1977

O(h^6) Accurate Finite Difference Approximation to Solutions of the Poisson Equation in Three Variables

Robert E. Lynch

Purdue University, rel@cs.purdue.edu

Report Number:
77-221


This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact epubs@purdue.edu for additional information.
$O(h^6)$ Accurate Finite Difference Approximation to Solutions of the Poisson Equation in Three Variables

Robert E. Lynch
Division of Mathematical Sciences
Purdue University
West Lafayette, Indiana 47907

February 15, 1977

Computer Science Department Report
CSD-TR 221
0(h^6) ACCURATE FINITE DIFFERENCE APPROXIMATION TO SOLUTIONS
OF THE POISSON EQUATION IN THREE VARIABLES

Robert E. Lynch*

Division of Mathematical Sciences
Purdue University
West Lafayette, Indiana 47907
February 15, 1977
CSD-TR 221

Abstract. Let \( u_{j,k,l} = u(jh, kh, lh) \) denote values of a
function on a cubic lattice. Let \( \sum_{r=2} u_{j,k,l} \) denote the sum
of values of \( u \) at lattice points a distance \( r \) from \( (jh, kh, lh) \).
Let

\[
L^h_{u,j,k,l} = \frac{-128u_{j,k,l}}{12} + 14 \sum_{r=2}^6 u_{j,k,l} + 3 \sum_{r=2}^6 u_{j,k,l} \\
+ \sum_{r=3}^6 u_{j,k,l} / (30h^6)
\]

\[
F^h_{f,j,k,l} = \frac{280f_{j,k,l}}{12} + 8 \sum_{r=2}^6 f_{j,k,l} + 48 \sum_{r=2}^6 f_{j,k,l} \\
+ \sum_{r=3}^6 f_{j,k,l} / 720
\]

\[
M^h = I + \left( \frac{h^2}{12} \nabla^2 + \frac{h^4}{360} \right) \nabla^2 + 2(\nabla^2 x^2 + \nabla^2 y^2 + \nabla^2 z^2)
\]

where \( I \) denotes the identity, \( \nabla^2 = \Delta = X^2 + Y^2 + Z^2 \) denotes the
Laplacian, and \( X = \partial / \partial x, Y = \partial / \partial y, Z = \partial / \partial z. \)

We show that if \( u \) has continuous eighth derivatives and
\( f = \nabla^2 u \), then \( L^h_{u,j,k,l} = M^h u_{j,k,l} + 0(h^6) = M^h f_{j,k,l} + 0(h^6) \)
\( = F^h_{f,j,k,l} + 0(h^6). \) Solutions of \( L^h_{u,j,k,l} = M^h f_{j,k,l} \) or

* Work supported, in part, by the National Science Foundation
grant MCS 75-10225.
subject to Dirichlet boundary conditions yield $O(h^6)$ estimates of $u$ at lattice points. If the region is a cartesian product of three intervals, then tensor product or Fast Fourier Transform techniques can be used to solve the discrete problem. Experimental results are given which confirm the $O(h^6)$ behavior of the discretization error.
0(h^6) ACCURATE FINITE DIFFERENCE APPROXIMATION TO SOLUTIONS
OF THE POISSON EQUATION IN THREE VARIABLES*

Robert E. Lynch
Division of Mathematical Sciences
Purdue University
West Lafayette, IN 47907

1. O(h^6) discretization to solutions of the Poisson equation in
terms of f and its derivatives. Consider the Poisson equation:
\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f \]

With X = \(9/3x\), Y = \(3/9y\), Z = \(3/3z\), the Laplacian can be written as
\[ \nabla^2 = X^2 + Y^2 + Z^2 \]

Taylor’s series representations of an analytic function can be written as
\[ u(x+h,y,z) = u(x,y,z) + hXu(x,y,z) + \left(\frac{h^2}{2}\right) X^2 u(x,y,z) + \ldots \]
\[ = e^{hX}u(x,y,z) \]
\[ u(x+h,y+h,z) = e^{h(X+Y)}u(x,y,z) \]
\[ u(x+h,y+h,z+h) = e^{h(X+Y+Z)}u(x,y,z) \]
and so on. Divided central differences can then be represented conveniently, for example
\[ \frac{\partial^2}{\partial x^2} u(x,y,z) = \frac{u(x-h,y,z) - 2u(x,y,z) + u(x+h,y,z)}{h^2} \]
\[ = \left[ x^2 + \frac{(h^2/12)}{h^4} x^4 + \frac{(h^4/360)}{h^6} x^6 \right] u(x,y,z) + O(h^6) \]

*This report records a derivation of a specific difference approximation. It is not intended for publication and is, therefore, not in polished form. The derivation is elementary and once the approximation is available, it can easily be verified directly.*
and, similarly,

\[
\delta_y^2 = y^2 + (h^2/12) y^4 + (h^4/360) y^6 + O(h^6)
\]

\[
\delta_z^2 = z^2 + (h^2/12) z^4 + (h^4/360) z^6 + O(h^6)
\]

\[
\delta_x^2 \delta_y^2 = x^2 y^2 + (h^2/12)(x^4 y^2 + x^2 y^4) + O(h^4)
\]

\[
\delta_x^2 \delta_y^2 \delta_z^2 = x^2 y^2 z^2 + O(h^2)
\]

and so on.

We use the operators \(A_h\), \(B_h\), and \(C_h\) defined by

\[
A_h = \delta_x^2 + \delta_y^2 + \delta_z^2 = (x^2 + y^2 + z^2) + (h^2/12)(x^4 + y^4 + z^4) + \ldots
\]

\[
= v^2 + (h^2/12)[v^4 - 2(x^2 v^2 + \ldots)]
\]

\[
+ (h^4/360)[v^6 - 3(x^4 v^2 + x^2 y^4 + x^4 z^2 + y^4 z^2 + 2^4 y^2 z^2) - 6 x^2 y^2 z^2]
\]

\[
+ O(h^6)
\]

\[
= v^2 + (h^2/12)[v^4 - 2(x^2 v^2 + \ldots)]
\]

\[
+ (h^4/360)[v^6 - 3(x^2 v^2 + \ldots)v^2 - 3 x^2 y^2 z^2] + O(h^6)
\]

\[
B_h = \delta_x^2 \delta_y^2 + \delta_y^2 \delta_z^2 + \delta_z^2 \delta_x^2 = (x^2 y^2 + \ldots)
\]

\[
+ (h^2/12)[(x^2 y^2 + \ldots)v^2 - 3 x^2 y^2 z^2] + O(h^4)
\]

\[
C_h = \delta_x^2 \delta_y^2 \delta_z^2 = x^2 y^2 z^2 + O(h^2)
\]

where here and below we use the abbreviation

\[
(x^2 y^2 + \ldots) = (x^2 y^2 + y^2 z^2 + z^2 x^2)
\]
We define operators $M_h$ and $L_h$ in terms of $A_h, B_h, C_h$ as

\begin{align*}
(1-1a) \quad M_h &= I + \left(\frac{h^2}{12}\right)\nabla^2 + \left(\frac{h^4}{360}\right)[\nabla^4 + 2(\nabla^2 + \ldots)] \\
(1-1b) \quad L_h &= A_h + \left(\frac{h^2}{6}\right) B_h + \left(\frac{h^4}{30}\right) C_h = M_h \nabla^2 + O(h^6)
\end{align*}

where $I$ denotes the identity operator.

To express the operator $L_h$ in terms of coefficients of a stencil, let $\sum_{|r|=r} u_{j+k+r}$ denote the sum of values of $u$ at lattice points a distance $r$ from $(jh, kh, lh)$. Then we have

\begin{align*}
A_h u_{j+k} &= \left[ -6u_{j+k} + \sum_{r \neq 0} u_{j+k+r} \right]/h^2 \\
B_h u_{j+k} &= \left[ -12u_{j+k} + \sum_{r^2=2h^2} u_{j+k+r} \right]/h^2 \\
&= \left[ 4A_h + \frac{h^2}{6} B_h \right] u_{j+k} \\
&= \left[ 4\nu^2 + \left(\frac{h^2}{12}\right)C \nu + 2(\nu^2 + \ldots) \right] + \ldots \ u_{j+k}, \\
C_h u_{j+k} &= \left[ -8u_{j+k} + \sum_{r^2=3h^2} u_{j+k+r} \right]/h^2 = \left[ 4A_h + 2h^2 B_h + \frac{h^4}{30} C_h \right] u_{j+k} \\
&= \left[ 4\nu^2 + \left(\frac{h^2}{12}\right)C \nu + 16(\nu^2 + \ldots) \right] + \ldots \ u_{j+k}, \\
L_h u_{j+k} &= \left[ 24A_h + 8B_h + C_h \right] u_{j+k} \\
(1-2) \\
&= \left[ -128u_{j+k} + 14 \sum_{r^2=h^2} u_{j+k+r} + 3 \sum_{r^2=2h^2} u_{j+k+r} + \sum_{r^2=3h^2} u_{j+k+r} \right]/(30h^2) \\
\end{align*}

**Theorem 1:** Let $R$ denote a connected domain made up of the union of cubes, each of which has volume $h^3$, with disjoint interiors and edges parallel to coordinate axes. Let $\partial R$ denote the boundary of $R$. 


Let one of the vertices of a cube be the origin. For an integer \( N \geq 2 \), let 
\( h = h_0/N \) and let \((j,h,k,h)\), with \( j,k,h \) integers, denote points of a cubic lattice with contains the vertices of the cubes as a sublattice. Let \( R_h \) denote the set of lattice points in \( R \) and \( 3R_h \) the set of lattice points in \( 3R \). Let \( u \) denote a function with continuous eighth derivatives and let \( f \) and \( g \) denote functions 
\[ f = \nabla^2 u, \quad (x,y,z) \in R, \quad \text{and} \quad g = u, \quad (x,y,z) \in 3R. \]

Let \( u^{(h)} \) denote the solution of
\[ L_h u^{(h)} \big|_{j,k,h} = M_h f_{j,k,h}, \quad (j,h,k,h) \in R_h \]
\[ u^{(h)} \big|_{j,k,h} = g_{j,k,h}, \quad (j,h,k,h) \in 3R_h \]

There exists a constant \( K \) which depends on \( u \) but not on \( h \) or \((j,h,k,h)\) such that
\[ |u_{j,k,h} - u^{(h)}_{j,k,h}| \leq K h^6, \quad (j,h,k,h) \in R_h \]

Furthermore, no other coefficients in (1-2) can give a higher order of accuracy.

Proof: The error, \( e = u - u^{(h)} \) satisfies (1-3a) with right side replaced with \( O(h^6) \) and zero boundary conditions (1-3b). For functions which are zero at the boundary, the operator \( L_h \) is of monotone type \( L_h v \leq 0 \implies v \geq 0 \), and \( L_h \) applied to \((kh^6/6)[x^2 + y^2 + z^2 - r^2] \) yields \( Kh^6 \). Hence, for sufficiently large \( r \) and \( K \), this function bounds the error. Hence (1-4) follows.

The last statement of the Theorem follows from the polynomial displayed in the proof of Theorem 11 in Birkhoff and Gulati [1974].

This polynomial is an eighth degree harmonic polynomial in two
independent variable which is \( O(h^8) \) on the nine points \((0,0), (\pm h,0), (0,\pm h), (\pm h,\pm h)\). Application of \( L_h \) to this polynomial gives, therefore, \( O(h^6) \), whereas the right side of (1-3a) is zero.

The coefficients in (1-2) are given by Mikeladze [1937]. He also displays the terms through \( O(h^2) \) in the \( N_h \). He did not realize, apparently, that (1-3a) yields \( O(h^6) \) accuracy, for he only proved \( O(h^4) \).
2. $O(h^6)$ discretization error for solutions in terms of values of $f$. Rather than evaluate derivatives of $f$ to obtain values of the right side of (1-3), we construct $O(h^6)$ difference approximation to the operator $M_h$.

One cannot obtain an $O(h^6)$ approximation by using only the 27 lattice points used for the operator $L_h$. This is because one obtains $v^2$ and $v^4$ only from $A_h$; they do not appear in $B_h$ or $C_h$. In $A_h$, these have coefficients $1$ and $h^2/12$, respectively, with ratio $12/h^2$, whereas the coefficients of these in $M_h$ are $h^2/12$ and $h^4/360$ with ratio $30/h^2$.

There are a number of disadvantages of using values of $f$ at other lattice points in addition to the 27 used in the operator $L_h$. These include the following: A linear combination of $A_h B_h C_h A_{2h} B_{2h} C_{2h}$ which gives an $O(h^6)$ approximation to the derivative terms in $M_h$ leads to negative coefficients of some of the values of $f$; there is a close relationship between the solution of (1-3) in terms of values of $f$ and quadrature and it is customary in quadrature formulas to use positive coefficients to reduce round-off error. [The connection is: elements of the inverse of the matrix associated with the system of difference equations has elements which approximate the Green's function of the Poisson equation problem, the product of the inverse and the vector of right-side-values of the difference equation gives components which approximate the integral of the Green's function and the right side of
the Poisson equation. For lattice points adjacent to the boundary values of $f$ must be evaluated outside the region. The coefficient in the error term is larger than if values of $f$ are taken in and on the cube of volume $8h^3$ centered at an interior lattice point.

Thus, we use additional values of $f$ in a cube of volume $8h^3$. There are a number of choices. We choose to use the operators $A_h, D_h, E_h$ defined in Section 2 and also $E_{h/2}$:

$$E_{h/2} = 4\pi^2 + (h^2/12)[\pi^4 + 4(x^2 + y^2 + z^2)] + O(h^4)$$

$$E_{h/2} f_{j,k} = 4\left[8 f_{j,k} + \sum_{r^2 = 3h^2/4} f_{j,k}\right]/h^2$$

This requires evaluation of $f$ at the eight half-lattice points: $(jh + h/2, kh + h/2, lh + h/2)$. Thus, $f$ is evaluated at points of a body-centered cubic lattice and there is on the average two evaluations of $f$ for each lattice point.

One obtains $O(h^6)$ approximation to $M_h$ by using $F_h$ defined by

$$F_h = I + (h^2/90)A_h + (h^2/60)E_{h/2} + (h^2/720)E_h$$

so that

$$F_h f_{j,k} = [280 f_{j,k} + 8 \sum_{r^2 = h^2} f_{j,k} + 48 \sum_{r^2 = 3h^2/4} f_{j,k}] + 0(h^6)$$

$$= M_h f_{j,k} + 0(h^6)$$
By changing the coefficients, one can include a term proportional to $D_h$ in the approximation, but this increases the amount of calculation required to evaluate $F_h$ applied to $f$.

Except for reference to specific equations, the proof of Theorem 1 is the same as the proof of the following.

THEOREM 2: The results stated in Theorem 1 hold when $(1-3a)$ is replaced with

\[(2-1) \quad L_{h(j)}^{(l)} f_{j,k,l} = F_h f_{j,k,l}, \quad (jh, kh, lh) \in R_h\]

For the case that $\partial u/\partial z \equiv 0$, the difference equation reduces to a two variable equation. The stencil for $L_h$ is

\[(2-2) \quad \frac{1}{(6h^2)} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix}\]

and the stencil at half-lattice points for $F_h$ is

\[
\begin{bmatrix}
1 & 0 & 4 & 0 & 1 \\
0 & 48 & 0 & 48 & 0 \\
(1/360) & 4 & 0 & 148 & 0 & 4 \\
0 & 48 & 0 & 48 & 0 \\
1 & 0 & 4 & 0 & 1
\end{bmatrix}
\]

Milne, p. 136 [1953] and others give expressions for the stencil in (2-2) in terms of derivatives. Milne is the only
source we know of which displays the term proportional to \( h^6 \). Using the result in Milne, we have

\[
L_h u = v^2 u + (h^2/12)v^4 u + (h^4/360)[v^6 u + 2X^2Y^2v^2 u] \\
+ (h^6/81)[12v^8 u + 64X^2Y^2v^4 u + 80X^4Y^4 u] + O(h^8)
\]

from which it is clear that the stencil in (2-2) cannot yield \( O(h^8) \) approximation to solutions of the Poisson equation in two variable, but can obtain \( O(h^6) \) approximation.

If \( \partial u/\partial y = 0, \partial u/\partial z = 0 \), then (2-1) reduces to

\[
(U_{j-1} - 2U_j + U_{j+1})/h^2
\]

(2-3)

\[
= [f_{j-1} + 16f_{j-1/2} + 26f_j + 16f_{j+1/2} + f_{j+1}] / 60
\]

In contrast to multi-dimensional problems involving elliptic second order partial differential equations in \( n \) independent variables, an approximation with finite difference operators made up of the cartesian product of \( n \) set of three points of a lattice along the coordinate directions, such as \( L_h \), there is no limit to the order of accuracy of approximation of second order ordinary differential operators. See Lynch and Rice [1976, 1977].

Rosser [1976] has given an \( O(h^6) \) scheme for the Poisson equation on a two dimensional region with \( f \) evaluated only at mesh points in the region; some of their coefficients are negative.
4. Evaluation by tensor product methods. If the domain $R$ is the cartesian product of three intervals of lengths $N_x h, N_y h, N_z h$ with $N_x, N_y, N_z$ integers, then tensor product methods—which are equivalent to separation of variables—yield very efficient computational schemes for solving either (1-3a) or (2-1) subject to Dirichlet conditions in (2-3b) as well as a variety of other standard boundary conditions. The use of tensor products for solving difference equations is discussed by a number of authors, see for example, Lynch, Rice and Thomas [1964a,1964b,1965]. Application to the difference approximation to the Poisson equation in three variables on a mesh with $N = N_x N_y N_z$ requires order $N^4$ operations. Since the discretization error is decreasing as $N^{-6}$, asymptotically, the error is halved with a 59% increase in work. Use of Fast Fourier Transforms reduces the work from order $N^4$ to order $N^3 \log_2 N$; for $N = 2, 4, 8, 16$, this gives a savings of factors of order 2, 2, 2.6, 4, respectively. For such techniques, see Hockney 1970.

Figure 1 shows experimental results for the Poisson equation subject to zero boundary conditions on the unit cube. The function $f$ was chosen so that the solution is

$$u(x,y,z) = x(x-1)y(y-1)z(z-1) \exp(x+y+z)$$

The maximum error is plotted versus $N$ and so is the solution time as well as $K/N^6$ and $CN^4$ for some constants $K$ and $C$. The calculation was done on Purdue University's CDC 6500 computer which uses floating point numbers accurate to about 1 part in $10^{15}$. For information which can be used to convert these times to other computers, we note that the solution time required for the solution of
50 linear algebraic equations with Gauss elimination, Crout reduction and one step of iterative refinement of the solution takes about 1.75 seconds.

Values used to plot the graphs in Figure 1 are given in Table 1.

Table 1

<table>
<thead>
<tr>
<th>N</th>
<th>(N-1)^3</th>
<th>maximum error</th>
<th>time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>4.36(-4)</td>
<td>0.018</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>5.54(-5)</td>
<td>0.053</td>
</tr>
<tr>
<td>4</td>
<td>27</td>
<td>9.92(-6)</td>
<td>0.126</td>
</tr>
<tr>
<td>5</td>
<td>64</td>
<td>2.34(-6)</td>
<td>0.272</td>
</tr>
<tr>
<td>6</td>
<td>125</td>
<td>8.28(-7)</td>
<td>0.526</td>
</tr>
<tr>
<td>7</td>
<td>216</td>
<td>3.37(-7)</td>
<td>0.947</td>
</tr>
<tr>
<td>8</td>
<td>343</td>
<td>1.49(-7)</td>
<td>1.582</td>
</tr>
<tr>
<td>9</td>
<td>521</td>
<td>7.24(-8)</td>
<td>2.499</td>
</tr>
<tr>
<td>10</td>
<td>729</td>
<td>3.94(-8)</td>
<td>3.756</td>
</tr>
</tbody>
</table>
Figure 1. Experimental result for \( u_{xx} + u_{yy} + u_{zz} = f \) and \( u = 0 \) on the surface of a unit cube. \( f \) chosen so that \( u(x,y,z) = x(x-1)y(y-1)z(z-1) \exp(x+y+z) \). \( dx = dy = dz = 1/N \).
References

Mikeladze, Sch. 1937, "Uber die numerische Lösung der Differentialgleichung $u_{xx} + u_{yy} + u_{zz} = \phi(x,y,z),"$ C.R. (Doklady) Acad. Sci. UCSR 14 177-179.
Hockney, R.W., 1970, "The potential calculation and some applications," Methods in Computational Physics, 9, 135-211.
Rosser, J.B., 1975 "Nine-point difference solutions for Poisson's equation," Comp. & Maths. with Appls, 1 351-360.