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Abstract

In this paper, we propose a novel feed-forward adaptive coding scheme called SACS (sample adaptive coding scheme) for the lossy compression of a discrete-time memoryless stationary source. SACS is based on a concept of adaptive quantization to the varying samples of the source and is very different from traditional adaptation techniques for non-stationary sources. SACS quantizes each source sample using a sequence of quantizers. Even when using scalar quantization in SACS, we can achieve performance comparable to vector quantization (with the complexity still of the order of scalar quantization). We also show that important lattice based vector quantizers can be constructed using scalar quantization in SACS. We mathematically analyze SACS and propose a simple algorithm to implement it. We numerically study SACS for independent and identically distributed Gaussian sources. Through our numerical study, we find that SACS using scalar quantizers achieves typical gains of 1–2 dB signal to noise ratio over the non-adaptive scheme based on the Lloyd-Max quantizer. We also show that SACS can be used in conjunction with vector quantizers to further improve the gains.

1. Introduction

Block source coding is a mapping from $\mathbb{R}^k$ into a finite subset that is called a code or a codebook. It is well known, from the Source Coding Theorem [6], that the average distortion of block source coding on a random vector can be decreased as the block (vector) size k gets large. The average distortion can be made to approach the distortion given by the corresponding rate distortion function [12]. We model the discrete-time memoryless stationary source as an i.i.d. sequence of random vectors $(\mathbf{X}_i)_{i=1}^m$, where $\mathbf{X}_i := (X_{1i}, \ldots, X_{ki})$ is a random vector in $\mathbb{R}^k$ and m is the *sample* size or the sequence size. Let F be defined as the distribution function (d.f.) of $\mathbf{X}_i$. F is continuous and $E\|\mathbf{X}_i\|^r < \infty$, where $\|\cdot\|^r$ denotes the rth power $\mathcal{L}^2$ norm to be used for the distortion measure.

Let $C_n := \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ denote an n-level codebook which has n codewords, and let $\mathcal{C}_n$ denote the class of all n-level codebooks that have real values in $\mathbb{R}^k$. The block source coder is then described by a function $Q_{C_n}(\mathbf{x})$ called vector quantization (VQ). This function is defined as

$$Q_{C_n}(\mathbf{x}) := \arg\min_{\mathbf{y}\in C_n} \|\mathbf{x} - \mathbf{y}\|^r, \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^k$ and $C_n \in \mathcal{C}_n$. Further, $D_r(C_n, F)$, the average distortion achieved when a random vector is quantized using codebook C_n, is given by

$$D_r(C_n, F) := \int \|\mathbf{x} - Q_{C_n}(\mathbf{x})\|^r dF(\mathbf{x}). \tag{1.2}$$

In (1.1), the bit rate[1] (defined as bits per source symbol) required is $(\log_2 n)/k$. In this paper, we focus only on block coding schemes that are based on fixed-length coding [29]. The quantity $\inf_{C\in\mathcal{C}_n} D_r(C, F)$ is called (n-level) F-optimal distortion and a codebook C_n^* that yields the F-optimal distortion is called an *(n-level)* F-optimal codebook, i.e., $D_r(C_n^*, F) = \inf_{C\in\mathcal{C}_n} D_r(C, F)$ if C_n^* exists. The corresponding quantizer, when k = 1, is called the Lloyd-Max quantizer [13],[18].

Let a sample be denoted by $(\mathbf{X}_i^\omega)_{i=1}^m$, where w is a sample point of the underlying sample space Ω and $\mathbf{X}_i^\omega$ is the ith vector of the sample. Typically, since the source $(\mathbf{X}_i)_{i=1}^m$ is a discrete-time memoryless stationary source, current coding schemes in the literature usually quantize each sample $(\mathbf{X}_i^\omega)_{i=1}^m$ by applying the same quantizer using an F-optimal codebook C_n^* to each $\mathbf{X}_i$ independently. The rationale for quantizing each random vector independently and identically is motivated by the fact that the random vectors $\mathbf{X}_i$ are themselves i.i.d. For such a scheme, the overall distortion is given as

$$E\left\{\frac{1}{m}\sum_{i=1}^{m} \|\mathbf{X}_i - Q_{C_n^*}(\mathbf{X}_i)\|^r\right\} = D_r(C_n^*, F), \tag{1.3}$$

which is the same as the F-optimal distortion of quantizing a single random vector $\mathbf{X}_i$. We call this type of coding scheme, the independent and identical coding scheme (IICS). A specific example of IICS for k = 1 (scalar quantization) is the Pulse Code Modulation (PCM) scheme, where an F-optimal codebook is applied independently and identically to each random variable [8]. The Differential PCM (DPCM) scheme, which is used in speech coding [5], also quantizes each difference sample based on IICS.

[1] **Note that encoding the quantized source outputs which could further decrease the bit rate is not considered in this paper.**

However, even if the source is i.i.d., independently and identically quantizing each random vector of $(\mathbf{X}_i)_{i=1}^m$ is just one of the many possible coding schemes. For a given bit rate, a natural question to ask is whether there exists a coding scheme that yields an average distortion that is less than the F-optimal distortion $D_r(C_n^*, \mathrm{F})$, the average distortion achieved by IICS for $(\mathbf{X}_i)_{i=1}^m$. From the source coding theorem, the following method is well known for outperforming IICS. If we represent a sample of $(\mathbf{X}_i)_{i=1}^m$ by a single index taking n^m values (in other words, if we use the km-dimensional VQ), then we can achieve a lower distortion, for the same bit rate, than with IICS [9]. The encoding complexity of the km-dimensional VQ is however extremely high, especially at high bit rates. In order to circumvent the encoding complexity of VQ, various modifications of the VQ structure have been conducted [10]. The tree-structured VQ, which is based on a tree search of the codewords, the classified VQ, the product VQ, where a large vector is partitioned into subvectors, the multistage VQ, and the lattice VQ techniques can be adopted for reducing the encoding complexity of the km-dimensional VQ. However, regardless of the techniques employed, since the quantization still has the km-dimensional VQ structure, the encoding complexity is high (of the order of km-dimensional VQ). Note that the performance of these techniques will fall between that the performance of the k-dimensional VQ and the performance of the km-dimensional VQ, due to the modifications made to reduce the encoding complexity.

Based on the above discussion, it would be interesting to see if one could develop a coding scheme for the source $(\mathbf{X}_i)_{i=1}^m$ which has the same structure as a k-dimensional VQ but could significantly improve the performance of IICS in k dimensions. In this paper, we propose such a coding scheme which we call the *sample-adaptive coding scheme* (SACS).

SACS is based on a new concept of adaptation to each sample $(\mathbf{X}_i^\omega)_{i=1}^m$. SACS quantizes each sample $(\mathbf{X}_i^\omega)_{i=1}^m$ using a sequence of m codebooks and periodically replaces the sequence of m codebooks from a finite set of codebooks. Note that, even for a memoryless source, the empirical distribution function [11] that is constructed using $(\mathbf{X}_i^\omega)_{i=1}^m$ may be substantially different from F, the distribution function of $\mathbf{X}_i$, especially for small values of m.[2] Hence, we expect the performance of SACS to be higher than that of IICS (which can be thought of as a *non-sample-adaptive coding scheme)*. In this paper we will also study a simplified version of SACS where the sequence of m codebooks used for adaptation are all the same. *It is important to note that SACS is very different from the traditional adaptive coding schemes that produce increased gains by replacing the quantizer depending on the varying statistical characteristics of a non-stationary source* [5],[8],[30],[32]. The main idea of SACS will be intuitively described by observing the *Voronoi partition* of the sample-adaptive quantization as a km-dimensional VQ. We will also show that using the scalar quantized version of SACS we can describe a number of important lattice based vector quantizers. We will formally study this coding scheme and propose a simple algorithm to implement it. Further, we will show via numerical analyses that our coding scheme significantly outperforms IICS and achieves VQ-level performance even for $k = 1$.

This paper is organized as follows. In Section II, we describe SACS and provide some mathematical

[2] **The empirical d.f. converges uniformly to F almost surely from the** *Glivenko-Cantelli Theorem* **[15]. However, if the sample size m is small, then the empirical d.f.s can be quite different from F, and the empirical d.f. for each sample point may also be different from sample to sample.**

definitions. In Section III we provide some mathematical observations which allow us to better understand the design principles of SACS. In Section IV, the relationship between SACS and several root lattices is investigated. A simple codebook design algorithm for SACS is provided in Section V. In Section VI, we conduct a numerical study using Gaussian i.i.d. samples, and compare SACS with IICS. In Section VII, we conclucle the paper and discuss future research. We also provide an appendix in which we describe in more detail the encoding complexity of SACS (including non-uniform codebook sizes).

2. Sample-Adaptive Coding Scheme (SACS)

A. Sample-Adaptive Quantization and m-SACS

In this section we describe our proposed adaptive coding scheme. For every sample, SACS employs a codebook sequence from a previously designed set of 2^η codebook sequences available at. both the encoder and the decoder, as shown in Fig. 2.1. In SACS, it is important to note is that the codebook sequence can be changed adaptively for each sample that contains m vectors. Assume that the samples of $(\mathbf{X}_i)_{i=1}^m$ sequentially enter the encoder. The encoder quantizes $\mathbf{X}_i^\omega$, vector i of a sample, using codebook C_n^i ($\in \mathcal{C}_n$ = class of all n-level codebooks), for $i = 1, \ldots, m$, and then calculates the m-codebook sample distance defined by

$$\frac{1}{m}\sum_{i=1}^{m} ||\mathbf{X}_i^\omega - Q_{C_n^i}(\mathbf{X}_i^\omega)||^r. \tag{2.1}$$

We call $(C_n^i)_{i=1}^m$ the codebook sequence and the SACS that is based on the m-codebook sample distance m-SACS. The distance in (2.1) is a random variable defined on the underlying sample space if $\mathbf{X}_1^\omega, \ldots, \mathbf{X}_m^\omega$ is replaced with the random vector $\mathbf{X}_1, \ldots, \mathbf{X}_m$. Note that in order to quantize these m random vectors, a sequence of m codebooks are employed as shown in the m-codebook sample distance. Also, even if the random vectors are i.i.d., the m codebooks in a codebook sequence can have different codebook sizes; this case is discussed in Appendix B. For each sample $(\mathbf{X}_i^\omega)_{i=1}^m$, the encoder finds a codebook sequence, from a finite class of codebook sequences, that yields the minimum distance given by (2.1). Let $(C_n^{i,j})_{i=1}^m$ denote the jth codebook sequence of a given class of codebook sequences and assume that the class has 2^η codebook sequences, where η is a positive integer. The resultant distortion of SACS, given by taking expectations in (2.1), is

$$E\left\{\frac{1}{m}\min_j \sum_{i=1}^{m} ||\mathbf{X}_i - Q_{C_n^{i,j}}(\mathbf{X}_i)||^r\right\} = \frac{1}{m}\int \min_j \sum_{i=1}^{m} ||\mathbf{x}_i - Q_{C_n^{i,j}}(\mathbf{x}_i)||^r dF^m, \tag{2.2}$$

where $j \in \{1, 2, \ldots, 2^\eta\}$, dF^m denotes $dF(\mathbf{x}_1)\cdots dF(\mathbf{x}_m)$, and $\int$ denotes a km-fold integral. The expected distortion given in (2.2) is called the sample-adaptive distortion. This distortion will later be compared with the F-optimal distortion achieved by IICS at the same bit rate.

In SACS, for each sample, the encoder transmits bits, for the codebook index with m quantized element indices, in the form of a feed-forward adaptive scheme. This makes it possible to replace different codebook sequences for each sample of $(\mathbf{X}_i)_{i=1}^m$. In other words, the encoder quantizes m vectors of a sample $(\mathbf{X}_i^\omega)_{i=1}^m$ using a codebook sequence of size m from 2^η codebook sequences and replaces the codebook sequence for each sample. Therefore the total bit rate in SACS is given by

$$\frac{m\log_2 n + \eta}{km} = \frac{\log_2 n}{k} + \frac{\eta}{km}, \tag{2.3}$$

where η are the additional bits required in our scheme as side information to indicate which codebook sequence is employed.

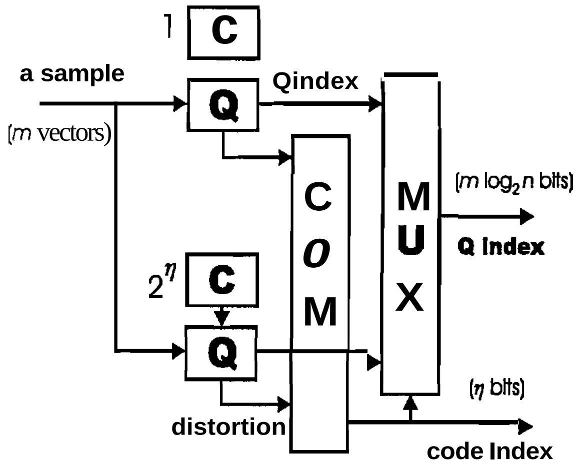


Figure 2.1: The sample-adaptive coding scheme (SACS). (Q: quantizer, C: m codebooks or a codebook, COM: comparator, MUX: multiplexer.)

B. Sample-Adaptive Quantization and 1-SACS

Note that m-SACS requires at most $m2^\eta$ different codebooks from $\mathcal{C}_n$. Hence, if m is large, the decoder needs a large memory for the codebooks and the codebook design complexity may be high. In order to reduce the required number of codebooks, one possibility is to use the *1-codebook sample distance* defined as

$$D_r(C, F_m^\omega) = \int ||\mathbf{x} - Q_C(\mathbf{x})||^r dF_m^\omega(\mathbf{x}) := \frac{1}{m}\sum_{i=1}^{m} ||\mathbf{X}_i^\omega - Q_C(\mathbf{X}_i^\omega)||^r, \tag{2.4}$$

for a codebook $C \in \mathcal{C}_n$. In (2.4), F_m^ω *is the empirical d.f. constructed by placing equal masses at each of the* m *vectors of the sample* $(\mathbf{X}_i^\omega)_{i=1}^m$ [11, p.268]. The empirical d.f. that is constructed for each sample is quite different from F and also quite different from sample to sample with high probability (especially if m is chosen to be small). This allows us to tailor an appropriate codebook for a given sample by choosing a codebook from an optimally designed finite class of codebooks, where the codebook minimizes the 1-codebook sample distance in (2.4) within the class. If we have 2^η codebooks for the sample adaptation, the resultant distortion of this simplified SACS is given by

$$E\left\{\min_{C\in\mathcal{C}_n^\eta} D_r(C, F_m)\right\}, \tag{2.5}$$

where the *codebook class* $\mathcal{C}_n^\eta \subset \mathcal{C}_n$ is a class of 2^η distinct codebooks. We call this simplified SACS (that is based on the 1-codebook sample distance) *1-SACS*. Note that the bit rate for 1-SACS is the same as that for m-SACS given in (2.3).

3. Theoretical Observations on SACS

A. Vector Quantization in km-Dimensional Space

Let us assume that we are using m-SACS, when our underlying quantization space: is k-dimensional. Then the principle of m-SACS can be explained in terms of a km-dimensional VQ. To do that we first describe the distortion for a general km-dimensional VQ for source $(\mathbf{X}_i)_{i=1}^m$.

Using the random variables in source $(\mathbf{X}_i)_{i=1}^m$, let km-tuples X $:= (X_{11}, X_{12}, \ldots, X_{km})$ denote a random vector in $\mathbb{R}^{km}$. Let $\mathcal{B}_u$ denote the class of all u-level codebooks that have real values in $\mathbb{R}^{km}$. The quantizer distortion in $\mathbb{R}^{km}$ is then given by

$$D_r^{km}(B, F^{km}) := \int \|\xi - Q_B^{km}(\xi)\|^r dF^{km}(\xi), \quad B \in \mathcal{B}_u, \tag{3.1}$$

where $\xi \in \mathbb{R}^{km}$, F^{km} is the joint d.f. of X, and the km-dimensional VQ is

$$Q_B^{km}(\xi) := \arg\min_{\zeta \in B} \|\xi - \zeta\|^r. \tag{3.2}$$

In this VQ, the bit rate is $(\log_2 u)/km$.

We next describe IICS as a quantizer in R^{km}. Let $\mathbf{C}_{m,v}$ denote the class of all v^m-level product codebooks $(\times_{i=1}^m C_v^i)$, where $C_v^i \in \mathcal{C}_v$. Since $\mathbf{C}_{m,v}$ includes all the possible product codebooks of $(\times_{i=1}^m C_v)$, where $C_v \in \mathcal{C}_v$, the v-level F-optimal distortion of IICS in (1.3) then satisfies the following relation

$$\inf_{B \in \mathbf{C}_{m,v}} D_r^{km}(B, F^{km}) \leq \inf_{C \in \mathcal{C}_v} D_r(C, \mathrm{F}). \tag{3.3}$$

Now we describe m-SACS as a quantizer in $\mathbb{R}^{km}$. Let $\mathbf{C}_{m,n}^\eta$ denote the class of all $n^m 2^\eta$-level product codebooks $\bigcup_{j=1}^{2^\eta} (\times_{i=1}^m C_n^{i,j})$, where $C_n^{i,j} \in \mathcal{C}_n$, then the sample-adaptive distortion of (2.2) can be rewritten as

$$E\left\{\frac{1}{m} \min_j \sum_{i=1}^m \|\mathbf{X}_i - Q_{C_n^{i,j}}(\mathbf{X}_i)\|^r\right\} = D_r^{km}(B, F^{km}), \tag{3.4}$$

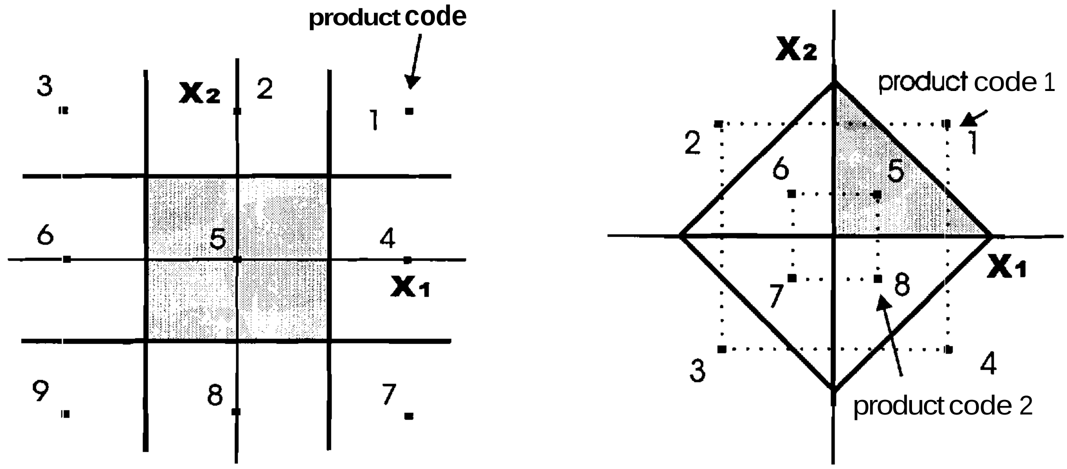
where B $:= \bigcup_{j=1}^{2^\eta} (\times_{i=1}^m C_n^{i,j}) \in \mathbf{C}_{m,n}^\eta$ and the bit rate is $(\log_2 n)/k + \eta/km$. Note that the product codebook of SACS are in the form of the unions of the sub-product codebooks $C_n^{i,j}$. Therefore, we have

$$\inf_{B \in \mathbf{C}_{m,n}^\eta} D_r^{km}(B, F^{km}) \leq \inf_{B \in \mathbf{C}_{m,v}} D_r^{km}(B, F^{km}) \tag{3.5}$$

for $v^m = n^m 2^\eta$, since $\mathbf{C}_{m,n}^\eta \supset \mathbf{C}_{m,v}$. In other words the average distortion in m-SACS is less than the average distortion in IICS. We now trivially obtain the following bounds.

Bound 1: Suppose that the bit rates of the km-dimensional VQ, IICS, and m-SACS are equal, i.e., $u = v^m = n^m 2^\eta$. Then for source $(\mathbf{X}_i)_{i=1}^m$ we have

$$\begin{aligned} &\inf_{B \in \mathcal{B}_u} D_r^{km}(B, F^{km}) \\ &\leq \inf_{\{(C_n^{i,j})_i \mid j=1,2,\ldots,2^\eta\}} E\left\{\frac{1}{m} \min_j \sum_{i=1}^m \|\mathbf{X}_i - Q_{C_n^{i,j}}(\mathbf{X}_i)\|^r\right\} \leq \inf_{C \in \mathcal{C}_v} D_r(C, F). \end{aligned} \tag{3.6}$$



(a) (b)

Figure 3.1: The Voronoi partitions of IICS and SACS in km-dimensional space. (a) Example of IICS ($m = 2, v = 3$). (b) Example of SACS ($m = 2, n = 2, \eta = 1$).

Proof: Since $\mathcal{B}_u \supset \mathbf{C}^{\eta}_{m,n}$, the first relation holds. Hence, from (3.3) and (3.5), Bound 1 follows. ∎

Bound 1 shows that the performance of m-SACS lies between that of k-dimensional VQ and *km*-dimensional VQ. In m-SACS, if $n = 1$ and $\eta = \log_2 u$, then $\mathbf{C}^{\eta}_{m,n} = \mathcal{B}_u$, i.e., the sample-adaptive distortion is equal to the distortion of the km-dimensional VQ. Note that increasing the sample size m in this case can decrease the quantizer distortion, which can approach the theoretically obtainable minimum distortion [6]. For the cases when $n > 1$ and $\eta > 0$, the performance of m-SACS is worse than the km-dimensional VQ but better than IICS. We can obtain further gains by increasing the sample size m for a fixed n. On the other hand, if $n = v$ and $\eta = 0$, then $\mathbf{C}^{\eta}_{m,n} = \mathbf{C}_{m,v}$; we cannot expect any gain from the sample size m.

We novr provide further intuition as to why we expect SACS to outperform IICS, by observing the Voronoi partitions of SACS and IICS as km-dimensional VQs. It is obvious that all the Voronoi regions that are generated by the product codebooks in $\mathbf{C}_{m,v}$ are rectangular in $\mathbb{R}^{km}$. On the other hand, in the SACS case, since the product codebook in $\mathbf{C}^{\eta}_{m,n}$ is composed of several different ***sub-product codebooks*** as in (3.4). In other words it is possible to make a Voronoi partition such that each Voronoi region yields lower average distortion than the rectangular region for a given volume of the Voronoi region. Thus, we expect that for a given bit rate, SACS will achieve a smaller average distortion than using IICS. Also note that the sub-product codebooks can be efficiently assigned to the joint d.f. of X, even if the random variables are dependent.[1] A more detailed discussion on the Voronoi regions will be introduced in Section IV by studying several root lattices.

[1]Note that VQ also obtains a gain based on a similar principle of Voronoi regions and codeword assignment [14],[19].

B. 1-SACS

Since the distribution functions of the random vectors in $(\mathbf{X}_i)_{i=1}^m$ are i.i.d., it seems that employing the 1-codebook sample distance in (2.4) may be good enough to quantize the source $(\mathbf{X}_i)_{i=1}^m$. In this way, the required number of codebooks is reduced from $m2^\eta$ in m-SACS to 2^η in 1-SACS. Hence, it may be advantageous to use 1-SACS if the coding scheme yields a sufficient gain over IICS. However, unlike in the m-SACS case, simply increasing the sample size m does not always guarantee a performance gain in this 1-SACS case. In other words, the second relation in (3.6) does not always hold, since the class of all possible product codebooks of 1-SACS does not include $C_{m,v}$. However, for the large codebook case, 1-SACS is asymptotically better than IICS for uniformly distributed data based on the root lattice analysis in Section IV. We will also provide numerical results in Section VI to show that under appropriate design parameters 1-SACS can significantly outperform IICS.

In the following discussion we will provide some asymptotic results which will help us better understand how to design an efficient sample adaptive coding scheme using 1-SACS. For our analysis, we define a new constant $\beta := m/n$. We call β the sample ratio since it is the ratio of the sample size to the maximum number of available codewords for a quantization of a sample in the 1-SACS case.

If C_n^* is an F-optimal codebook, and if the codebook class $\mathcal{C}_n^\eta$ includes C_n^*, then $\min_{C\in\mathcal{C}_n^\eta} D_r(C, F_m^\omega) \leq D_r(C_n^*, F_m^\omega)$ holds for every w. Remember that (as described earlier) F_m^ω is the empirical distribution constructed by placing equal masses at each of the m vectors of the sample $(\mathbf{X}_i^\omega)_{i=1}^m$. Hence, since $E\{D_r(C_n^*, F_m)\} = D_r(C_n^*, \mathrm{F})$ (from [31, Appendix]), we have

$$E\left\{\min_{C\in\mathcal{C}_n^\eta} D_r(C, F_m)\right\} < \inf_{C\in\mathcal{C}_n} D_r(C, \mathrm{F}), \tag{3.7}$$

for $\eta \geq 1$. The inequality in (3.7) suggests that the minimum of the sample-adaptive distortion of 1-SACS is less than the n-level F-optimal distortion.

In order to provide a lower bound on the sample-adaptive distortion of 1-SACS we first describe the empirically optimal distortion introduced in [7]. In the sample-adaptive distortion given by (2.5), let us replace the codebook class $\mathcal{C}_n^\eta$ with $\mathcal{C}_n$. Then, the sample-adaptive distortion is changed to the empirically optimal distortion

$$E\left\{\min_{C\in\mathcal{C}_n} D_r(C, F_m)\right\}, \tag{3.8}$$

which is always less than (2.5). We call this distortion the *F_m-optimal* distortion.

If $\beta = 1$, i.e., the codebook size n is equal to the sample size m, then for all sample points $w \in \Omega$, $D_r(C^\omega, F_m^\omega) = 0$ holds if codebook $C^\omega = \{\mathbf{X}_1^\omega, \ldots, \mathbf{X}_n^\omega\}$ is chosen for each F_m^ω. Hence, the F_m-optimal distortion is obviously equal to zero. In the special case when n = 1 and $\beta \geq 1$, we obtain the well known relation $E\{\min_{C\in\mathcal{C}_n} D_r(C, F_m)\} = (\mathrm{m}-1)/m \cdot \mathrm{Var}(\mathbf{X}_i)$ which implies that the mean distortion is a biased estimator of the variance. However, for the $\beta > 1$ case, an explicit derivation of (3.8) is usually difficult in general. It will be shown that this F_m-optimal distortion is the infimum of the sample-adaptive distortion of 1-SACS for all q.

Proposition 1: For an increasing sequence $\mathcal{C}_n^1 \subset \mathcal{C}_n^2 \subset \cdots$, sequence $(E\{\min_{C \in \mathcal{C}_n^\eta} D_r(C, F_m)\})_{\eta=1}^{\infty}$ is monotonically decreasing and

$$\lim_{\eta \to \infty} E\left\{\min_{C \in \mathcal{C}_n^\eta} D_r(C, F_m)\right\} = E\left\{\min_{C \in \mathcal{C}_n} D_r(C, F_m)\right\} \tag{3.9}$$

holds for every positive integer m and n.

Proof: See Appendix A for the proof.

Using the above proposition, Bound 1, and (3.7) we now have both an upper and lower bound to the 1-SACS distortion.

Bound 2: Suppose that the bit rates of the km-dimensional VQ and 1-SACS are equal, i.e., $u = n^m 2^\eta$. Then

$$\begin{aligned} \max\left\{\inf_{B \in \mathcal{B}_u} D_r^{km}(B, F^{km}), E\left\{\min_{C \in \mathcal{C}_n} D_r(C, F_m)\right\}\right\} \\ \leq \inf_{\mathcal{C}_n^\eta \subset \mathcal{C}_n} E\left\{\min_{C \in \mathcal{C}_n^\eta} D_r(C, F_m)\right\} \leq \inf_{C \in \mathcal{C}_n} D_r(C, F) \end{aligned} \tag{3.10}$$

hold for the source $(\mathbf{X}_i)_{i=1}^m$, if there exists an n-level F-optimal codebook.

In [16] and [25], the consistency problem of the F_m-optimal distortion is investigated based on the convergence of probability measure in metric space and the Glivenko-Cantelli Theorern. From these investigations we obtain an asymptotic result (as the sample size m gets large).

Proposition 2: Suppose that there exists an F-optimal codebook. Then

$$\lim_{m \to \infty} \inf_{\mathcal{C}_n^\eta \subset \mathcal{C}_n} E\left\{\min_{C \in \mathcal{C}_n^\eta} D_r(C, F_m)\right\} = \inf_{C \in \mathcal{C}_n} D_r(C, F) \tag{3.11}$$

for fixed integer n and η.

Proof: See Appendix A for the proof.

The above proposition tells us that as the sample size m increases to infinity compared to the codebook size n, the minimum distortion in 1-SACS simply converges to the n-level F-optimal distortion. In other words, if we increase m for fixed n and η (increase the sample ratio β), then the gain decreases and Proposition 2 follows. Moreover, as m increases, the bit rate also decreases to $(\log_2 n)/k$ from (2.3). Further, for a fixed n and a fixed ratio η/m, the bit rate is also fixed, and the sample-adaptive distortion of 1-SACS still converges to the n-level F-optimal distortion from Bound 2 and Proposition 2 even if there is a constant side information η/m. This is an expected result since it implies that the adaptation occurs over larger and larger time intervals compared to the codebook size. The above discustiion indicates that there is no gain to be had in 1-SACS by increasing the sample size m for a fixed codebook size n. However, as will be demonstrated in Section VI, increasing m while keeping the sample ratio $\beta = m/n$ small (i.e.

increasing the codebook size n as well), will allow for very efficient adaptation, and performance that is virtually identical to the m-SACS case.

As an aside, note that in the m-SACS case, the sample ratio is always $m/mn = 1/n \leq 1$, since the maximum number of available codewords for a codebook sequence is mn. In other words, the number of codewords, which will describe the source, is greater than or equal to the sample size, since the number of codebooks in m-SACS increases as m grows. Therefore, unlike 1-SACS, in the m-SACS case, the ratio $\beta = m/n$ does not affect the performance of SACS.

4. SACS and Root Lattices

We will show that several important lattice based vector quantizers in m-dimensional space can be constructed using SACS. These lattices are important because vector quantizers of certain dimensions based on these lattices result in optimal or close to optimal mean square distortion. Rennember our earlier discussion on Voronoi regions in Section III. There we provided an intuitive reason why we expected SACS to outperform IICS, by being able to change the shape of the Voronoi regions. Here, we explicitly show that the Voronoi regions of many important lattices can be also constructed via SACS with k = 1.

The lattice VQ [27] is a uniform quantizer whose output is a truncated root lattice [1]. An m-dimensional lattice is defined by a set of points in $\mathbb{R}^{m'}$,

$$\Lambda := \{\mathbf{x} \mid \mathbf{x} = U\mathbf{p}, \mathbf{p} \in \mathbb{Z}^m\}, \tag{4.1}$$

where the m' x m matrix $U = (\mathbf{u}_1 \ldots \mathbf{u}_m)$ is a generator matrix for Λ, $\mathbf{u}_i$ ($\in \mathbb{R}^{m'}$) are linearly independent vectors, and m $\leq$ m'. Let the Voronoi regions that are constructed by the lattice Λ have the shape of some polytope $\mathcal{P}$ with centroid $\mathbf{x}_o$. Then $G(\Lambda)$, the normalized second moment of $\mathcal{P}$ is defined as

$$G(\Lambda) := \frac{1}{m} \frac{\int_{\mathcal{P}} \|\mathbf{x} - \mathbf{x}_o\|^2 d\mathbf{x}}{\left(\int_{\mathcal{P}} d\mathbf{x} \right)^{(m+2)/m}}. \tag{4.2}$$

The quantity $G(\Lambda)$ determines the performance of a lattice VQ using the mean square error distortion measure [14],[19]. For example, the hexagonal lattice A_2 has $G(A_2) \approx$ 0.0802 and is the optimal 2-dimensional lattice quantizer for uniformly distributed data, while the rectangular lattice has $12^{-1} \approx$ 0.0833. Conway and Sloane [19],[23] have calculated the second moments of various lattices that yield close values to $\min_\Lambda G(\Lambda)$ for various dimensions [23, Table I]. We will show that VQ based on many of the important lattices described in [23] can in fact be constructed based on SACS. Our results are summarized in Table I. We next describe how we obtain these results.

For sets $\Lambda^{i,j} \subset \mathbb{R}$, i $= 1, \ldots,$ m, j $= 1, 2, \ldots, 2^\eta$, assume that $\text{card}(\Lambda^{i,j}) = \text{card}(\mathbb{Z})$ (remember that $\text{card}(H)$ denotes the cardinality of a set H) and $\mathbf{L}_m^\eta$ is the class of all sets that have the product form $\bigcup_{j=1}^{2^\eta}(\times_{i=1}^m \Lambda^{i,j})$. Note that $(\times_{i=1}^m \Lambda^{i,j})$ is a coset of a rectangular lattice and that the *sub-product codebook* ofm-SACS is a subset of $(\times_{i=1}^m \Lambda^{i,j})$. Throughout this discussion we say that two lattices H and **I** are said to be equivalent, written H $\cong$ **I**, if they differ only by a rotation and possibly a change of scale [23].

The first type of lattice that we investigate are the A_m lattices. Note that the A_2 lattice has been shown to be optimal (i.e. VQ based on this lattice minimizes the mean squared distortion) in two dimensions for uniformly distributed data. As described in Table I, we will show that the A_2 and A_3 lattice vector quantizers can be described by m-SACS with $\eta = 1$. Specifically we will show that there exists rotated versions of A_2 and A_3 that belong to $\mathbf{L}_2^1$ and $\mathbf{L}_3^1$, respectively. For $m \geq 1$, A_m is the m-dimensional lattice consisting of the points $\mathbf{x} = (x_0, x_1, \ldots, x_m)$ having integer coordinates that sum to zero. In, other words,

$$A_m := \left\{ \mathbf{x} \in \mathbb{Z}^{m+1} \mid \sum_{i=0}^{m} x_i = 0 \right\}. \tag{4.3}$$

We let $A_m^\perp$ denote the dual lattice [19] of A_m. Then $A_1 \cong A_1^\perp \cong \mathbb{Z}$. A_2 ($\cong A_2^\perp$) is the hexagonal lattice and can be generated by the basis vectors $\mathbf{u}_1 = (1,-1,0)$ and $\mathbf{u}_2 = (1,0,-1)$. Since $\mathbf{u}_1$ and $(2\mathbf{u}_2 - \mathbf{u}_1)$ are orthogonal, $\mathbf{u}_1$ and $2\mathbf{u}_2$ generate a rotated rectangular lattice Λ. Thus,

$$A_2 = \Lambda \bigcup \left(\mathbf{u}_2 + \Lambda \right), \tag{4.4}$$

it follows that there exists a lattice Λ satisfying $A_2 \cong \Lambda \in \mathbf{L}_2^1$. In the 3-dimensional case, A_3 is the faced-centered cubic lattice and has three basis vectors, $(1,-1,0,0)$, $(1,0,-1,0)$, and $(1,0,0,-1)$ [17]. Applying the same idea as in the A_2 case, there is a lattice Λ such that $A_3 \cong \Lambda \in \mathbf{L}_3^1$. To summarize, the lattice VQ based on A_2 and A_3 can be described by m-SACS (m = 2 and 3, respectively), with $\eta = 1$. By increasing η we can also describe A_m for larger values of m.

The next lattice shown in Table I is $A_3^\perp$, which can also be shown to be an element of $\mathbf{L}_3^1$. It turns out that for m = 3, the $D_m^\perp$ lattice to be discussed later is equivalent to the $A_3^\perp$ lattice, and hence we will show $A_3^\perp \in \mathbf{L}_3^1$ as part of our discussion on the $D_m^\perp$ lattice.

Another important type of lattices are the D_m lattices. For $m \geq 2$, D_m consists of the points $(x_1, \ldots, x_m)$ having integer coordinates with an even sum. The generator matrix U_D for lattice D_m is

$$U_D := \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{4.5}$$

Hence, $D_m = \{\mathbf{x} \mid \mathbf{x} = U_D\mathbf{p}, \mathbf{p} \in \mathbb{Z}^m\}$. Let $D_{m,\ell}$ denote a sublattice of D_m defined as

$$D_{m,\ell} := \{\mathbf{x} \mid \mathbf{x} = U_D\ \mathbf{p}', \mathbf{p}' = (p_1, 2p_2 + \ell_0, 2p_3 + \ell_1, \ldots, 2p_m + \ell_{m-2}), p_1, \ldots, p_m \in \mathbb{Z}\}, \tag{4.6}$$

where $\ell = \ell_{m-2}2^{m-2} + \ell_{m-3}2^{m-3} + \ldots + \ell_0 2^0$ and $\ell = 0, 1, 2, \cdots, 2^{m-1} - 1$. The lattice D_m can then be rewritten as

$$\begin{aligned} D_m &= \bigcup_{\ell=0}^{(2^{m-1}-1)} D_{m,\ell} \\ &= \bigcup_{\ell=0}^{(2^{m-1}-1)} \Big(U_D(0, \ell_0, \ell_1, \ldots, \ell_{m-2})^T \\ &\quad + \{\mathbf{x} \mid \mathbf{x} = U_D(p_1, 2p_2, 2p_3, \cdots, 2p_m)^T, p_1, \ldots, p_m \in \mathbb{Z}\} \Big) \\ &= \bigcup_{\ell=0}^{(2^{m-1}-1)} \Big(\mathbf{r}_\ell + \{\cdots, -4, -2, 0, 2, 4, \cdots\}^m \Big), \end{aligned} \tag{4.7}$$

where $\mathbf{r}_\ell := U_D(0, \ell_0, \ell_1, \ldots, \ell_{m-2})$. Since from (4.7), we can represent D_m as the union of 2^{m-1} cosets of rectangular sub-lattices, and since one sub-lattice corresponds to a sub-product codebook in SACS, we obtain $2^{m-1} = 2^\eta$. Hence, the side information required in this case is $\eta = m - 1$ which corresponds to the number of 1s in the diagonal of U_D. This also means that $D_m \in \mathbf{L}_m^{m-1}$.

Another important lattice discussed in [23] and also listed in Table I is the $D_m^\perp$ lattice. For $m \geq 2$, $D_m^\perp$ is the dual of the lattice D_m defined as

$$D_m^\perp := \mathbb{Z}^m \bigcup \left((\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})^T + \mathbb{Z}^m \right) \tag{4.8}$$

By its definition, it is clear that $D_m^\perp \in \mathbf{L}_m^1$. This implies that 1-SACS can construct the $D_m^\perp$ lattice with only $\eta = 1$. Further, it also tells us that 1-SACS with $\eta = 1$ can construct the optimal lattice in 3 dimensions, since the $D_3^\perp$ lattice (or equivalently the lattice $A_3^\perp$) is a body-centered cubic lattice and optimal in 3-dimensions [22].

The next lattices shown in Table I are E_7 and E_8. VQ based on E_7 and E_8 result in mean square distortion that is very close to the lower bound on the distortion in 7 and 8 dimensions, respectively [26]. Note that the diagonals of generator matrixes of E_7 and E_8 have 3 and 4 1s, respectively. Thus, in a similar way to the D_m case, $E7 \in \mathbf{L}_7^3$ and $E_8 \in \mathbf{L}_8^4$.

Lattice vector quantizers have a highly regular structure based on a lattice, which forms a regularly spaced array of points in $\mathbb{R}^m$. Due to this regularity, the encoding operation also becomes easy. Conway and Sloane developed encoding techniques based on the special properties of particular lattices [20],[23]. Sayood, *et. al.*, presented an algorithm for constructing the root lattice generation matrix and a multistep searching technique for encoding [24]. SACS, however, can be viewed as a unifying encoding scheme for several root lattices, as shown previously. For example, the encoding process of the lattice VQ based on the lattice $D_m^\perp$ can be described by 1-SACS with $\eta = 1$, since $D_m^\perp \in \mathbf{L}_m^1$. Note that 1-SACS with $\eta = 1$ requires only two scalar codebooks. In other words, in this scheme, two different scalar quantizers quantize m points of a sample independently and the index of the quantizer that yields the minimum distortion will be transmitted. For example, if the codebook size is 4 the two codebooks for the scalar quantizers can be $\{1, 2, 3, 4\}$ and $\{1.5, 2.5, 3.5, 4.5\}$.

To summarize, vector quantization based on many different root lattices can be treated by the very simple operation of scalar quantization in SACS. The encoding complexity as described in detail in Appendix B is very low, of the order of $\mathcal{O}(2^\eta R)$. We next describe a codebook design algorithm for implementing SACS.

5. Codebook Class Design Algorithm

For a given bit rate R = $(\log_2 n)/k + \eta/mk$ and vector dimension k, designing an optimal SACS is finding (optimal) values for m, n, η, and the codebook sequences $(C_n^{i,j})_{i=1}^m$, j = $1, 2, \ldots, 2^\eta$, which minimize the sample-adaptive distortion. However, the sample-adaptive distortion is not a known function of m, n, and η. Thus, we will try to find an optimal class of codebook sequences or codebooks that minimizes the sample-adaptive distortion. In this section, we focus only on the codebook design problem for 1-SACS, since the codebook design for m-SACS follows in much the same way. The codebook design problem is to find an optimal codebook class $\mathcal{C}_n^\eta$ that achieves the following sample-adaptive distortion

$$\inf_{\mathcal{C}_n^\eta \subset \mathcal{C}_n} E\left\{\min_{C \in \mathcal{C}_n^\eta} D_r(C, F_m)\right\}, \tag{5.1}$$

for given values of m, n, and η.

In order to find an optimal codebook class, we have developed a codebook class design algorithm that uses a large number of samples as a *training sequence* (TS). In a similar way, this algorithm can be extended to designing the $m2^\eta$ codebooks for the m-SACS case. Let $(\mathbf{x}_{1\ell}, \ldots, \mathbf{x}_{m\ell})$ denote the ℓth sample in a given set of M samples, where a sample has m training vectors. The first part of our algorithm quantizes m training vectors in each sample using 2^η different codebooks and then selects a codebook that yields the minimum 1-codebook sample distance (given in (2.4)) for a sample. The second part of the algorithm updates the codebooks using the partitioned TS in the quantization process of the first part. These two parts are then iteratively applied to the given TS. The algorithm is described below.

Algorithm

1. Initialization (i = 0): Given a codebook size n, sample size m, number of codebooks 2^η, distortion threshold $\epsilon \geq 0$, initial codebook class $\mathcal{C}_{n,0}^\eta$, and TS $\left((\mathbf{x}_{1\ell}, \ldots, \mathbf{x}_{m\ell})\right)_{\ell=1}^M$, set $d_{-1} = \infty$.

2. Given $\mathcal{C}_{n,i}^\eta$, find n27 partitions of each training vectors in the TS for the corresponding $n2^\eta$ codewords, where each training vector's codeword is determined by

$$C_{(\ell)} := \arg\min_{C \in \mathcal{C}_{n,i}^\eta} \frac{1}{m}\sum_{j=1}^m \|\mathbf{x}_{j\ell} - Q_C(\mathbf{x}_{j\ell})\|^r \quad \text{for } \ell = 1, \ldots, M. \tag{5.2}$$

   Next, compute the average distortion d_i given by

$$d_i := \frac{1}{M}\sum_{\ell=1}^M \frac{1}{m}\sum_{j=1}^m \|\mathbf{x}_{j\ell} - Q_{C_{(\ell)}}(\mathbf{x}_{j\ell})\|^r. \tag{5.3}$$

3. If $(d_{i-1} - d_i)/d_i \leq \epsilon$, stop. $\mathcal{C}_{n,i}^\eta$ is the final codebook class. Otherwise continue.

4. Increase i by 1. Compute a centroid for each of the $n2^\eta$ partitions and replace the codewords in $\mathcal{C}_{n,i}^\eta$ by the new $n2^\eta$ centroids. Go to Step 1.

Note that the average distortion on the TS in (5.3) is an estimate of the sample-adaptive distortion in (2.5). It can be shown using similar techniques as in the case of the Lloyd-Max: algorithm or the k-means algorithm [3] that d_i is a decreasing sequence. Thus, d_i converges to a (local:) minimum, which depends on the initial codebook class $\mathcal{C}^{\eta}_{n,0}$. In the case of the Lloyd-Max algorithm, we can obtain the global optimum if the input source has a log-concave density as in the case of the Gaussian source [21]. However, even for Gaussian sources, in the case of our algorithm, convergence to the global optimum is not guaranteed. Therefore, it is especially important to choose an appropriate initial codebook class. We next outline a "split method" using an F-optimal (or local optimal) codebook to calculate the initial codebook class. The notion of this algorithm is intuitively based on the decomposed lattices in Section IV.

Initial *Codebook* Class Algorithm (Split Method)

1. Initialization (i = 0): We are given a codebook size n, the number of codebooks to be generated is 2^{η}, the split constant ϵ' and TS $\left((\mathbf{x}_{1\ell}, \ldots, \mathbf{x}_{m\ell}) \right)_{\ell=1}^{M}$. The initial codebook class $\mathcal{C}^{0}_{n,0}$, contains only one codebook which is the F-optimal (or suboptimal) codebook.

2. If i $\geq \eta$ stop. $\mathcal{C}^{i}_{n,0}$ is the initial codebook class for the algorithm.

3. Increase i by 1. Construct a new codebook class $\mathcal{C}^{i}_{n,0}$, by doubling the number of codebooks in the class $\mathcal{C}^{i-1}_{n,0}$ as follows. The first 2^{i-1} codebooks of $\mathcal{C}^{i}_{n,0}$ are given by subtracting ϵ' from all the elements of the codewords in $\mathcal{C}^{i-1}_{n,0}$. The next 2^{i-1} codebooks of $\mathcal{C}^{i}_{n,0}$ are given by adding ϵ' to all the elements of the codewords in $\mathcal{C}^{i-1}_{n,0}$.

4. Given $\mathcal{C}^{i}_{n,0}$, find $n2^i$ partitions of training vectors according the quantization

$$\min_{C\in\mathcal{C}^{i}_{n,0}} \frac{1}{m}\sum_{j=1}^{m} \|\mathbf{x}_{j\ell} - Q_C(\mathbf{x}_{j\ell})\|^r, \quad \text{for } \ell = 1, \ldots, M \tag{5.4}$$

   Compute the centroids of $n2^i$ partitions, respectively and replace the codewords in $\mathcal{C}^{i}_{n,0}$ by the new $n2^i$ centroids. Go to 1.

In this Initial *Codebook* Class Algorithm, Step 2 doubles the number of codebooks in the codebook class by adding to and subtracting from each element of the previous codewords a small constant ϵ'. This doubling scheme is based on a principle that the all sub-product codebooks in 1-SACS rnust be symmetric with respect to the line $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_m \in \mathbb{R}^k$ in km-dimensional space.

6. Simulation Results and Discussions

For our experimental results, we use synthetic data that represents a Gaussian i.i.d. source for all elements of the source $(\mathbf{X}_i)_{i=1}^m$. In other words, all the random variables in $(\mathbf{X}_i)_{i=1}^m$ are independent and normally distributed with the same mean and variance. In our numerical study, to ensure a good codebook design for the Gaussian source, we have used more than 5000 training vectors per codeword. The mean square distortion measure is employed (i.e., r = 2 in (1.1)). In Fig. 6.1 we illustrate how the algorithm converges to a (local) minimum distortion (here m = 2, n = 2, and $\eta = 2$), where the split constant is $\epsilon' = 0.001$. Since the initial codebook is the Lloyd-Max quantizer at n = 2, the starting distortion d_0 in (5.3) is less that that of the Lloyd-Max, 0.363. The distortion sequence of d_i monotonically decreases with each iteration in both the m-SACS and 1-SACS cases. In the m-SACS case in Fig. 6.1, the distortions for the first several iterations follow that of 1-SACS, since the ***Initial*** *Codebook* Class *Algorithm*, which is for 1-SACS, is also employed for m-SACS.

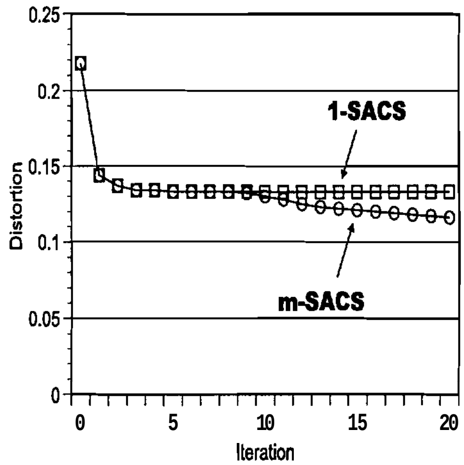
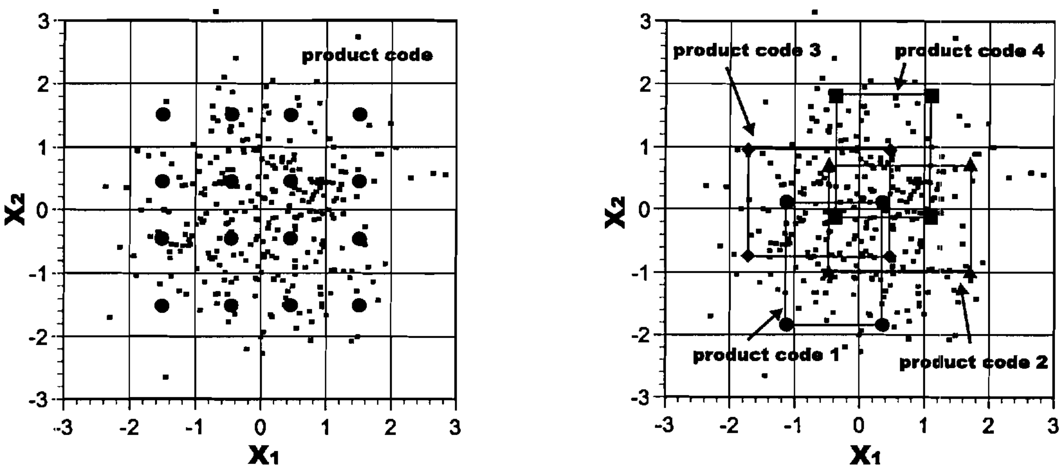


Figure 6.1: Sample-adaptive distortion d, with respect to iteration a in the algorithm at the bit rate 2. (The source has Gaussian distribution with variance 1, m = 2, n = 2, and $\eta = 2$. m-SACS: sample-adaptive distortion of m-SACS, 1-SACS: sample-adaptive distortion of 1-SACS.)

For understanding the principle of SACS, the product codebooks of IICS and m-SACS are illustrated in Fig. 6.2(a) and (b), respectively. In the IICS case, since the 16 (= v^m) codewords in 2-dimensional space are the elements of the product codebook $C_v \times C_v$, the Voronoi regions are rectangular as shown in Fig. 6.2(a). On the other hand, the sub-product codebooks in the m-SACS case can include the IICS case and further, can make non-rectangular Voronoi regions, which yield lower distortion than the IICS case, as shown in Fig. 6.2(b) (according to the relation in Bound 1). In contrast, in the 1-SACS case, all the sub-product codebooks must be symmetric with respect to the line $\mathbf{X}_2 = \mathbf{X}_1$ as shown in Fig. 6.3. This means that the product codebooks of 1-SACS cannot include all the product codebooks of IICS, and that the sample-adaptive distortion of 1-SACS is not always guaranteed to be less than the IICS case. In the examples of Fig. 6.2 and Fig. 6.3, IICS yields 9.30dB of SNR at the bit rate of 2. m-SACS increases the SNR to 9.56dB but, 1-SACS decreases the SNR to 8.75dB. However, we will show that 1-SACS can significantly do better than IICS depending on choosing appropriate values of n and β. Note that SNR is defined as

$$\text{SNR} = -10 \log \frac{(\text{Distortion})}{\sigma^2} \text{ (dB)},$$

where σ^2 is the variance of $\mathbf{X}_i$.



(a) (b)

Figure 6.2: The product codebooks of IICS and m-SACS at a bit rate of 2. (The source has Gaussian distribution with variance 1. m = 2) (a) Product codebook in IICS ($v = 4$, and SNR= 9.30dB). (b) Sub-product codebooks in m-SACS ($n = 2, \eta = 2$, and SNR= 9.56dB).

In Fig. 6.4, for a bit rate of 3, we compare the sample-adaptive distortions of m-SACS, 1-SACS, and IICS for increasing values of the sample size m at a fixed value of n = 4 (or equivalently for increasing values of the sample ratio β). As expected, m-SACS always yields better results than IICS and 1-SACS. In the m-SACS case, increasing m for a fixed value of n yields more gain over the IICS case. However, for 1-SACS, the increases in SNR can be seen to diminish for large values of m, and will eventually decrease and converge to that of the 4-level Lloyd-Max quantizer (i.e. to 9.30dB for a bit rate of 3). Therefore, to obtain gains in 1-SACS for a given bit rate, it is important to use as large a value for m (and n) as possible, while keeping the sample ratio β small (note that since increasing n increases the total bit rate, this implies that for a given bit rate the side information η should be accordingly decreased).

We know that m-SACS is always better than 1-SACS. However, observe Fig. 6.5. The SNR of 1-SACS is nearly the same as that of m-SACS, especially for $n \geq 4$. In fact, we have found through extensive simulation studies that for fixed values of β and bit rate, increasing n results in each of the m codebooks of a codebook sequence in m-SACS to approach a single codebook (i.e., become equal to one another). Therefore, for a relatively large n (compared to η) and a fixed value of β, it is advantageous to use 1-SACS, since its performance will closely approximate that of m-SACS. Using these design guidelines allows us to reduce the memory requirement by a factor of m (from m-SACS to 1-SACS) without a major compromise on performance. In the next few simulation studies, we will show results under the above

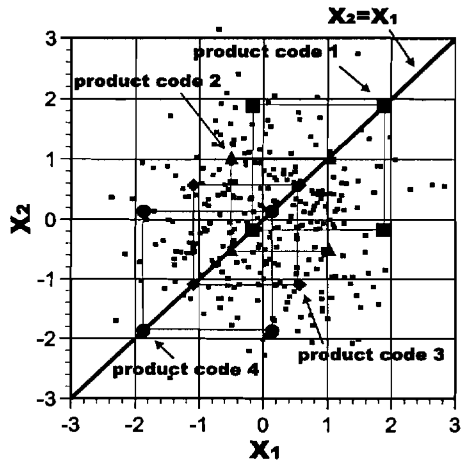


Figure 6.3: The product codebooks of 1-SACS at a bit rate of 2. (The source has a Gaussian distribution with variance 1. $m = 2$, $n = 2$, $\eta = 2$, and SNR=8.75dB.)

design guidelines, and hence only show comparisons between IICS and 1-SACS.

In Fig. 6.6 we depict the results obtained by simulations for the scalar quantization case (k=1) using the Gaussian i.i.d. source. In Fig. 6.6(a) and (b) we plot the SNR versus the bit rate for $n = 8$, and $n = 16$, respectively. In each case, we plot curves for $\eta = 1, \ldots, 4$ and compare the performance of 1-SACS to IICS, i.e. the Lloyd-Max quantizer. In all the cases in Fig. 6.6(a) and (b), 1-SACS can be seen to significantly outperform IICS. For example, for $\beta = 1$ and $\eta = 4$ in Fig. 6.6(b) (i.e., at a bit rate of 4.25), the SNR curve of 1-SACS shows a 1.8dB improvement over the SNR curve of IICS.

In Fig. 6.7, the performance benefits of using 1-SACS is also demonstrated in the VQ case, with $k = 2$. As before, 1-SACS obtains higher SNR than the VQ case. As we can see in both Fig. 6.6 and Fig. 6.7, increasing η provides a greater improvement in the quantizer performance than is obtained by increasing the codebook size n. In other words, the sample-adaptive distortion curve for each sample ratio β always shows higher slope than the F-optimal distortion case.

We now provide an example to illustrate the encoding complexity of SACS. For illustration, focus on Fig. 6.6(a), for the $\beta = 1$ case which results in a bit rate of 3.125 and an SNR of 16.1dB. At this bit rate, we can achieve a similar performance (actually an SNR of 16.0dB at the bit rate of 3.124, i.e., the codebook size 76) using VQ with vector dimension $\kappa = 2$. For our scheme, the required number of additions and multiplications per symbol are 33.1 and 16, respectively (see Appendix B). However, the VQ case requires 114 additions and 76 multiplications; the complexity is substantially greater than SACS [28]. Note also that the complexity in the VQ case increases faster than the proposed scheme as the bit rate increases. If the distance measures are stored in a lookup table, our scheme requires a total of 4 096 bits when the word length $b = 8$, while VQ requires 1 048 576 bits. Further, 1-SACS requires 16 words for the memory of the codebooks while VQ requires 64 words.

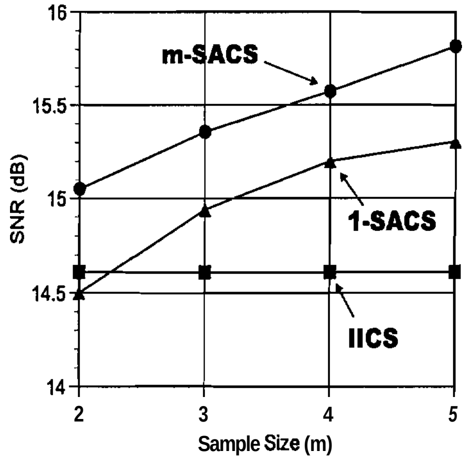


Figure 6.4: SNR versus sample size m for IICS, m-SACS, and 1-SACS. (The source has a Gaussian distribution with variance 1. Bit rate= 3, $k = 1$, and $\eta = m$. $v = 8$ for IICS and $n = 4$ for SACS.)

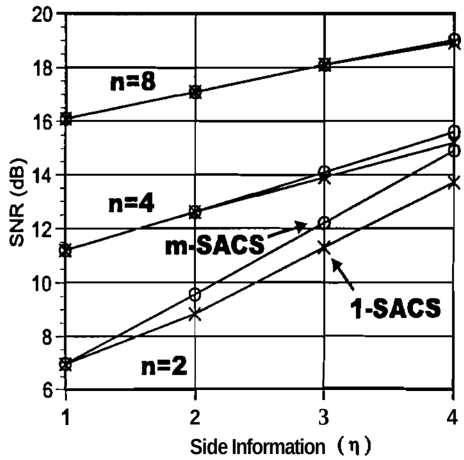


Figure 6.5: SNR of m-SACS and 1-SACS for different values of η. (The source has Gaussian distribution with variance 1 and $\beta = 1$.)

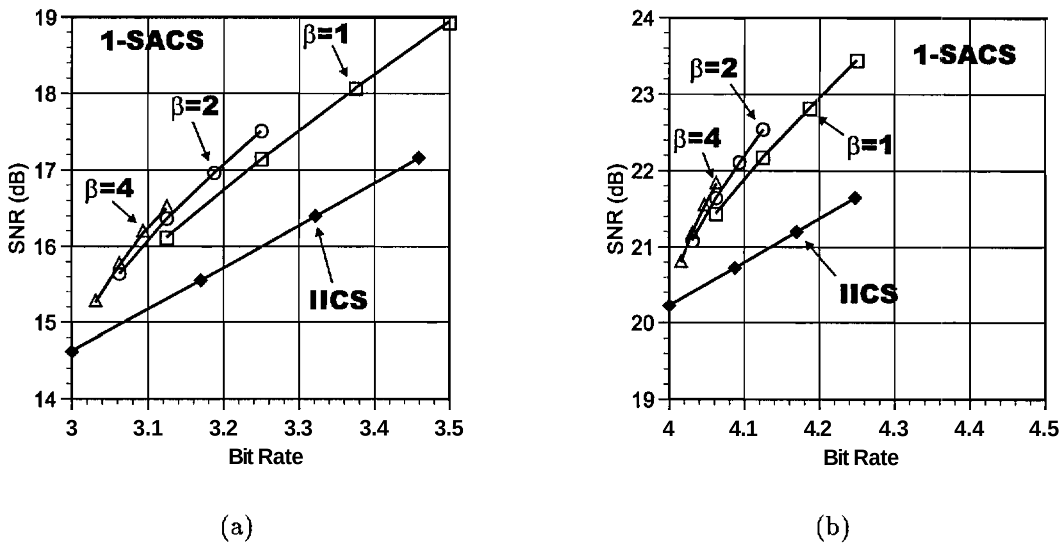


Figure 6.6: Proposed scheme in scalar quantizations of a Gaussian i.i.d. source. The results are obtained by varying η ($\eta = 1, 2, 3,$ and 4). for each β. IICS: IICS using Lloyd-Max quantizer, 1-SACS: sample-adaptive distortion of 1-SACS. (a) n = 8. (b) n = 16.

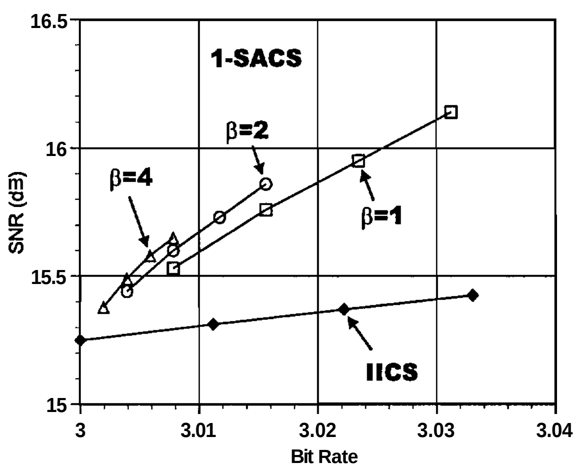


Figure 6.7: Proposed scheme in VQ of a Gaussian i.i.d. source. The results are obtained by varying η ($\eta = 1, 2, 3,$ and 4). for each β. The codebook size n = 64. IICS: IICS using VQ of k = 2, 1-SACS: sample-adaptive distortion of 1-SACS for k = 2.

7. Conclusion

In this paper we have proposed a novel coding scheme called the ***sample-adaptive** coding scheme* (SACS), for a discrete-time memoryless stationary source. This scheme uses more than one codebook and, adapting to each sample, selects an appropriate codebook from the previously designed codebook class available at both the encoder and decoder. The main principle of SACS is adaptation to each sample. Based on this new concept, we proposed two coding schemes; m-SACS, which requires $m2^{\eta}$ codebooks, and 1-SACS, a simplified version of m-SACS which requires 2^{η} codebooks. The performance of m-SACS is always better than 1-SACS. However, under appropriate design guidelines, the performance of 1-SACS can be made to closely approximate that of m-SACS. The complexity of both schemes is $\mathcal{O}(2^R)$ of the order of the scalar quantizer, if $k = 1$. One can obtain comparable performance to SACS in the scalar quantization case, by using a κ-dimensional VQ ($1 < \kappa < m$). However, for the same performance, the encoding complexity of the κ-dimensional **VQ** would be significantly higher ($\mathcal{O}(2^{\kappa R})$). We also show that a number of important lattice based vector quantizers can be constructed using scalar quantization in SACS.

In order to implement SACS, we have proposed and simulated a simple iterative algorithm for the design of the codebooks. Through numerical studies, we find that, for scalar quantization, the sample adaptive coding scheme significantly outperforms the Lloyd-Max based quantizer for an i.i.d. Gaussian source, with typical gains in SNR being between 1–2 dB. In general, we also show that based on the k-dimensional **VQ** structure, the performance of SACS can be made comparable to $k\kappa$-dimensional **VQ.**

Since most commercial data compression systems for video and audio are based on scalar quantization and IICS, applying SACS to those systems has the potential of significant performance gains while maintaining the scalar structure.

APPENDIX A

Proof of Proposition 1: Let $\mathcal{C}'_n$ denote the direct limit of the increasing sequence $(\mathcal{C}^{\eta}_n)^{\infty}_{\eta=1}$, i.e., $\mathcal{C}'_n = \lim \mathrm{dir}_{\ell\to\infty} \mathcal{C}^{\eta}_n$. Note that $\mathrm{card}(\mathcal{C}^{\eta}_n)$, the cardinality of $\mathcal{C}^{\eta}_n$, is equal to 2^{η}. Given n and m, since the sequence $\left(\min_{C\in\mathcal{C}^{\eta}_n} D_r(C, F^{\omega}_m) \right)^{\infty}_{\eta=1}$ is monotonically decreasing and bounded,

$$\lim_{\eta\to\infty} \min_{C\in\mathcal{C}^{\eta}_n} D_r(C, F^{\omega}_m) = \min_{C\in\mathcal{C}'_n} D_r(C, F^{\omega}_m), \quad \text{for every } w. \tag{A 1}$$

Also every subsequence of $(\mathcal{C}^{\eta}_n)^{\infty}_{\eta=1}$ satisfies $\mathrm{card}(\mathcal{C}^{\eta}_n) \to \mathrm{card}(\mathcal{C}'_n) = \mathrm{card}(\mathcal{C}_n)$, where $\mathrm{card}(\mathcal{C}'_n)$ is the cardinal number of the power set of the natural number set, i.e., the cardinal number of the continuum. Thus, since $\mathcal{C}'_n$ contains all the possible codebooks, it follows that

$$\min_{C\in\mathcal{C}'_n} D_r(C, F^{\omega}_m) = \min_{C\in\mathcal{C}_n} D_r(C, F^{\omega}_m), \quad \text{for every } w. \tag{A 2}$$

Therefore from the *Dominated Convergence Theorem* [2, p.110], the proposition follows. ∎

Proof of Proposition 2: It is clear that

$$\min_{C\in\mathcal{C}_n} D_r(\mathrm{C}, F^{\omega}_m) \leq \min_{C\in\mathcal{C}^{\eta}_n} \mathrm{D}, (\mathrm{C}, F^{\omega}_m) \leq \mathrm{D}, (C^{*}_n, F^{\omega}_m) \tag{A 3}$$

holds for every $w \in$ St, when $\mathcal{C}_n^\eta$ includes the F-optimal codebook C_n^* (if it exists). From [16, Consistency Theorem] and [25, Theorem 1],

$$\lim_{m\to\infty} \min_{C\in\mathcal{C}_n} D_r(C, F_m) = \inf_{C\in\mathcal{C}_n} D(C, F), \quad \text{almost surely.} \tag{A 4}$$

It follows that since $D_r(C_n^*, F_m)$ converges to the F-optimal distortion almost surely from the *Strong Law of Large Numbers* [4, p.204], the sequence of the sample-adaptive distortions converges to the F-optimal distortion almost surely for fixed n and η, and the proposition follows ∎.

APPENDIX B
ENCODING COMPLEXITY OF SACS

In this Appendix, we analyze in detail the complexity of SACS for both non-uniform (e.g., the m codebooks in a codebook sequence of m-SACS may have different sizes) and uniform codebook sizes.

Let n_i represent the ith codebook size. Using codebooks C^i ($\in \mathcal{C}_{n_i}$), i = 1, ..., m., the m-codebook sample distance in (2.1) can be replaced by

$$\frac{1}{m}\sum_{i=1}^{m} ||\mathbf{X}_i^\omega - Q_{C^i}(\mathbf{X}_i^\omega)||^r. \tag{B 1}$$

Hence, the average distortion in SACS is given by

$$E\left\{\min_j \frac{1}{m}\sum_{i=1}^{m} ||\mathbf{X}_i - Q_{C^{i,j}}(\mathbf{X}_i)||^r\right\}, \tag{B 2}$$

where $j \in \{1, 2, \ldots, 2^\eta\}$ and $C^{i,j} \in \mathcal{C}_{n_i}$, and the total bit rate is

$$\rho + \frac{\eta}{km} := \sum_{i=1}^{m} \rho_i + \frac{\eta}{km}, \tag{B 3}$$

where $\rho_i := \log_2 n_i/km$.

Consider the calculation of the distortion in (B 2) for the mean squared error distortion measure (i.e. $r = 2$). Now, (B 2) without the expectation can be rewritten as

$$\frac{1}{m}\min_j \sum_{i=1}^{m}\left(\min_{\mathbf{y}\in C^{i,j}} ||\mathbf{x}-\mathbf{y}||^2\right), \tag{B 4}$$

for a given input $\mathbf{x} \in \mathbb{R}^k$. We first expand $||\mathbf{x}-\mathbf{y}||^2$ to obtain

$$||\mathbf{x}-\mathbf{y}||^2 = \mathbf{x}^T\mathbf{x} - 2\left(\mathbf{x}^T\mathbf{y} - \frac{\mathbf{y}^T\mathbf{y}}{2}\right). \tag{B 5}$$

In (B 5), the $\mathbf{x}^T\mathbf{x}$ term is independent of the codewords to be compared, while the $\gamma := \mathbf{y}^T\mathbf{y}/2$ term is given by the codebook. Therefore, for an input $\mathbf{x}$, we need to calculate $\mathbf{x}^T\mathbf{y}$ and add γ that was previously obtained for a given C^i. Note that this set of operations requires k additions and k multiplications. We then find the codeword index that minimizes $\mathbf{x}^T\mathbf{y} + \gamma$. In order to compare two constants, 1 addition (subtraction) is required. Hence, $(k+1)n_i$ additions and kn_i multiplications are required for finding the codeword index in the term enclosed by parentheses in (B 4).

Since for each value of i in the sum given by (B 4), we need $(k+1)n_i$ additions and kn_i multiplications, we need a total of $(k+1)\sum_{i=1}^{m} n_i$ additions and $k\sum_{i=1}^{m} n_i$ multiplications to calculate the m-codebook sample distance in the term enclosed by brackets in (B 4).

Now we focus on searching through the index j in (B 4). We have 2^η different code sequences. Thus, we require $(k+1)\sum_{i=1}^{m} n_i \cdot 2^\eta$ additions, and $(k+1)\sum_{i=1}^{m} n_i \cdot 2^\eta$ multiplications are required to calculate the distances, and 2^η additions are required for the comparison of 2^η distances. In summary, in order to perform the SACS encoding, we need a total of

$$\frac{2^\eta}{m}\sum_{i=1}^{m} n_i + \frac{2^\eta}{k}\left(1+\frac{1}{m}\right) \tag{B 6}$$

additions per element and

$$\frac{2^\eta}{m}\sum_{i=1}^{m} n_i \tag{B 7}$$

multiplications per element are required.

We will now provide closed form expressions (without summations) for two special cases: the first one is the unifbrm codebook size case described in the paper, and the second one is for scalar quantization with either non-uniform or uniform codebook size.

In the uniform codebook size case, substituting $n_i = 2^{km\rho_i}$ into (B 6) and (B 7) and then the minima of (B 6) and (B 7) are obtained when ρ_i is constant for all i, i.e., n_i is constant under the constraints of fixed $\rho = \sum_{i=1}^{m} \rho_i$ and η. Hence, the required additions and multiplications for SACS with uniform codebook size for a given bit rate $R = \rho + \eta/km$ and η are

$$\left(1+\frac{1}{k}\right)2^{Rk+\eta-\eta/m} + \frac{2^\eta}{km} \quad \text{and} \quad 2^{Rk+\eta-\eta/m} \tag{B 8}$$

respectively. Consequently, as R increases, the encoding complexity of SACS is $\mathcal{O}(2^{Rk})$. However, for a comparablle performance of $k\kappa$-dimensional VQ at the same bit rate R, where $1 < \kappa < m$, the encoding complexity is $\mathcal{O}(2^{Rk\kappa})$ [28]. Note that in terms of complexity, the difference between m-SACS and 1-SACS is the required memory for the codebooks, i.e., m-SACS requires $m2^\eta$ codebooks but, 1-SACS requires only 2^η codebooks.

For the special case of scalar quantization (i.e., $k = 1$), the codeword searching process can be conducted based on the *tree-structured search.* Hence, the required number of multiplications are reduced to

$$\frac{2^\eta}{m}\sum_{i=1}^{m}\log_2(2^{m\rho_i}) = 2^\eta\rho = 2^\eta(R-\eta/m). \tag{B 9}$$

In other words, the encoding complexity of SACS is $\mathcal{O}(2^\eta R)$.

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