathbf{v} - \mathbf{v}^T\mathbf{D}\frac{d}{dt}(\mathbf{v}), \end{aligned}$$

where  $\mathbf{v} = \mathbf{J}^{-1}\mathbf{G}^T$ . For any vector  $\boldsymbol{\chi} \in \mathbb{R}^{\hat{m}}$ , we can write

$$\boldsymbol{\chi}^T(\dot{\mathbf{D}}^* - 2\mathbf{C}^*)\boldsymbol{\chi} = \boldsymbol{\chi}^T\mathbf{v}^T(\dot{\mathbf{D}} - 2\mathbf{C})\mathbf{v}\boldsymbol{\chi} + \boldsymbol{\chi}^T(\dot{\mathbf{M}} - 2\mathbf{N}_2)\boldsymbol{\chi} + \boldsymbol{\chi}^T\dot{\mathbf{v}}^T\mathbf{D}\mathbf{v}\boldsymbol{\chi} - \boldsymbol{\chi}^T\mathbf{v}^T\mathbf{D}\dot{\mathbf{v}}\boldsymbol{\chi}.$$

Using properties P2 and P3 we get,

$$\boldsymbol{\chi}^T(\dot{\mathbf{D}}^* - 2\mathbf{C}^*)\boldsymbol{\chi} = \boldsymbol{\chi}^T\dot{\mathbf{v}}^T\mathbf{D}\mathbf{v}\boldsymbol{\chi} - \boldsymbol{\chi}^T\mathbf{v}^T\mathbf{D}\dot{\mathbf{v}}\boldsymbol{\chi} = 0.$$

As  $\boldsymbol{\chi}^T\mathbf{v}^T\mathbf{D}\dot{\mathbf{v}}\boldsymbol{\chi}$  is a scalar ( $\boldsymbol{\chi}^T\mathbf{v}^T\mathbf{D}\dot{\mathbf{v}}\boldsymbol{\chi} = (\boldsymbol{\chi}^T\mathbf{v}^T\mathbf{D}\dot{\mathbf{v}}\boldsymbol{\chi})^T = \boldsymbol{\chi}^T\dot{\mathbf{v}}^T\mathbf{D}\mathbf{v}\boldsymbol{\chi}$ ).

**Lemma 2 (Due to Barbalat):**

If the differentiable function  $\mathbf{f}(t)$  has a finite limit as  $t \rightarrow \infty$ , and if  $\dot{\mathbf{f}}(t)$  is uniformly continuous, then  $\dot{\mathbf{f}}(t) \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$ .

The proof of Barbalat's lemma can be found in Popov [6].

Now we state without proof a corollary of Barbalat's Lemma taken from **Sastry and Bodson** (p. 19, [7]).

**Corollary 1:**

If  $g, \mathbf{g} \in L_\infty$ , and  $\dot{g} \in L_p$ , for some  $p \in [1, \infty)$ , then  $\dot{g}(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**3. Control Law for Flexible joint manipulators**

**3.1 Design of Position Control**

We now define several variables which will be used in the design of the controller. Let us assume that we desire the carried load to track a **desired** trajectory which is four times differentiable,  $\mathbf{x}_d(t) \in C^4$ , then there exists a **corresponding** actuator trajectory  $\mathbf{q}_{md}(t) \in C^2$  for the assumed dynamic model. This allows us to define the load position error,  $\mathbf{e}_o \in R^m$ , and the actuator position errors,  $\mathbf{e}_m \in R^n$ , as

$$\mathbf{e}_o = \mathbf{x} - \mathbf{x}_d \quad \text{and} \quad \mathbf{e}_m = \mathbf{q}_m - \mathbf{q}_{md}. \tag{27}$$

Let  $\mathbf{s}_o \in R^{\hat{m}}$  denote the augmented trajectory error for the load; and let  $\mathbf{s}_m \in R^n$  denote the augmented trajectory error for the actuator,

$$\mathbf{s}_o = \dot{\mathbf{e}}_o + \Lambda_o \mathbf{e}_o = \dot{\mathbf{x}} - \dot{\mathbf{x}}_r \quad \text{and} \quad \mathbf{s}_m = \dot{\mathbf{e}}_m + \Lambda_m \mathbf{e}_m \tag{28}$$

where the reference velocity vector for the load,  $\dot{\mathbf{x}}_r$ , is  $\dot{\mathbf{x}}_r = \dot{\mathbf{x}}_d - \Lambda_o \mathbf{e}_o$ . The matrices  $\Lambda_o$  and  $\Lambda_m$  are positive definite diagonal gain matrices, and  $\mathbf{x}_d$  and  $\mathbf{q}_{md}$  are the desired trajectories for the load and the motors respectively.

**Proposition 1:**

The control law  $\mathbf{u}$  given by (29) ensures global asymptotic stability of the load and actuators positions (i.e.,  $\mathbf{e} \rightarrow \mathbf{0}$  and  $\mathbf{e}_m \rightarrow \mathbf{0}$ , as  $t \rightarrow \infty$ ) if  $\mathbf{q}_{md}$  is chosen such that (30) holds.

$$\mathbf{u} = -\boldsymbol{\tau}_2 + \mathbf{K}_s \mathbf{q}_{md}(t) - \mathbf{K}_m \mathbf{s}_m \tag{29}$$

$$\mathbf{G}\mathbf{J}^{-T} \mathbf{K}_s \mathbf{q}_{md}(t) = -\boldsymbol{\tau}_1 - \mathbf{K}_o \mathbf{s}_o \tag{30}$$

where.



$$\tau_1 = -G^* - D^* \ddot{x}_r - C^* \dot{x}_r + GJ^{-T} \tau_x \quad (31)$$

$$\tau_2 = -D_m(\ddot{q}_{md} - \Lambda_m \dot{e}_m) - \tau_x \quad (32)$$

$$\text{and, } \tau_x = K_s[-q(0) + q_m(0) - q_{md}(0) - \int_{r=0}^t (J^{-1}G^T \dot{x}_r + \Lambda_m e_m) dr]. \quad (33)$$

The matrices  $K_o$  and  $K_m$  are  $n \times n$  positive definite matrices.

**Proof:** Consider the following Lyapunov like positive function in terms of the augmented error variables  $s$  and  $s_m$ ,

$$V(t) = 1/2 s_o^T D^* s_o + 1/2 s_m^T D_m s_m + 1/2 \int_{r=0}^t (J^{-1}G^T s_o - s_m)^T dr K_s \int_{r=0}^t (J^{-1}G^T s_o - s_m) dr.$$

The integral term in  $V$  is motivated by the work of Brogliato and Lozano-Leal [5] they utilized a term which has similar functions for single flexible joint robots. Now if we differentiate  $V$  with respect to time and use Lemma 2, we get,

$$\begin{aligned} \dot{V} &= s_o^T (D^* \dot{s}_o + 1/2 \dot{D}^* s_o) + s_m^T D_m \dot{s}_m + (J^{-1}G^T s_o - s_m)^T K_s \int_0^t (J^{-1}G^T s_o - s_m) dt \\ &= s_o^T [D^* \dot{s}_o + C^* s_o + GJ^{-T} K_s \int_{r=0}^t (J^{-1}G^T s_o - s_m) dr] + s_m^T [D_m \dot{s}_m - K_s \int_{r=0}^t (J^{-1}G^T s_o - s_m) dr]. \end{aligned} \quad (34)$$

In the following we will expand some of the terms in equation (34). Notice that,

$$J^{-1}G^T s_o = J^{-1}G^T \dot{x} - J^{-1}G^T \dot{x}_r = \dot{q} - J^{-1}G^T \dot{x}_r. \quad (35)$$

Hence,

$$\begin{aligned} &D^* \dot{s}_o + C^* s_o + GJ^{-T} K_s \int_{r=0}^t (J^{-1}G^T s_o - s_m) dr \\ &= D^* (\ddot{x} - \ddot{x}_r) + C^* (\dot{x} - \dot{x}_r) + GJ^{-T} K_s \int_{r=0}^t (\dot{q} - J^{-1}G^T \dot{x}_r - \dot{q}_m + \dot{q}_{md} - \Lambda_m e_m) dr \\ &= D^* (\ddot{x} - \ddot{x}_r) + C^* (\dot{x} - \dot{x}_r) + GJ^{-T} K_s [q(t) - q(0) - q_m(t) + q_m(0) + q_{md}(t) - q_{md}(0)] \end{aligned}$$

$$\begin{aligned}
& - \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s \int_{r=0}^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\dot{\mathbf{x}}_r + \Lambda_m \mathbf{e}_m) dr \\
& = - \mathbf{D}^* \dot{\mathbf{x}}_r - \mathbf{C}^* \dot{\mathbf{x}}_r - \mathbf{G}^* + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s \mathbf{q}_{md}(t) \\
& + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s [-\mathbf{q}(0) + \mathbf{q}_m(0) - \mathbf{q}_{md}(0) - \int_{r=0}^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\dot{\mathbf{x}}_r + \Lambda_m \mathbf{e}_m) dr] .
\end{aligned}$$

Thus we have,

$$\mathbf{D}^* \dot{\mathbf{s}}_o + \mathbf{C}^* \mathbf{s}_o + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s \int_{r=0}^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\mathbf{s}_o - \mathbf{s}_m) dr = \boldsymbol{\tau}_1 + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s \mathbf{q}_{md}(t) . \quad (36)$$

Now expanding the second term of equation (34) and using equation (3) we get,

$$\begin{aligned}
\mathbf{D}_m \dot{\mathbf{s}}_m - \mathbf{K}_s \int_0^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\mathbf{s}_o - \mathbf{s}_m) dr & = \mathbf{D}_m (\ddot{\mathbf{q}}_m - \ddot{\mathbf{q}}_{md} + \Lambda_m \dot{\mathbf{e}}_m) \\
& - \mathbf{K}_s [\mathbf{q}(t) - \mathbf{q}(0) - \mathbf{q}_m(t) + \mathbf{q}_m(0) + \mathbf{q}_{md}(t) - \mathbf{q}_{md}(0)] + \mathbf{K}_s \int_{r=0}^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\dot{\mathbf{x}}_r + \Lambda_m \mathbf{e}_m) dr \\
& = \mathbf{u} - \mathbf{D}_m (\ddot{\mathbf{q}}_{md} - \Lambda_m \dot{\mathbf{e}}_m) - \mathbf{K}_s [-\mathbf{q}(0) + \mathbf{q}_m(0) - \mathbf{q}_{md}(0) - \int_{r=0}^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\dot{\mathbf{x}}_r + \Lambda_m \mathbf{e}_m) dr] - \mathbf{K}_s \mathbf{q}_{md}(t) . \\
& = \mathbf{u} + \boldsymbol{\tau}_2 - \mathbf{K}_s \mathbf{q}_{md}(t) . \quad (37)
\end{aligned}$$

Combining equations (34), (36) and (37), we get,

$$\dot{\mathbf{V}} = \mathbf{s}_o^{\mathbf{T}} [\boldsymbol{\tau}_1 + \mathbf{G}\mathbf{J}^{-\mathbf{T}}\mathbf{K}_s \mathbf{q}_{md}(t)] + \mathbf{s}_m^{\mathbf{T}} [\mathbf{u} + \boldsymbol{\tau}_2 - \mathbf{K}_s \mathbf{q}_{md}(t)] .$$

Using equations (29) and (30)  $\mathbf{V}$  becomes,

$$\dot{\mathbf{V}} = -\mathbf{s}_o^{\mathbf{T}} \mathbf{K}_o \mathbf{s}_o - \mathbf{s}_m^{\mathbf{T}} \mathbf{K}_m \mathbf{s}_m .$$

Hence for  $\|\mathbf{s}_m\| > 0$  and  $\|\mathbf{s}_o\| > 0$ ,  $\mathbf{V} > 0$  and  $\dot{\mathbf{V}} \leq 0$ . Therefore we have  $\mathbf{s}_o \in \mathbf{L}_2 \cap \mathbf{L}_\infty$ ,

$\mathbf{s}_m \in \mathbf{L}_2 \cap \mathbf{L}_\infty$  and,  $\int_0^t (\mathbf{J}^{-1}\mathbf{G}^{\mathbf{T}}\mathbf{s}_o - \mathbf{s}_m) dr \in \mathbf{L}_\infty$ . It will be shown that

$\ddot{\mathbf{V}} = -2\mathbf{s}_o^{\mathbf{T}} \mathbf{K}_o \dot{\mathbf{s}}_o - \mathbf{s}_m^{\mathbf{T}} \mathbf{K}_m \dot{\mathbf{s}}_m$  is bounded, and hence  $\mathbf{V}$  is uniformly continuous. Then the errors  $\mathbf{e}_o$ ,  $\dot{\mathbf{e}}_o$ ,  $\mathbf{e}_m$  and  $\dot{\mathbf{e}}_m \in \mathbf{L}_2 \cap \mathbf{L}_\infty$ . Therefore we will be able to conclude from corollary one that  $\mathbf{e}_o$  and  $\mathbf{e}_m \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$ .

In the following, we will show that  $\mathbf{s}_o \rightarrow \mathbf{0}$ ,  $\dot{\mathbf{s}}_o \rightarrow \mathbf{0}$ ,  $\mathbf{s}_m \rightarrow \mathbf{0}$  and  $\dot{\mathbf{s}}_m \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$ . From equations (27), (28), and because  $\mathbf{s}_o$  is bounded we conclude that  $\mathbf{e}_o$  is

bounded, hence,  $x$  and  $\dot{x}$  is bounded, as  $x_d \in C^4$ . Therefore,  $q$  and  $\dot{q}$  is bounded. Note that  $q_{md}$  is a function of the following variables,  $x_d, \dot{x}_d, \ddot{x}_d, x, \dot{x}, y, q$ , and  $(GJ^{-T}K_s \int_{r=0}^t (J^{-1}G s_o - s_m) dr)$  all these were shown to be bounded. Thus, we can write in terms of an appropriate function  $F_1(\cdot)$ ,  $q_{md} = F_1(x_d, \dot{x}_d, \ddot{x}_d, q, \dot{q}, e_m) \in L_\infty$ , further as  $s_m$  is bounded then  $q_m$  is also bounded. From equations (26), (29), (30), and (31), we have

$$\begin{aligned} GJ^{-T}K_s(q - q_m) &= GJ^{-T}D_m\ddot{q}_m - GJ^{-T}u = GJ^{-T}D_m\ddot{q}_m - GJ^{-T}(-\tau_2 + K_s q_{md} - K_m s_m) \\ &= GJ^{-T}D_m\ddot{q}_m + GJ^{-T}\tau_2 + \tau_1 + K_o s_o + GJ^{-T}K_m s_m \\ &= GJ^{-T}D_m\dot{s}_m - D^* \ddot{x}_r - C^* \dot{x}_r - G^* + K_o s_o + GJ^{-T}K_m s_m. \end{aligned} \quad (37)$$

Because  $D_m$  is a positive definite matrix (i.e.,  $D_m^{-1}$  exists and is bounded), we can conclude that  $\dot{s}_m$  is bounded, hence,  $\ddot{e}_m \in L_1$  as,  $\dot{e}_m \in L_\infty$ . Now combine equations (25) and equations (37), to get,

$$D^* \dot{s}_o + C^* s_o + K_o s_o + GJ^{-T}D_m \dot{s}_m + GJ^{-T}K_m s_m = 0. \quad (38)$$

Because  $D^*$  is a positive definite matrix (i.e.,  $D^{*-1}$  exists and is bounded), we can conclude that  $\dot{s}_o$  is bounded. Thus from corollary 1, we can conclude that  $s_o$  and  $s_m \rightarrow 0$  as  $t \rightarrow \infty$  (as  $s_o, s_m \in L_2 \cap L_\infty$  and  $\dot{s}_o, \dot{s}_m \in L_\infty$ ). From the above we see that  $\dot{e}_o \in L_1$  and therefore,  $\ddot{q}, \ddot{x} \in L_\infty$ . Differentiating the expression for  $q_{md}$  we obtain,  $\dot{q}_{md} = F_2(x_d, \dot{x}_d, \ddot{x}_d, \dot{x}_d, q, \dot{q}, \ddot{q}, e_m, \dot{e}_m)$  (where  $F_2(\cdot)$  is the time derivative of  $F_1(\cdot)$ ), thus,  $\dot{q}_{md} \in L_\infty$  and  $\dot{q}_m \in L_1$ . Now differentiating equation (37), we have,

$$\begin{aligned} \frac{d}{dt}(GJ^{-T})K_s(q - q_m) + GJ^{-T}K_s(\dot{q} - \dot{q}_m) &= \frac{d}{dt}(GJ^{-T})D_m\dot{s}_m + GJ^{-T}D_m\ddot{s}_m \\ &\quad - \dot{D}^* \ddot{x}_r - D^* x_r^{(3)} - \dot{C}^* \dot{x}_r - C^* \ddot{x}_r - \dot{G}^* + K_o \dot{s}_o + \frac{d}{dt}(GJ^{-T})K_m s_m \\ &\quad + GJ^{-T}K_m \dot{s}_m. \end{aligned}$$

Thus, we can conclude that  $\ddot{s}_m$  and  $e_m^{(3)}$  are bounded. Now differentiating equation (38), we obtain

$$\begin{aligned} D^* \ddot{s}_o + (\dot{D}^* + C^* + K_o) \dot{s}_o + \dot{C}^* s_o + \frac{d}{dt}(GJ^T) D_m \dot{s}_m + GJ^T D_m \ddot{s}_m \\ + \frac{d}{dt}(GJ^T) K_m s_m + GJ^T K_m \dot{s}_m = 0. \end{aligned}$$

Hence, we can **conclude** that  $\ddot{s}_o$  and  $e_o^{(3)}$  are bounded. Using Lemma 2, we can conclude that  $e_o \rightarrow 0$ ,  $\dot{e}_o \rightarrow 0$ ,  $\ddot{e}_o \rightarrow 0$ ,  $e_x \rightarrow 0$  as  $t \rightarrow \infty$ . **Therefore,  $s_r \rightarrow 0$ ,  $\dot{s}_r \rightarrow 0$ ,  $s_m \rightarrow 0$  and  $\dot{s}_m \rightarrow 0$  as  $t \rightarrow \infty$ .** Thus, the controller ensures the global asymptotic tracking of the multi-robot system load trajectory. Note that by differentiating the expression for  $q_{md}$  twice we notice that  $\ddot{q}_{md} \in L_\infty$ , (as all the variables on which  $\ddot{q}_{md}$  depend is also bounded).

### 3.2 Design of the Internal Force Control Law

We can write equation (30) as,

$$K_s q_{md} = -J^T G^+ (K_o s_o + \tau_1) + J^T \tau_f \quad (39)$$

where  $G\tau_f = 0$  and  $G^+ = G^T(GG^T)^{-1}$ . Now if we replace  $\tau_1$  by its value from equation (31-33) and use equations (22), (23) and (24), we get,

$$K_s q_{md} = A_1 \ddot{x}_r + A_2 \dot{x}_r + A_3 - \tau_x - J^T G^+ K_o s_o + J^T \tau_f = \phi_1 + J^T \tau_f \quad (40)$$

where,

$$A_1 = DJ^{-1}G^T + J^T G^+ M, \quad A_2 = D \frac{d}{dt}(J^{-1}G^T) + CJ^{-1}G^T + J^T G^+ N_2$$

$$\text{and, } A_3 = G_r + J^T G^+ G_l \text{ with, } \phi_1 = A_1 \ddot{x}_r + A_2 \dot{x}_r + A_3 - \tau_x - J^T G^+ K_o s_o. \quad (41)$$

Now equation (29) can be expanded to,

$$u = D_m (\ddot{q}_{md} - \Lambda_m \dot{e}_m) - K_m s_m + A_1 \ddot{x}_r + A_2 \dot{x}_r + A_3 - J^T G^+ K_o s_o + J^T \tau_f$$

letting,  $\phi_2 = -D_m \Lambda_m \dot{e}_m - K_m s_m + A_1 \ddot{x}_r + A_2 \dot{x}_r + A_3 - J^T G^+ K_o s_o$ , then

$$u = D_m \ddot{q}_{md} + \phi_2 + J^T \tau_f. \quad (42)$$

If we replace  $\ddot{q}_{md}$  by its value obtained from differentiating equation (40) twice, we get,

$$u = D_m K_s^{-1} \ddot{\phi}_1 + D_m K_s^{-1} \frac{d^2}{dt^2}(J^T \tau_f) + \phi_2 + J^T \tau_f. \quad (43)$$

Our goal is to find the **term  $\tau_f$**  such that  $e_f \rightarrow 0$  as  $t \rightarrow \infty$ . Motivated by the work of Hsu et al. [4], we will **choose  $\tau_f$**  to be,

$$\tau_f = F_{el_d} - K_f \int_{r=0}^t e_f dr, \quad (44)$$

where  $K_f \in \mathbb{R}^{n \times n}$  is a positive semi-definite diagonal matrix.

**Theorem:**

The control law given by equation (43) and (44) ensures the global asymptotic stability of the actuator and load position and the asymptotic convergence of the internal forces (i.e.,  $e_m \rightarrow 0$ ,  $e_0 \rightarrow 0$ , and  $e_f \rightarrow 0$  as  $t \rightarrow \infty$ .)

**Proof:**

First note that

$$G\tau_f = G(F_{el_d} - K_f \int_{r=0}^t e_f dr) = 0. \quad (45)$$

Also note that the choice of  $u$  guarantees that  $e_0 \rightarrow 0$  and  $e_m \rightarrow 0$  as  $t \rightarrow \infty$  (from proposition 1), thus we need only to prove is that  $e_f \rightarrow 0$  as  $t \rightarrow \infty$ .

We will start off by determining the equation of the closed loop system involving  $F_f$ . Solving for  $F_e$  from equation (7), we get,

$$F_e = G^+(M\ddot{x} + N_2\dot{x} + G_1) + F_{el}. \quad (46)$$

Now combining equations (2) and (46), we get,

$$D\ddot{q} + C\dot{q} + G_r + K_s(q - q_m) + J^T G^+(M\ddot{x} + N_2\dot{x} + G_1) + J^T F_{el} = 0,$$

replacing  $\dot{q}$  and  $q$  by their values from (16) and (17) we get,

$$\begin{aligned} (DJ^{-1}G^T + J^T G^+ M)\ddot{x} + [D \frac{d}{dt}(J^{-1}G^T) + CJ^{-1}G^T + J^T G^+ N_2]\dot{x} \\ + G_r + J^T G^+ G_1 + K_s(q - q_m) + J^T F_{el} = 0. \end{aligned} \quad (47)$$

Using the definitions of  $A_1$ ,  $A_2$  and  $A_3$  given by equations (41), the above equation can be written as follows,

$$A_1\ddot{x} + A_2\dot{x} + A_3 + K_s(q - q_m) + J^T F_{el} = 0. \quad (48)$$

Combining the dynamics of the motors, equation (3) and equation (42), we get.

$$K_s(q - q_m) = D_m\ddot{q}_m - D_m\ddot{q}_{ind} - \phi_2 - J^T \tau_f = D_m\ddot{e}_m - \phi_2 - J^T \tau_f \quad (49)$$

Combining equations (48) and (49), we get,

$$A_1\ddot{x} + A_2\dot{x} + A_3 + J^T F_{el} + D_m\ddot{e}_m - \phi_2 - J^T \tau_f = 0. \quad (50)$$

Now replacing  $\phi_2$  by its value from (42), and  $\tau_f$  by its value in (44), we get

$$A_1 \dot{s}_o + A_2 s_o + J^T(e_f + K_f \int_{r=0}^t e_f dr) + D_m \dot{s}_m + K_m s_m + J^T G^+ K_o s_o = 0. \quad (51)$$

**Equation** (51) represents the multi-robot closed loop system dynamics. We proved previously that  $e_o \rightarrow 0$ ,  $\dot{e}_o \rightarrow 0$ ,  $e_m \rightarrow 0$ ,  $\dot{e}_m \rightarrow 0$ ,  $s_o \rightarrow 0$ ,  $\dot{s}_o \rightarrow 0$ ,  $s_m \rightarrow 0$  and  $\dot{s}_m \rightarrow 0$  as  $t \rightarrow \infty$ . **Thus**, we have

$$J^T(e_f + K_f \int_{r=0}^t e_f dr) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (52)$$

and as  $J^T$  is not a singular matrix, thus  $e_f + K_f \int_{r=0}^t e_f dr \rightarrow 0$  as  $t \rightarrow \infty$ . Hence,

$e_f \rightarrow 0$  as  $t \rightarrow \infty$ , because  $K_f$  is selected as a positive semi-definite matrix.

**Remark:**

We notice that we need to know the desired internal force, its first derivative and its second derivative in order to compute the control law. We also need to know the actual internal force and its derivatives. We also need to know the desired load position, its first, second, third and fourth derivatives. **The** actual load position and its **first**, second, and third derivative is also necessary for control calculation. The control law in equation (43) involves up to the second derivative of  $\tau_f$  as a result  $F_{eI}$  and  $\dot{F}_{eI}$  are needed. The first **term** in the control law involves the second derivative of  $\phi_1$  which involves  $\ddot{x}_r = \ddot{x}_d + \Lambda_o \dot{e}_o$  as a result  $x^{(3)}$ ,  $x^{(2)}$ ,  $x^{(1)}$  and  $x$  is needed in the control law.

### 3.3 Computation of $\ddot{x}$ , $x^{(3)}$ , $F_{eI}$ and $\dot{F}_{eI}$ from Joint Positions and Velocities.

Note that **from** the forward kinematics, we can write,  $x = \alpha_0(q)$ . Also from equation (14), we can write,  $x = \alpha_1(q, \dot{q})$ . We can solve for  $x$ , from equation (25), we get,

$$\ddot{x} = -D^{*-1}[C^* \dot{x} + G^* + GJ^T K_s(q - q_m)]. \quad (53)$$

Here  $D^*$  is a function of  $q$  only;  $C^*$  is a function of  $q$  and  $\dot{q}$  and  $GJ^T K_s(q - q_m)$  is a function of  $q$  and  $q_m$ . Thus we can write,  $x = \alpha_2(q, \dot{q}, q_m)$ . We can solve for  $F_{eI}$  from equation (48),  $F_{eI} = -J^T[A_1 x + A_2 \dot{x} + A_3 + K_s(q - q_m)]$ . Thus we see that  $F_{eI}$  can be computed from measurements of  $q$ ,  $\dot{q}$  and  $q_m$ .

$$F_{eI} = J^T(q)[A_1(q)\alpha_2(q, \dot{q}, q_m) + A_2(q, \dot{q})\alpha_1(q, \dot{q}) + A_3(q) + K_s(q - q_m)] = \gamma_0(q, \dot{q}, q_m). \quad (54)$$

If we differentiate equation (53), we obtain,  $x^{(3)} = \alpha_3(q, \dot{q}, q_m, \dot{q}_m)$ . We then can

differentiate equation (54) and replace  $\mathbf{x}^{(3)}$  by its value, we obtain,

$$\dot{\mathbf{F}}_{eI} = \gamma_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_m, \dot{\mathbf{q}}_m). \quad (55)$$

We can conclude that  $\mathbf{F}_{eI}$  and  $\dot{\mathbf{F}}_{eI}$  can be computed from measurements of  $\mathbf{q}$ ,  $\dot{\mathbf{q}}$ ,  $\mathbf{q}_m$  and  $\dot{\mathbf{q}}_m$ . Finally we can show that,  $\ddot{\phi}_1 = \beta_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_m, \dot{\mathbf{q}}_m)$  and  $\ddot{\phi}_2 = \beta_2(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_m, \dot{\mathbf{q}}_m)$ . Therefore we can conclude that the **control**  $u$  can be computed from the **measurements** of the joints and motors positions and velocities (i.e.,  $u = u(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_m, \dot{\mathbf{q}}_m)$ ). Thus because of the boundedness of  $\mathbf{q}$ ,  $\dot{\mathbf{q}}$ ,  $\mathbf{q}_m$  and  $\dot{\mathbf{q}}_m$ , we can conclude that the **Control** law  $u$  is bounded. We can also show that  $\mathbf{q}_{md}$ ,  $\dot{\mathbf{q}}_{md}$  and  $\mathbf{x}^*$  are bounded.

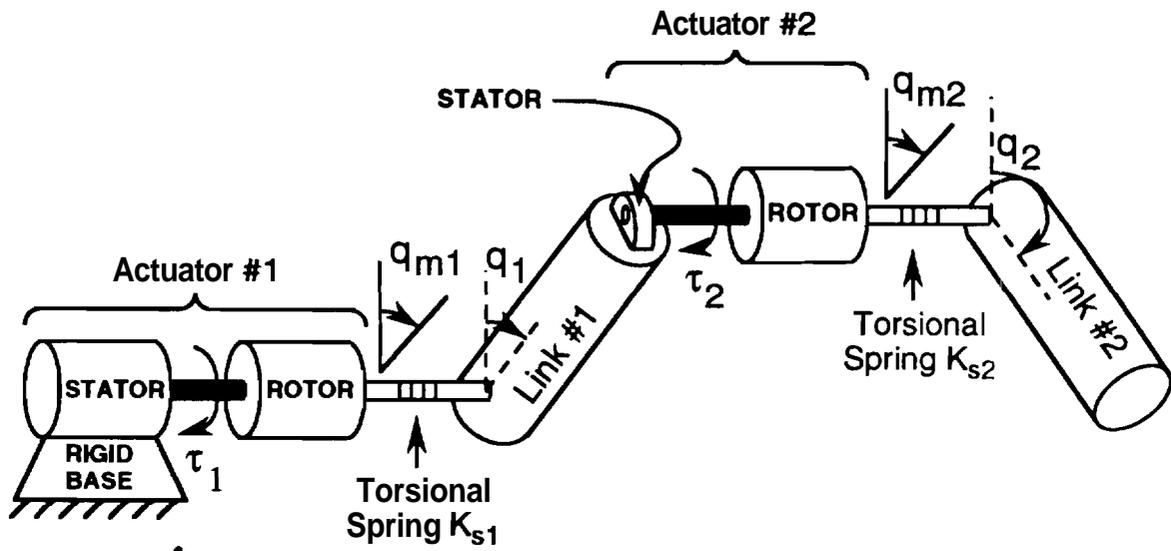
#### 4. Conclusion

In this paper, we addressed the problem of controlling multiple flexible joint robots manipulating a load cooperatively. We assumed that the grasp is rigid. We **first** obtained a combined model which takes into account the dynamics of the load and the manipulators. We then derived few properties for the combined **model**; these properties were used later to derive the control law. Finally we **proposed** a **controller** that ensures the asymptotic convergence of the load position, and the internal forces to **their** desired values. Force sensors mounted on the robot wrist are not needed for the tracking of the desired internal forces trajectories. We have also shown that the control law only requires the measurements of the joints and motors **positions** and velocities. The controller derived in this paper requires the exact knowledge of the robot and load parameters. Future work must **be carried** to develop an adaptive version of this **control** law and to study the robustness of the control law to parameter uncertainty and modeled actuator dynamics. There are other models for **flexible** joint robots ( see [8]) further work needs to **be** carried out to develop multi-robot control schemes for those models.

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Conceptual diagram of a flexible robot.

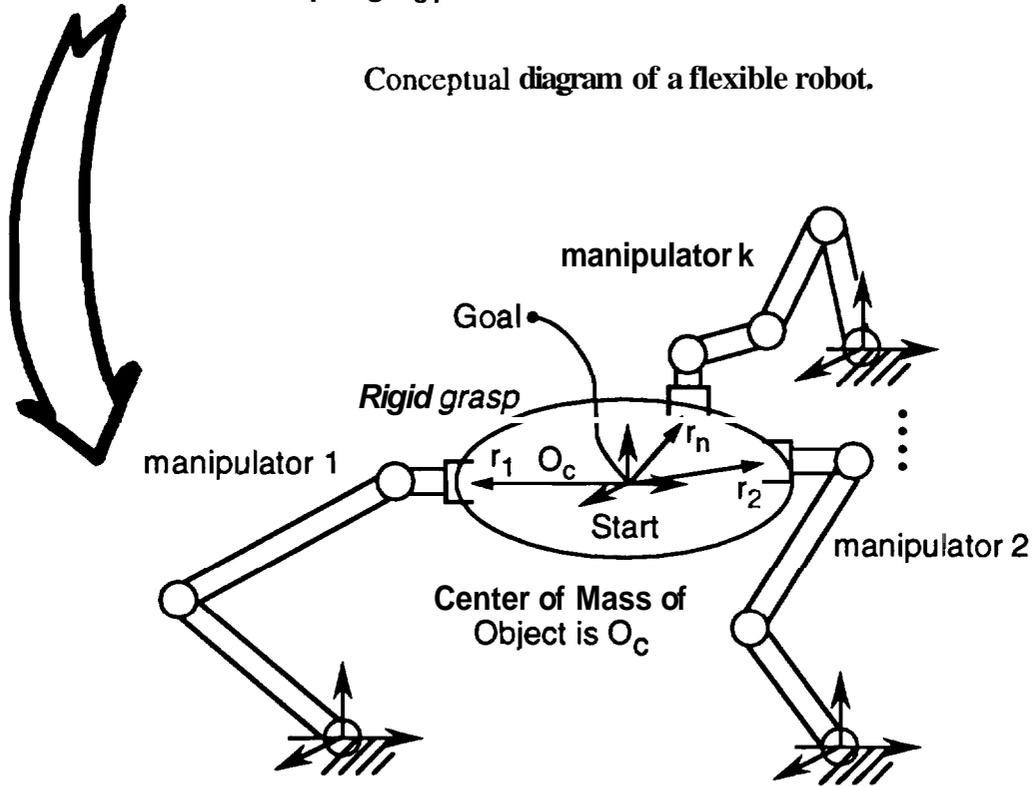


Figure 1. Multirobot system organization, with desired trajectory

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