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Numerical Methods for Flood Routing Problems

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NUMERICAL METHODS FOR FLOOD-ROUTING PROBLEMS

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ABSTRACT

A number of papers have appeared in the past years that use characteristic methods and finite difference schemes to solve the flood routing problem, [1,4]. The finite element methods on the other hand have only recently been applied to the two-dimensional shallow-water flow equations [5].

The purpose of this report is to present a survey of numerical methods used in solving the flood-routing problem. These methods are compared to a finite element scheme developed according to the Galerkin method.

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1. Introduction

Unsteady flow in rivers and channels is commonplace. It occurs during the flooding and recession phase of a river. It also occurs in the head and tail races of hydropower schemes in direct response to the changes of the load on the power station. Finally, it is present in the "dam-break" problem, where the effects of a partial or total collapse of a dam are of importance, for the assessment of the associated hazard and appropriate organization of the defense of inhabitants and structures in the downstream valley.

The equations of unsteady flow in open channels are derived from the principle of hydraulic continuity and momentum equation. They form a system of nonlinear hyperbolic partial differential equations known as the St. Venant equations. Depth changes and other geometric and physical changes (friction coefficient) encountered in natural channels make any attempt to find closed form solutions unpractical. In the present report, two numerical schemes are proposed for the resolution of the St. Venant equations based on the finite element method (FEM). The one-dimensional geometric space (along the axis of the channel) is approximated by finite-elements. Their number depends on the variability of the cross-section of the channel. The time is discretized in constant time-steps (finite-difference scheme). The FEM techniques used are the Gallerkin method and the collocation method. Both these schemes are compared with the method of characteristics as applied by Streeter (9), for flood routing problems, and a finite difference explicit scheme suggested by Viessman (10). First, the physical phenomenon is defined, and the governing St. Venant equations developed. Also an extensive list of boundary conditions is presented.
2. DEFINITION OF A FLOOD ROUTING PROBLEM

The physical phenomenon is described by the following two governing equations:

THE MOMENTUM EQUATION

\[-g \frac{\partial v}{\partial x} + g(S_F - S_o) + 2v \frac{\partial v}{\partial x} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial t} + \frac{\partial v}{\partial t} = 0 \]  \hspace{1cm} (2.1)

THE CONTINUITY EQUATION

\[\frac{v}{A} \frac{\partial A}{\partial x} + \frac{1}{A} \frac{\partial A}{\partial t} + \frac{\partial v}{\partial x} = 0 \] \hspace{1cm} (2.2)

where:  
- \( v \) = mean velocity of cross section (unknown)
- \( y \) = free surface elevation (unknown)
- \( S_F \) = friction slope
- \( S_o \) = Bed slope
- \( A \) = Area of the cross section
- \( B \) = the width of the channel

Multiplying Eq. 2.2 by \( v \)

\[\frac{v^2}{A} \frac{\partial A}{\partial x} + \frac{v}{A} \frac{\partial A}{\partial t} + \frac{\partial v}{\partial x} = 0 \]  \hspace{1cm} (2.3)

Subtracting from Eq. 2.1 we get

\[-g \frac{\partial v}{\partial x} + g(S_F - S_o) + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0 \] \hspace{1cm} (2.4)

Equation 2.2 may be written as

\[\frac{v}{A} B \frac{\partial y}{\partial x} + \frac{B}{A} \frac{\partial y}{\partial t} + \frac{\partial v}{\partial x} = 0 \]

or

\[v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + \frac{A}{B} \frac{\partial v}{\partial x} = 0 \]

The final expressions are

\[v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = -g \frac{\partial y}{\partial x} + g(S_o - S_F) \] \hspace{1cm} (2.4)

\[\frac{\partial y}{\partial t} + \frac{A}{B} \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial x} = 0 \] \hspace{1cm} (2.5)
3. Boundary Conditions

The importance of the boundary conditions in unsteady open-channel flow can be visualized in the fact that unsteadiness of flow is precisely generated from a boundary condition. Examples of boundary conditions are the following:

(1) A downstream estuary whose level fluctuates with the tides.
(2) An upstream catchment providing flows that vary with rainfall producing flood or storm surges.
(3) Upstream or downstream weirs.
(4) Upstream or downstream sluice gates.
(5) Downstream reservoirs with characteristic slow backwater fluctuations.
(6) Upstream or downstream spillways.
(7) Junctions.

The level history in a tidal estuary can be measured and is often available from records, so that data values of depth at the end of the channel can be read in at \( \Delta t \) time-step intervals, or interpolated from a reading array.

Flood or storm surge upstream conditions are given by the inflow hydrograph \( Q_t \), (time history of inflowing rates). Then, for a broad or rectangular in cross-section channel, the mean velocity at the upstream cross-section will be given by:

\[
\nu_p = \frac{Q_t}{B \cdot y_p} 
\]  

(3-1)
Upstream or downstream weir conditions are specified by the weir empirical formula:

\[ Q_w = 1.72 \cdot B \cdot (E_w - h_w)^{1.5} \quad \text{in} \quad [m^3/s] \quad (3-2) \]

where: \( E_w = y_w + \frac{v_w^2}{2g} \) is the specific energy just upstream of the weir, and \( h_w \) is the height of the weir crest above channel bed level.

Similar empirical relations exist for sluice-gate boundary conditions and spillway conditions.

Two conditions are necessary for the case of a junction. First, the depth is the same for all channels joining there. Second, the inflow to the junction must equal the outflow from it (continuity). For flows at other than small Froude numbers however, it will be necessary to include local losses and kinetic energy terms. The continuity equation is as follows:

\[ \sum_i \text{sign}(i) \cdot B_i \cdot y_{pi} \cdot v_{pi} = 0 \quad (3-3) \]

where subscript \( i \) denotes \( i \)-th channel at junction \( p \), and \( \text{sign}(i) \) is conventionally positive if channel \( i \) conveys water to the junction and is negative otherwise.
4. Collocation Method

The finite element collocation procedure that is proposed is based on the Hermite cubic elements: let \( \Delta = (x_i)_{i=1}^{N+1} \) be a partition of the spatial interval \([0,L]\). Denoting by \( H_\Delta \) the \((2N+2)\)-dimensional vector space of all continuously differentiable functions which reduce to polynomials of degree at most 3 over each subinterval of \( \Delta \), any function in \( H_\Delta \) is expressed as a linear combination of \( 2N+2 \) basis functions \( B_1, \ldots, B_{2N+2} \). For example, if \( \Delta \) is uniform and \( h = x_{i+1} - x_i \), then such a basis is given by

\[
B_{2i-1}(x) = \phi \left( \frac{x-x_i}{h} \right), \quad B_{2i}(x) = h \psi \left( \frac{x-x_i}{h} \right), \quad 1 \leq i \leq N+1,
\]

where:

\[
\phi(x) = \begin{cases} 
(1-x)^2 (1+2x) & 0 \leq x \leq 1 \\
(1+x)^2 (1-2x) & -1 \leq x \leq 0 \\
0 & \text{otherwise,}
\end{cases}
\]

\[
\psi(x) = \begin{cases} 
(x(1-x))^2 & 0 \leq x \leq 1 \\
(x(1+x))^2 & -1 \leq x \leq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Use is made of the Gauss-Legendre points in each subinterval \([x_i, x_{i+1}]\) given by

\[
\sigma_{2i+j} = .5 \left[ x_{i+1} + x_i + \frac{(-1)^j}{\sqrt{3}} (x_{i+1} - x_i) \right], \quad j=1,2
\]
The basis functions, evaluated at these points, give rise to a $2N \times (2N+2)$ matrix $G = (B_k(a_t))$. In the case of a uniform partition $\Delta$, one easily finds

$$G = \begin{bmatrix}
\alpha & \beta & \bar{\alpha} & -\bar{\beta} \\
\bar{\alpha} & \bar{\beta} & \alpha & -\beta \\
\alpha & \beta & \bar{\alpha} & -\bar{\beta} \\
\bar{\alpha} & \bar{\beta} & \alpha & -\beta \\
\vdots \\
\alpha & \beta & \bar{\alpha} & -\bar{\beta} \\
\bar{\alpha} & \bar{\beta} & \alpha & -\beta 
\end{bmatrix} \quad \text{(4-3)}$$

where: $\alpha = 0.5 + 2\sqrt{3}/9$, $\beta = 1/12 + \sqrt{3}/36$,

$\bar{\alpha} = 0.5 - 2\sqrt{3}/9$, $\bar{\beta} = 2/23 - \sqrt{3}/36$.

In the collocation method an approximate solution in the space $H_\Delta$ is sought. That is, we seek to determine approximations $v, y$ to $v, y$ in the form

$$v(x,t) = \sum_i V_i(t) B_i(x), \quad y(x,t) = \sum_i Y_i(t) B_i(x) \quad \text{(4-4)}$$

where here and in the sequel all summations extend from 1 to $2N+2$.

The unknown coefficients (functions of $t$) $V_i, Y_i$ are determined by a system of ordinary differential equations, as follows: expressions (4-4) are substituted into the partial differential equations (2-1). The resulting equations are forced to be satisfied at all the collocation points (in our method, the Gauss-Legendre points) $x=\sigma_l, \ l=1,...,2N$. Then, one easily finds

$$\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \ddot{\xi}(t) = F(\xi(t)) \quad \text{(4-5)}$$

where $G$ is given by (4-3), the dot denotes time derivative and
\[ Z(t) = [V_1(t), \ldots, V_{2N+2}(t), Y_1(t), \ldots, Y_{2N+2}(t)]^T \]

\[ F = [F_{11}, \ldots, F_{1,2N}, F_{21}, \ldots, F_{2,2N}]^T \]

\[ F_{1\&}(Z(t)) = - \sum_{i,k} B_i(\sigma_k) B_k(\sigma_k) V_i(t) V_k(t) - \]

\[ - g \sum_{i} B_i(\sigma_k) Y_i(t) + g(S_0 - S_1) \]

\[ F_{2\&}(Z(t)) = - \frac{A}{B} \sum_{i} B_i(\sigma_k) V_i(t) - \sum_{i,k} B_i(\sigma_k) B_k(\sigma_k) V_i(t) Y_j(t) \]

\[ \lambda = 1, \ldots, 2N. \]

Note that, in the expression for \( F_{2\&} \), the area \( A \) depends on \( Y(t) \).

System (4-5) relates \( 2(2N+2) \) unknown functions by \( 4N \) equations. Consequently, the number of unknown functions must be reduced and this is done by use of the boundary conditions. As an illustration, consider the boundary conditions at the left end

\[ v(0,t) = v_0(t) \]

\[ y(0,t) = y_0(t) \]

where \( v_0, y_0 \) are given functions, which in general do not belong to \( H_\Delta \).

Nevertheless, \( v_0, y_0 \) are readily approximated by elements \( \overline{v}_0, \overline{y}_0 \) of \( H_\Delta \):

\[ \overline{v}_0(t) = \sum_{i=1}^{2M+2} v_i B_i(t), \quad \overline{y}_0(t) = \sum_{i=1}^{2M+2} y_i B_i(t) \]  

(4-7)

where the basis functions \( B_i \), now correspond to a partition \( \{t_i\}_{i=1}^{m+1} \) of the time interval. The coefficients \( v_i, y_i \) are given by

\[ v_{2i-1} = \overline{v}_0(t_i), \quad v_{2i} = \frac{d}{dt} \overline{v}_0(t_i), \quad y_{2i-1} = \overline{y}_0(t_i), \quad y_{2i} = \frac{d}{dt} \overline{y}_0(t_i) \]

It is well-known that (with \( w=v_0 \) or \( w=y_0 \))

\[ ||w-\overline{w}||_\infty \leq c(\max_i |t_{i+1} - t_i|)^4 \]

from which it is seen that (4-7) constitute satisfactory approximations.
Now the boundary conditions (4-6) are approximated by

\[ v_d(0,t) = \overline{v}_0(t) \]
\[ y_d(0,t) = \overline{y}_0(t) \]  

(4-8)

Substituting (4-4) and (4-7) into (4-8) and taking into account that \( B_i(0) = 0 \) if \( i \neq 0 \) we find

\[ V_1(t) = \overline{v}_0(t) \]
\[ Y_1(t) = \overline{y}_0(t) \]  

(4-9)

thus eliminating \( V_1, Y_1 \) from the unknown functions.

A similar procedure may be followed for a boundary condition at the right end. System (4-5), supplied with appropriate initial conditions, is now solvable.

5. **Gallerkin Method**

_The Finite Element Procedure_ is applied directly on the P.D.E.'s making use of Gallerkin's Weighted Residual Method in the following computational steps:
STEP 1  
**The UNIDIMENSIONAL FLOW FIELD is discretized into n elements**

For each element the following linear **SHAPE FUNCTIONS** are chosen:

\[ N_i^e = \frac{x - x_i}{x_j - x_i}, \quad N_j^e = \frac{x_j - x}{x_j - x_i} \]  \hspace{1cm} (5.6)

where: \( (e) \) denotes the element number

\( i, j \) are the nodes of the element respectively the left and right nodes.

Equations (5.6) can also be viewed as **INTERPOLATION FUNCTIONS**.

\begin{center}
\begin{tabular}{c c c c}
1 & 2 & 3 & 4 & 5
\end{tabular}
\end{center}

**ILLUSTRATION OF THE LINEAR SHAPE FUNCTIONS**

**GEOMETRIC CONFIGURATION OF THE ELEMENTS**

\begin{align*}
\mathbf{v} &= \sum_{e=1}^{n} N^e \cdot v^e \quad (5.7) \\
\mathbf{y} &= \sum_{e=1}^{n} N^e \cdot y^e \quad (5.8)
\end{align*}

where: \( v^e \) and \( y^e \) are the unknown nodal values of the depth and the velocity.

According to Gallerkin's procedure the Residual must be orthogonal to the Shape Functions. Thus for each element equations 5.4 and 5.5 become
\[
\int \left( N^e \frac{\partial v^e}{\partial t} \right) dx \cdot N_i + \int \left( N^e \frac{\partial v^e}{\partial x} \right) dx \cdot N_i + g \int \left( N^e \frac{\partial v^e}{\partial x} \right) dx \cdot N_i + \\
+ g \int \left( N^e \left( S_f - S_0 \right) \right) dx \cdot N_i = 0 \quad (5.9)
\]

and
\[
\int \left( N^e \frac{v^e}{t} \right) dx \cdot N_i + \int \left[ N^e \frac{\partial A^e}{\partial x} \right] N^e \frac{v^e}{x} \ dx \cdot N_i + \\
+ \int \left( N^e \frac{v^e}{x} \right) \left[ N^e \frac{\partial v^e}{\partial x} \right] dx \cdot N_i = 0 \quad (5.10)
\]

We obtain similar expressions for the function \( N_j \). At this stage we expand term by term the above two equations for the case of the shape function \( N_i \) and Eq. 5.9.

\[
\int_{x_i}^{x_j} \left[ \frac{x - x_i}{x_j - x_i}, \frac{x - x_i}{x_j - x_i} \right] \left\{ \frac{\partial v_i}{\partial t} \right\} \left\{ \frac{x - x_i}{x_j - x_i} \right\} dx = \int_{x_i}^{x_j} \left\{ \frac{\partial v_i}{\partial t} + (x_j - x) (x - x_i) \right\} dx
\]

Then
\[
\int_{x_i}^{x_j} \left[ \frac{x - x_i}{x_j - x_i}, \frac{x - x_i}{x_j - x_i} \right] \left\{ \frac{\partial v_i}{\partial x} \right\} \left\{ \frac{\partial v_i}{\partial x} \right\} \left\{ \frac{\partial v_j}{\partial x} \right\} \left\{ \frac{\partial v_j}{\partial x} \right\} dx = \\
= \int_{x_i}^{x_j} \left\{ \frac{x - x_i}{x_j - x_i} v_i + \frac{x - x_i}{x_j - x_i} v_j \right\} \left\{ \frac{\partial v_i}{\partial x} \right\} \left\{ \frac{\partial v_i}{\partial x} \right\} \left\{ \frac{\partial v_j}{\partial x} \right\} \left\{ \frac{\partial v_j}{\partial x} \right\} dx
\]

(5.12)
Also

\[ \int_{x_i}^{x_j} \frac{3}{3x} \left[ \frac{x - x_i}{x_j - x_i}, \frac{x_j - x}{x_j - x_i} \right] \left\{ \begin{array}{c} y_i \\ y_j \end{array} \right\} \frac{x - x_i}{x_j - x_i} \cdot dx = \]

\[ = \int_{x_i}^{x_j} \frac{3}{3x} \left[ \frac{x - x_i}{x_j - x_i} \cdot y_i \right] + \frac{3}{3x} \left[ \frac{(x_j - x)}{(x_j - x_i)} \cdot y_i \right] \frac{x - x_i}{x_j - x_i} \cdot dx \quad (5.13) \]

Also

\[ \int_{x_i}^{x_j} \left[ \frac{x - x_i}{x_j - x_i}, \frac{x_j - x}{x_j - x_i} \right] \left[ S_{fi} - S_{oi} \right] \left[ S_{fj} - S_{oj} \right] \frac{x_i - x}{x_j - x_i} \cdot dx = \]

\[ = \int_{x_i}^{x_j} \left\{ \left( \frac{x - x_i}{x_j - x_i} \right)^2 \left( S_{fi} - S_{oi} \right) + \frac{(x_j - x)(x_i - x)}{(x_j - x_i)^2} \left( S_{fj} - S_{oj} \right) \right\} \cdot dx \quad (5.14) \]

The expressions (5.11) through (5.14) are developed in the following pages using a forward DIFFERENCE SCHEME for the integration in the time domain. Therefore:

\[ \frac{\partial \nu}{\partial t} = \frac{\nu^n - \nu^{n-1}}{\Delta t} \quad \text{and} \quad \frac{\partial \nu}{\partial t} = \frac{\nu^n - \nu^{n-1}}{\Delta t} \]

and expression (5.11) becomes, having in mind that \( \nu^n \) and \( \nu^n \) are the velocity and depth at time \( n \), while \( \nu^{n-1} \) and \( \nu^{n-1} \) are the velocity and depth at time \( n-1 \):
\[ \int_{x_i}^{x_j} \left\{ \frac{(x-x_i)^2}{L^2} \cdot \frac{v_j^n - v_i^{n-1}}{\Delta t} + \frac{(x-x_j)(x_j-x)}{L^2} \cdot \frac{v_j^n - v_j^{n-1}}{\Delta t} \right\} \, dx = \]

\[= \frac{v_i^n}{L^2 \cdot \Delta t} \int_{x_i}^{x_j} (x^2-x_i^2-2xx_i) \, dx - \frac{v_i^{n-1}}{L^2 \cdot \Delta t} \int_{x_i}^{x_j} (x^2-x_i^2-2xx_i) \, dx + \]

\[+ \frac{v_j^n}{L^2 \cdot \Delta t} \int_{x_i}^{x_j} (x^2-x_i^2-2x_jx) \, dx - \frac{v_j^{n-1}}{L^2 \cdot \Delta t} \int_{x_i}^{x_j} (x^2-x_i^2-2x_jx) \, dx \]

The integration is easily performed and the following result is obtained

\[= \frac{v_i^n}{3 \cdot L^2 \cdot \Delta t} (x_j-x_i)^3 - \frac{v_i^{n-1}}{3 \cdot L^2 \cdot \Delta t} (x_j-x_i)^3 + \frac{v_j^n}{6 \cdot L^2 \cdot \Delta t} (x_j-x_i)^3 - \frac{v_j^{n-1}}{6 \cdot L^2 \cdot \Delta t} (x_j-x_i)^3 \]

Expression (5.12) becomes

\[\int_L \left\{ \frac{x-x_i}{L} \cdot v_i^{n-1} + \frac{x-x_j}{L} \cdot v_j^{n-1} \right\} \left\{ \frac{a}{\Delta x} \cdot \frac{x-x_i}{L} \cdot v_j^n + \frac{a}{\Delta x} \cdot \frac{x-x_j}{L} \cdot v_j^n \right\} \frac{x-x_i}{L} \, dx \]

The following simplification was made for the nonlinear term

\[v \cdot \frac{\partial v}{\partial x} = v^{n-1} \cdot \frac{\partial v^n}{\partial x} \]

existing in the original expression (5.12). Then

\[\int_L \left\{ \left( \frac{x-x_i}{L} \right)^n \cdot v_i^{n-1} + \left( \frac{x-x_j}{L} \right)^n \cdot v_j^{n-1} \right\} \left\{ \frac{1}{L} \cdot v_j^n - \frac{\partial v_j^n}{\partial x} \right\} \frac{x-x_i}{L} \, dx \]

\[\int_L \left\{ \frac{1}{L} \cdot \left( \frac{x-x_i}{L} \right)^2 \cdot v_i^{n-1} \cdot v_i^n + \frac{(x_j-x)(x-x_i)}{L^3} \cdot v_j^{n-1} \cdot v_i^n - \frac{(x-x_i)^2}{L^3} \cdot v_i^{n-1} \cdot v_j^n \right\} \frac{x-x_i}{L} \, dx = \]

\[= \frac{L^3}{6} \frac{v_i^{n-1} \cdot v_i^n}{6} - \frac{v_i^{n-1} \cdot v_j^n}{3} - \frac{v_j^{n-1} \cdot v_j^n}{6} \]
Expression (5.13) becomes, after considering a LEAP FROG scheme

\[ g \int \left[ \frac{3}{a \lambda} \frac{x-x_i}{L} y_i^{n-1} + \frac{3}{a \lambda} \frac{x_j-x}{L} y_j^{n-1} \right] \frac{x-x_i}{L} \cdot dx = \]

again assuming \( \frac{x-x_i}{L} \frac{a y_i}{a x} = \frac{x_j-x}{L} \frac{a y_j}{a x} = 0 \)

\[ = \frac{g}{2} \left[ y_i^{n-1} - y_j^{n-1} \right] \]

Finally expression (5.14) becomes

\[ g \int \left\{ \frac{(x-x_i)^2}{L^2} \left( S_{f_1-S_{o_1}} \right) + \frac{(x_j-x)(x-x_i)}{L^2} \left( S_{f_2-S_{o_2}} \right) \right\} \cdot dx = \]

\[ = g \left[ \frac{L^3}{3L^2} \left( S_{f_1-S_{o_1}} \right) + \frac{L^3}{6L^2} \left( S_{f_2-S_{o_2}} \right) \right] = g L \left[ \frac{S_{f_1-S_{o_1}}}{3} + \frac{S_{f_2-S_{o_2}}}{6} \right] \]

Collecting the terms obtained from expressions (5.11-5.14) we obtain the momentum equation multiplied by \( N_i \).

\[ \frac{L}{3\Delta t} v_i^n - \frac{L}{6\Delta t} v_i^{n-1} + \frac{L}{6\Delta t} v_j^n - \frac{L}{6\Delta t} v_j^{n-1} + \frac{v_i^{n-1}}{3} + \frac{v_j^{n-1}}{6} v_i^n - \frac{v_i^{n-1}}{3} - \frac{v_j^{n-1}}{6} v_j^n + \]

\[ + \frac{g}{2} \left[ y_i^{n-1} - y_j^{n-1} \right] + g L \left[ \frac{S_{f_1-S_{o_1}}}{3} + \frac{S_{f_2-S_{o_2}}}{6} \right] = 0 \]  \hspace{1cm} (5.15)

Now multiplying the momentum equation by \( N_j \) a similar equation to (5.15) can be obtained

\[ \frac{L}{6\Delta t} v_i^n - \frac{L}{3\Delta t} v_i^{n-1} + \frac{L}{3\Delta t} v_j^n - \frac{L}{3\Delta t} v_j^{n-1} + \frac{v_i^{n-1}}{6} v_i^n - \frac{v_j^{n-1}}{3} v_j^n - \frac{v_i^{n-1}}{6} v_j^n + \]

\[ + \frac{v_j^{n-1}}{3} v_j^n + \frac{g}{2} \left[ y_i^{n-1} - y_j^{n-1} \right] + g L \left[ \frac{S_{f_1-S_{o_1}}}{6} + \frac{S_{f_2-S_{o_2}}}{3} \right] \]  \hspace{1cm} (5.16)

The LEAPFROG technique was used to obtain the decoupling of the St. Venant's equations thus obtaining two independent systems of equations. Indeed as it can be observed Eqs. 5.15 and 5.16 represent the first system of equations with \( v_i^n \) and \( v_j^n \) the unknown velocities at time \( n \).
The Matrix form of these equations will be:

\[
\begin{bmatrix}
\frac{L}{3\Delta t} + \frac{v_i^{n-1}}{3} + \frac{v_j^{n-1}}{6} & \frac{L}{6\Delta t} - \frac{v_i^{n-1}}{3} - \frac{v_j^{n-1}}{6} \\
\frac{L}{6\Delta t} + \frac{v_i^{n-1}}{6} + \frac{v_j^{n-1}}{3} & \frac{L}{3\Delta t} - \frac{v_i^{n-1}}{6} - \frac{v_j^{n-1}}{3}
\end{bmatrix}
\begin{bmatrix}
v_i^n \\
v_j^n
\end{bmatrix} = \begin{bmatrix}
\frac{L}{\Delta t} \left( \frac{v_i^{n-1}}{3} + \frac{v_j^{n-1}}{6} \right) - \frac{g}{2} (y_i^{n-1} - y_j^{n-1}) - g L \left[ \frac{S_{fi}^{n-1} - S_{oi}}{3} + \frac{S_{fj}^{n-1} - S_{oi}}{6} \right] \\
\frac{L}{\Delta t} \left( \frac{v_i^{n-1}}{6} + \frac{v_j^{n-1}}{3} \right) - \frac{g}{2} (y_i^{n-1} - y_j^{n-1}) - g L \left[ \frac{S_{fi}^{n-1} - S_{oi}}{6} + \frac{S_{fj}^{n-1} - S_{oi}}{3} \right]
\end{bmatrix}
\tag{5.17}
\]

Still there is a need to define a system of equations with respect to \( y_i^n \) and \( y_j^n \) the depths. This is obtained multiplying the CONTINUITY EQUATION (5.5) by the shape functions \( N_i \) and \( N_j \) respectively. Multiplying (5.5) by \( N_i \) according to Gallerkin's principle we obtain:

\[
\int_{x_i}^{x_j} \left[ \frac{(x-x_i)^2}{L^2} \cdot \frac{y_i^n - y_i^{n-1}}{\Delta t} + \frac{(x-x_i)(x_j-x)}{L^2} \cdot \frac{y_j^n - y_j^{n-1}}{\Delta t} \right] dx + \\
\int_{x_i}^{x_j} \left[ \frac{(x-x_i)^2}{L^2} A_i + \frac{(x_j-x)(x-x_i)}{L^2} A_j \right] \left[ \frac{v_i^{n-1}}{L} - \frac{v_i^{n-1}}{L} \right] dx + \\
\int_{x_i}^{x_j} \left[ \frac{(x-x_i)^2}{L^2} v_i^{n-1} + \frac{(x_j-x)(x-x_i)}{L^2} v_j^{n-1} \right] \left[ \frac{1}{L} (y_i^n) - \frac{1}{L} (y_j^n) \right] dx = 0 \tag{5.18}
\]
After performing the integrations, equation 18 becomes

\[
\begin{align*}
\frac{L}{3\Delta t} y_i^n - \frac{L}{3\Delta t} y_i^{n-1} + \frac{L}{6\Delta t} y_j^n - \left(\frac{1}{3} \frac{A_i}{B_i} + \frac{1}{3} \frac{A_j}{B_j}\right) \left(v_i^{n-1} - v_j^{n-1}\right) + \\
\frac{v_j^{n-1}}{6} y_i^n + \frac{v_i^{n-1}}{3} y_j^n + \frac{v_i^n}{3} y_j^n + \frac{v_j^n}{3} y_j^n = 0
\end{align*}
\]  
\text{(5.19)}

Multiplying now (5.5) by \(N_j\) and according to Gallerkin’s procedure we obtain after integration the following expression:

\[
\begin{align*}
\frac{L}{3\Delta t} y_i^n - \frac{L}{3\Delta t} y_i^{n-1} + \frac{L}{3\Delta t} y_j^n - \frac{L}{3\Delta t} y_j^{n-1} - \left(\frac{1}{3} \frac{A_i}{B_i} + \frac{1}{3} \frac{A_j}{B_j}\right) \left(v_i^{n-1} - v_j^{n-1}\right) + \\
\frac{v_j^{n-1}}{3} y_i^n + \frac{v_i^{n-1}}{6} y_j^n + \frac{v_i^n}{6} y_j^n + \frac{v_j^n}{3} y_j^n = 0
\end{align*}
\]  
\text{(5.20)}

Rearranging equations 5.19 and 5.20 we obtain the following linear system of equations:

\[
\begin{bmatrix}
\frac{L}{3\Delta t} - \frac{v_i^{n-1}}{3} - \frac{v_j^{n-1}}{6} & \frac{L}{6\Delta t} + \frac{v_i^{n-1}}{3} + \frac{v_j^{n-1}}{6} \\
\frac{L}{6\Delta t} - \frac{v_i^{n-1}}{6} - \frac{v_j^{n-1}}{3} & \frac{L}{3\Delta t} + \frac{v_i^{n-1}}{6} + \frac{v_j^{n-1}}{3}
\end{bmatrix}
\begin{bmatrix}
y_i^n \\
y_j^n
\end{bmatrix}
= 
\begin{bmatrix}
\frac{L}{\Delta t} \left(\frac{y_i^{n-1}}{3} + \frac{y_j^{n-1}}{6}\right) + \left(\frac{A_i}{3B_i} + \frac{A_j}{6B_j}\right) \left(v_i^{n-1} - v_j^{n-1}\right) \\
\frac{L}{\Delta t} \left(\frac{y_i^{n-1}}{6} + \frac{y_j^{n-1}}{3}\right) + \left(\frac{A_i}{6B_i} + \frac{A_j}{3B_j}\right) \left(v_i^{n-1} - v_j^{n-1}\right)
\end{bmatrix}
\]  
\text{(5.21)}

The unknowns are \(y_i^n\) and \(y_j^n\), the depths at time \(n\).
6. **EXAMPLE OF APPLICATION**

The following flood routing problem is considered: a twenty feet wide rectangular channel, 10560 feet long, having uniform flow of 6 ft. depth is subjected to an upstream increase in flow to 2000 cfs in a period of 20 min. Thus flow then decreases uniformly to the initial flow depth in an additional period of 40 min. The following physical and geometric characteristics are given: the Manning's coefficient is estimated to be 0.02 and the bottom slope is 0.0015 ft/ft.

The total channel's length is divided into twenty elements (each element being 528 feet long). The physical parameters characterizing the flow are provided at each of the twenty-one nodes of the finite element partition. The downstream boundary is free and the wave perturbation is given at the upstream boundary. The sequence of the performed computations are described in the flowchart of Figure 5.1.

Table 6.1 provides the partial results obtained at time 2.01, 3.01, 4.01 and 5.0 minutes respectively at each node.

The results of program REWAVE are conformal with our common engineering sense, however the present version needs how to use an economic subroutine to solve the nonsymmetric system of equations.

The results obtained from program FEWAVE, for the particular example treated in Section 5, are compared with two widely used numerical schemes. More specifically the Gallerkin Finite Element method is compared with the method of characteristics as developed by Streeter, V. (9) and a finite difference explicit scheme suggested by Viessman, W. (10).
Table 6 shows the maximum values of the computed velocities and depths. The order of magnitude of these physical quantities is the same for the three schemes. The results of the Gallerkin scheme is closer to the method of characteristics (Figures 2 to 4).

A more systematic study is however, required in order to assess the reliability of each method and identify the parameters affecting the accuracy of the computations.

In conclusion the St. Venant equations can be effectively solved by the proposed Gallerkin, finite element scheme.
VALUES OF VELOCITIES AND DEPTHS AT 21 LOCATIONS
OF THE CHANNEL \( \Delta x = 528 \) FT.

<table>
<thead>
<tr>
<th>TIME</th>
<th>VELOCITIES (ft/s)</th>
<th>DEPTHS (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.016666667</td>
<td>7.167</td>
<td>6.496</td>
</tr>
<tr>
<td></td>
<td>7.279</td>
<td>6.496</td>
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<td></td>
<td>7.043</td>
<td>6.294</td>
</tr>
<tr>
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<td>6.946</td>
<td>6.946</td>
</tr>
<tr>
<td></td>
<td>6.946</td>
<td>6.946</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>VELOCITIES (ft/s)</th>
<th>DEPTHS (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.016666667</td>
<td>7.315</td>
<td>6.467</td>
</tr>
<tr>
<td></td>
<td>7.444</td>
<td>6.467</td>
</tr>
<tr>
<td></td>
<td>7.279</td>
<td>6.294</td>
</tr>
<tr>
<td></td>
<td>7.015</td>
<td>6.095</td>
</tr>
<tr>
<td></td>
<td>6.946</td>
<td>6.946</td>
</tr>
<tr>
<td></td>
<td>6.946</td>
<td>6.946</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>VELOCITIES (ft/s)</th>
<th>DEPTHS (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.016666667</td>
<td>7.446</td>
<td>6.086</td>
</tr>
<tr>
<td></td>
<td>7.592</td>
<td>6.086</td>
</tr>
<tr>
<td></td>
<td>7.326</td>
<td>6.294</td>
</tr>
<tr>
<td></td>
<td>7.181</td>
<td>6.095</td>
</tr>
<tr>
<td></td>
<td>6.946</td>
<td>6.946</td>
</tr>
<tr>
<td></td>
<td>6.946</td>
<td>6.946</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>VELOCITIES (ft/s)</th>
<th>DEPTHS (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.016666667</td>
<td>7.546</td>
<td>6.080</td>
</tr>
<tr>
<td></td>
<td>7.699</td>
<td>6.080</td>
</tr>
<tr>
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<td>7.539</td>
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<tr>
<td></td>
<td>7.333</td>
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<tr>
<td></td>
<td>7.221</td>
<td>6.095</td>
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<tr>
<td></td>
<td>7.087</td>
<td>6.095</td>
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</table>

Table 6.1 Partial Results of Program FEWAVE
MAIN PROGRAM OF THE FLOOD ROUTING PROBLEM

READ INPUT
GEOMETRIC AND
PHYSICAL
CHARACTERISTICS

INITIAL+B. CONDITIONS
T=0

CALL FORMEQ
FORM SYSTEM OF
EQ. FOR VELOCITIES (V)

CALL SOLVE
FIND VELOCITIES AT
NEW TIME INCREMENT

T=T+ΔT

CALL FORMEQ
FORM SYSTEM OF
EQ. FOR DEPTHS (Y)

CALL SOLVE
FIND DEPTHS AT
NEW TIME INCR.

CHECK NUMBER
OF TIME INCREMENTS

STOP

Figure 6.1 Flowchart of Program FEWAVE
<table>
<thead>
<tr>
<th>LOCATION</th>
<th>VARIABLE'S SPECIFICATION</th>
<th>METHOD OF CHARACTERISTICS (ft/sec)</th>
<th>FINITE DIFFERENCE SCHEME (ft)</th>
<th>FINITE ELEMENT SCHEME</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max. Velocity</td>
<td>9.42</td>
<td>9.43</td>
<td>9.2</td>
</tr>
<tr>
<td></td>
<td>Max. Depth</td>
<td>10.88</td>
<td>10.68</td>
<td>10.87</td>
</tr>
<tr>
<td></td>
<td>Time Min.</td>
<td>24.</td>
<td>22.</td>
<td>29.</td>
</tr>
<tr>
<td></td>
<td>Max. Velocity</td>
<td>9.20</td>
<td>9.14</td>
<td>9.18</td>
</tr>
<tr>
<td></td>
<td>Max. Depth</td>
<td>10.40</td>
<td>10.21</td>
<td>10.44</td>
</tr>
<tr>
<td></td>
<td>Time</td>
<td>30.</td>
<td>31.</td>
<td>39.</td>
</tr>
<tr>
<td></td>
<td>Max. Velocity</td>
<td>8.5</td>
<td>8.46</td>
<td>8.74</td>
</tr>
<tr>
<td></td>
<td>Max. Depth</td>
<td>10.33</td>
<td>10.16</td>
<td>10.30</td>
</tr>
<tr>
<td></td>
<td>Time</td>
<td>36.</td>
<td>35.</td>
<td>48.</td>
</tr>
</tbody>
</table>

Table 6.1 Comparison of Obtained Peak Values
Figure G.2 Results Obtained by the Method of Characteristics
Figure 6.3 Results Obtained by the Finite Difference Scheme
Figure 6.4 Results Obtained by the Finite Element Scheme
Figure 6.5 Comparison of Results
<table>
<thead>
<tr>
<th>TIME (MIN)</th>
<th>DISCHARGE</th>
<th>VELOCITY</th>
<th>DISCHARGE</th>
<th>VELOCITY</th>
<th>DISCHARGE</th>
<th>VELOCITY</th>
<th>DISCHARGE</th>
<th>VELOCITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
</tr>
<tr>
<td>2.00</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
</tr>
<tr>
<td>3.00</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
</tr>
<tr>
<td>4.00</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
</tr>
<tr>
<td>5.00</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
</tr>
</tbody>
</table>

Table 6.3 Results of the Finite Difference Scheme
<table>
<thead>
<tr>
<th>TIME IN MINUTES</th>
<th>VELOCITIES AND DEPTHS AT 1000 FOOT INTERVALS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>V1</td>
</tr>
<tr>
<td>540</td>
<td>0.076</td>
</tr>
<tr>
<td>2,179</td>
<td>0.076</td>
</tr>
<tr>
<td>4,647</td>
<td>0.076</td>
</tr>
<tr>
<td>8,079</td>
<td>0.076</td>
</tr>
<tr>
<td>7,847,327</td>
<td>0.076</td>
</tr>
<tr>
<td>9,409</td>
<td>0.076</td>
</tr>
<tr>
<td>11,342</td>
<td>0.076</td>
</tr>
<tr>
<td>13,094</td>
<td>0.076</td>
</tr>
<tr>
<td>14,826</td>
<td>0.076</td>
</tr>
<tr>
<td>16,629</td>
<td>0.076</td>
</tr>
<tr>
<td>17,681</td>
<td>0.076</td>
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<tr>
<td>18,773</td>
<td>0.076</td>
</tr>
<tr>
<td>19,915</td>
<td>0.076</td>
</tr>
<tr>
<td>21,096</td>
<td>0.076</td>
</tr>
<tr>
<td>22,327</td>
<td>0.076</td>
</tr>
<tr>
<td>23,595</td>
<td>0.076</td>
</tr>
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<td>24,915</td>
<td>0.076</td>
</tr>
<tr>
<td>26,396</td>
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<tr>
<td>27,937</td>
<td>0.076</td>
</tr>
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</table>
Bibliography


PROGRAM FEUAW (INPUT, OUTPUT)

******************************************************************************
VERSION INF478
******************************************************************************

FINITE ELEMENT PROGRAM TO SOLVE FLOOD WAVE PROBLEM
ONE DIMENSIONAL LINEAR ELEMENTS USED IN CONNECTION
WITH GALLERKINS PROCEDURE.
A FINITE DIFFERENCE SCHEME IS USED FOR THE TIME
DOMAIN + LEAPFROG TECHNIQUE TO COMPUTE THE
VELOCITIES AND DEPTHS OF THE FLOW AT PARTICULARS
NODES

-------------------------------------------------
MAXTIM = MAX. TIME
MNODE = MAXIMUM NUMBER OF NODES
NEL = MAXIMUM NUMBER OF ELEMENTS

NODE = NODE NUMBER
X(NODE) = X COORDINATE
Y(NODE) = Y COORDINATE
ID(NODE) = INITIAL BOUNDARY CONDITION

N = ELEMENT NUMBER
NI = NUMBER OF NODE I
NJ = NUMBER OF NODE J
SF(N) = SLOPE FRICTION
A(N) = CROSS SECTION
B(N) = FREE SURFACE WIDTH
PH(N) = HYDRAULIC PERIMETER

-------------------------------------------------
DIMENSION U(40), F1(2), Y(40), F2(2), AA(13), SM(40), YN(40), FL(10), FD(40), UN(40), A(40), B(40), SF(40), SQ(40), X(40), IDB(40), NZ(30), NJ(30), C(20,20)

REAL K(21,21), KL(2,2), R(40)

PRINT 140
READ 141, II, (AA(I), I=1,13)
IF (II.EQ.0) GO TO 130
PRINT 140, II, (AA(I), I=1,13)
PRINT 142
PRINT 143

READ MESH CHARACTERISTICS

NEL=20
DT=1.
MNODE=NEL+1
IBCO=1

TLGTH=10550.
DX=TLGTH/VEL
DY=.0

XX=0.
DO 101 I=1,NNODE
  X(I)=XX+DX
  Y(I)=YY+DY
  XX=XX+1.
101 CONTINUE

DO 102 I=1,NNODE
DO 105 I=1,NNODE
   IBD(I)=0
   A(I)=120.
   B(I)=20.
   SO(I)=0.0015
102 CONTINUE
   IBD(I)=1
C
   DO 103 I=1,NEL
   NI(I)=I
   NJ(I)=I+1
103 CONTINUE
C
   INITIALIZE MATRICES
   DO 104 I=1,NNODE
      FI(I)=0.
      FD(I)=0.
104 CONTINUE
C
   DO 104 J=1,NNODE
      K(I,J)=0.
104 CONTINUE
C
   READ ELEMENT INFORMATION
T'=0.
   IDEBUG=1
C
   PRINT 142
   PRINT 144
   PRINT 142
C
   READ INITIAL VALUES
EFF=2./3.
   CMN=0.02
   YO=8.
   CM=1.486/CMN*SQRT(SO(I))
C
   Q0=CM*B(1)*YO*(B(1)*YO/(B(1)+2.*YO))*EFF
   VO=Q0/(B(1)*YO)
   PRINT *,Q0,VC0,VO
C
   DO 105 I=1,NNODE
      U(I)=YO
105 CONTINUE
C
   PRINT *,INITIAL VALUES=
   DO 106 I=1,NNODE
      PRINT *,VELOCITIES, DEPTHS, U(I), Y(I)
106 CONTINUE
C
   READ 145, QMAX,TTP,THAX,TAP
C
   PRINT *,QMAX,TTP,QMAX,TTP
   PRINT *,THAX,TAP,THAX,TAP
C
   KOUNT=0
T'=0
C
   DTPRIT=60.
   UPRINT=60.
   TPRINT=60.
C
107 CONTINUE
T'=T+DT
   N=1
   DO 108 I=1,NNODE
      EF=4./3.
CONTINUE
DO 117 I=1,NNODE
117 U(I)="'UN(1)

IF (N.EQ.NEL) GO TO 114

MAIN TIME LOOP

CALL FORMED (NI,NJ,N,X,Y,NNODE,U,DT,KL,F1,1,F2,II,JJ,SO,SSF,A,B)

FORMATION OF GLOBAL MATRIX

K(II,II)=K(II,II)+KL(1,1)
K(II,JJ)=K(II,JJ)+KL(1,2)
K(JJ,JJ)=K(JJ,JJ)+KL(2,2)
K(JJ,II)=K(JJ,II)+KL(2,1)

FU(II)=FU(II)+FI(1)
FU(JJ)=FU(JJ)+FI(2)

IF (IBD(II).EQ.0) GO TO 112
FU(II)=U(II)

DO 111 J=1,NNODE
111 K(II,J)=0.
112 IF (IBD(JJ).EQ.0) GO TO 113
FU(JJ)=FU(JJ)-K(II,JJ)·U(JJ)
K(JJ,JJ)=1.
113 CONTINUE

IF (N.EQ.NEL) GO TO 114
N=N+1
GO TO 110

PRINT *,#STIFNESSMATRIX#
DO 115 I=1,NNODE
115 PRINT 147, (K(I,J),J=1,NNODE)

PRINT 147, (K(I,J),J=1,NNODE)

CALL SOLVE (FU,K,NNODE,NB,UN,R)

DO 117 I=1,NNODE
117 U(I)="'UN(1)
L1=X(2)-X(1)
Y(1)=Y(1)+(DT/L1)*(U(1)*(Y(1)-Y(2))+Y(1)-(U(1)-U(2)))
T=T+TTP
IF (T) 118,118,119
118 Q=Q0-(QMAX-Q0)/TTP*T
GO TO 122
119 IF (T-TAP) 120,120,121
120 Q=QMAX-(QMAX-Q0)/TAP*T
GO TO 122
121 Q=Q0
122 CONTINUE
U(I)=Q/(B(I)*Y(I))

C
IF (T-UPRINT+0.002) 124,123,123
123 UPRINT=UPRINT+DT/60.
C
PRINT 146
PRINT: *TIME=*,TT
PRINT: *VELOCITIES*
C
PRINT 147, (U(I),I=1,NNODE)

124 CONTINUE
N=1
T=T+DT
C
DO 125 I=1,NNODE
DO 125 J=1,NNODE
FD(J)=0.
C
125 K(I,J)=0.
126 CONTINUE
C
C CALL FORMEQ (MI,MJ,NX,NY,NNODE,V,DT,KL,F1,0,F2,II,JJ,SD,SF,A,B)
C
C FORMATION OF GLOBAL MATRIX
C
K(II,II)=K(II,II)+KL(1,1)
K(II,JI)=K(II,JI)+KL(1,2)
K(JJ,JI)=K(JJ,JI)+KL(2,2)
K(JJ,II)=K(JJ,II)+KL(2,1)
C
FD(II)=FD(II)+F1(I)
FD(JJ)=FD(JJ)+F1(J)
C
IF (IBD(II).EQ.0) GO TO 128
FD(II)=Y(II)
DO 127 J=1,NNODE
127 K(II,J)=0.
C
K(II,II)=1.
128 IF (IBD(JJ).EQ.0) GO TO 129
FD(JJ)=FD(JJ)-K(II,JJ)*Y(JJ)
K(JJ,JJ)=1.
C
129 CONTINUE
C
IF (N.EQ.MEL) GO TO 130
N=N+1
GO TO 126
CONTINUE

CALL SOLVE (FD,K,NMODE,MB,YN,R)

DO 131 I = 1,NMODE

Y(I) = YN(I)

TD = T - TTP

IF (TD) 132, 132, 133

Q = Q0 + (QMAX - Q0) / TTP * T

GO TO 136

133 IF (TD - TAP) 134, 134, 135

Q = QMAX - (QMAX - Q0) / TAP * TD

GO TO 136

135 Q = Q0

CONTINUE

L1 = X(2) - X(I)

Y(I) = Q / (B(I) * U(I))

IF (T - TPRINT + 0.002) 138, 137, 137

PRINT , #DEPHTS#

PRINT 147, (Y(I), I = 1,NMODE)

IF (T .LT. TMAX) GO TO 107

STOP

FORMAT (1X)

FORMAT (12, 13A6)

FORMAT (2X, 47H==========================================,/

FORMAT (///)

FORMAT (2X, 25HELEMENTS CHARACTERISTICS,///)

FORMAT (4F10.0)

FORMAT (////2X)

FORMAT (2X, 10(F10.3, 2X))

END

SUBROUTINE FORME (NI, NJ, N, X, Y, NNODE, U, JT, KL, F1, M1, F2, II, JJ, SO, SF, 1A, B)

REAL KL(2, 2), X(NNODE), Y(NNODE), U(NNODE), V(NNODE), F1(2), F2(2), SO(NNODE), SF(NNODE), A(NNODE), B(NNODE)

INTEGER NI(NNODE), NJ(NNODE)

II = NI(N)

JJ = NJ(N)

XJ = X(JJ)

YJ = Y(JJ)

L = XJ - XI

IF (M1 .EQ. 0) GO TO 101

CALL LINEAR (KL, II, JJ, L, U, V, Y, SF, SO, A, B, NNODE, JT, M1, F1)
SUBROUTINE LINEAR (KL, JJ, L, U, V, SF, SO, A, B, NNODE, DT, N1, F1)

CALL LINEAR (KL, JJ, L, U, V, SF, SO, A, B, NNODE, DT, N1, F1)

DO 102 I=1, 2
F2(I) = F1(I)
RETURN
END

SUBROUTINE LINEAR (KL, JJ, L, U, V, SF, SO, A, B, NNODE, DT, N1, F1)

CALL GAUSS (NB, NB, KK, BB, R)

DO 102 I=1, NNODE
RN(I) = 0.
CALL GAUSS (NB, NB, KK, BB, R)
DO 102 I=1, NNODE
RN(I) = R(I)
RETURN
END