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Malliavin Calculus in the Canonical Levy Process: White Noise Theory and Financial Applications.

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MALLIAVIN CALCULUS IN THE CANONICAL LÉVY PROCESS: WHITE NOISE THEORY AND FINANCIAL APPLICATIONS

A Dissertation

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"Stay hungry, stay foolish!"
Steve Jobs (1955-2011)

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TABLE OF CONTENTS

Page

ABSTRACT

Navarro, Rolando, Jr. D. PhD, Purdue University, December 2015. Malliavin Calculus in the Canonical Lévy Process: White Noise Theory and Financial Applications. Major Professor: Frederi G. Viens.

We constructed a white noise theory for the Canonical Lévy process by Solé, Utzet, and Vives. The construction is based on the alternative construction of the chaos expansion of square integrable random variable. Then, we showed a Clark-Ocone theorem in $L^2(P)$ and under the change of measure. The result from the Clark-Ocone theorem was used for the mean-variance hedging problem and applied it to stochastic volatility models such as the Barndorff-Nielsen and Shepard model model and the Bates model. A Donsker Delta approach is employed on a Binary option to solve the mean-variance hedging problem. Finally, we are able to derive the Delta and Gamma for a barrier and lookback options for an exp-Lévy process using the methodology of Bernis, Gobet, and Kohatsu-Higa by employing a dominating process.

1. INTRODUCTION

1.1 Motivation

Financial modeling of risky assets is assumed to follow the classical Black-Scholes-Merton model where the log-returns risky asset follows a normal distribution. However, stylized facts suggests that the Black-Scholes-Merton model is inadequate. There is a growing interest that suggests that financial modeling under a Lévy process is better suited in capturing market behavior. This includes skewness and long-tailed distribution of the asset returns, presence of jumps, and implied volatility smile [21], [74].

The classical Canonical space for a Lévy process is constucted from the σ-field of cylinder sets and a probaility measure using the Kolmogorov extension theorem [73], [7]. However, Solé, Utzet and Vives [77] has formulated another construction of the Canonical space for the Lévy process to be able to obtain an interpretation the Malliavin derivative for the Lévy process $D_{t,z}$. The derivative $D_{t,0}$ is associated with the Malliavin derivative with respect to the Wiener process while $D_{t,z}$, $z \neq 0$ is the Malliavin derivative with respect to the pure jump process that has a form of an increment quotient. We shall refer to the *Canonical Lévy process* to the Canonical space constructed by Solé, Utzet, and Vives. [77].

White noise theory was first introduced by Hida for Wiener process which has origins in quantum physics [45]. Subsequently, white noise theory was extended in the pure jump Lévy process [1], [64], [24]. This was done by incorporating generalized function spaces related to $L^2(P)$ in a natural way [46]. This includes the dual spaces $(\mathcal{G}, \mathcal{G}^*)$ and the Hida dual spaces $((\mathcal{S}), (\mathcal{S})^*)$ with the following inclusions: $(\mathcal{S}) \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (\mathcal{S})^*$ [27]. We extend this theory for the Canonical Lévy space by first deriving an alternative chaos expansion of square integrable random variable and give some important characterizations such as the Wick-Skorohod identity, then prove the Clark-Ocone theorem for $L^2(P)$.

The Clark-Ocone theorem is the explicit representation of the Itô representation theorem in terms of the Malliavin derivative. The univariate version of the Clark-Ocone theorem in $\mathbb{D}^{1,2}$ for the Canonical Lévy process can be stated as follows:

Theorem 1.1.1 *[78] Let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_T$-measurable, then*

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t^-}]M(dt,dz) \tag{1.1}$$

where M is independent measure given by (2.53).

The Clark-Ocone representation can be weakened to a representation for $F \in L^2(P)$ using white noise analysis with the same form (1.1). However, the Malliavin derivative $D_{t,z}$ and the expectation E will be generalized to a *stochastic gradient* and *generalized expectation* respectively. An example of a contingent claim F that is not in $\mathbb{D}^{1,2}$ but belong to $L^2(P)$ is a binary option. We will evaluate the generalized conditional expectation $E[D_{t,z}F|\mathbb{F}_{t^-}]$ using the Donsker Delta of an Itô-Lévy process [26].

Under the change of equivalent measure $Q \sim P$, Ocone [63] and Huenhe [49], we were able to derive the Clark-Ocone theorem under the change in measure under $\mathbb{D}^{1,2}$ for the Wiener and Pure Jump Lévy processes. Suzuki has further extended this representation for the Canonical Lévy processes [80].

Using white noise theory, the Clark-Ocone theorem under the change of measure was proven by Okur in the Wiener case [66], pure-jump Lévy case [67], and the combination of Wiener and pure jump Lévy case [67]. Let $u(t)$ and $\theta(t,z)$ be the drift terms for the Wiener process $W(t)$ and pure jump process $\tilde{N}(dt,dz)$ such that $dW^Q = dW(t) + u(t)dt$ is a Q-Brownian motion and $\tilde{N}^Q(dt,dz) = \tilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt$ is a Q-compensated Poisson random measure. The Lévy process is in general an incomplete model. Hence, the Q measure is not unique. Nevertheless, there are some ways of finding drift parameters to obtain a unique equivalent measure Q by some selection criterion such as the Föllmer-Schweizer minimal measure and the minimal martingale measure [7].

Okur [67] assumed that $u(t)$ and $\theta(t, z)$ is either deterministic or driven by Brownian and compensated Poisson random measure respectively. However, this model is in general not adequate to obtain a Clark-Ocone theorem for stochastic volatility models. Hence, we will generalize the drift vectors $u(t)$ and $\theta(t, z)$ to be driven possibly by multivariate independent Wiener and Poisson noise sources. One example is the BNS model with drift under the minimal martingale measure. In this model, the drift parameter $u(t)$ is driven by a compensated Poisson random measure. Another example is the Bates model which is driven by another independent Wiener process.

As an application to financial modeling in Lévy processes, following the methodology of Benth, et al., [15], the hedging portfolio by minimizing the quadratic hedging error under the martingale measure can be expressed in terms of the representation of the Clark-Ocone theorem.

Another financial application of Malliavin calculus considered in the study is the evaluation of the sensitivities or so-called Greeks for exotic options under the exp-Lévy process. The Greeks are used in risk-management to hedge against changes in the parameters on the option price. For a Lévy process, a closed form of Greeks is in general not available. However, there are numerical methods in evaluating Greeks such as finite-difference, likelihood ratio, and pathwise approach [34].

Greeks using Malliavin calculus was first derived by Fournie, et al., [31]. One advantage of using the Malliavin calculus approach is it doesn't require the density function in contrast to the likelihood ratio approach. Moreover, Bernis, Gobet, and Kohatsu-Higa was able to extend Fournie's result for a class of exotic options which includes barrier and lookback options using a dominating process [16], [38] for a discrete and continuous monitoring case. We extend their result for an exp-Lévy process and find a suitable dominating process for discrete and continuous monitoring case.

1.2 Overview of the Dissertation

The dissertation is organized as follows.

Chapter 2 presents a background review of the stochastic calculus of Lévy process, then we discuss the Malliavin calculus for the Canonical Lévy processes.

We present the white noise theory for Canonical Lévy process in Chapter 3. First, we present the construction of the Canonical Lévy white noise process. Then, we show the alternative chaos expansion of a square-integrable random variable under the Canonical Lévy processes and introduce the white noise Lévy process and Lévy white noise field. From this framework, we extend the white noise theory for a Canonical Lévy process to prove a Clark-Ocone theorem for $L^2(P)$. Finally, we shall present the multivariate extensions.

For readers interested in the financial applications of the Canonical Lévy process, readers can proceed immediately to Chapter 3.11 for an overview of important definitions and characterizations of the white noise theory extended on the multivariate version on the first reading. Likewise, for those who are interested in the characterization of the white noise theory for the Canonical Lévy processes, we invite the reader to explore Chapter 3 on its entirety.

We derive a Clark-Ocone formula under the change of equivalent measure $Q \sim P$ in Chapter 4. Then, we shall present an application to mean-variance hedging portfolio under the martingale measure Q. Specific applications are presented for geometric Lévy processes and for the stochastic volatility model such as the BNS model, and the Bates model (Heston volatility with jumps). We present the Donsker Delta approach in Chapter 5 in evaluating the generalized conditional expectation $E[D_{t,z}F|\mathbb{F}_{t^-}]$ and apply the technique for the binary option.

In Chapter 6, we derive the Delta and Gamma for a barrier and lookback options for an exp-Lévy process using the methodology of Bernis, Gobet, and Kohatsu-Higa by employing suitable dominating processes.

1.3 Main Results

We briefly discuss the main results and contributions of this dissertation.

Chapter 3 - *Canonical Lévy White Noise Processes*

- We have shown the alternative chaos expansion for $F \in L^2(P)$ the Canonical Lévy process in Proposition 3.3.4. The proof of the the chaos expansion uses the chaotic representation property in Theorem 3.2.4 by Nualart and Schoutens [61]. From the results of Solé, Utzet, and Vives in Theorems 3.3.1-3.3.3, [78], we are able to construct the alternative chaos expansion Canonical Lévy process. In addition, we have shown the isometry relation in Theorem 3.3.1 for this chaos expansion.

  This alternative chaos expansion for the Canonical Lévy process is new. From this expansion, we characterize the white noise theory using some family of fuction spaces of stochastic test functions and distribution functions. This characterization is an extension of the Wiener case [44] and the Poisson case [64], [27].

- Let $X(t)$ be the square-integrable Lévy process given by (2.49). Then, in Chapter 3.5 we introduce the white noise Lévy process $\dot{X}(t)$ and show that $\dot{X}(t)$ is the derivative of $X(t)$ in $(\mathcal{S})^*$. Also, we introduce the Lévy white noise field $\dot{M}(t,x)$ and we show the Radon-Nikodym derivative relation $M(dt,dx) = \dot{M}(t,x)\mu(dt,dx)$ in $(\mathcal{S})^*$ in (3.163).

- The concepts of white noise theory to the Canonical Lévy space is presented in Chapter 3.4 - 3.11. Moreover, these concepts has a parallel analog Wiener and Poisson cases [27], [24], [66], [67].

  - Closability of the stochastic derivative $D_{t,z}$ (Theorem 3.7.1).
  - $F \in L^2(P)$ implies $D_{t,z}F \in \mathcal{G}^*$ (Theorem 3.7.2)
  - Fundamental Theorem of stochastic calculus in $\mathcal{G}^*$ (Theorem 3.9.1)
  - Wick-Skorohod identity (Theorem 3.9.5).

  - Clark-Ocone theorem for Wick polynomials (Theorem 3.10.3) and $L^2(P)$ (Theorem 3.10.5)

- We give a multivariate extension to the white theory for Canonical Lévy space (Chapter 3.11).

Chapter 4 - *Clark-Ocone Theorem Under The Change of Measure and Mean-Variance Hedging*

- We show Clark-Ocone theorem under the change of measure (Theorem 4.2.2) for $F \in L^2(P) \cap L^2(Q)$ is $\mathcal{F}_T$-measurable and $FZ(T) \in L^2(P)$.

- We show the mean-variance hedging portfolio with partial information under the martingale measure (Theorem 4.3.1). Furthermore, we give some specific examples of finding the mean-variance hedging portfolio in the following models:

  - Geomteric Lévy Processes (Chapter 4.3.3)
  - BNS model (Chapter 4.3.4)
  - Bates model (Chapter 4.3.5)

Chapter 5 - *Donsker Delta And Its Application to Finance*

- The generalized conditional expectation $E[D_{t,z}F|\mathbb{F}_{t^-}]$ is evaluated using the Donsker Delta approach (Chapter 5.2).

- From this result, we give an example of evaluating the mean-variance hedging porfolio for a binary option under the Merton model (Chapter 5.3.1) and the continuous case (Chapter 5.3.2).

Chapter 6 - *Evaluating Greeks In Exotic Options*

- We derive the Delta (Theorem 6.5.1) and Gamma (Theorem 6.6.1) for a barrier and lookback options for an exp-Lévy process using the methodology of Bernis, Gobet, and Kohatsu-Higa. A suitable dominating process was constructed for continuous and discrete monitoring case (Chapter 6.7).

2. PRELIMINARIES

2.1 Lévy Processes

We present some background on Lévy processes [7], [27], [73], [78]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A Lévy process $X = \{X(t) : t \geq 0\}$ is a stochastic process with the following properties:

1. $X(0) = 0, \quad P - a.s.,$
2. $X(t)$ has independent increments,
3. $X(t)$ has stationary increments,
4. $X(t)$ is stochastically continuous, that is, $X(s) \xrightarrow{p} X(t)$, as $s \to t$.

The Poisson random measure also known as the jump measure $N : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{N}_0$ is a counting measure defined as

$$N(A) = \sum_{s \in (0,t]} \mathbf{1}_{\{s:(s,\Delta X(s)) \in A\}}, \quad A \in \mathfrak{B}([0, T] \times \mathbb{R}_0) \tag{2.1}$$

where $\mathbb{R}_0 = \mathbb{R} \backslash 0$ and $\Delta X(t) = X(t) - X(t^-)$ is the jump of X at time t. The Lévy measure ν of X is defined as the expectation of N as follows:

$$\nu(B) = E[N((0, 1] \times B)] = E\left[\sum_{s \in (0,1]} \mathbf{1}_{\{s:\Delta X(s) \in B\}}\right], \quad B \in \mathfrak{B}(\mathbb{R}_0). \tag{2.2}$$

The Lévy measure is a σ-finite measure and satisfies

$$\int_{\mathbb{R}_0} (1 \wedge z^2)\nu(dz) < \infty. \tag{2.3}$$

The compensated Poisson random measure also known as the compensated jump measure $\tilde{N} : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ is given by

$$\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz). \tag{2.4}$$

The Lévy process $X(t)$ has integral representation known as the Lévy-Itô decomposition theorem.

Theorem 2.1.1 *Lévy-Itô decomposition theorem*
Let $X(t) \in \mathbb{R}$ be a Lévy process, then there exists a triplet (a, σ^2, ν) such that for all $t \geq 0$

$$X(t) = at + \sigma W(t) + \int_{[0,t]\times\{|z|\geq 1\}} zN(ds, dz) + \int_{[0,t]\times\{|z|<1\}} z\tilde{N}(ds, dz). \tag{2.5}$$

The triplet (a, σ^2, ν) is known as the Lévy triplet or the characteristic triplet. Likewise, we can write the Lévy process representation as follows:

$$X(t) = bt + \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(ds, dz) \tag{2.6}$$

where

$$b = a + \int_{|z|\geq 1} z\nu(dz). \tag{2.7}$$

The characteristic function of the Lévy process is given by the Lévy Khintchine formula [27].

Theorem 2.1.2 *Lévy Khintchine formula*
Let $X(t) \in \mathbb{R}$ be a Lévy process in law then a necessary and sufficient condition that its characteristic function is given as

$$E[\exp(iuX(t))] = \exp(\Psi(u)t) \tag{2.8}$$

where $\Psi(u)$ is the characteristic exponent given by

$$\Psi(u) = iau - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}_0} (\exp(iuz) - 1 - iuz\mathbf{1}_{\{|z|<1\}})\nu(dz) \tag{2.9}$$

where $\alpha \in \mathbb{R}$, $\sigma^2 \leq 0$ are constants and $\nu = \nu(dz)$, $z \in \mathbb{R}_0$ is σ-finite measure in $\mathfrak{B}(\mathbb{R}_0)$ satisfying

$$\int_{\mathbb{R}_0} (1 \wedge z^2)\nu(dz) < \infty. \tag{2.10}$$

From the Lévy-Itô representation theorem, it is natural to consider an Itô-Lévy process of the form of

$$X(t) = x + \int_0^t \alpha(s) + \int_0^t \beta(s)dW(s) + \int_{[0,t]\times\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz). \tag{2.11}$$

In short-hand SDE form, we have the following:

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz), \quad X(0) = x. \tag{2.12}$$

If the coefficients $\alpha(s)$, β, and $\gamma(s,x) > -1$ are predictable for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}_0$ such that

$$\int_{\mathbb{R}} \left(|\mu(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\nu(dz) \right) ds < \infty, \qquad P \quad a.s. \tag{2.13}$$

Then the stochastic integrals in (2.11) are local martingale. Furthermore, if we impose the following square-integrablilty condition

$$E\left[\int_{\mathbb{R}} \left(|\mu(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\nu(dz) \right) ds \right] < \infty. \tag{2.14}$$

Then the stochastic integrals in (2.11) are martingales. Now, we present Itô's lemma for Itô-Lévy proceses .

Theorem 2.1.3 *[27] Let $X(t)$ be an Itô-Lévy process given by* (2.12) *and let $F \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ and define $Y(t) = F(t, X(t))$, then $Y(t)$is a Itô-Lêvy process with SDE*

$$\begin{aligned} &dY(t) \\ &= \frac{\partial}{\partial t} F(t,X(t))dt + \frac{\partial}{\partial x} F(t,X(t)) \left(\alpha(t)dt + \beta(t)dW(t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2} F(t,X(t))\beta^2(t)dt \\ &+ \int_{\mathbb{R}_0} \left[F(t,X(t) + \gamma(t,z) - F(t,X(t)) - \frac{\partial}{\partial x} F(t,X(t))\gamma(t,z) \right] \nu(dz)dt \\ &+ \int_{\mathbb{R}_0} \left[F(t,X(t) + \gamma(t,z) - F(t,X(t)^-) \right] \tilde{N}(dt,dz). \end{aligned} \tag{2.15}$$

Extending the Itô-Lêvy in the multidimensional case, $X(t) = (X_1(t), \cdots, X_n(t))^T$, we have following form

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(ds,dz), \quad X(0) = x. \tag{2.16}$$

where $\alpha(t) \in \mathbb{R}^n$, $\beta(t) \in \mathbb{R}^{n\times d}$, and $\gamma(t,z) \in \mathbb{R}^{n\times l}$ are predicable processes, $W(t) = (W_1(t), \cdots, W_d(t))^T$ is a vector of d-dimensional independent Wiener process and $\tilde{N}(dt,dz) = \left(\tilde{N}_1(dt,dz_1), \cdots, \tilde{N}_1(dt,dz_1)\right)^T$ is a vector of l-dimensional independent compensated Poisson random measures. That is, the SDE for $X_i(t)$ is given as follows:

$$\begin{aligned} dX_i(t) =& \alpha_i(t) + \sum_{j=1}^{d} \beta_{ij}(t) dW_j(t) + \int_{\mathbb{R}_0} \gamma_{ij}(t,z_j) \tilde{N}_j(dt,dz_j) \\ X_i(0) =& x_i, \quad i \in \{1, \cdots, n\}. \end{aligned} \tag{2.17}$$

Then, we have the following Itô's lemma for the multidimensional case.

Theorem 2.1.4 *[27] Let $X(t)$ be a -Itô-Lévy process given by* (2.17) *and let $F \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ and define $Y(t) = F(t, X(t))$, then $Y(t)$ is a Itô-Lêvy process with SDE*

$$\begin{aligned} & dY(t) \\ =& \frac{\partial}{\partial t} F(t,X(t)) dt + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(t,X(t)) \alpha_i(t) dt \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{d} \frac{\partial}{\partial x_i} \beta_{ij}(t) dW_j(t) + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_i} F(t,X(t)) (\beta\beta^T)_{ij}(t) dt \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}_0} \left(F(t, X(t) + \gamma^j(t,z)) - F(t,X(t)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(t,X(t)) \gamma_{ij}(t,z_j) \right) \nu(dz_j) dt \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}_0} \left(F(t, X(t) + \gamma^j(t,z)) - F(t, X(t^-)) \right) \tilde{N}_j(dt, dz_j). \end{aligned} \tag{2.18}$$

where γ^j is the j^{th} column of γ.

The following theorem states criterion for a Lévy process concerning to the variation process and moments.

Theorem 2.1.5 *[73] Let $X = \{X(t)\}_{t\geq 0}$ with characteristic triplet (a, σ^2, ν).*

(i) X has a finite variation process iff

$$\sigma = 0, \qquad \int_{|z|<1} |z| \nu(dz) < \infty. \tag{2.19}$$

(ii) X has a finite n^{th} absolute moment, where $n \in \mathbb{N}$, that is,

$$E[|X(t)|^n] < \infty, \quad \forall t > 0 \Leftrightarrow \int_{|z|\geq 1} |z|\nu(dz). \tag{2.20}$$

(ii) X has a finite exponential moment $E[e^{uX(t)}]$ where $u \in \mathbb{R}$ and $\forall t > 0$ iff

$$\int_{|z|\geq 1} e^{uz}\nu(dz) < \infty. \tag{2.21}$$

In this case,

$$E[e^{uX(t)}] = e^{t\Psi(-iu)}. \tag{2.22}$$

2.2 Moment Inequalities

We introduce some moment inequalities that will be useful in finding upper bound of moments for both continuous and pure jump case [7], [53]. Let F be a square-integrable, adapted process. Denote the following Wiener integral as follows:

$$M(t) = \int_0^t F(s)dW(s). \tag{2.23}$$

Then M is a square-intergrable martingale. From the Burkholder's inequality, followed by Doob's martingale inequality, then for any $p \geq 2$, there exists $C_p > 0$ such that

$$E\left[\sup_{s\in[0,t]} |M(s)|^p\right] \leq C_p E\left[M, M\right]_t^{p/2}. \tag{2.24}$$

On the other hand, let H be a predictable process and denote the following compensated Poisson integral as follows:

$$I(t) = \int_{[0,t]\times A} H(s,z)\tilde{N}(ds,dz) \tag{2.25}$$

where $A \in \mathfrak{B}(\mathbb{R}_0)$. Then for $p \geq 2$, there exists $D_p > 0$ such that

$$\begin{aligned} &E\left[\sup_{s\in[0,t]} |I(t)|^p\right] \\ \leq& D_p\left(E\left[\left(\int_{[0,t]\times A} |H(s,z)|^2\nu(dz)ds\right)^{p/2}\right] + E\left[\int_{[0,t]\times A} |H(s,z)|^p\nu(dz)ds\right]\right). \end{aligned} \tag{2.26}$$

2.3 Geometric Lévy Processes

We let $S_s, s \in [0, T]$ be a risky-asset (i.e. stock) price process modeled as geometric Lévy Process of the form

$$\begin{aligned} dS(s) =& S(s)\left(\mu(s)ds + \sigma(s)dW(s) + \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}(ds,dz)\right), \quad s \in [t,T], \\ S(t) =& z. \end{aligned} \tag{2.27}$$

We denote

$$Y(s) = \log S(s), \quad s \in [0,T] \tag{2.28}$$

to be the log-returns. Then, from Itô's lemma by taking $f(t,x) = \log x$, we obtain, a Itô-Lévy process of the form of

$$\begin{aligned} dY(s) =& \alpha(s)ds + \beta(s)dW(s) + \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz), \quad s \in [t,T], \\ Y(t) =& y \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} \alpha(s) =& \left(\mu(s) - \frac{\sigma^2(s)}{2}\right) + \int_{\mathbb{R}_0} [\log(1+\theta(s,z)) - \theta(s,z)]\nu(dz)ds, \\ \beta(s) =& \sigma(s), \\ \gamma(s,z) =& \log(1+\theta(s,z)), \\ y =& \log x \end{aligned} \tag{2.30}$$

where y is a constant, $\alpha(s)$, $\beta(s)$, and $\gamma(s,z) > -1$ are deterministic for all $(s,z) \in [0,T] \times \mathbb{R}_0$ such that

$$\int_0^T \left[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\right] ds < \infty. \tag{2.31}$$

Then we have the following conditional characteristic function as stated in the following lemma.

Lemma 2.3.1

$$E[\exp(iuY(T))|\mathcal{F}_t] = \exp\left(iuY(t) + \int_t^T \left(\alpha(s) - \frac{u^2\beta(s)}{2}\right) ds \right.$$
$$\left. + \int_{[t,T]\times\mathbb{R}_0} (\exp(iu\gamma(s,z)) - 1 - iu\gamma(s,z))\, \nu(dz)ds\right) \tag{2.32}$$

Proof We let

$$F = F(s,x) = \exp(iuY(s)). \tag{2.33}$$

Then, from Itô's lemma for $F_s = F(s, Y(s))$

$$dF_s = F_s\left(a(s)ds + b(s)dW(s) + \int_{\mathbb{R}_0} c(s,z)\tilde{N}(ds,dz)\right) \tag{2.34}$$

where

$$\begin{aligned} a(s) &= \left(\alpha(s) - \frac{u^2\beta(s)}{2}\right) + \int_{\mathbb{R}_0} (\exp(iu\gamma(s,z)) - 1 - iu\gamma(s,z))\, \nu(dz), \\ b(s) &= iu\beta(s), \\ c(s,z) &= \left(e^{iu(s,z)} - 1\right). \end{aligned} \tag{2.35}$$

Integrating (2.34) we get

$$F_T = F_t + \int_t^T a(s)F_s ds + \int_t^T b(s)F_s dW(s) + \int_{[t,T]\times\mathbb{R}_0} c(s,z)\tilde{N}(ds,dz). \tag{2.36}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ gives us

$$E[F_T|\mathcal{F}_t] = F_t + \int_t^T a(s)E[F_s|\mathcal{F}_t]ds. \tag{2.37}$$

We let $m(s) = E[F_s|\mathcal{F}_t]$, then by differentiating the above equation, we obtain the following ODE

$$\begin{aligned} dm(s) &= a(s)m(s), \qquad s \in [t,T], \\ m(t) &= F_t \end{aligned} \tag{2.38}$$

Solving the ODE gives us

$$m(s) = F_t \exp\left(\int_t^s a(u)du\right). \tag{2.39}$$

Hence, we finally obtain the desired result.

■

2.4 Stochastic Differential Equations

We present the conditions for the existence of the strong solutions for Lévy processes namely: Lipschitz and growth conditions.

Let X be a cadlag semimartingale with the following SDE

$$dX(t) = \alpha(t, X(t))dt + \beta(t, X(t))dW(t) + \int_{\mathbb{R}_0} \gamma(t, X(t), z)\tilde{N}(ds, dz) \tag{2.40}$$

where $\alpha, \beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly measurable and $\mathcal{F}_t$-adapted, $\gamma : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ be jointly measurable and $\mathcal{F}_t$ predictable. We say that the SDE in (2.40) has a strong solution if its $X(t)$ pathwise unique $\mathcal{F}_t$ adapted solution.

To ensure a strong solution, the following conditions should be satisfied [76]:

(i) Growth conditions :

$$\begin{aligned} |\alpha(t,x)| &\leq c(t)(1+|x|), \\ |\beta(t,x)|^2 + \int_{\mathbb{R}_0} |\gamma(t,x,z)|^2 \nu(dz) &\leq c(t)(1+|x|^2) \end{aligned} \tag{2.41}$$

where $c(t) \geq 0$ is some deterministic function such that

$$C(T) \equiv \int_0^T c(t)dt < \infty \qquad \forall T > 0, \tag{2.42}$$

(ii) Lipschitz conditions:

$$\begin{aligned} |\alpha(t,x) - \alpha(t,y)| &\leq c(t)|x-y|, \\ |\beta(t,x) - \beta(t,y)|^2 + \int_{\mathbb{R}_0} |\gamma(t,x,z) - \gamma(t,y,z)|^2 \nu(dx) \leq K_1|x-y|^2 &\leq c(t)|x-y|^2 \end{aligned} \tag{2.43}$$

(iii) Initial conditions:

$$X(0) \in \mathcal{F}_0, \qquad E[X^2(0)] < \infty. \tag{2.44}$$

The existence of the strong solution implies that

$$E\left[\sup_{t\in[0,T]} X^2(t)\right] \leq k(T) < \infty \tag{2.45}$$

where $k(T)$ depends on T and $C(T)$ only.

2.5 Canonical Lévy Space

The usual Canonical Lévy space is constructed from the set of cadlag functions with the σ-field generated by the cylinders and with the measure given by the Kolmogorov extension theorem [73]. The alternative construction of the Canonical Lévy space by Solé, Utzet, and Vives [77] was constructed to provide a probabilistic interpretation of the Malliavin derivative $D_{t,x}$. In their construction, the gradient operator becomes the sum of a derivative and increment quotient operators [5].

Consider the Canonical Lévy Process

$$(\Omega, \mathcal{F}, P) = (\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, P_W \otimes P_J) \tag{2.46}$$

where $(\Omega_W, \mathcal{F}_W, P_W)$ is the Canonical Wiener space and $(\Omega_J, \mathcal{F}_J, P_J)$ is the Canonical jump Lévy space. If $X(t)$ is a Lévy Process with triplet (a, σ^2, ν). From the Lévy-Itô decomposition [27], $X(t)$ can be expressed as follows:

$$X(t) = bt + \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(ds, dz) \tag{2.47}$$

where

$$b = a + \int_{|z|\geq 1} z\nu(dz). \tag{2.48}$$

Let $X(t)$ be a centered, square-integrable Lévy process, then $X(t)$ can be written as follows:

$$X(t) = \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(ds, dz). \tag{2.49}$$

Its characteristic function is given by

$$E\left(\exp(iuX(t))\right] = \exp\left[\left(-\frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}_0} (\exp(iuz) - 1 - iuz)\nu(dz)\right) t\right]. \tag{2.50}$$

From the the moment theorem [7], for $p \geq 1$,

$$E[|X(t)|^p] < \infty \Leftrightarrow \int_{|z|\geq 1} |z|^p \nu(dz) < \infty. \tag{2.51}$$

Hence, from the square-integrable assumption of $X(t)$, (2.51) and (2.3) implies

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty. \tag{2.52}$$

Itô [50] has extended the centered square-integrable Lévy process X to an independent measure M on $(\mathbb{R}_+ \times \mathbb{R}, \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R}))$ can be constructed as follows

$$M(E) = \sigma \int_{E_0} dW(t) + \int_{E'} z d\tilde{N}(dt, dz) \tag{2.53}$$

where $E \in \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R})$, $E_0 = \{t \in \mathbb{R}_+ : (t, 0) \in E\}$ and $E' = E \setminus E_0$. Then for $E_1, E_2 \in \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mu(E_1) < \infty, \mu(E_2) < \infty$

$$E[M(E_1)M(E_2)] = \mu(E_1 \cap E_2) \tag{2.54}$$

where μ is a measure on $([0, T] \times \mathbb{R}, \mathfrak{B}([0, T] \times \mathbb{R})$ and

$$\mu(E) = \sigma^2 \int_{E_0} dt + \int_{E'} z^2 d\nu(z) dt, \qquad E \in \mathfrak{B}([0, T] \times \mathbb{R}). \tag{2.55}$$

In differential form, we have

$$\mu(dt, dz) = \sigma^2 d\delta_0(z) dt + z^2(1 - \delta_0(z)) d\nu(z) dt = \lambda(dt)\eta(dz) \tag{2.56}$$

where $\lambda(dt) = dt$ is the Lebesgue measure and

$$\eta(dz) = \sigma^2 d\delta_0(z) dt + z^2(1 - \delta_0(x)) d\nu(z). \tag{2.57}$$

2.6 Iterated Lévy-Itô Integral

Let $f \in L^2(\mu^n) = L^2((\lambda \times \eta)^n) = L^2(([0, T] \times \mathbb{R})^n)$ be a deterministic function such that

$$||f||^2_{L^2(\mu^n)} = \int_{[0,T]\times\mathbb{R}} \cdots \int_{[0,T]\times\mathbb{R}} |f\left((t_1, z_1) \cdots (t_n, z_n)\right)|^2 \mu(dt_1, dz_1) \cdots \mu(dt_n, dz_n) < \infty. \tag{2.58}$$

The symmetrization of f denoted by $f^\wedge$ over the pairs $(t_1, x_1), \cdots, (t_n, x_n)$ is given by

$$f^\wedge((t_1, z_1), \cdots (t_n, z_n)) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f((t_{\sigma(1)}, z_{\sigma(1)}), \cdots (t_{\sigma(n)}, z_{\sigma(n)}) \tag{2.59}$$

where $\sigma = (\sigma(1), \cdots, \sigma(n))$ is a permutation of $\{1, \cdots, n\}$ and $\mathfrak{S}_n$ is the set of permutations of $\{1, \cdots, n\}$.

Denote $S_n = \{(t_1, z_1), \cdots (t_n, z_n) : 0 < t_1, \cdots t_n < T, x_i \in \mathbb{R}, i \in \{1, \cdots n\}\}$. For $f \in L^2(\mu^n)$ define the n-fold iterated integral by over S_n as follows:

$$J_n(f) \equiv \int_{[0,T]\times\mathbb{R}} \int_{[0,t_n^-)\times\mathbb{R}} \cdots \int_{[0,t_2^-)\times\mathbb{R}} f\left((t_1, z_1)\cdots(t_n, z_n)\right) \\ M(dt_1, dz_1)\cdots M(dt_{n-1}, dz_{n-1})M(dt_n, dz_n). \tag{2.60}$$

Also, for $f \in L^2(\mu^n)$, define the n-fold iterated integral over $([0,T]\times\mathbb{R})^n$ as follows:

$$I_n(f) \equiv \int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}} \cdots \int_{[0,T]\times\mathbb{R}} f\left((t_1, z_1)\cdots(t_n, z_n)\right) \\ M(dt_1, dz_1)\cdots M(dt_{n-1}, dz_{n-1})M(dt_n, dz_n). \tag{2.61}$$

Denote $L_s^2(\mu^n)$ be the subspace of symmetric functions in $L^2(\mu^n)$. Then, for $f \in L_s^2(\mu^n)$, we have the following identity:

$$I_n(f) = n! J_n(f). \tag{2.62}$$

The integrated integral I_n has the following properties [78]:

1. Symmetrization

$$I_n(f) = I_n(f^\wedge), \qquad f \in L^2(\mu^n), \tag{2.63}$$

2. Linearity

$$I_n(af + bg) = aI_n(f) + bI_n(g), \qquad f, g \in L^2(\mu^n), \quad a, b \in \mathbb{R}, \tag{2.64}$$

3. Isometry

$$E[I_n(f)I_m(g)] = n!\, \langle f^\wedge, g^\wedge\rangle_{L^2(\mu^n)}\, \delta_{mn}, \qquad f \in L^2(\mu^n), \quad g \in L^2(\mu^m). \tag{2.65}$$

Itô has has shown the following chaos expansion for the Lévy space.

Theorem 2.6.1 *[50] Let $F \in L^2(P)$, then F has chaos expansion given by*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.66}$$

where we set $I_0(f_0) = E[F]$. The chaos expansion is unique if $f_n \in L_s^2(\mu^n)$ for all $n \in \mathbb{N}$. Furthermore, we have the following isometry relation:

$$||F||^2_{L^2(P)} = \sum_{n=0}^{\infty} n!||f_n||^2_{L^2(\mu^n)}, \qquad f_n \in L_s^2(\mu^n). \tag{2.67}$$

2.7 Skorohod Integral

Definition 2.7.1 *[77], [78] Let $F \in L^2(P \times \mu)$ with chaos expansion of the form of*

$$F(t,z) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,(t,z))) \tag{2.68}$$

such that

$$\sum_{n=0}^{\infty}(n+1)!||\tilde{f}||^2_{L^2(\mu^{n+1})} < \infty \tag{2.69}$$

where $\tilde{f}_n \in L^2_s(\mu^{n+1})$. Then we define the Skorohod integral of F with respect to M as follows:

$$\delta(F) = \int_{\mathbb{R}_+\times\mathbb{R}} F(t,x)M(\delta t, dx) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n). \tag{2.70}$$

We say that F is Skorohod integrable if it converges in $L^2(P)$, that is,

$$||\delta(F)||^2 = \sum_{n=0}^{\infty}(n+1)!||\tilde{f}_n||^2_{L^2(\mu^{n+1})} < \infty. \tag{2.71}$$

Definition 2.7.2 *[77], [78], [80] Let $F \in L^2(P)$ with chaos expansion of the form of* (2.66). *Denote the set $\mathbb{D}^{1,2} \equiv Dom\ D$ is the set $F \in L^2(P)$ such that*

$$\sum_{n=1}^{\infty} nn!||f_n||^2_{L^2(\mu^n)} < \infty. \tag{2.72}$$

For $F \in Dom\ D$, the Malliavin derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,z))\right). \tag{2.73}$$

with convergence in $L^2(P \times \mu)$. Moreover, we have the following:

$$||D_{t,z}F||^2_{L^2(P\times\mu)} = \sum_{n=1}^{\infty} nn!||f_n||^2_{L^2(\mu^n)} < \infty. \tag{2.74}$$

Dom D is a Hilbert space with scalar product of $F, G \in Dom\ D$

$$< F, G >= E[FG] + E\left[\int_{[0,T]\times\mathbb{R}} D_{t,z}FD_{t,z}G\mu(dt,dz)\right] \tag{2.75}$$

and D is a closed operator from Dom D to $L^2(P \times \mu)$.

For $f : ([0,T] \times \mathbb{R})^n \to \mathbb{R}$, we have the following decomposition:

$$\int_{([0,T]\times\mathbb{R})^n} f d\mu^{\otimes n} = \sigma^2 \int_{[0,T]\times([0,T]\times\mathbb{R})^{n-1}} f(\cdot,(t,0)) dt d\mu^{\otimes^{n-1}} + \int_{[0,T]\times\mathbb{R}_0\times([0,T]\times\mathbb{R})^{n-1}} f(\cdot,(t,z)) z^2 \nu(dz) dt d\mu^{\otimes^{n-1}} \quad (2.76)$$

We define the spaces $DomD^0$ and $DomD^1$ as follows.

Definition 2.7.3 *[77], [78], [80] Let $F \in L^2(P)$ with chaos expansion of the form of* (2.66).

(i) $DomD^0$ is the set of $F \in L^2(P)$ such that $\sigma > 0$,

$$\sum_{n=1}^{\infty} nn! \int_0^T ||f_n(\cdot,(t,0))||^2_{L^2(\mu^{n-1})} \sigma^2 dt < \infty. \quad (2.77)$$

For $F \in Dom\ D^0$, we define

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,0))\right) \quad (2.78)$$

with convergence in $L^2(P \times \lambda)$.

(ii) $DomD^1$ is the set of $F \in L^2(P)$ such that $\nu \neq 0$ and

$$\sum_{n=1}^{\infty} nn! \int_{[0,T]\times\mathbb{R}_0} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} z^2 \nu(dz) dt < \infty. \quad (2.79)$$

For $F \in Dom\ D^1$, we define

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,z))\right), \quad z \neq 0 \quad (2.80)$$

with convergence in $L^2(P \times z^2\nu(dz)dt)$.

Remark 2.7.1 *If $\sigma > 0$ and $\nu \neq 0$, then $Dom\ D = Dom\ D^0 \cap\ Dom\ D^1 \subset L^2(P)$.*

Theorem 2.7.2 *[80] Chain Rule*
Let $F = (F_1, \cdots, F_n)$, $F_i \in \mathbb{D}^{1,2}$ for $i \in \{1, \cdots, n\}$ and $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$. Suppose that

(i) $\varphi(F) \in L^2(P)$,

(ii) $\sum_{k=1}^{n} \frac{\partial \varphi(F)}{\partial x_k} D_{t,0}F \in L^2(P \times \lambda)$,

(iii) $\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z} \in L^2(P \times z^2\nu(dz)dt)$,

then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$\begin{aligned} D_{t,z}\varphi(F) =& \sum_{k=1}^{n} \frac{\partial \varphi(F)}{\partial x_k} D_{t,0}F_k \mathbf{1}_{\{z=0\}} \\ =& + \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \mathbf{1}_{\{z \neq 0\}}. \end{aligned} \tag{2.81}$$

Definition 2.7.4 *[77], [78] The space $\mathbb{L}^{1,2}$*
Let $F \in L^2(P \times \mu)$ with chaos expansion of the form of (2.66). such that $F(t,z) \in \mathbb{D}^{1,2}$ for all $(t,z) \in [0,T] \times \mathbb{R}$ μ-a.e., $DF \in L^2(P \times \mu^{\otimes 2})$. Then the chaos expansion of F is equivalent to

$$\sum_{n=1}^{\infty} nn! ||\hat{f}_n||^2_{L^2(\mu^{n+1})} < \infty. \tag{2.82}$$

Remark 2.7.3 *The above chaos expansion implies $\mathbb{L}^{1,2} \subset \mathbb{D}^{1,2}$.*

We state some of the important characterization of $\mathbb{L}^{1,2}$

(i) Let $F, G \in \mathbb{L}^{1,2}$, then

$$\begin{aligned} E[\delta(F)\delta(G)] =& E\left[\int_{[0,T]\times\mathbb{R}} F(t,z)G(t,z)\mu(dt,dz)\right] \\ &+ E\left[\int_{([0,T]\times\mathbb{R})^2} D_{t,z}F(s,x)D_{t,x}G(s,x)\mu(ds,dx)\mu(dt,dz)\right]. \end{aligned} \tag{2.83}$$

(ii) Let $F \in \mathbb{L}^{1,2}$ such that $D_{t,z}F \in Dom\ \delta$ for all $(t,z) \in [0,T] \times \mathbb{R}$ μ-a.e. Then $\delta(F) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}\delta(F) = F(t,z) + \delta(D_{t,z}F) \tag{2.84}$$

for all $(t,z) \in [0,T] \times \mathbb{R}$ μ-a.e.

2.8 Predictable Process

Definition 2.8.1 *[27] Predictable Process*

A predictable process is a stochastic process measurable with respect to the σ-field generated by

$$A \times (s,t] \times B, \qquad A \in \mathcal{F}_s, \quad 0 \le s < t, B \in \mathcal{B}(\mathbb{R}_0). \tag{2.85}$$

Note: Any measurable $\mathcal{F}$-adapted and left-continuous (with respect to t) process is predictable [27].

We shall present some important theorems related to predictable process. The first theorem is the isometry relation prsented by the following theorem.

Theorem 2.8.1 *[7], [78] Let F and G be μ-square integrable predictable processes, then*

$$\begin{aligned} &E\left[\int_{[0,T]\times\mathbb{R}} F(t,z)M(dt,dz)\int_{[0,T]\times\mathbb{R}} F(t,z)M(dt,dz)\right] \\ =&E\left[\int_{[0,T]\times\mathbb{R}} F(t,z)G(t,z)\mu(dt,dz)\right]. \end{aligned} \tag{2.86}$$

Theorem 2.8.2 *[78] Let F be μ-square integrable predictable processes, then*

$$\int_{[0,T]\times\mathbb{R}} F(t,z)M(dt,dz) = \sigma\int_0^T F(t,0)dW(t) + \int_{[0,T]\times\mathbb{R}_0} zF(t,z)\tilde{N}(dt,dz). \tag{2.87}$$

Theorem 2.8.3 *[78] Let $F \in L^2(P\times\mu)$ be a predictable processes. Then $F \in Dom(\delta)$ and*

$$\delta(F) = \int_{[0,T]\times\mathbb{R}} F(t,z)M(dt,dz). \tag{2.88}$$

Finally, we have the Clark-Ocone theorem in $\mathbb{D}^{1,2}$ stated as follows.

Theorem 2.8.4 *[78] Let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_T$-measurable, then*

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t^-}]M(dt,dz) \tag{2.89}$$

where M is independent measure given by (2.53).

3. CANONICAL LÉVY WHITE NOISE PROCESSES

3.1 Construction of Canonical Lévy White Noise Process

We construct the Canonical Lévy white noise process [56] using a parallel procedure in deriving Wiener and Poisson white noise process [46]. Let $\mathcal{S} \equiv S(\mathbb{R})$ be the Schwartz space of test functions which consists of rapidly decreasing smooth functions $f \in C^{\infty}(\mathbb{R})$ such that

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^{\alpha} f^{(\beta)}(x)| < \infty. \tag{3.1}$$

In addition, $S(\mathbb{R})$ is a Fréchet space with respect to the seminorm $\|f\|_{\alpha,\beta}$. Its dual $\mathcal{S}' \equiv S'(\mathbb{R})$ is the Schwartz space of tempered distribution functions endowed with a weak* topology. The action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$ given by the mapping $w : \mathcal{S}(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$

$$w(\phi, \omega) =< \omega, \phi > . \tag{3.2}$$

Moreover, we have the following inclusions:

$$S(\mathbb{R}) \subset L^2(P) \subset S'(\mathbb{R}). \tag{3.3}$$

We construct the Canonical Lévy white noise process on the $\Omega = S'(\mathbb{R})$ using the Bochner-Minlos theorem which is stated as follows:

Theorem 3.1.1 *A necessary and sufficient condition for the existence of a probability measure P on $S'(\mathbb{R})$ such that*

$$g(\phi) = E[e^{i<\omega,\phi>}] = \int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) \tag{3.4}$$

satisfies the following conditions:

a.) $g(0) = 1$,

b.) g is positive definite, that is, for $\phi_i \in S'(\mathbb{R})$ *and* $c_i \in \mathbb{C}$ *such that* $\mathbf{c} = (c_1 \cdots c_n)^T \neq \mathbf{0}_n, i \in \{1, \cdots, n\}, \forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j g(\phi_i - \phi_j) > 0, \tag{3.5}$$

c.) g is continuous in Fréchet Topology.

In our construction, we let

$$g(\phi) = \exp\left(\int_{\mathbb{R}} \Psi(\phi(y)) dy \right) \tag{3.6}$$

where

$$\Psi(u) = -\frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}_0} (e^{iuz} - iuz - 1)\nu(dz). \tag{3.7}$$

Claim The functional g satisfies the Bochner-Minlos theorem.

Proof We can express g as the product

$$g(\phi) = f(\phi)h(\phi) \tag{3.8}$$

where

$$f(\phi) = \exp\left(-\frac{\sigma^2}{2} \int_{\mathbb{R}} |\phi(y)|^2 dy \right) = \exp\left(-\frac{\sigma^2}{2} \|\phi\|^2_{L^2(\mathbb{R})} \right) \tag{3.9}$$

$$h(\phi) = \exp\left(\int_{\mathbb{R}} \int_{\mathbb{R}_0} (e^{i\phi(y)z} - i\phi(y)z - 1)\nu(dz)dy \right). \tag{3.10}$$

Then, f and h satisfies the Bochner-Minlos theorem corresponding to the Wiener and the compensated Poisson case respectively [46]. Clearly, g satisfies conditions (a) and (c) of the Bochner-Minlos condition. It is suffice to check (b) to prove our assertion. Define the following $n \times n$ matrices:

$$G_n = \{g(\phi_i - \phi_j)\}, \quad F_n = \{f(\phi_i - \phi_j)\}, \quad H_n = \{h(\phi_i - \phi_j)\}. \tag{3.11}$$

From (3.8) and (3.11),

$$G_n = \{g(\phi_i - \phi_j)\}_{i,j \in \{1, \cdots n\}} = \{f(\phi_i - \phi_j)h(\phi_i - \phi_j) = F_n \odot H_n. \tag{3.12}$$

where $\odot$ denotes the Hadamard product. Since f and h are positive definite, so does the matrices F_n and H_n is also positive definite for all $n \in \mathbb{N}$. By the Schur's product theorem [47] implies G_n is positive definite. Since this holds for all $n \in \mathbb{N}$, then g is positive definite. Thus, this proves our assertion. ■

By taking $\phi(y) = t\varphi(y)$ with $t \in \mathbb{R}$ fixed then from (3.7), we obtain

$$\begin{aligned} &E[e^{it<\omega,\varphi>}] \\ &= \exp\left(-\frac{\sigma^2 t^2}{2}\int_{\mathbb{R}} |\varphi(y)|^2 dy + \int_{\mathbb{R}}\int_{\mathbb{R}_0} (\exp(itz\varphi(y)) - itz\varphi(y) - 1)\nu(dz)dy\right). \end{aligned} \tag{3.13}$$

Claim Let $\varphi \in S(\mathbb{R})$, then

$$E[<\omega,\varphi>] = 0, \tag{3.14}$$

$$E[<\omega,\varphi>^2] = \zeta \int_{\mathbb{R}} \varphi(y)dy \tag{3.15}$$

where

$$\zeta = \sigma^2 + \int_{\mathbb{R}_0} z^2 \nu(dz). \tag{3.16}$$

Proof By density argument, it is suffice to show the identity for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Let the Lévy density $\nu \in [-r, r]\setminus\{0\}$ for some $r > 0$. Then by expanding the terms in (3.13) by Taylor series expansion, we obtain:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{i^n t^n}{n!} E[<\omega,\varphi>^n] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_{\mathbb{R}}\left(-\frac{\sigma^2 t^2 \varphi^2(y)}{2} + \int_{\mathbb{R}_0}\sum_{k=2}^{\infty}\frac{i^k t^k z^k \varphi^k(y)}{k!}\nu(dz)\right)dy\right)^n. \end{aligned} \tag{3.17}$$

Collecting the t and t^2 coefficients yields the desired result. ■

We extend the definition of $<\omega,\phi>$ from $\phi \in \mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, then for $\varphi \in L^2(\mathbb{R})$ arbitrary, there exists $\varphi_n \in \mathcal{S}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R})$. By completeness of $L^2(\mathbb{R})$, as $m, n \to \infty$

$$|<\omega,\varphi_n> - <\omega,\varphi_m>| = |<\omega,\varphi_n - \varphi_m>| \to 0. \tag{3.18}$$

Hence, $\{<\omega,\varphi_n>: n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathbb{R}$ and its limit is $<\omega,\varphi>$. Then, define $\tilde{X}(t,\omega)\equiv<\omega,\chi_{[0,t]}>$ where $\chi_{[0,t]}\in L^2(\mathbb{R})$ as follows:

$$\chi_{[0,t]}=\begin{cases}1, & s\in[0,t], t\geq 0\\ -1, & s\in[-t,0), t<0\\ 0, & otherwise.\end{cases} \tag{3.19}$$

Taking the characteristic function of $\tilde{X}(t)$ yields

$$\begin{aligned}&E[\exp(iu\tilde{X}(t))]=E[\exp(iu<\omega,\chi_{[0,t]}>)]\\ &=\exp\left[\int_{\mathbb{R}}\left(-\frac{\sigma^2u^2\chi^2_{[0,t]}(y)}{2}+\int_{\mathbb{R}_0}\big(\exp(iuz)\chi_{[0,t]}(y))-iuz\chi_{[0,t]}(y)-1\big)\,\nu(dz)\right)dy\right]\\ &=\exp\left[\left(-\frac{\sigma^2u^2}{2}+\int_{\mathbb{R}_0}(\exp(iuz)-iuz-1)\,\nu(dz)\right)t\right].\end{aligned} \tag{3.20}$$

By the Lévy-Khinchine theorem, $X(t)$ is a Lévy process and there exists a càdlàg modification of $\tilde{X}(t)$, say $X(t)$ which is a Lévy process [7]. The smoothed white noise process for the Canonical Lévy process is given by:

$$<\omega,\phi>=\int_{\mathbb{R}}\phi(t)dX(t,\omega),\qquad \omega\in\Omega,\qquad \phi\in L^2(\mathbb{R}) \tag{3.21}$$

where $X(t)$ has the following representation:

$$X(t)=\sigma\int_0^t dW(t)+\int_{[0,t]\times\mathbb{R}_0}z\tilde{N}(ds,dz). \tag{3.22}$$

We define the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ for the white noise Canonical Lévy process where $\mathcal{F}=\mathfrak{B}(\mathcal{S}'(\mathbb{R}))$ and $\mathcal{F}_t=\mathcal{F}_t^X\vee\mathcal{N}$ where $\mathcal{F}_t^X=\sigma\{X(s):s\in[0,t]\}$ is the σ-field generated by X up to time t and $\mathcal{N}$ are the P-null sets.

3.2 Construction of Alternative Chaos Expansion for Canonical Lévy processes

Nualart-Schoutens Chaos Decomposition

We assume that the Lévy measure ν satisfies the so-called Nualart-Schoutens assumption [61]: for all $\varepsilon > 0$ there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}_0 \setminus (-\varepsilon,\varepsilon)} \exp(\lambda|z|)\nu(dz) < \infty. \tag{3.23}$$

This assumption covers some important classes in Lévy processes such as the normal (Gaussian), Poisson, gamma, negative binomial, and Meixner processes. This assumption implies the following implications:

1. The absolute moments are greater than or equal to 2 with respect to ν is finite, that is, for all $p \geq 2$,
$$\int_{\mathbb{R}_0} |z|^p \nu(dz) < \infty \tag{3.24}$$
and thus, $X(t)$ has moments of all orders for all $p \geq 2$.

2. The characteristic function $E[\exp(iuX(t))]$ is analytic in the neighborhood of zero and the polynomials are dense in $L^2(\mathbb{R}, P \circ X(t)^{-1})$.

Power Jump Processes

Definition 3.2.1 Power Jump Processes $X^{(i)} = \{X^{(i)}(t) : t \geq 0\}$, $i \in \mathbb{N}$

$$X^{(i)}(t) = \begin{cases} \sum_{s\in(0,t]} \Delta(X(t))^i, & i > 1, \\ X(t), & i = 1. \end{cases} \tag{3.25}$$

Then, from the representation in (2.47) we can express $X^{(i)}$ in integral form follows:

$$X^{(i)}(t) = \begin{cases} \int_{[0,t]\times\mathbb{R}_0} z^i N(ds,dz), & i > 1 \\ bt + \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z^i \tilde{N}(ds,dz), & i = 1. \end{cases} \tag{3.26}$$

The power jump processes $X^{(i)}$ is also a Lévy processes. In general,

$$X(t) \neq \sum_{s \in (0,t]} \Delta X(s) \tag{3.27}$$

and the equality only holds for a pure jump processes ($\sigma = 0$) with bounded variation. Taking the expectation yields

$$E[X^{(i)}(t)] = m_i(t) \tag{3.28}$$

where

$$m_i = \begin{cases} \int_{\mathbb{R}_0} x^i \nu(dx), & i > 1 \\ b, & i = 1. \end{cases} \tag{3.29}$$

Definition 3.2.2 *Compensated Power Jump Processes* $Y^{(i)} = \{Y^{(i)}(t) : t \geq 0\}, i \in \mathbb{N}$ *given by*

$$Y^{(i)}(t) = X^{(i)}(t) - E\left[X^{(i)}(t)\right]. \tag{3.30}$$

$Y^{(i)}$ is referred to as the Teugels martingale of order i, and it is a normal martingale. Alternatively, we can express $Y^{(i)}$ in integral form follows:

$$Y^{(i)}(t) = \begin{cases} \int_{[0,t]\times\mathbb{R}_0} z^i \tilde{N}(ds, dx), & i > 1 \\ \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z^i \tilde{N}(ds, dz), & i = 1. \end{cases} \tag{3.31}$$

Moreover, the quadratic covariation and the predicatable covariation processes for the Teugels martingales $Y^{(i)}$ are as follows:

$$\begin{aligned} [Y^{(i)}, Y^{(j)}]_t =& \sigma^2 t \mathbf{1}_{\{i=j=1\}} + \int_{[0,t]\times\mathbb{R}_0} z^{i+j} N(ds, dz) \\ =& \sigma^2 t \mathbf{1}_{\{i=j=1\}} + X^{(i+j)}(t), \end{aligned} \tag{3.32}$$

$$\begin{aligned} < Y^{(i)}, Y^{(j)} >_t =& \left(\sigma^2 t \mathbf{1}_{\{i=j=1\}} t + \int_{\mathbb{R}_0} z^{i+j} \nu(dz)\right) t \\ =& (\sigma^2 \mathbf{1}_{\{i=j=1\}} + m_{i+j}) t. \end{aligned} \tag{3.33}$$

The Spaces S_1 and S_2 [61]

Let S_1 be the space of real polynomials in $\mathbb{R}_+$, that is,

$$S_1 = \left\{ \sum_{k=0}^{n} c_k z^{k-1} : c_k \in \mathbb{R}, z \in \mathbb{R}_+, k \in \{1, \cdots n\}, n \in \mathbb{N} \right\} \tag{3.34}$$

endowed with the inner product $<< \cdot, \cdot >>_1$ given by

$$\begin{aligned} << P, Q >>_1 =& \sigma^2 P(0)Q(0) + \int_{\mathbb{R}_0} P(z)Q(z)z^2\nu(dz) \\ =& \int_{\mathbb{R}} P(z)Q(z)\eta(dz) =< P, Q >_{L^2(\eta)} \end{aligned} \tag{3.35}$$

where $P, Q \in S_1$. Note that

$$\begin{aligned} << z^{i-1}, z^{j-1} >>_1 =& \sigma^2 \mathbf{1}_{\{i=j=1\}} + \int_{\mathbb{R}_0} z^{i+j}\nu(dx) \\ =& \sigma^2 \mathbf{1}_{\{i=j=1\}} + m_{i+j}. \end{aligned} \tag{3.36}$$

Let $\{p_i(z)\}_{i\in\mathbb{N}}$ be the orthogonalization of $\{1, z, z^2, \cdots\}$ in S_1. From the Gram-Schmidt orthogonality procedure, we have the following:

$$\begin{aligned} p_1(z) =& 1, \\ p_i(z) =& z^{i-1} - \sum_{j=1}^{i-1} \frac{<< p_j(z), z^{i-1} >>_1}{\|p_j(z)\|_1^2} p_j(z) = \sum_{j=1}^{i} a_{ij} z^{j-1} \end{aligned} \tag{3.37}$$

where

$$a_{ij} = \begin{cases} -\dfrac{<< p_j(z), z^{i-1} >>_1}{\|p_j(z)\|_1^2} = \dfrac{\int_{\mathbb{R}} p_j(z) z^{i-1}\eta(dz)}{\int_{\mathbb{R}} p_j^2(z)\eta(dz)}, & j \in \{1, \cdots, i-1\}, \\ 1, & j = i. \end{cases} \tag{3.38}$$

Example For $i = 2$, $p_2(z) = a_{21} + a_{22}z$, where $a_{22} = 1$ and

$$a_{21} = -\frac{\int_{\mathbb{R}} z\eta(dz)}{\int_{\mathbb{R}} \eta(dz)} = -\frac{\int_{\mathbb{R}_0} z^3\nu(dx)}{\sigma^2 + \int_{\mathbb{R}_0} z^2\nu(dx)}.$$

On the other hand, let S_2 be the space of linear transformations of Teugels martingales of the Lévy Processes, that is,

$$S_2 = \left\{ \sum_{k=0}^{n} c_k Y^{(k)} : c_k \in \mathbb{R}, k \in \{1, \cdots n\}, n \in \mathbb{N} \right\} \tag{3.39}$$

endowed with the inner product $<< \cdot, \cdot >>_2$ given by

$$<< Y^{(i)}, Y^{(j)} >>_2 = E[[Y^{(i)}, Y^{(j)}]_1] = \sigma^2 \mathbf{1}_{\{i=j=1\}} + m_{i+j}. \tag{3.40}$$

Then $x^{i-1} \leftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2.

Let $\{H^{(i)}\}_{i\in\mathbb{N}}$ be the orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \cdots\}$ in S_2. Then, $\{H^{(i)}\}_{i\in\mathbb{N}}$ are strongly orthogonal martingales. From the Gram-Schmidt orthogonality procedure, we have the following:

$$\begin{aligned} H^{(1)} =& Y^{(1)}, \\ H^{(1)} =& Y^{(i)} - \sum_{j=1}^{i-1} \frac{<< H^{(j)}, Y^{(i)} >>_2}{||H^{(j)}||_2^2} = \sum_{j=1}^{i-1} a^*_{ij} Y^{(i)} \end{aligned} \tag{3.41}$$

where

$$a^*_{ij} = \begin{cases} -\dfrac{<< H^{(j)}, Y^{(i)} >>_2}{||H^{(j)}||_2^2} = -\dfrac{E[[H^{(j)}, Y^{(i)}]_1]}{E[[H^{(j)}]_1]}, & j \in \{1, \cdots, i-1\} \\ 1, & j = i. \end{cases} \tag{3.42}$$

Lemma 3.2.1 *The Gram-Schmidt coefficients in S_1 and S_2 coincide, that is,*

$$a_{ij} = a^*_{ij}, \qquad j \in \{1, \cdots, i\}, \quad i \in \mathbb{N}. \tag{3.43}$$

Proof We shall prove this lemma by induction.

Base step: Since

$$p_1(x) = a_{11} = 1 \tag{3.44}$$

$$H^{(1)} = a^*_{11} Y^{(1)} = Y^{(1)}. \tag{3.45}$$

Hence,

$$a_{11} = a^*_{11} = 1. \tag{3.46}$$

Inductive step: Suppose that $a_{ij} = a^*_{ij}, \quad j \in \{1, \cdots, i\}$. Then, from the Gram-Schmidt procedure,

$$p_{i+1}(z) = \sum_{j=1}^{i+1} a_{i+1,j} z^{j-1} \tag{3.47}$$

where

$$a_{i+1,j} = \begin{cases} -\dfrac{<< p_j(z), z^i >>_1}{\|p_j(z)\|_1^2}, & j \in \{1, \cdots, i\}, \\ \\ 1, & j = i+1. \end{cases} \tag{3.48}$$

On the other hand,

$$H^{(i+1)} = \sum_{j=1}^{i+1} a^*_{i+1,j} Y^{(i)} \tag{3.49}$$

where

$$a^*_{i+1,j} = \begin{cases} -\dfrac{<< H^{(j)}, Y^{(i+1)} >>_2}{||H^{(j)}||_2^2}, & j \in \{1, \cdots, i\}, \\ \\ 1, & j = i+1. \end{cases} \tag{3.50}$$

From the isometry relation between S_1 and S_2 and by induction hypothesis, we obtain

$$\begin{aligned} << p_j(z), z^i >>_1 &= \sum_{k=1}^{j} a_{jk} << z^{k-1}, z^i >>_1 \\ &= \sum_{k=1}^{j} a^*_{jk} << Y^{(k)}, Y^{(i+1)} >>_2 = << H^{(j)}, Y^{(i+1)} >>_2, \end{aligned} \tag{3.51}$$

$$\begin{aligned} \|p_j(z)\|_1^2 &= \sum_{k=1}^{j} \sum_{l=1}^{j} a_{jk} a_{jl} << x^{k-1}, x^{l-1} >>_1 \\ &= \sum_{k=1}^{j} \sum_{l=1}^{j} a^*_{jk} a^*_{jl} << Y^{(k)}, Y^{(l)} >>_2 = \left\| H^{(j)} \right\|_2^2 \end{aligned} \tag{3.52}$$

then from (3.48), (3.50), (3.51), (3.52), yields

$$a_{i+1,j} = a^*_{i+1,j}, \qquad j \in \{1, \cdots, i+1\}. \tag{3.53}$$

■

Since $\left\{H^{(i)}\right\}_{i \in \mathbb{N}}$ are pairwise strongly orthogonal martingales which forms a linear combination of $Y^{(j)}$, $j \in \{1, \cdots, i\}$ of the form of

$$H^{(i)} = \sum_{j=1}^{i} a^*_{ij} Y^{(j)}(t). \tag{3.54}$$

For $i \neq j$, the product $H^{(i)}H^{(j)}$ and the quadratic covariation process $[H^{(i)}, H^{(j)}]$ are both uniformly integrable martingales [69]. Since $\langle H^{(i)}, H^{(i)} \rangle$ is a predictable covariation process such that $H^{(i)}H^{(j)} - \langle H^{(i)}, H^{(i)} \rangle$ is a martingale, then

$$\langle H^{(i)}, H^{(j)} \rangle_t = 0, \qquad i \neq j. \tag{3.55}$$

Moreover, we have following quadratic covariation and predictable covariation process for $\{H^{(i)}\}_{i \in \mathbb{N}}$

$$\begin{aligned} [H^{(i)}, H^{(j)}]_t =& \sum_{k=1}^{i} \sum_{l=1}^{j} a_{ik}^* a_{jl}^* [Y^{(i)}, Y^{(j)}]_t \\ =& \sigma^2 t + \sum_{k=1}^{i} \sum_{l=1}^{j} a_{ik}^* a_{jl}^* X^{(k+l)}, \end{aligned} \tag{3.56}$$

$$\begin{aligned} < H^{(i)}, H^{(j)} >_t =& \sum_{k=1}^{i} \sum_{l=1}^{j} a_{ik}^* a_{jl}^* < Y^{(i)}, Y^{(j)} >_t \\ =& \left(\sum_{k=1}^{i} \sum_{l=1}^{j} a_{ik}^* a_{jl}^* m_{k+l} + \sigma^2 \right) t\delta_{ij} = q_i t \delta_{ij} \end{aligned} \tag{3.57}$$

where

$$q_i = \sigma^2 + \sum_{k=1}^{i} \sum_{l=1}^{i} a_{ik}^* a_{il}^* m_{k+l}. \tag{3.58}$$

Theorem 3.2.2 *[54]*

$$< p_i(x), p_j(x) >_{L^2(\eta)} = \int_{\mathbb{R}} p_i(z) p_j(z) \eta(dz) = q_i \delta_{ij} \tag{3.59}$$

where $q_i = \|p_i\|^2_{L^2(\eta)}$ *is given by* (3.58).

Proof Since

$$<< p_i(z), p_j(z) >>_1 = \int_{\mathbb{R}} p_i(z) p_j(z) \eta(dz). \tag{3.60}$$

From the isometry relation of S_1 and S_2, we obtain

$$\langle \langle z^{k-1}, z^{l-1} \rangle \rangle_1 = \langle \langle Y^{(k)}, Y^{(l)} \rangle \rangle_2 . \tag{3.61}$$

Then, from the preceding lemma, since the Gram-Schmidt coefficients in S_1 and S_2 coincide then, we have the following:

$$<< p_i(z), p_j(z) >>_1 = \left\langle\left\langle \sum_{k=1}^{i} a_{ik} z^{k-1}, \sum_{k=1}^{j} a_{jl} z^{k-1} \right\rangle\right\rangle_1$$
$$= \left\langle\left\langle \sum_{k=1}^{i} a^*_{ik} Y^{(k)}, \sum_{k=1}^{j} a^*_{jl} Y^{(l)} \right\rangle\right\rangle_2 =<< H^{(i)}, H^{(j)} >>_2 . \quad (3.62)$$

For $i \neq j$, $H^{(i)}$ and $H^{(j)}$ are strongly orthogonal, then the quadratic covariation process $[H^{(i)}, H^{(j)}]$ is a martingale hence,

$$<< H^{(i)}, H^{(j)} >>_2 = E[[H^{(i)}, H^{(j)}]_1] = E[[H^{(i)}, H^{(j)}]_0] = 0. \quad (3.63)$$

On the other hand, for $i = j$, since

$$[Y^{(k)}, Y^{(l)}]_t = \sigma^2 t \mathbf{1}_{\{k=l=1\}} + X^{(k+l)}(t) \quad (3.64)$$

then

$$[H^{(i)}, H^{(i)}]_t = \left[\sum_{k=1}^{i} a^*_{ik} Y^{(l)}, \sum_{l=1}^{i} a^*_{il} Y^{(l)} \right]_t$$
$$= \sum_{k=1}^{i} \sum_{l=1}^{i} a^*_{ik} a^*_{ik} \left(\sigma^2 t \mathbf{1}_{\{k=l=1\}} + X^{(k+l)}(t)\right)$$
$$= \sigma^2 t + \sum_{k=1}^{i} \sum_{l=1}^{i} a^*_{ik} a^*_{ik} X^{(k+l)}(t). \quad (3.65)$$

Finally, taking the expectation at $t = 1$ gives us

$$<< p_i(z), p_i(z) >>_1 = E[[H^{(i)}, H^{(i)}]_1] = \sigma^2 + \sum_{k=1}^{i} \sum_{l=1}^{i} a^*_{ik} a^*_{il} m_{k+l} = q_i. \quad (3.66)$$

■

Chaotic and Predictable Representation Properties

Denote the following multiple integral for $f \in L^2([0,T]^n)$ with respect to the orthogonal martinagles $H^{(i)}$'s:

$$J_n^{(i_1,\cdots,i_n)}(f)$$
$$= \int_0^T \int_0^{t_n^-} \cdots \int_0^{t_2^-} f(t_1,\cdots t_{n-1},t_n)dH^{(i_1)}(t_1)\cdots dH^{(i_{n-1})}(t_{n-1})dH^{(i_n)}(t_n). \tag{3.67}$$

Leon et al., [54] has shown orthogonality relationship between different multi-indices $(i_1,\cdots,i_n)$ stated in the following theorem.

Theorem 3.2.3 *[54] Let $f \in L^2([0,T]^n)$ and $g \in L^2([0,T]^m)$, then*

$$E[J_n^{(i_1,\cdots,i_n)}(f)J_n^{(j_1,\cdots,j_m)}(g)] = \begin{cases} q_{i_1}\cdots q_{i_n}\int_{\Sigma_n} f(t_1,\cdots,t_n)g(t_1,\cdots,t_n)dt_1\cdots dt_n \\ \qquad for \quad n=m, \quad (i_1,\cdots,i_n)=(j_1,\cdots,j_n), \\ 0, \qquad\qquad otherwise \end{cases} \tag{3.68}$$

where

$$\Sigma_n = \{(t_1,\cdots,t_n) : 0 < t_1 < \cdots t_n \le T\} \tag{3.69}$$

is the positive simplex of $[0,T]^n$.

Proof We prove the theorem by induction as well as the identity

$$< H^{(i)}, H^{(j)} >_t = q_i t\delta_{ij}. \tag{3.70}$$

Note that $J_n^{(i_1,\cdots,i_n)}$ can be written recursively as follows:

$$J_n^{(i_1,\cdots,i_n)}(f) = \int_0^T \alpha_n dH^{(i_n)}(t_n),$$
$$J_n^{(j_1,\cdots,j_m)}(g) = \int_0^T \beta_m dH^{(i_m)}(t_m) \tag{3.71}$$

where

$$\alpha_k = \int_0^{t_k^-} \alpha_{k-1} dH^{(i_{k-1})}(t_{k-1}), \quad k \in \{2, \cdots n\}, \qquad \alpha_1 = f(t_1, \cdots, t_n),$$

$$\beta_k = \int_0^{t_k^-} \beta_{k-1} dH^{(i_{k-1})}(t_{k-1}), \quad k \in \{2, \cdots m\}, \qquad \beta_1 = g(t_1, \cdots, t_m). \tag{3.72}$$

Case I: $(m = n)$.

$$E[J_n^{(i_1,\cdots,i_n)}(f) J_n^{(j_1,\cdots,j_n)}(g)] = q_{i_n} \delta_{i_n,j_n} \int_0^T E[\alpha_n \beta_n] dt_n. \tag{3.73}$$

Then, the desired result is obtained by induction.

Case II: $(m \neq n)$. Without loss of generality, we assume that $m < n$. Then, by induction

$$E[J_n^{(i_1,\cdots,i_n)}(f) J_n^{(j_1,\cdots,j_m)}(g)] = q_{i_{n-m+1}} \cdots q_{i_n} \delta_{i_{n-m+1},j_1} \cdots \delta_{i_n,j_m} \cdot \int_0^T \int_0^{t_{n-1}^-} \cdots \int_0^{t_{n-m+2}^-} E[\alpha_{n-m+1} \beta_1] dt_{n-m+2} \cdots dt_{n-1} dt_n. \tag{3.74}$$

Since

$$\begin{aligned} & E[\alpha_{n-m+1}\beta_1] \\ =& E\left[\int_0^{t_{n-m+1}^-} \cdots \int_0^{t_2^-} f(t_1, \cdots, t_n) dH^{(i_1)}(t_1) dH^{(i_{n-m+2})}(t_{n-m+2}) g(t_1, \cdots, t_m)\right] \\ =& E\left[\int_0^{t_{n-m+1}^-} \cdots \int_0^{t_2^-} f(t_1, \cdots, t_n) g(t_1, \cdots, t_m) dH^{(i_1)}(t_1) dH^{(i_{n-m+2})}(t_{n-m+2})\right] \\ =& 0 \end{aligned} \tag{3.75}$$

then, we obtain the desired result. ■

Nualart and Schoutens [61] have shown the for every $F \in L^2(P)$ can be represented in terms of the iterated integrals in terms of $H^{(i)}$.

Theorem 3.2.4 *Chaotic Representation Property (CRP) [61]*

Every random variable $F \in L^2(P)$ has a representation of the form of

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n \geq 1} J_n^{(j_1,\cdots,j_n)}(f_{j_1,\cdots,j_n}) \tag{3.76}$$

As a corollary to the CRP, they have shown a predictable representation in terms of in terms of $H^{(i)}$.

Theorem 3.2.5 *Predictable Representation Property (PRP) [61]*
Every random variable $F \in L^2(P)$ has a representation of the form of

$$F = E[F] + \sum_{n=1}^{\infty} \int_0^T \phi^{(n)}(s) dH^{(n)}(s) \tag{3.77}$$

where $\phi^{(j)}(s)$ is a predictable process.

3.3 Alternative Chaos Expansion for Canonical Lévy processes

We present some important results all based in Solé, et al., [78] which is crucial in finding the alternative chaos expansion for the Canonical Lévy space.

Theorem 3.3.1 *[78] Let $g = \{g(t) : t \in [0, T]\}$ be a predictable process such that*

$$E\left[\int_0^T g^2(t)dt\right] < \infty. \tag{3.78}$$

Then, $g(t)p_i(x)$ is integrable with respect to M and

$$\int_0^T g(t)dH^{(i)}(t) = \int_{[0,T]\times\mathbb{R}} g(t)p_i(x)M(dt, dx). \tag{3.79}$$

Proof For $i = 1$, $p_1(x) = 1$ and

$$H^{(1)} = Y^{(1)} = \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz). \tag{3.80}$$

Then

$$\begin{aligned}\int_0^T g(t)dH^{(1)}(t) &= \int_0^T g(t)\left(\sigma dW(t) + \int_{\mathbb{R}_0} z\tilde{N}(dt, dz)\right)\\ &= \int_{[0,T]\times\mathbb{R}} g(t)p_1(z)M(dt, dz). \end{aligned}\tag{3.81}$$

For $i > 1$, since $g(t)$ is predictable and so is $g(t)z^i$. From Itô isometry, we obtain.

$$\begin{aligned}E\left[\left(\int_{[0,T]\times\mathbb{R}_0} g(t)z^i\tilde{N}(ds, dz)\right)^2\right] =&E\left[\int_{[0,T]\times\mathbb{R}_0} g^2(t)z^{2i}dt\nu(dz)\right]\\ =&\int_{[0,T]} z^2\nu(dx)\cdot E\left[\int_0^T g^2(t)dt\right] < \infty\end{aligned}\tag{3.82}$$

Hence, $g(t)z^i$ is square-integrable with respect to $\tilde{N}$ and thus, integrable with respect to $\tilde{N}$. In addition, with the square-integrability condition in (3.78) implies integrability with respect to W and thus to M. From the integral form of the compensated power jump process $Y^{(j)}$ for $j > 1$,

$$\int_0^T g(t)dY^{(j)}(t) = \int_{[0,T]\times\mathbb{R}_0} g(t)z^j \tilde{N}(dt, dz) = \int_{[0,T]\times\mathbb{R}} g(t)x^{j-1}M(dt, dz). \tag{3.83}$$

Hence, from the preceding equation, we finally obtain

$$\begin{aligned}\int_0^T g(t)dH^{(i)}(t) &= \sum_{j=1}^{i} a_{ij} \int_0^T g(t)dY^{(j)}(t) \\ &= \int_{[0,T]\times\mathbb{R}} g(t) \sum_{j=1}^{i} a_{ij}z^{i-1}M(dt, dz) \\ &= \int_{[0,T]\times\mathbb{R}} g(t)p_i(z)M(dt, dz).\end{aligned} \tag{3.84}$$

■

Corollary 3.3.2 *[78] $H^{(i)}(t)$ can be expressed as the follows:*

$$H^{(i)}(t) = \int_{[0,T]\times\mathbb{R}} p_i(z)M(dt, dz). \tag{3.85}$$

Proof Take $g = 1$ from the preceding theorem. ■

Theorem 3.3.3 *[78] Let $f \in L^2([0,T]^n)$, then*

$$J_n^{(j_1,\cdots,j_n)}(f) = I_n(f(t_1,\cdots,t_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)). \tag{3.86}$$

Proof Using the preceding corollary, we have as follows:

$$\begin{aligned}&{J_n}^{(j_1,\cdots,j_n)}(f) \\ &= \int_0^T \int_0^{t_n^-} \cdots \int_0^{t_2^-} f(t_1,\cdots,t_n)dH^{(j_1)}(t_1)\cdots dH^{(j_{n-1})}(t_{n-1})dH^{(j_n)}(t_n) \\ &= \int_0^T \int_{\mathbb{R}} \int_0^{t_n^-} \int_{\mathbb{R}} \cdots \int_0^{t_2^-} \int_{\mathbb{R}} f(t_1,\cdots,t_n)p_{j_1}(x_1)\cdots p_{j_n}(z_n)M(dt_1, dz_1)\cdots M(dt_n, dz_n) \\ &= \int_{([0,T]\times R)^n} f(t_1,\cdots,t_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)M(dt_1, dz_1)\cdots M(dt_n, dz_n) \\ &=I_n(f(t_1,\cdots,t_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)).\end{aligned} \tag{3.87}$$

■

We follow Benth, et al., [15] approach in comparing the relationship between Itô's chaos expansion and the CRP. However, their approach was only limited with respect to chaos expansion with respect to the iterated integral of the compensated Poisson random measure $\tilde{N}$. With their result, Di Nunno was able to derive the alternative expansion in the Poisson case [24]. A discussion of the alternative chaos expansion for the Wiener and Poisson case is referred to the Appendix.

With the results of the preceding section, we are able to use this relationship for the Canonical Lévy case. From Itô's chaos expansion [50] of the Canonical Lévy process we have

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n) \quad f_n \in L_s^2(\mu^n). \tag{3.88}$$

On the other hand, from Nualart-Schoutens CRP and from the preceding theorem 3.86, we obtain

$$\begin{aligned} F - E[F] =& \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n \geq 1} J_n^{(j_1,\cdots,j_n)} \left(f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) \right) \\ =& \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n \geq 1} I_n \left(f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) p_{j_1}(z_1) \cdots p_{j_n}(z_n) \mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n) \right) \\ =& \sum_{n=1}^{\infty} I_n \left(\sum_{j_1,\cdots,j_n \geq 1} f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) p_{j_1}(z_1) \cdots p_{j_n}(z_n) \mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n) \right). \end{aligned} \tag{3.89}$$

We let

$$g_n((t_1,z_1),\cdots(t_n,z_n)) = \sum_{j_1,\cdots,j_n \geq 1} f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) p_{j_1}(z_1) \cdots p_{j_n}(z_n) \mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n). \tag{3.90}$$

Then, by the uniqueness of the chaos expansion, we obtain

$$f_n = g_n^{\wedge}, \qquad \forall n \in \mathbb{N}. \tag{3.91}$$

Since

$$f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) \mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n) \in L^2(\lambda^n) \tag{3.92}$$

and the Hermite functions[1] $\{e_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\lambda)$, then we can express (3.92) as follows:

$$f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) = \sum_{i_1,\cdots,i_n\geq 1} \gamma_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)} e_{i_1}(t_1)\cdots e_{i_n}(t_n). \tag{3.93}$$

We let

$$\pi_i(z) = \frac{p_i(z)}{\|p_i\|_{L^2(\eta)}}, \qquad i \in \mathbb{N} \tag{3.94}$$

then, from (3.59), $\{\pi_i\}_{i\in\mathbb{N}}$ are orthonormal basis functions in $L^2(\eta)$. Denote

$$c_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)} = \|p_{i_1}\|_{L^2(\eta)}\cdots\|p_{i_n}\|_{L^2(\eta)}\,\gamma_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)}. \tag{3.95}$$

Hence, we can express (3.90) in terms of orthonormal basis functions in $L^2(\mu^n)$ as follows:

$$g_n((t_1,z_1),\cdots,(t_n,z_n)) = \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)} e_{i_1}(t_1)\pi_{j_1}(z_1)\cdots e_{i_n}(t_n)\pi_{j_n}(z_n). \tag{3.96}$$

Since the symmetrization operator is linear operator, that is, $(af+bg)^\wedge = af^\wedge + bg^\wedge$ where $a, b, \in \mathbb{R}$, then

$$g_n^\wedge((t_1,z_1),\cdots(t_n,x_n)) = \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)} \left(e_{i_1}(t_1)\pi_{j_1}(z_1)\cdots e_{i_n}(t_n)\pi_{j_n}(z_n)\right)^\wedge. \tag{3.97}$$

We want to express (3.97) in terms of orthogonal functions in $L_s^2(\mu^n)$. We shall adapt the same trick Di Nunno applied in the Poisson case using the Cantor diagonalization technique [24]. Denote the Cantor diagonalization mapping $\kappa : \mathbb{N}\times\mathbb{N}$ as follows:

$$\kappa(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}. \tag{3.98}$$

Let $k = \kappa(i,j)$ and

$$\delta_k(t,z) = e_i(t)\pi_j(z) \tag{3.99}$$

[1]See Appendix for a discussion of the Hermite function e_n and its relationship to the Hermite polynomial h_n.

then, $\{\delta_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mu)$. Then from (3.97) and (3.99)

$$\begin{aligned} & g_n^{\wedge}((t_1,z_1),\cdots,(t_n,z_n)) \\ & = \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c_{i_1,\cdots,i_n}^{(j_1,\cdots,j_n)}\left(\delta_{\kappa(i_1,j_1)}(t_1,z_1)\cdots\delta_{\kappa(i_n,j_n)}(t_n,z_n)\right)^{\wedge}. \end{aligned} \tag{3.100}$$

Denote the following multi-indices given by

$$\alpha=(\alpha_1,\alpha_2,\cdots),\qquad \alpha_i\in\mathbb{N}_0,\quad i\in\mathbb{N} \tag{3.101}$$

with compact support and $\mathcal{I}$ by the set of α in (3.101). Also, we denote the following:

$$Index(\alpha)=max\{i:\alpha_i\neq 0\} \tag{3.102}$$

$$|\alpha|=\sum_{i=1}^{m}\alpha_i,\qquad \alpha!=\prod_{i=1}^{m}\alpha_i,\qquad m=Index(\alpha). \tag{3.103}$$

Suppose that $m=Index(\alpha)$ and $n=|\alpha|$, define the following tensor product as follows:

$$\begin{aligned} & \delta^{\otimes\alpha}((t_1,z_1)\cdots(t_n,z_n)) \\ & =\delta_1^{\otimes\alpha_1}\otimes\cdots\otimes\delta_m^{\otimes\alpha_m}((t_1,z_1)\cdots(t_n,z_n)) \\ & =\delta_1(t_1,z_1)\cdots\delta_1(t_{\alpha_1},z_{\alpha_1})\delta_2(t_{\alpha_1+1},z_{\alpha_1+1})\cdots\delta_2(t_{\alpha_1+\alpha_2},z_{\alpha_1+\alpha_2}) \\ & \quad\cdots\qquad \delta_m(t_{n-\alpha_m+1},z_{n-\alpha_m+1})\cdots\delta_m(t_n,x_n) \end{aligned} \tag{3.104}$$

with the convention $\delta_i^{\otimes 0}=1$, $i\in\{1,\cdots,m\}$. Also, we denote the symmetrized tensor product as follows:

$$\begin{aligned} \delta^{\hat{\otimes}\alpha}((t_1,z_1)\cdots(t_n,z_n)) &= \left(\delta^{\otimes\alpha}((t_1,z_1)\cdots(t_n,z_n))\right)^{\wedge} \\ &=\delta_1^{\hat{\otimes}\alpha_1}\hat{\otimes}\cdots\hat{\otimes}\delta_m^{\hat{\otimes}\alpha_m}((t_1,z_1)\cdots(t_n,z_n)). \end{aligned} \tag{3.105}$$

Now, since

$$g_n^{\wedge}\in\overline{span\{\delta^{\hat{\otimes}\alpha}:|\alpha|=n,\alpha\in\mathcal{I}\}} \tag{3.106}$$

then $g_n^{\wedge}$ has of the form of

$$f_n=g_n^{\wedge}=\sum_{|\alpha|=n}c_\alpha\delta^{\hat{\otimes}\alpha}. \tag{3.107}$$

By taking $I_0(\delta^{\hat{\otimes}\alpha}) = 1$ and $c_0 = E[F]$, then by (3.88), (3.91), and (3.107) we obtain

$$F = E[F] + \sum_{n=1}^{\infty} I_n(g_n^{\wedge}) = E[F] + \sum_{n=1}^{\infty} I_n \left(\sum_{|\alpha|=n} c_\alpha \delta^{\hat{\otimes}\alpha} \right) = \sum_{\alpha \in \mathcal{I}} c_\alpha I_{|\alpha|} \left(\delta^{\hat{\otimes}\alpha} \right).$$

We denote

$$\mathbb{K}_\alpha = I_{|\alpha|} \left(\delta^{\hat{\otimes}\alpha} \right) \tag{3.108}$$

then we have the following chaos expansion stated in the theorem below.

Theorem 3.3.4 *Let $F \in L^2(P)$, then it has a unique chaos expansion of the form of*

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{3.109}$$

Moreover, we shall establish an isometry relation to this chaos expansion by proving the following lemma.

Lemma 3.3.5

$$E[\mathbb{K}_\alpha \mathbb{K}_\beta] = \alpha! \cdot \mathbf{1}_{\{\alpha=\beta\}} \tag{3.110}$$

Proof We let

$$m_\alpha = Index(\alpha), \quad n_\alpha = |\alpha|, \qquad m_\beta = Index(\beta), \quad n_\beta = |\beta|. \tag{3.111}$$

Then, by isometry (2.65),

$$\begin{aligned} E[\mathbb{K}_\alpha \mathbb{K}_\beta] =& E\left[I_{n_\alpha}\left(\delta^{\hat{\otimes}\alpha}\right) I_{n_\beta}\left(\delta^{\hat{\otimes}\beta}\right) \right] \\ =& n_\alpha! \left\langle \delta^{\hat{\otimes}\alpha}, \delta^{\hat{\otimes}\beta} \right\rangle \delta_{n_\alpha n_\beta} \\ =& n_\alpha! \int_{([0,T]\times\mathbb{R})^{n_\alpha}} \delta^{\hat{\otimes}\alpha} \delta^{\hat{\otimes}\beta} d\mu^{\otimes n} \delta_{n_\alpha n_\beta}. \end{aligned} \tag{3.112}$$

For $n_\alpha \neq n_\beta$, the (3.112) vanishes. Throughout the remainder of the proof, it suffice to evaluate for the case $n = n_\alpha = n_\beta$. Denote $m = m_\alpha$, and consider the tensor product in (3.104). There are $n!$ terms in the symmetrization on $\delta^{\hat{\otimes}\alpha}$ as well as $\delta^{\hat{\otimes}\beta}$ with each term of these symmetrized tensor product has a factor of $1/n!$. Since $\{\delta_k\}_{k\in\mathbb{N}}$ forms an orthonomal basis in $L^2(\mu)$, then for $\alpha \neq \beta$, (3.112) vanishes.

Consider the case $\alpha = \beta$, for each $n!$ terms in $\mathbb{K}_\alpha$, one can get a non-zero expectation term with a product on a term in $\mathbb{K}_\beta$ by permuting the terms in (3.104) by permuting the first α_1 terms, then permuting the next α_2 terms, and so forth and finally, permuting the last α_m terms. There are $\alpha! = \alpha_1! \cdots \alpha_m!$ possible combinations in this procedure each with the weight of one by orthonormality of $\delta_{k\{k\in\mathbb{N}\}}$ in $L^2(\mu)$. Hence, we obtain

$$E[\mathbb{K}_\alpha \mathbb{K}_\beta] = n! \cdot \frac{1}{(n!)^2} \cdot n! \cdot \alpha! \cdot \mathbf{1}_{\{\alpha=\beta\}} = \alpha! \cdot \mathbf{1}_{\{\alpha=\beta\}}. \tag{3.113}$$

■

Proposition 3.3.1 *(Isometry) Let $F \in L^2(P)$, with a chaos expansion of the form of* (3.109)*, then*

$$\|F\|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!. \tag{3.114}$$

Proof From the preceding lemma,

$$\|F\|^2_{L^2(P)} = E[F^2] = E\left[\sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha \sum_{\beta\in\mathcal{I}} c_\beta \mathbb{K}_\beta\right] = \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} c_\alpha c_\beta E[\mathbb{K}_\alpha \mathbb{K}_\beta] = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!. \tag{3.115}$$

■

3.4 Stochastic Test and Distribution Function

Consider the following formal expansion

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{3.116}$$

If the following growth condition holds,

$$\sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha! < \infty, \tag{3.117}$$

then $F \in L^2(P)$. We relax this growth condition to obtain a family of generalized function spaces of stochastic test functions and stochastic distribution functions that relates to $L^2(P)$ naturally [46].

3.4.1 The spaces $\mathcal{G}$ and $\mathcal{G}^*$

The stochastic test function $\mathcal{G}$ and the stochastic distribution function $\mathcal{G}^*$ was first investigated by Pothoff and Timpel in the Wiener case [68]. A parallel definition was carried out by Di Nunno [24] in the Poisson case. We extend these definitions for the Canonical Lévy space.

Definition 3.4.1 *(i) Let $\mathcal{G}_q$, $q \in \mathbb{R}$ be the space of formal expansion*

$$F = \sum_{n \in \mathbb{N}_0} I_n(f_n) \tag{3.118}$$

such that

$$\|F\|_{\mathcal{G}_q} = \left(\sum_{n \in \mathbb{N}_0} n! \, \|f_n\|^2_{L^2(\mu^n)} \, e^{2qn} \right)^{1/2} < \infty. \tag{3.119}$$

For every $q \in \mathbb{R}$, $\mathcal{G}_q$ is a Hilbert space with inner product

$$< X, Y >_{\mathcal{G}_r} = \sum_{n \in \mathbb{N}_0} n! < f_n, g_n >_{L^2(\mu^n)} e^{2qn} \tag{3.120}$$

where F and G has the following formal sum:

$$F = \sum_{n \in \mathbb{N}_0} I_n(f_n), \qquad G = \sum_{n \in \mathbb{N}_0} I_n(g_n). \tag{3.121}$$

Define the stochastic test function $\mathcal{G}$ as

$$\mathcal{G} = \bigcap_{q>0} \mathcal{G}_q \tag{3.122}$$

endowed with the projective topology, that is, as $n \to \infty$

$$F_n \to F \text{ on } \mathcal{G} \Leftrightarrow \|F_n - F\|_{\mathcal{G}_q} \to 0 \quad \forall q > 0. \tag{3.123}$$

(ii) Define the stochastic distribution function $\mathcal{G}^*$ as

$$\mathcal{G}^* = \bigcup_{q>0} \mathcal{G}_{-q} \tag{3.124}$$

endowed with the inductive topology, that is, as $n \to \infty$

$$G_n \to G \text{ on } \mathcal{G}^* \Leftrightarrow \exists q > 0 \text{ such that } \|G_n - G\|_{\mathcal{G}_{-q}} \to 0. \tag{3.125}$$

Note that $\mathcal{G}^*$ is a dual of $\mathcal{G}$. Let $F \in \mathcal{G}$ and $G \in \mathcal{G}^*$ with the formal expansion of F and G of the form of (3.121). The action of G on F is given by:

$$< G, F >_{\mathcal{G},\mathcal{G}^*} = \sum_{n \in \mathbb{N}_0} n! < f_n, g_n >_{L^2(\mu^n)} . \tag{3.126}$$

Also, we can express the $\mathcal{G}_q$-norm, $q \in \mathbb{R}$ in terms of the chaos expansion (3.109) as follows:

$$\|F\|^2_{\mathcal{G}_q} = \sum_{\alpha \in \mathcal{I}} c_\alpha \alpha! e^{2qn}. \tag{3.127}$$

3.4.2 Kontratiev and Hida spaces

Similarly, we extend the definitions of Kontratiev and Hida space [46] to the Canonical Lévy space.

We let $\alpha \in \mathcal{I}$ and suppose that $Index(\alpha) = m$, then we denote

$$(2\mathbb{N})^{\alpha k} = \prod_{j=1}^{m} (2j)^{\alpha_j k} \tag{3.128}$$

where $k \in \mathbb{Z}$. In particular, if $\alpha = \varepsilon^{(m)} = (0, \cdots, 0, 1, 0, \cdots)$, that is, $\varepsilon^{(m)}$ is a multi-index with all zeros except for the m-th component which contains one, then $(2\mathbb{N})^{\varepsilon^{(m)} k} = (2m)^k$.

Definition 3.4.2 *(i) Let $p \in [0, 1]$. Suppose that F has a formal expansion of the form (3.121). Then, F belongs to the space $(\mathcal{S})_{p,q}$, $q \in \mathbb{R}$ if*

$$\|F\|^2_{p,q} = \sum_{\alpha \in \mathcal{I}} a^2_\alpha (\alpha!)^{1+p} (2\mathbb{N})^{\alpha q} < \infty. \tag{3.129}$$

Define the Kontratiev test function $(\mathcal{S})_p$ as

$$(\mathcal{S})_p = \bigcap_{q>0} (\mathcal{S})_{p,q} \tag{3.130}$$

endowed with the projective topology.

(ii) Let $q \in \mathbb{R}$, then the formal expansion

$$G = \sum_{\alpha \in \mathcal{I}} b_\alpha \mathbb{K}_\alpha \tag{3.131}$$

belongs to the space $(\mathcal{S})_{-q}$ *if*

$$\|G\|^2_{-p,-q} = \sum_{\alpha\in\mathcal{I}} b^2_\alpha (\alpha!)^{1-p}(2\mathbb{N})^{-\alpha q} < \infty. \tag{3.132}$$

Define the Kontratiev distribution function $(\mathcal{S})^*$ *as*

$$(\mathcal{S})_{-p} = \bigcup_{q>0} (\mathcal{S})_{-p,-q} \tag{3.133}$$

endowed with the inductive topology.

Note that $(\mathcal{S})_{-p}$ is a dual of $(\mathcal{S})_p$. The action of $G \in (\mathcal{S})_{-p}$ on $F \in (\mathcal{S})_p$, with the formal expansion of F and G of the form (3.121) is given by

$$< G, F >= \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha. \tag{3.134}$$

The Hida spaces are the special cases of the Kondratiev spaces. The Hida test function $(\mathcal{S})$ and Hida distribution function $(\mathcal{S})^*$ is given by $(\mathcal{S}) = (\mathcal{S})_0$ and $(\mathcal{S})^* = (\mathcal{S})_{-0}$ respectively. From the above definitions, we have the following inclusions for $p \in [0, 1]$:

$$(\mathcal{S})_1 \subset (\mathcal{S})_p \subset (\mathcal{S})_0 \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-p} \subset (\mathcal{S})_{-1}. \tag{3.135}$$

3.5 White Noise Processes from Canonical Lévy Processes

We extend the concept of white noise processes in the Canonical Lévy space. Consider the chaos expansion of $X(t)$ in (2.49)

$$\begin{aligned} X(t) =& I_1(1) = I_1\left(\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \langle 1, e_i\rangle_{L^2(\lambda)} \langle 1, \pi_j\rangle_{L^2(\nu)} e_i(s)\pi_j(z)\right) \\ =& \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \int_0^t e_i(s)ds \int_{\mathbb{R}} \pi_j(s)\eta(dz) I_1\left(e_i(s)\pi_j(z)\right). \end{aligned} \tag{3.136}$$

Now since

$$\begin{aligned} \mathbb{K}_{\varepsilon^{\kappa(i,j)}} =& I_1(\delta^{\otimes\varepsilon^{\kappa(i,j)}}) = I_1\left(e_i(s)\pi_j(z)\right) \\ =& \int_0^t \int_{\mathbb{R}} e_i(s)\pi_j(z)ds\eta(dz) \end{aligned} \tag{3.137}$$

and $\{\pi_j\}_{j\in\mathbb{N}}$ is orthonormal with respect to $L^2(\eta)$, then

$$\int_{\mathbb{R}} \pi_j(z)\eta(dz) = \int_{\mathbb{R}} \pi_1(z)\pi_j(z)\eta(dz) = \int_{\mathbb{R}} \eta(dz)\cdot \mathbf{1}_{\{j=1\}} = \zeta\mathbf{1}_{\{j=1\}} \tag{3.138}$$

where

$$\zeta = \sigma^2 + \int_{\mathbb{R}_0} z^2\nu(dz). \tag{3.139}$$

Alternatively, we can write $X(t)$ as follows:

$$X(t) = \zeta \sum_{i\in\mathbb{N}} \int_0^t e_i(s)ds\mathbb{K}_{\varepsilon^{\kappa(i,1)}}. \tag{3.140}$$

Lemma 3.5.1 *For $i,\ j \in \mathbb{N}$,*

$$\kappa(i,j) \geq i. \tag{3.141}$$

Proof Since

$$\kappa(i,j) = j + \frac{1}{2}(i+j-2)(i+j-1) \tag{3.142}$$

then

$$\kappa(i,j) - i = \frac{1}{2}\left[i^2 + (2j-5)i + (j^2 - j + 2)\right]. \tag{3.143}$$

Let $j \in \mathbb{N}$ and consider the quadratic equation in i as follows:

$$F_j(i) = i^2 + (2j-5)i + (j^2 - j + 2). \tag{3.144}$$

To prove the lemma, it suffice to show that $F_j(i) \geq 0$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$.

- Case I: $(j > 1)$ The discriminant in this case is

$$\Delta = (2j-5)^2 - 4(j^2 - j + 2) = -16j + 17 < 0. \tag{3.145}$$

  Since F_j is concave upwards then, $F_j(i) > 0$ for all $i \in \mathbb{N}$.

- Case II: $(j = 1)$ In this case, we have

$$F_1(i) = i^2 - 3i + 2 = (i-1)(i-2). \tag{3.146}$$

  and it is concave upwards. Thus, for all $i \in \mathbb{N}$, $F_1(i) \geq 0$.

■

Definition 3.5.1 *White Noise Lévy Process* $\dot{X}(t)$

$$\dot{X}(t) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(t)\int_{\mathbb{R}} \pi_j(z)\eta(dz)\mathbb{K}_{\varepsilon^{\kappa(i,j)}} = \zeta\sum_{i\in\mathbb{N}} e_i(t)\mathbb{K}_{\varepsilon^{\kappa(i,1)}}. \tag{3.147}$$

Lemma 3.5.2

$$\dot{X}(t) \in (\mathcal{S})^*. \tag{3.148}$$

Proof Since $\kappa(i,1) \geq i$ and $\sup_{t\in\mathbb{R}} |e_n(t)| = O(n^{-1/12})$ [46], then,

$$\begin{aligned}\left\|\dot{X}(t)\right\|_{-q}^2 =&\zeta^2\sum_{i\in\mathbb{N}} \varepsilon^{\kappa(i,1)}!e_i^2(t)(2\mathbb{N})^{-\varepsilon^{\kappa(i,1)}q}\\ =&\zeta^2\sum_{i\in\mathbb{N}} e_i^2(t)(2\kappa(i,1))^{-q}\\ \leq&\zeta^2\sum_{i\in\mathbb{N}} e_i^2(t)(2i)^{-q}.\end{aligned} \tag{3.149}$$

Hence, the series converges for $q \geq 2$ and thus proves our claim. ■

Lemma 3.5.3

$$\dot{X}(t) = \frac{dX(t)}{dt} \quad in \quad (\mathcal{S})^*. \tag{3.150}$$

That is, there exists $q > 0$ such that

$$\left\|\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right\|_{-q}^2 \to 0 \quad as \quad h \to 0. \tag{3.151}$$

Proof Note that from (3.140) and (3.147), we have the following:

$$\begin{aligned}\frac{X(t+h)-X(t)}{h} - \dot{X}(t) =&\zeta\sum_{i\in\mathbb{N}}\frac{1}{h}\int_t^{t+h}(e_i(s)-e_i(t))ds\mathbb{K}_{\varepsilon^{\kappa(i,1)}}\\ =&\zeta\sum_{i\in\mathbb{N}} a_i(h)\mathbb{K}_{\varepsilon^{\kappa(i,1)}}\end{aligned} \tag{3.152}$$

where

$$a_i(h) = \frac{1}{h}\int_t^{t+h}(e_i(s)-e_i(t))ds. \tag{3.153}$$

Since $\sup_{t\in\mathbb{R}}|e_n(t)| = O(n^{-1/12})$ then, $\sup_{i\in\mathbb{N}}|a_i(h)| < \infty$ for all $h \in [0,1]$. Furthermore, since $\kappa(i,1) \geq 1$ then,

$$\left\|\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right\|_{-q}^2 = \zeta^2 \sum_{i\in\mathbb{N}} \varepsilon^{(i,1)}!|a_i(h)|^2(2\kappa(i,1))^{-q} \\ \leq \zeta^2 \sum_{i\in\mathbb{N}} |a_i(h)|^2(2i)^{-q}. \tag{3.154}$$

Now, since $a_i(h) \to 0$ as $h \to 0$ for all $i \in \mathbb{N}$ then, for all $q \geq 2$ from the dominated convergence theorem,

$$\sum_{i\in\mathbb{N}} |a_i(h)|^2(2i)^{-q} \to 0 \quad \text{as} \quad h \to 0. \tag{3.155}$$

Hence, from the bounded convergence theorem, we obtain

$$\left\|\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right\|_{-q}^2 \to 0 \quad \text{as} \quad h \to 0. \tag{3.156}$$

■

Definition 3.5.2 *Lévy White Noise Field* $\dot{M}(t,z)$

$$\dot{M}(t,z) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(t)\pi_j(z)\mathbb{K}_{\varepsilon^{\kappa(i,j)}}. \tag{3.157}$$

Lemma 3.5.4 *For* $i,\ j \in \mathbb{N}$,

$$\sqrt{ij} \leq \kappa(i,j). \tag{3.158}$$

Proof For $i,\ j \in \mathbb{N}$, $i+j-2 \geq 0$, hence $\kappa(i,j) = j + \frac{(i+j-2)(i+j-1)}{2} \geq j$. On the other hand, since $\kappa(i,j) \geq i$. Hence, from the above arguments, we have $\sqrt{ij} \leq \kappa(i,j)$. ■

Lemma 3.5.5 $\dot{M}(t,z) \in (\mathcal{S})^*$ $\mu - a.e.$

Proof Since

$$\left\|\dot{M}(t,z)\right\|_{-q}^2 = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i^2(t)\pi_j^2(z)(2\kappa(i,j))^{-q}. \tag{3.159}$$

Then, from the preceding lemma and by orthonormality of $\{e_i\}_{i\in\mathbb{N}}$ and $\{\pi_j\}_{j\in\mathbb{N}}$ is orthonormal with respect to $L^2(\lambda)$ and $L^2(\eta)$ respectively, then

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} \left\|\dot{M}(t,z)\right\|_{-q}^2 \mu(dt,dz) &\leq \int_{\mathbb{R}_+\times\mathbb{R}} \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i^2(t)\pi_j^2(z)(2\sqrt{ij})^{-q} dt\eta(dz) \\ &= \sum_{i\in\mathbb{N}}(\sqrt{2i})^{-q}\int_{\mathbb{R}_+} e_i^2(t)dt \sum_{j\in\mathbb{N}}(\sqrt{2j})^{-q}\int_{\mathbb{R}} \pi_j^2(z)\eta(dz) \\ &= \sum_{i\in\mathbb{N}}(\sqrt{2i})^{-q}\sum_{j\in\mathbb{N}}(\sqrt{2j})^{-q}. \qquad (3.160)\end{aligned}$$

Hence, the above equation converges for $q > 2$, thus proves our claim.

■

Remark 3.5.6 *Radon-Nikodym Interpretation of the Lévy White Noise Field*

Let $t \in \mathbb{R}_+$ and $A \in \mathfrak{B}(\mathbb{R})$, then

$$M(t,A) = \int_0^t \int_A M(ds,dz). \qquad (3.161)$$

Likewise, we can express $M(t,A)$ as follows:

$$\begin{aligned} M(t,A) =& \sigma \int_0^t dW(s) + \int_0^t \int_A z\tilde{N}(ds,dz) \\ =& I_1(\mathbf{1}_{[0,t]}(s)\mathbf{1}_A(z)) \\ =& I_1\left(\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left\langle \mathbf{1}_{[0,t]}, e_i\right\rangle_{L^2(\lambda)} \left\langle \mathbf{1}_A, \pi_j\right\rangle_{L^2(\nu)} e_i(s)\pi_j(z)\right) \\ =& \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left\langle \mathbf{1}_{[0,t]}, e_i\right\rangle_{L^2(\lambda)} \left\langle \mathbf{1}_A, \pi_j\right\rangle_{L^2(\nu)} I_1\left(e_i(s)\pi_j(z)\right) \\ =& \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \int_0^t e_i(s)ds \int_A \pi_j(z)\eta(dz)\mathbb{K}_{\varepsilon^{\kappa(i,j)}} \\ =& \int_0^t\int_A \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(s)\pi_j(z)\mu(ds,dz)\mathbb{K}_{\varepsilon^{\kappa(i,j)}} \\ =& \int_0^t\int_A \dot{M}(t,z)\mu(ds,dz). \qquad (3.162)\end{aligned}$$

Hence, from (3.161) and (3.162), $\dot{M}(s,z)$ as the Radon-Nikodym derivative in $(S)^*$ is as follows:

$$M(dt,dz) = \dot{M}(t,z)\mu(dt,dz). \qquad (3.163)$$

3.6 Wick Product and Hermite Transform

The Wick Product was a first introduced by Wick in 1950 as a renormalization tool in quantum field theory and its application in stochastic analysis was introduced by Hida and Ikeda in 1965 [46]. We state some of its properties which are similar to the Wiener and Poisson white noise theory.

Definition 3.6.1 *Wick Product*
Let $F = \sum_{\alpha \in \mathcal{I}} a_\alpha \mathbb{K}_\alpha \in (\mathcal{S})_{-1}$ *and* $G = \sum_{\beta \in \mathcal{I}} b_\beta \mathbb{K}_\beta \in (\mathcal{S})_{-1}$, *then the Wick Product of* X *and* Y *denoted by* $X \diamond Y$ *is defined as*

$$X \diamond Y = \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} a_\alpha b_\beta \mathbb{K}_{\alpha+\beta} = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \mathbb{K}_\gamma. \tag{3.164}$$

We define the Wick powers of $X \in (\mathcal{S})_{-1}$ as follows:

$$X^{\diamond n} = X^{\diamond(n-1)} \diamond X, \quad n \in \mathbb{N}, \qquad X^{\diamond 0} = 1. \tag{3.165}$$

Moreover, if $f : \mathbb{C} \to \mathbb{C}$ is entire, given by the following Taylor series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{3.166}$$

then, we define the following Wick version $f^\diamond(X)$, $X \in (\mathcal{S})_{-1}$ given as

$$f^\diamond(X) = \sum_{n=0}^{\infty} a_n X^{\diamond n}. \tag{3.167}$$

Moreover, we define the the Wick exponential of $X \in (\mathcal{S})_{-1}$ denoted as

$$\exp^\diamond(X) = \sum_{n=0}^{\infty} \frac{X^{\diamond n}}{n!}. \tag{3.168}$$

whenever it is convergent in $(\mathcal{S})_{-1}$. Let $\beta \in L^2(\mathbb{R})$ and $\gamma \in L^2(\mathbb{R} \times \mathbb{R}_0)$ deterministic, then we have the following Wick exponentials with respect to the Wiener and the compensated Poisson random measure respectively [27]:

$$\exp^\diamond \left(\int_{\mathbb{R}_+} \beta(t) dW(t) \right) = \exp \left(\int_{\mathbb{R}_+} \beta(t) dW(t) - \frac{1}{2} \int_{\mathbb{R}_+} \beta^2(t) dt \right), \tag{3.169}$$

$$
\begin{aligned}
&\exp^{\diamond}\left(\int_{\mathbb{R}_+\times\mathbb{R}_0}\gamma(t,z)\tilde{N}(dt,dz)\right)\\
=&\exp\left(\int_{\mathbb{R}_+\times\mathbb{R}_0}(\log(1+\gamma(t,z))-\gamma(t,z))\nu(dz)dt+\int_{\mathbb{R}_+\times\mathbb{R}_0}\log(1+\gamma(t,z))\tilde{N}(dt,dz)\right).
\end{aligned}
\tag{3.170}
$$

Some of the important properties of the Wick Product [46].

1. The Wick product is closed in the following spaces: $(\mathcal{S})_{-1}$, $(\mathcal{S})^*$, $(\mathcal{S})$, $(\mathcal{S})_1$, $\mathcal{G}^*$, $\mathcal{G}^*$ However, the Wick product is in general, not closed in $L^2(P)$.

2. If either X or Y are deterministic, then

$$X \diamond Y = X \cdot Y. \tag{3.171}$$

3. Let X, Y, and $Z \in (\mathcal{S})_{-1}$, then,

$$
\begin{aligned}
X \diamond Y =& Y \diamond X,\\
(X \diamond Y) \diamond Z =& X \diamond (Y \diamond Z),\\
X \diamond (Y + Z) =& (X \diamond Y) + (X \diamond Z)
\end{aligned}
\tag{3.172}
$$

that is, the commutative, associative, and distributive law holds respectively.

4. Wick algebra follows the same rules as ordinary algebra. For example,

$$
\begin{aligned}
(X+Y)^{\diamond 2} =& X^{\diamond 2} + 2X \diamond Y + Y^{\diamond 2},\\
\exp^{\diamond}(X+Y) =& \exp^{\diamond}(X) \diamond \exp^{\diamond}(Y).
\end{aligned}
\tag{3.173}
$$

5. Expectation Properties

   (a) Let X, Y, $X \diamond Y \in L^1(P)$, then

$$E[X \diamond Y] = E[X] \cdot E[Y]. \tag{3.174}$$

   Note: Independence of X and Y is not required.

(b) Let $X \in L^1(P)$, then

$$E[\exp^{\diamond} X] = \exp(E[X]). \tag{3.175}$$

6. Wick Chain Rule: Let $X(\cdot) : \mathbb{R} \to (\mathbb{S})_{-1}$ continuously differentiable, and let $f : \mathbb{C} \to \mathbb{C}$ be entire such that $f(\mathbb{R}) \subset \mathbb{R}$, then

$$\frac{d}{dt} f^{\diamond}(X(t)) = (f')^{\diamond}(X(t)) \diamond \frac{d}{dt} X(t) \tag{3.176}$$

in $(\mathcal{S})_{-1}$.

Definition 3.6.2 *[46] Let $F = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha \in (\mathcal{S})_{-1}$, then the Hermite Transform denoted by $\mathcal{H}F$ or $\tilde{F}$ is defined by*

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_{\alpha \in \mathcal{I}} a_\alpha z^\alpha \in \mathbb{C}. \tag{3.177}$$

where $z = (z_1, z_2, \cdots, z_n, \cdots) \in \mathbb{C}^{\mathbb{N}}$, and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$, $\alpha \in \mathcal{I}$, and $z_j^0 \in 1$, $\forall j \in \mathbb{N}$.

Some of the important properties of the Hermite transform which will enable us to manipulate the Wick product is stated below.

Theorem 3.6.1 *[46] Let $F, G \in (\mathcal{S})_{-1}$, then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z). \tag{3.178}$$

In addition, let $f : \mathbb{C} \to \mathbb{C}$, be an entire function such that $f(\mathcal{R}) \subset \mathcal{R}$, and $f^{\diamond}(F) \in (\mathcal{S})_{-1}$, then

$$\mathcal{H}(f^{\diamond}(F))(z) = f(\mathcal{H}F(z)). \tag{3.179}$$

whenever it converges in $\mathbb{C}$.

Lemma 3.6.2 *[46] Suppose $X(t,\omega)$ and $F(t,\omega)$ are $(S)_{-1}$ processes such that*

(i) $\frac{d\tilde{X}(t,z)}{dt} = \tilde{F}(t,z), \qquad \forall t \in (a,b), \quad z \in \mathbb{K}_q(\delta)$

(ii) $\tilde{F}(t,z)$ *is a bounded function of* $(t,z) \in (a,b) \times \mathbb{K}_q(\delta)$*, continuous in* $t \in (a,b)$ *for each* $z \in \mathbb{K}_q(\delta)$*.*

where

$$\mathbb{K}_q(\delta) = \{z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq = 0} |z|^{\alpha} (2\mathbb{N})^{q\alpha} < \delta^2\}. \tag{3.180}$$

Then $X(t,\omega)$ is a differentiable $(S)_{-1}$ process and

$$\frac{dX(t,z)}{dt} = F(t,z), \qquad \forall t \in (a,b). \tag{3.181}$$

3.7 Stochastic Derivative

Consider the formal sum

$$F = \sum_{n \in \mathbb{N}_0} I_n(f_n) = \sum_{\alpha \in \mathcal{I}} c_\alpha \mathbb{K}_\alpha \tag{3.182}$$

where

$$\mathbb{K}_\alpha = I_{|\alpha|}\left(\delta^{\hat{\otimes}\alpha}\right), \tag{3.183}$$

$$f_n = \sum_{|\alpha|=n} c_\alpha \delta^{\hat{\otimes}\alpha}. \tag{3.184}$$

If $F \in \mathbb{D}^{1,2}$, then we have the Malliavin derivative in $F \in \mathbb{D}^{1,2}$ as follows,

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(., (t,z))). \tag{3.185}$$

Let us relax the $\mathbb{D}^{1,2}$ case and let us define a stochastic derivative in $F \in \mathcal{G}^*$ with the same form as (3.185). This is well-defined if $D_{t,z}F$ converges in $\mathcal{G}^*$. We employ this same strategy of Øksendal and Proske [64] in taking the stochastic derivative in $F \in \mathcal{G}^*$ in the Poisson case. Similarly, we can define a stochastic derivative in $F \in (\mathcal{S})^*$ whenever $D_{t,z}F$ converges in $(\mathcal{S})^*$. In the Wiener case, the stochastic derivative corresponds to the Hida-Malliavin derivative [27].

From (3.184), we have as follows:

$$f_n(.,(t,z)) = \sum_{|\alpha|=n} c_\alpha \delta^{\hat{\otimes}\alpha}(.,(t,z)). \tag{3.186}$$

Let $p = Index(\alpha)$, then $\alpha_i = 0$ for $i > p$ and $\varepsilon_i = (0, \cdots, 1, \cdots, 0)^T$, a unit vector with a 1 in the i^{th} component and zero otherwise. Then, $\delta^{\hat{\otimes}\alpha}(., t, z)$ can be computed as follows:

$$\begin{aligned}\delta^{\hat{\otimes}\alpha}(.,t,z) =& \frac{1}{|\alpha|!}\sum_{i=1}^{p} \alpha_i |\alpha - \varepsilon_i|! \delta^{\hat{\otimes}(\alpha-\varepsilon_i)} \delta^{\hat{\otimes}\varepsilon_i}(t,z) \\ =& \frac{1}{|\alpha|}\sum_{i\in\mathbb{N}} \alpha_i \delta^{\hat{\otimes}(\alpha-\varepsilon_i)} \delta^{\hat{\otimes}\varepsilon_i}(t,z). \end{aligned} \tag{3.187}$$

Then, from (3.185), (3.186), and (3.187), we obtain the stochastic derivative

$$\begin{aligned} D_{t,z}F =& \sum_{n=1}^{\infty} I_n \left(n \sum_{|\alpha|=n} c_\alpha \sum_{i\in\mathbb{N}} \alpha_i \delta^{\hat{\otimes}(\alpha-\varepsilon_i)} \delta^{\hat{\otimes}\varepsilon_i}(t,z) \right) \\ =& \sum_{n=1}^{\infty} \sum_{|\alpha|=n} \sum_{i\in\mathbb{N}} c_\alpha \alpha_i I_n \left(\delta^{\hat{\otimes}(\alpha-\varepsilon_i)} \right) \delta^{\hat{\otimes}\varepsilon_i}(t,z) \\ =& \sum_{\alpha\in\mathcal{I}} \sum_{i\in\mathbb{N}} c_\alpha \alpha_i \mathbb{K}_{\alpha-\varepsilon_i} \delta^{\hat{\otimes}\varepsilon_i}(t,z). \end{aligned} \tag{3.188}$$

Note that if $F \in \mathbb{D}^{1,2}$, then the Malliavin derivative in (3.185) and the stochastic derivative in (3.188) coincide. Since κ is bijective, then for any $i \in \mathbb{N}$, $\exists (k,m) \in \mathbb{N}\times\mathbb{N}$ such that $i = \kappa(k,m)$. Hence, we can also express (3.188) as follows:

$$D_{t,z}F = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}} c_\alpha \alpha_{\kappa(k,m)} \mathbb{K}_{\alpha-\varepsilon_{\kappa(k,m)}} \delta^{\hat{\otimes}\varepsilon_{\kappa(k,m)}}(t,z) \tag{3.189}$$

$$= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}} c_\alpha \alpha_{\kappa(k,m)} \mathbb{K}_{\alpha-\varepsilon_{\kappa(k,m)}} e_k(t) \pi_m(z) \tag{3.190}$$

$$= \sum_{\beta\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}} c_{\beta+\varepsilon_{\kappa(k,m)}} (\beta_{\kappa(k,m)} + 1) \mathbb{K}_\beta e_k(t) \pi_m(z). \tag{3.191}$$

Theorem 3.7.1 *Closability of Stochastic Derivatives*

Let $F_m, F \in \mathcal{G}^*$ such that as $m \to \infty$

(i) $F_m \to F$ in $\mathcal{G}^*$,

(ii) $D_{t,z}F_m$ converges in $\mathcal{G}^*$

Then, $D_{t,z}F_m \to D_{t,z}F$ in $\mathcal{G}^*$.

Proof We follow a parallel arguments in showing closability in the $\mathbb{D}^{1,2}$ case [27]. Consider the formal expansion

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha, \tag{3.192}$$

$$F_m = \sum_{\alpha\in\mathcal{I}} c_\alpha^m \mathbb{K}_\alpha \tag{3.193}$$

such that $F_m \to F$ in $\mathcal{G}^*$, then there exists $q > 0$ such that

$$||F_m - F||^2_{\mathcal{G}^*} = \sum_{\alpha\in\mathcal{I}} \alpha! |c_\alpha^m - c_\alpha|^2 e^{-2q|\alpha|} \to 0. \tag{3.194}$$

Hence, $c_\alpha^m \to c_\alpha$. Since the stochastic derivative of $D_{j,t,z}F$ is given as

$$D_{t,z}F_m = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} c_\alpha^m \alpha_{\kappa(k,l)} \mathbb{K}_{\alpha-\varepsilon_{\kappa(k,l)}} e_k(t)\pi_l(z). \tag{3.195}$$

and since $D_{t,z}F_m$ converges in $\mathcal{G}^*$ then by the Cauchy criterion, there exists $r > 0$ such that

$$||D_{t,z}F_m - D_{t,z}F_n||^2_{\mathcal{G}_{-r}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} |c_\alpha^m - c_\alpha^n|^2 \alpha^2_{\kappa(k,l)} (\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} \to 0. \tag{3.196}$$

From Fatou's lemma,

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} |c_\alpha^m - c_\alpha|^2 \alpha^2_{\kappa(k,l)} (\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}!|} \\
&= \lim_{m\to\infty} \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} \liminf_{n\to\infty} |c_\alpha^m - c_\alpha^n|^2 \alpha^2_{\kappa(k,l)} (\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} \\
&\le \lim_{m\to\infty} \liminf_{n\to\infty} \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} |c_\alpha^m - c_\alpha^n|^2 \alpha^2_{\kappa(k,l)} (\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} = 0.
\end{aligned} \tag{3.197}$$

Hence,

$$||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-r}} \to 0. \tag{3.198}$$

So therefore,

$$D_{t,z}F_m \to D_{t,z}F \in \mathcal{G}_{-r} \subset \mathcal{G}^*. \tag{3.199}$$

■

Theorem 3.7.2 *Let*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^* \tag{3.200}$$

where $f_n \in L^2_s(\mu^n)$*. Then,* $D_{t,z}F \in \mathcal{G}^*$*,* μ *a.e. given by*

$$D_{t,z}F = \sum_{n=1} nI_n(f_{n-1}(\cdot,(t,z))). \tag{3.201}$$

Proof We follow parallel arguments of [67] in the Poisson case. Since $F \in L^2(P)$ and define its partial sum as

$$F_m = \sum_{n=0}^{m} I_n(f_n) \tag{3.202}$$

then $F_m \to F$ in $\mathcal{G}^*$ as $m \to \infty$. Pick $q > 0$ be arbitrary, then

$$||F_m - F||^2_{\mathcal{G}_{-q}} = \sum_{n=m+1}^{\infty} n!||f_n||^2_{L^2(\mu^n)}e^{-2qn} \to 0. \tag{3.203}$$

Since $q > 0$ is arbitrary, then $F \in \mathcal{G}^*$. Note that

$$||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-q}} = \sum_{n=m+1}^{\infty} nn!||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})}e^{-2q(n-1)}. \tag{3.204}$$

Integrating both sides and as $m \to \infty$ yields

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{R}} ||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-q}} \mu(dt,dz) \\
&\quad = \int_{[0,t]\times\mathbb{R}} \sum_{n=m+1}^{\infty} nn! ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} \mu(dt,dz) \\
&\quad . = \sum_{n=m+1}^{\infty} nn! \int_{[0,t]\times\mathbb{R}} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} \mu(dt,dz) \\
&\quad = \sum_{n=m+1}^{\infty} nn! ||f_n||^2_{L^2(\mu^n)} e^{-2q(n-1)} \\
&\quad \leq K \sum_{n=m+1}^{\infty} n! ||f_n||^2_{L^2(\mu^n)} \to 0
\end{aligned} \tag{3.205}$$

for some $K > 0$. Thus, verifies our claim. ■

We define the counterpart of Dom D, Dom D^0, and Dom D^1 in the space $\mathfrak{G}^*$ denoted by $\mathfrak{G}$, $\mathfrak{G}^0$, and $\mathfrak{G}^1$ respectively.

Definition 3.7.1 *Let $F \in \mathcal{G}^*$ with chaos expansion of the form* (3.182).
$\mathfrak{G}$ is the set $F \in \mathcal{G}^*$ such that there exists $q > 0$ such that there exists $q > 0$ such that

$$\sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2(\mu^n)} e^{-2q(n-1)} < \infty. \tag{3.206}$$

For $F \in \mathfrak{G}$, the stochastic derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,z))\right). \tag{3.207}$$

with convergence in $\mathcal{G}^* \times L^2(\mu)$. Moreover, we have the following:

$$\begin{aligned}
||D_{t,z}F||^2_{\mathcal{G}_{-q}\ L^2(\mu)} &= \int_{[0,T]\times\mathbb{R}} ||D_{t,z}F||^2_{\times L^2(\mu)} \mu(dt,dz) \\
&= \sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2(\mu^n)} e^{-2q(n-1)} < \infty.
\end{aligned} \tag{3.208}$$

Definition 3.7.2 *Let $F \in \mathcal{G}^*$ with chaos expansion of the form* (3.182).

(i) $\mathfrak{G}^0$ *is the set of* $F \in \mathcal{G}^*$ *such that* $\sigma > 0$ *and there exists* $q > 0$ *such that*

$$\sum_{n=1}^{\infty} nn! \int_0^T ||f_n(\cdot,(t,0))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} \sigma^2 dt < \infty. \tag{3.209}$$

For $F \in \mathfrak{G}^0$, *we define*

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,0))\right). \tag{3.210}$$

with convergence in $\mathcal{G}^* \times L^2(\lambda)$. *Moreover, we have the following:*

$$\begin{aligned} ||D_{t,0}F||^2_{\mathcal{G}_{-q}\times L^2([0,T])} &= \int_0^T ||D_{t,0}F||^2_{\mathcal{G}_{-q}} \sigma^2 dt \\ &= \sum_{n=1}^{\infty} nn! \int_0^T ||f_n(\cdot,(t,0))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} \sigma^2 dt < \infty. \end{aligned} \tag{3.211}$$

(ii) $\mathfrak{G}^1$ *is the set of* $F \in \mathcal{G}^*$ *such that* $\nu \neq 0$ *and there exists* $q > 0$ *such that*

$$\sum_{n=1}^{\infty} nn! \int_{[0,T]\times\mathbb{R}_0} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} z^2 \nu(dz)dt < \infty. \tag{3.212}$$

For $F \in \mathfrak{G}^1$, *we define*

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,(t,z))\right), \quad z \neq 0. \tag{3.213}$$

with convergence in $\mathcal{G}^* \times L^2(z^2\nu(dz)dt)$. *Moreover, we have the following:*

$$\begin{aligned} ||D_{t,z}F||^2_{\mathcal{G}_{-q}\times L^2(\mathbb{R}_0)} &= \int_{[0,T]\times\mathbb{R}} ||D_{t,z}F||^2_{\mathcal{G}_{-q}} z^2\nu(dz)dt \\ &= \sum_{n=1}^{\infty} nn! \int_{[0,T]\times\mathbb{R}_0} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)} z^2 \nu(dz)dt < \infty. \end{aligned} \tag{3.214}$$

If $\sigma > 0$ *and* $\nu \neq 0$, *then* $\mathfrak{G} = \mathfrak{G}^0 \cap \mathfrak{G}^1 \subset \mathcal{G}^*$.

We state a chain rule in $\mathfrak{G}$. The proof of this chain rule is analogous to Dom D case [36], [80] by weakening the assumption from $L^2(P)$ to $\mathcal{G}^*$.

Theorem 3.7.3 *Chain Rule*

Let $F = (F_1, \cdots, F_n)$*,* $F_i \in \mathfrak{G}$ *for* $i \in \{1, \cdots, n\}$ *and* $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$*. Suppose that*

(i) $\varphi(F) \in \mathcal{G}^*$*,*

(ii) there exists $q_0 > 0$ *such that*

$$\left\| \sum_{k=1}^{n} \frac{\partial \varphi(F)}{\partial x_k} D_{t,0} \right\|_{\mathcal{G}_{-q_0}} \in L^2(\lambda),$$

(iii) there exists $q_1 > 0$ *such that*

$$\left\| \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \right\|_{\mathcal{G}_{-q_1}} \in L^2(z^2\nu(dz)dt).$$

Then, $\varphi(F) \in \mathfrak{G}$ *and*

$$\begin{aligned} D_{t,z}\varphi(F) =& \sum_{k=1}^{n} \frac{\partial \varphi(F)}{\partial x_k} D_{t,0} F_k \mathbf{1}_{\{z=0\}} \\ &+ \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \mathbf{1}_{\{z \neq 0\}}. \end{aligned} \tag{3.215}$$

Corollary 3.7.4 *Product Rule*

Let $F, G \in L^2(P)$ *and suppose that* $F^2, G^2, FG \in \mathcal{G}^*$*, then*

$$D_{t,z}(FG) = FD_{t,z}G + GD_{t,z}F + zD_{t,z}FD_{t,z}G. \tag{3.216}$$

Lastly, we state the chain rule under a Wick polynomial g that is entire.

Theorem 3.7.5 *[25] Let* $F \in (\mathcal{S})^*$ *and* $g : \mathbb{C} \to \mathbb{C}$ *be entire, then*

$$D_{t,z}g^{\diamond}(F) = (g')^{\diamond}(F) \diamond D_{t,z}F. \tag{3.217}$$

3.8 Generalized Expectation and Generalized Conditional Expectation

Definition 3.8.1 *Generalized Expectation and Generalized Conditional Expectation in* $(\mathcal{S})^*$

Let $F = \sum_{\alpha \in \mathcal{I}} c_\alpha \mathbb{K}_\alpha \in (S)^*$*, we define the generalized expectation* $E[F]$ *is given by*

$$E[F] = c_0. \tag{3.218}$$

We define generalized conditional expectation $E[F|\mathcal{F}_A]$ with respect to $\mathcal{F}_A$, $A \in \mathfrak{B}(\mathbb{R}_+)$ is given by

$$E[F|\mathcal{F}_A] = \sum_{\alpha \in \mathcal{I}} c_\alpha E[\mathbb{K}_\alpha|\mathcal{F}_A] \tag{3.219}$$

whenever it converges in $(S)^*$.

Remark 3.8.1 *If $F \in L^2(P) \subset (\mathcal{G})^*$, then the generalized expectation coincides with the usual conditional expectation.*

Theorem 3.8.2 *[27] Properties of conditional expectation in $(S)^*$*

(i) Suppose that F, G, $E[F|\mathcal{F}_t]$, and $E[G|\mathcal{F}_t]$ belongs to $(S)^*$, then

$$E[F \diamond G|\mathcal{F}_A] = E[F|\mathcal{F}_A] \diamond E[G|\mathcal{F}_A]. \tag{3.220}$$

In addition, if F, G, $\in L^1(P)$, then

$$E[F \diamond G] = E[F] \cdot E[G]. \tag{3.221}$$

(ii.) Let $F \in (S)^*$, and suppose that $\exp^\diamond(F)$, $E[F|\mathcal{F}_t]$, $\exp^\diamond(E[F|\mathcal{F}_A]) \in (S)^*$ then

$$E[\exp^\diamond F|\mathcal{F}_A] = \exp^\diamond(E[F|\mathcal{F}_A]). \tag{3.222}$$

In addition, if $F \in L^1(P)$, then

$$E[\exp^\diamond F] = \exp^\diamond(E[F]). \tag{3.223}$$

Theorem 3.8.3 *[27] Suppose that $u(s,x)$ is Skorohod integrable and $E[u(s,x)|\mathcal{F}_t] \in (S)^*$ for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$, then*

$$\begin{aligned} E\left[\int_{\mathbb{R}_+\times\mathbb{R}} u(s,x)M(\delta s,dx)\middle|\mathcal{F}_t\right] &= \int_{[0,t]\times\mathbb{R}} E\left[u(s,x)|\,\mathcal{F}_t\right] M(\delta s,dx), \\ E\left[\int_{(t,\infty)\times\mathbb{R}} u(s,x)M(\delta s,dx)\middle|\mathcal{F}_t\right] &= 0. \end{aligned} \tag{3.224}$$

Definition 3.8.2 *Generalized Expectation and Generalized Conditional Expectation in $\mathcal{G}^*$*
Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$, we define the generalized expectation $E[F]$ in $\mathcal{G}^*$ is given by

$$E[F] = I_0(f_0) \tag{3.225}$$

and we define the conditional expectation of F with respect to $A \in \mathfrak{B}([0,T])$ is given by

$$E[F|\mathcal{F}_A] = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n}). \tag{3.226}$$

The generalized conditional expectation in $\mathcal{G}^*$ is more tractable to handle compared to the generalized conditional expectation in $(S)^*$.

Remark 3.8.4 *If $F \in L^2(P) \subset \mathcal{G}^*$, then the generalized expectation coincides with the usual conditional expectation.*

Lemma 3.8.5 *[14], [27] Basic properties of conditional expectation in $\mathcal{G}^*$*

(i) Closure under $\mathcal{G}^*$
Let $F \in \mathcal{G}^*$ and $A \in \mathfrak{B}([0,T])$, then $E[F|\mathcal{F}_A] \in \mathcal{G}^*$ and for some $q > 0$.

$$\|E[F|\mathcal{F}_A]\|_{\mathcal{G}_{-q}} \leq ||F||_{\mathcal{G}_{-q}}. \tag{3.227}$$

(ii) Closure under $L^2(P)$
Let $F \in \mathcal{G}^*$ and $A \in \mathfrak{B}([0,T])$, then $E[F|\mathcal{F}_A] \in L^2(P)$ and

$$\|E[F|\mathcal{F}_A]\|_{L^2(P)} \leq ||F||_{L^2(P)}. \tag{3.228}$$

(iii) Linearity
Let $F, G \in \mathcal{G}^*$, $a, b \in \mathbb{R}$, and $A \in \mathfrak{B}([0,T])$, then

$$E[aF + bG|\mathcal{F}_A] = aE[F|\mathcal{F}_A] + bE[G|\mathcal{F}_A]. \tag{3.229}$$

(iv) *Tower Property*

Let $F \in \mathcal{G}^*$*, and* $A, B \in \mathfrak{B}([0,T])$ *such that* $A \subset B$*, then*

$$E[E[F|\mathcal{F}_A]\mathcal{F}_B] = E[F|\mathcal{F}_A] = E[E[F|\mathcal{F}_B]\mathcal{F}_A]. \tag{3.230}$$

Proof We denote the following formal expansions:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \qquad G = \sum_{n=0}^{\infty} I_n(g_n) \tag{3.231}$$

where $f_n, g_n \in L_s^2(\mu^n)$ for all $n \in \mathbb{N}_0$.

(i) Since $F \in \mathcal{G}^*$, then $||F||_{\mathcal{G}_{-q}} < \infty$ for some $q > 0$. Hence,

$$\|E[F|\mathcal{F}_A]\|_{\mathcal{G}_{-q}}^2 = \sum_{n=0}^{\infty} n! \left\|f_n \mathbf{1}_A^{\otimes n}\right\|_{L^2(\mu^n)}^2 e^{-2qn} \leq \|F\|_{\mathcal{G}_{-q}}^2 < \infty. \tag{3.232}$$

(ii) Since $F \in L^2(P)$, then $||F||_{L^2(P)} < \infty$. Hence,

$$\|E[F|\mathcal{F}_A]\|_{F \in L^2(P)}^2 = \sum_{n=0}^{\infty} n! \left\|f_n \mathbf{1}_A^{\otimes n}\right\|_{L^2(\mu^n)} \leq \|F\|_{L^2(P)}^2 < \infty. \tag{3.233}$$

(iii) Using Cauchy-Schwartz inequality, we can show that $aF + bG \in \mathcal{G}^*$. Then, we have the following expansion

$$\begin{aligned} E[aF + bG|\mathcal{F}_A] =& \sum_{n=0}^{\infty} I_n((af_n + bg_n)\mathbf{1}_A^{\otimes n}) \\ =& a\sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n}) + b\sum_{n=0}^{\infty} I_n(g_n \mathbf{1}_A^{\otimes n}) = aE[F|\mathcal{F}_A] + bE[G|\mathcal{F}_A]. \end{aligned} \tag{3.234}$$

(iv) Since $A \subset B$, then $\mathbf{1}_A \mathbf{1}_B = \mathbf{1}_B \mathbf{1}_A = \mathbf{1}_{A \cap B} = \mathbf{1}_A$. Hence,

$$E[E[F|\mathcal{F}_A]\mathcal{F}_B] = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n} \mathbf{1}_B^{\otimes n}) == \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_B^{\otimes n} \mathbf{1}_A^{\otimes n}) = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n}). \tag{3.235}$$

That is,

$$E[E[F|\mathcal{F}_A]\mathcal{F}_B] = E[F|\mathcal{F}_A] = E[E[F|\mathcal{F}_B]\mathcal{F}_A]. \tag{3.236}$$

■

Theorem 3.8.6 *[27], [66] Let $F \in \mathcal{G}^*$ and $A \in \mathfrak{B}([0,T])$, then*

$$D_{t,z}E[F|\mathcal{F}_A] = E[D_{t,z}F|\mathcal{F}_A]\mathbf{1}_A(t). \tag{3.237}$$

Proof

$$D_{t,z}E[F|\mathcal{F}_A] = D_{t,z}\sum_{n=0}^{\infty} I_n(f_n\mathbf{1}_A^{\otimes n}) = \sum_{n=1}^{\infty} I_{n-1}(f_n\mathbf{1}_A^{\otimes(n-1)})\mathbf{1}_A = E[D_{t,z}F|\mathcal{F}_A]\mathbf{1}_A(t). \tag{3.238}$$

■

Corollary 3.8.7 *[27] Let $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{G}^*$ be an $\mathcal{F}$-predictable process, then*

(i) $D_{t,z}u(s,x)$ is $\mathcal{F}$-predictable process for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,

(ii) $D_{t,z}u(s,x) = 0$, for $s < t$, $z \in \mathbb{R}$.

Proof The assertion holds in (i) and (ii) by applying previous theorem

$$D_{t,z}u(s,x) = E[u(s,x)|\mathcal{F}_{s^-}]\mathbf{1}_{[0,s)}(t) = E[u(s,x)|\mathcal{F}_{s^-}]\mathbf{1}_{\{t>s\}}. \tag{3.239}$$

■

Theorem 3.8.8 *[27] Properties of conditional expectation in $\mathcal{G}^*$*

(i) Let $F, G \in \mathcal{G}^*$, and $A \in \mathfrak{B}([0,T])$, then

$$E[F \diamond G|\mathcal{F}_A] = E[F|\mathcal{F}_A] \diamond E[G|\mathcal{F}_A]. \tag{3.240}$$

(ii) Let F, $\exp^\diamond F \in \mathcal{G}^*$, and $A \in \mathfrak{B}([0,T])$ then

$$E[\exp^\diamond F|\mathcal{F}_A] = \exp^\diamond(E[F|\mathcal{F}_A]). \tag{3.241}$$

3.9 Skorohod Integration on $\mathcal{G}^*$

Definition 3.9.1 *Skorohod Integral in* $\mathcal{G}^*$

Let $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{G}^*$ *with the formal expansion given by*

$$u(t,z) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, z)) \tag{3.242}$$

such that

$$\sum_{n=0}^{\infty} (n+1)! ||\tilde{f}||^2_{L^2(\mu^{n+1})} e^{-2q(n+1)} < \infty \tag{3.243}$$

where $\tilde{f}_n \in L^2_s(\mu^{n+1})$, *and for some* $q > 0$. *Then we define the Skorohod integral of* u *with respect to* M *as follows:*

$$\delta(u) = \int_{\mathbb{R}_+ \times \mathbb{R}} u(t,x) M(\delta t, dx) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n). \tag{3.244}$$

We say that u *is Skorohod integrable if* $\delta(u) \in \mathcal{G}^*$, *that is, there exists some* $q > 0$ *such that*

$$||\delta(u)||^2_{\mathcal{G}_{-q}} = \sum_{n=0}^{\infty} (n+1)! ||\tilde{f}_n||^2_{L^2(\mu^{n+1})} e^{-2q(n+1)} < \infty. \tag{3.245}$$

Theorem 3.9.1 *Fundamental Theorem of Stochastic Calculus*

Let $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{G}^*$ *be a random field satisfying the following conditions:*

(i) $u \in L^2(P \times \mu)$,

(ii) $D_{t,z}u$ *is Skorohod integrable for all* $(t,z) \in [0,T] \times \mathbb{R}$,

(iii) $D_{t,z}\delta(u) \in \mathcal{G}^*$ *and* $\delta(D_{t,z}u) \in \mathcal{G}^*$, *for all* $(t,z) \in [0,T] \times \mathbb{R}$ *and there exists* $q > 0$ *such that*

$$\int_{[0,T]\times\mathbb{R}} ||D_{t,z}\delta(u)||^2_{\mathcal{G}_{-q}} \mu(dt,dz) < \infty, \tag{3.246}$$

$$\int_{[0,T]\times\mathbb{R}} ||\delta(D_{t,z}u)||^2_{\mathcal{G}_{-q}} \mu(dt,dz) < \infty. \tag{3.247}$$

Then,

$$D_{t,z}\left(\delta(u)\right) = u(t,z) + \delta(D_{t,z}u), \tag{3.248}$$

that is,

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} u(s,x)M(\delta s, dx) = u(t,z) + \int_{[0,T]\times\mathbb{R}} D_{t,z}u(s,x)M(\delta s, dx). \tag{3.249}$$

Proof First, suppose the base case where $u(s,x)$ has of the form

$$u(s,x) = I_n(f_n(\cdot,(s,x))) \tag{3.250}$$

then

$$\delta(u) = I_{n+1}(\tilde{f}_n) \tag{3.251}$$

where

$$\tilde{f}_n = \tilde{f}_n((t_1,z_1),\cdots,(t_{n+1},z_{n+1})) = \frac{1}{n+1}\left[f_n(\cdot,(t_1,z_1)) + \cdots f_n(\cdot,(t_{n+1},z_{n+1}))\right]. \tag{3.252}$$

Since

$$\tilde{f}_n(\cdot,(t,z)) = \frac{1}{n+1}\left[f_n(\cdot,(t_1,z_1),(t,z)) + \cdots f_n(\cdot,(t_n,z_n),(t,z)) + f_n(\cdot,\cdot,(t,z))\right] \tag{3.253}$$

then

$$\begin{aligned} D_{t,z}\delta(u) =&(n+1)I_n(\tilde{f}(\cdot,(t,z))) \\ =&I_n(f_n(\cdot,(t_1,z_1),(t,z))) + \cdots I_n(f_n(\cdot,(t_n,z_n),(t,z))) + I_n(f_n(\cdot,\cdot,(t,z))) \\ =&I_n(f_n(\cdot,(t_1,z_1),(t,z))) + \cdots I_n(f_n(\cdot,(t_n,z_n),(t,z))) + u(t,z) \end{aligned} \tag{3.254}$$

and also,

$$D_{t,z}u(s,x) = nI_{n-1}(f_n(\cdot,(t,z),(s,x))). \tag{3.255}$$

Then from (ii), its Skorohod integral is given as

$$\begin{aligned}\delta(D_{t,z}u) &= \int_{[0,t]\times\mathbb{R}} D_{t,z}u(s,x)M(\delta s, dx) \\ &= \int_{[0,t]\times\mathbb{R}} nI_{n-1}(f_n(\cdot,(t,z),(s,x))M(\delta s, dx) \\ &= nI_n(\tilde{f}_n(\cdot,(t,z),\cdot)) \end{aligned} \tag{3.256}$$

where

$$\tilde{f}_n(\cdot,(t,z),\cdot) = \frac{1}{n}\left[f_n(\cdot,(t,z),(t_1,z_1)) + \cdots + f_n(\cdot,(t,z),(t_n,z_n))\right] \tag{3.257}$$

is the symmetrization with respect to $(t_1,z_1),\cdots,(t_n,z_n)$. Hence, from (3.256) and (3.257) yields

$$\delta(D_{t,z}u) = I_n(f_n(\cdot,(t_1,z_1),(t,z))) + \cdots I_n(f_n(\cdot,(t_n,z_n),(t,z))) \tag{3.258}$$

Then from (3.254) and (3.258) yields (3.248).

On the other hand, consider the general case of $u(s,x)$ has of the form

$$u(s,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,(s,x))). \tag{3.259}$$

Then, we have the following:

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{3.260}$$

$$D_{t,z}\delta(u) = \sum_{n=0}^{\infty}(n+1)I_n(\tilde{f}_n(\cdot,(t,z))). \tag{3.261}$$

Consider the partial sum

$$u_m(s,x) = \sum_{n=0}^{m} I_n(f_n(\cdot,(s,x))). \tag{3.262}$$

From (i) and by isometry,

$$\begin{aligned}||u||^2_{L^2(P\times\mu)} &= \int_{[0,T]\times\mathbb{R}} \sum_{n=0}^{\infty} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^n)}\mu(dt,dz) \\ &= \sum_{n=0}^{\infty} ||f_n||^2_{L^2(\mu^{n+1})} < \infty. \end{aligned} \tag{3.263}$$

So therefore, we have the following convergence as $m \to \infty$,

$$||u - u_m||^2_{L^2(P\times\mu)} = \sum_{n=m+1}^{\infty} ||f_n||^2_{L^2(\mu^{n+1})} \to 0. \tag{3.264}$$

Hence, $u_m \to u$ in $L^2(P \times \mu)$. Applying the result from base case, we obtain

$$D_{t,z}(\delta(u_m)) = \delta(D_{t,z}u_m) + u_m(t, z). \tag{3.265}$$

To show (3.248), we need to show the following as $m \to \infty$,

$$D_{t,z}(\delta(u_m)) \to u(t, z) + \delta(D_{t,z}u), \tag{3.266}$$

$$D_{t,z}(\delta(u_m)) \to D_{t,z}(\delta(u)) \tag{3.267}$$

in $\mathcal{G}^* \times L^2(\mu)$. To show (3.266), we have we have the following:

$$D_{t,z}u(s, x) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, (t, z), (s, x)))$$

$$\delta(D_{t,z}u(s, x)) = \sum_{n=1}^{\infty} nI_n(\tilde{f}_n(\cdot, (t, z), \cdot)) = \sum_{n=0}^{\infty} I_n(n\tilde{f}_n(\cdot, (t, z), \cdot)). \tag{3.268}$$

where the last equation is from (ii). Then from (iii), there exists $q > 0$ such that

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}} ||\delta(D_{t,z}u)||^2_{\mathcal{G}_{-q}}\mu(ds, dx) \\
&= \int_{[0,T]\times\mathbb{R}} \sum_{n=0}^{\infty} n!||n\tilde{f}_n(\cdot, (t, z), \cdot)||^2_{L^2(\mu^n)} e^{-2qn}\mu(dt, dz) \\
&= \sum_{n=0}^{\infty} n!n^2 \int_{[0,T]\times\mathbb{R}} ||\tilde{f}_n(\cdot, (t, z), \cdot)||^2_{L^2(\mu^n)}\mu(dt, dz)e^{-2qn} \\
&= \sum_{n=0}^{\infty} n!n^2||\tilde{f}_n||^2_{L^2(\mu^{n+1})}e^{-2qn} < \infty.
\end{aligned} \tag{3.269}$$

Hence, as $m \to \infty$, we obtain

$$\int_{[0,T]\times\mathbb{R}} ||\delta(D_{t,z}u) - \delta(D_{t,z}u_m)||^2_{\mathcal{G}_{-q}}\mu(ds, dx) = \sum_{n=0}^{\infty} n!n^2||\tilde{f}_n||^2_{L^2(\mu^{n+1})}e^{-2qn} \to 0. \tag{3.270}$$

This implies as $m \to \infty$,

$$\delta(D_{t,z}u_m) \to \delta(D_{t,z}u), \qquad \mathcal{G}^* \times L^2(\mu). \tag{3.271}$$

From (3.265), we have the following:

$$D_{t,z}(\delta(u_m)) \to u(t,z) + \delta(D_{t,z}u), \quad \mathcal{G}^* \times L^2(\mu). \tag{3.272}$$

On the other hand, to show (3.267), note that

$$\begin{aligned}(n+1)\tilde{f}(\cdot,(t,z)) =& f_n(\cdot,(t_1,z_1),(t,z)) + \cdots + f_n(\cdot,(t_n,z_n),(t,z)) + f_n(\cdot,\cdot,(t,z))\\ =& n\tilde{f}(\cdot,(t,z),\cdot) + f_n(\cdot,\cdot,(t,z))\end{aligned} \tag{3.273}$$

then we have,

$$\tilde{f}(\cdot,(t,z)) = \frac{n}{n+1}\tilde{f}_n(\cdot,(t,z),\cdot) + \frac{1}{n+1}f_n(\cdot,\cdot,(t,z)). \tag{3.274}$$

From the parallelogram inequality

$$||\tilde{f}_n(\cdot,(t,z))||^2_{L^2(\mu^n)} \leq \frac{2n^2}{(n+1)^2}||\tilde{f}(\cdot,(t,z),\cdot)||^2_{L^2(\mu^n)} + \frac{2}{(n+1)^2}||f_n(\cdot,\cdot,(t,z))||^2_{L^2(\mu^n)}. \tag{3.275}$$

Hence,

$$\begin{aligned}&||\tilde{f}_n||^2_{L^2(\mu^{n+1})}\\ =&\int_{[0,T]\times\mathbb{R}} ||\tilde{f}_n(\cdot,(t,z))||^2_{L^2(\mu^n)}\mu(dt,dz)\\ \leq&\int_{[0,T]\times\mathbb{R}} \left(\frac{2n^2}{(n+1)^2}||\tilde{f}_n(\cdot,(t,z),\cdot)||^2_{L^2(\mu^n)} + \frac{2}{(n+1)^2}||f_n(\cdot,\cdot,(t,z))||^2_{L^2(\mu^n)}\right)\mu(dt,dz)\\ =&\frac{2n^2}{(n+1)^2}||\tilde{f}_n||^2_{L^2(\mu^{n+1})} + \frac{2}{(n+1)^2}||f_n||^2_{L^2(\mu^{n+1})}.\end{aligned} \tag{3.276}$$

So therefore,

$$\begin{aligned}&\int_{[0,T]\times\mathbb{R}} ||D_{t,z}\delta(u)||^2_{\mathcal{G}_{-q}}\mu(dt,dz)\\ &\qquad\leq 2\sum_{n=0}^{\infty} n^2 n!||\tilde{f}_n||^2_{L^2(\mu^{n+1})}e^{-2qn} + 2\sum_{n=0}^{\infty} n!||f_n||^2_{L^2(\mu^{n+1})}e^{-2qn}\\ &\qquad\leq 2\int_{[0,T]\times\mathbb{R}} ||D_{t,z}\delta(u)||^2_{\mathcal{G}_{-q}}\mu(dt,dz) + 2||u||^2_{L^2(P\times\mu)} < \infty.\end{aligned} \tag{3.277}$$

The last term is finite from (i) and (iii). Then finally, we have the following expression as $m \to \infty$,

$$\int_{[0,T]\times\mathbb{R}} ||D_{t,z}\delta(u) - D_{t,z}\delta(u_m)||^2_{\mathcal{G}_{-q}}\mu(dt,dz) \leq 2\sum_{n=m+1}^{\infty} n^2 n!||\tilde{f}_n||_{L^2(\mu^{n+1})}e^{-2qn} + 2\sum_{n=m+1}^{\infty} n!||f_n||_{L^2(\mu^{n+1})}e^{-2qn} \to 0. \quad (3.278)$$

Hence, as $m \to \infty$

$$D_{t,z}(\delta(u_m)) \to u(t,z) + D_{t,z}(\delta(u)), \quad \mathcal{G}^* \times L^2(\mu). \quad (3.279)$$

Finally, the limits of the integral in (3.249) is a consequence of (3.272). ■

The special case of the theorem if u is predictable then by applying Corollary 3.8.7, we have the following corollary.

Corollary 3.9.2 *Let u satisfies the conditions of the preceding theorem and in addition, suppose that it is also predictable, then we have the following identity:*

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} u(s,x)M(ds,dx) = u(t,z) + \int_{[t,T]\times\mathbb{R}} D_{t,z}u(s,x)M(ds,dx). \quad (3.280)$$

Corollary 3.9.3 *Let u satisfies the conditions of the preceding corollary, then*

$$D_{t,z}\int_{[0,T]} u(s,0)dW(s) = \sigma^{-1}u(t,0)\mathbf{1}_{\{z=0\}} + \int_{[t,T]} D_{t,z}u(s,0)dW(s), \quad (3.281)$$

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} u(s,x)x\tilde{N}(ds,dx) = \sigma^{-1}u(t,z)\mathbf{1}_{\{z\neq 0\}} + \int_{[t,T]\times\mathbb{R}_0} D_{t,z}u(s,x)\tilde{N}(ds,dx). \quad (3.282)$$

Remark 3.9.4 *From the corollary, we have the following identities:*

(i) For $z = 0$,

$$D_{t,0}\int_{[0,T]} u(s,0)dW(s) = \sigma^{-1}u(t,0) + \int_{[t,T]} D_{t,z}u(s,0)dW(s), \quad (3.283)$$

$$D_{t,0}\int_{[0,T]\times\mathbb{R}_0} u(s,x)x\tilde{N}(ds,dx) = \int_{[t,T]\times\mathbb{R}_0} D_{t,z}u(s,x)\tilde{N}(ds,dx), \quad (3.284)$$

(ii) For $z \neq 0$,

$$D_{t,z} \int_{[0,T]} u(s,0) dW(s) = \int_{[t,T]} D_{t,z} u(s,0) dW(s), \tag{3.285}$$

$$D_{t,z} \int_{[0,T]\times\mathbb{R}_0} u(s,x) x \tilde{N}(ds,dx) = u(t,z) + \int_{[t,T]\times\mathbb{R}_0} D_{t,z} u(s,x) x \tilde{N}(ds,dx) \tag{3.286}$$

Proof From the independent measure

$$M(ds,dx) = \sigma dW(t)\delta_0(x) + x\tilde{N}(ds,dx)(1-\delta_0(x)) \tag{3.287}$$

and from (3.281), we have following:

$$\begin{aligned} &D_{t,z} \int_{[0,T]\times\mathbb{R}} u(s,x) M(ds,dx) \\ =&\sigma D_{t,z} \int_{[0,T]} u(s,0) dW(s) + D_{t,z} \int_{[0,T]\times\mathbb{R}_0} u(s,x) x \tilde{N}(ds,dx) \end{aligned} \tag{3.288}$$

and

$$\begin{aligned} u(t,z) + &\int_{[t,T]\times\mathbb{R}} D_{t,z} u(s,x) M(ds,dx) \\ &= u(t,z)\mathbf{1}_{\{z=0\}} + \sigma \int_{[0,T]} D_{t,z} u(s,0) dW(s) \\ &+ u(t,z)\mathbf{1}_{\{z\neq 0\}} + \int_{[0,T]\times\mathbb{R}_0} D_{t,z} u(s,x) x \tilde{N}(ds,dx). \end{aligned} \tag{3.289}$$

Separating the continuous and the jump term, we obtain the desired identity. ■

We extend the concept of $(S)^*$ integrability [45], [64] in the Canonical Lévy space.

Definition 3.9.2 *$(S)^*$ integrability*

The random field $u : \mathbb{R}_+ \times \mathbb{R}$ is $(S)^*$-integrable if the action of u for all $F \in (S)^*$ satisfies

$$< u, F > \in L^1(\mu) \tag{3.290}$$

The $(S)^*$-integral denoted by

$$I \doteq \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z) \mu(dt,dz) \tag{3.291}$$

is a unique element in $(S)^*$ *such that*

$$\left\langle \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)\mu(dt,dz), F \right\rangle = \int_{\mathbb{R}_+\times\mathbb{R}} \langle u(t,z), F\rangle \, \mu(dt,dz). \tag{3.292}$$

Theorem 3.9.5 *Wick-Skorohod Identity*

Let u *be Skorohod integrable with respect to* M*, then* $u(t,z) \diamond \dot{M}(t,x)$*, for all* $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$ *is* $(S)^*$ *integrable and*

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(t,x)M(\delta t, dz) = \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z) \diamond \dot{M}(t,z)\mu(dt,dz). \tag{3.293}$$

Proof Since $\mathcal{G} \in (S)^*$, then it remains to show the identity in (3.293). Likewise, since u is Skorohod-integrable with respect to M, then it has a representation of the form

$$u(t,z) = \sum_{\alpha\in\mathcal{I}} c_\alpha(t,z)\mathbb{K}_\alpha = \sum_{n=0}^{\infty} I_n(f_n(\cdot,(t,z))) \tag{3.294}$$

where $f_n(\cdot,(t,z)) \in L^2_s(\mu^n)$. The right-hand side of (3.293) yields the following:

$$\begin{aligned}
&\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z) \diamond \dot{M}(dt,dz)\mu(dt,dz) \\
&\quad= \int_{\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha\in\mathcal{I}} c_\alpha(t,x)\mathbb{K}_\alpha \diamond \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} e_k(t)\pi_m(z)\mathbb{K}_{\epsilon^{\kappa(k,m)}}\mu(dt,dz) \\
&\quad= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} \int_{\mathbb{R}_+\times\mathbb{R}} c_\alpha(t,x)e_k(t)\pi_m(z)\mu(dt,dx)\mathbb{K}_{\alpha+\epsilon^{\kappa(k,m)}} \\
&\quad= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} < c_\alpha, e_k p_m >_{L^2(\mu)} \mathbb{K}_{\alpha+\epsilon^{\kappa(k,m)}}.
\end{aligned} \tag{3.295}$$

Now since

$$f_n(\cdot,(t,z)) = \sum_{|\alpha|=n} c_\alpha(t,z)\delta^{\hat{\otimes}\alpha} \tag{3.296}$$

Then, $f_n(\cdot,(t,z))$ has the following orthonormal expansion

$$f_n(\cdot,(t,z)) = \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\sum_{|\alpha|=n} < c_\alpha, e_k\pi_m >_{L^2(\mu)} \delta^{\hat{\otimes}\alpha} e_k(t)\pi_m(z). \tag{3.297}$$

Hence, the left-hand side of (3.293) yields the following:

$$\begin{aligned}
&\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)M(\delta t,dz)\\
&\quad=\int_{\mathbb{R}_+\times\mathbb{R}}\sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x))M(\delta t,dz)\\
&\quad=\int_{\mathbb{R}_+\times\mathbb{R}}\sum_{n=0}^{\infty} I_n\left(\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\sum_{|\alpha|=n} < c_\alpha, e_k\pi_m >_{L^2(\mu)} \delta^{\hat{\otimes}\alpha} e_k(t)\pi_m(z)\right) M(\delta t,dz)\\
&\quad=\sum_{n=0}^{\infty}\int_{\mathbb{R}_+\times\mathbb{R}} I_n\left(\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\sum_{|\alpha|=n} < c_\alpha, e_k\pi_m >_{L^2(\mu)} \delta^{\hat{\otimes}\alpha}\otimes\delta^{\otimes\epsilon_{\kappa(k,m)}}\right) M(\delta t,dz)\\
&\quad=\sum_{n=0}^{\infty} I_{n+1}\left(\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\sum_{|\alpha|=n} < c_\alpha, e_k\pi_m >_{L^2(\mu)} \delta^{\hat{\otimes}(\alpha+\epsilon_{\kappa(k,m)})}\right)\\
&\quad=\sum_{n=0}^{\infty}\sum_{|\alpha|=n}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} < c_\alpha, e_k\pi_m >_{L^2(\mu)} I_{n+1}\left(\delta^{\hat{\otimes}(\alpha+\epsilon_{\kappa(k,m)})}\right)\\
&\quad=\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} < c_\alpha, e_k p_m >_{L^2(\mu)} \mathbb{K}_{\alpha+\epsilon^{\kappa(k,m)}}. \qquad (3.298)
\end{aligned}$$

Finally, form (3.295) and (3.298) gives us the desired identity. ■

3.10 Clark-Ocone Theorem in $L^2(P)$

With the framework concepts presented for the white noise theory for Canonical Lévy processes, our goal is to show a Clark-Ocone theorem in $L^2(P)$ with respect to the independent random measure M.

The steps in proving the Clark-Ocone theorem in $L^2(P)$ is similar to the Wiener and Poisson white noise cases [27] by first showing the Clark-Ocone theorem for a Wick polynomial then establish an auxiliary lemma (Lemma 3.10.4) that will prove the Clark-Ocone theorem in $L^2(P)$.

Denote the following polynomial:

$$P(x)=\sum_{\alpha\in\mathcal{I}} c_\alpha x^\alpha \qquad ,x\in\mathbb{R}^{\mathbb{N}}, c_\alpha\in\mathbb{R} \qquad (3.299)$$

where $x^\alpha \doteq x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and $x_j^0 = 1, j \in \mathbb{N}$. Denote its Wick version of the polynomial at $X = (X_1, \cdots X_n)^T$ by

$$P^\diamond(X) = \sum_{\alpha \in \mathcal{I}} c_\alpha X^{\diamond \alpha}. \tag{3.300}$$

Throughout this section, we assume that a process $u : \mathbb{R}_+ \to (S)^*$ is differentiable in the $(S)^*$ sense. define the following processes in $(S)^*$ as follows:

$$\begin{aligned} X_{k,m} &= \int_{\mathbb{R}_+ \times \mathbb{R}} e_k(x) \pi_m(s) M(ds, dx) = \mathbb{K}_{\epsilon^{\kappa(k,m)}}, \\ X_{k,m}^{(t)} &= \int_{[0,t] \times \mathbb{R}} e_k(x) \pi_m(s) M(ds, dx) \doteq \mathbb{K}_{\epsilon^{\kappa(k,m)}}^{(t)}. \end{aligned} \tag{3.301}$$

from the relation (3.163) in $(S)^*$, then from the Wick-Skorohod identity, we have

$$\begin{aligned} X_{k,m} &= \int_{\mathbb{R}_+ \times \mathbb{R}} e_k(x) \pi_m(s) \dot{M}(ds, dx) \mu(ds, dx), \\ X_{k,m}^{(t)} &= \int_{[0,t] \times \mathbb{R}} e_k(x) \pi_m(s) \dot{M}(ds, dx) \mu(ds, dx). \end{aligned} \tag{3.302}$$

Then, from the have the following derivative in $(S)^*$

$$\frac{d}{dt} X_{k,m}^{(t)} = e_k(x) L_m(t) \tag{3.303}$$

where

$$L_m(t) = \int_{\mathbb{R}} \pi_m(t) \dot{M}(ds, dx) \eta(dx). \tag{3.304}$$

We let $P(x)$ be a polynomial in $\mathbb{R}^n$, that is, $P(x)$ can be written as follows:

$$P(x) = \sum_{\alpha \in \mathcal{I}} c_\alpha x^\alpha \qquad x \in \mathbb{R}^n, \quad c_\alpha \in \mathbb{R}, \quad \mathcal{I} = \mathbb{N}^n. \tag{3.305}$$

and let

$$\begin{aligned} X &= (X_{k_1,m_1}, \cdots, X_{k_n,m_n})^T, \\ X^{(t)} &= (X_{k_1,m_1}^{(t)}, \cdots, X_{k_n,m_n}^{(t)})^T, \\ \alpha &= (\alpha_{\kappa(k_1,m_1)}, \cdots, \alpha_{\kappa(k_n,m_n)})^T \end{aligned} \tag{3.306}$$

where $k_i, m_i \in \mathbb{N}$, for all $i \in \{1, \cdots, n\}$. Then, its Wick version of the polynomial at $X = (X_1, \cdots X_n)^T$ by

$$P^{\diamond}(X) = \sum_{\alpha \in \mathcal{I}} c_\alpha X^{\diamond \alpha}. \tag{3.307}$$

Moreover, we have the following identities:

$$\begin{aligned} X^{\diamond\alpha} &= (X_{k_1,m_1})^{\diamond\kappa(k_1,m_1)} \diamond \cdots \diamond \left(X_{k,m_n}\right)^{\diamond\kappa(k_n,m_n)} = \mathbb{K}_\alpha, \\ \left(X^{(t)}\right)^{\diamond\alpha} &= \left(X^{(t)}_{k_1,m_1}\right)^{\diamond\kappa(k_1,m_1)} \diamond \cdots \diamond \left(X^{(t)}_{k,m_n}\right)^{\diamond\kappa(k_n,m_n)} = \mathbb{K}^{(t)}_\alpha. \end{aligned} \tag{3.308}$$

If $F = P^{\diamond}(X) \in \mathcal{G}^*$, then its generalized conditional expectation in $\mathcal{G}^*$ is given by

$$E[F|\mathcal{F}_t] = \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(t)}\right)^{\diamond\alpha} = P^{\diamond}\left(X^{(t)}\right). \tag{3.309}$$

We define the concept of $\mathcal{F}_T$ measurablity in $\mathcal{G}^*$ in the following definition.

Definition 3.10.1 *[27] Let $T > 0$ be a constant, we say that $F \in \mathcal{G}^*$ is $\mathcal{F}_T$ measurable if*

$$E[F|\mathcal{F}_T] = F. \tag{3.310}$$

Lemma 3.10.1 *[27] $F \in \mathcal{G}^*$ is $\mathcal{F}_T$ measurable iff F can be written as*

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond\alpha}. \tag{3.311}$$

Lemma 3.10.2 *Differentiation of the Wick Polynomial*

(i)

$$D_{t,z}P(X) = \sum_{i=1}^{n} \sum_{\alpha \in \mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\alpha - \epsilon^{\kappa(k_i,m_i)}} e_{k_i}(t)\pi_{m_i}(z), \tag{3.312}$$

$$D_{t,z}P^{\diamond}(X) = \sum_{i=1}^{n} \sum_{\alpha \in \mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\diamond(\alpha - \epsilon^{\kappa(k_i,m_i)})} e_{k_i}(t)\pi_{m_i}(z). \tag{3.313}$$

(ii)

$$\frac{d}{dt} P^{\diamond}(X^{(t)}) = \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond e_{k_i}(t) L_{m_i}(z). \tag{3.314}$$

Proof (i) From the chain rule,

$$D_{t,z}P(X) = \sum_{i=1}^{n} \frac{\partial P(X)}{\partial x_i} D_{t,z}X_{k_i,m_i}, \tag{3.315}$$

$$D_{t,z}P^{\diamond}(X) = \sum_{i=1}^{n} \frac{\partial P^{\diamond}(X)}{\partial x_i} D_{t,z}X_{k_i,m_i}. \tag{3.316}$$

Since

$$\frac{\partial P(X)}{\partial x_i} = \sum_{\alpha \in \mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\alpha - \epsilon^{\kappa(k_i,m_i)}}, \tag{3.317}$$

$$\frac{\partial P^{\diamond}(X)}{\partial x_i} = \sum_{\alpha \in \mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\diamond(\alpha - \epsilon^{\kappa(k_i,m_i)})} \tag{3.318}$$

and

$$X_{k_i,m_i} = \int_{\mathbb{R}_+ \times \mathbb{R}} e_{k_i}(s)\pi_{m_i}(x) M(ds,dx) = I_1(e_{k_i}\pi_{m_i}) \tag{3.319}$$

Then,

$$D_{t,z}X_{k_i,m_i} = e_{k_i}(t)\pi_{m_i}(x). \tag{3.320}$$

Plugging (3.317) – (3.320) into (3.315) – (3.316), yields the desired result.

(ii) From the Wick chain rule and (3.319), we obtain

$$\begin{aligned}\frac{d}{dt}P^{\diamond}(X^{(t)}) &= \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond \frac{d}{dt} X^{(t)}_{k_i,m_i} \\ &= \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond e_{k_i}(t) L_{m_i}(z). \end{aligned} \tag{3.321}$$

■

To show the Clark-Ocone theorem in $L^2(P)$, we first establish a Clark-Ocone theorem for polynomials.

Theorem 3.10.3 *Clark-Ocone Theorem for Polynomials*
Let $F \in \mathcal{G}^*$ be an $\mathcal{F}_T$ measurable Wick polynomial of degree n, then

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}] M(dt,dz). \tag{3.322}$$

Proof Since F Wick polynomial of degree n, then it has of the form

$$F = P^{\diamond}(X) = \sum_{\alpha \in \mathcal{I}} c_\alpha X^{\diamond \alpha} \tag{3.323}$$

where $P(x)$ is a polynomial in $\mathbb{R}^n$. Moreover, since F is an an $\mathcal{F}_T$ measurable, then

$$\begin{aligned} F =& E[F|\mathcal{F}_T] = E\left[\sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond \alpha} \middle| \mathcal{F}_T \right] \\ =& \sum_{\alpha \in \mathcal{I}} c_\alpha E\left[\left(X^{(T)}\right)^{\diamond \alpha} \middle| \mathcal{F}_T \right] \\ =& \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond \alpha}. \end{aligned} \tag{3.324}$$

The expansion F and $E[D_{t,z}F|\mathcal{F}_{t-}]$ consists of finite number of terms. Hence, both processes are Skorohod integrable. Then from the Wick-Skorohod identity and from the preceding lemma,

$$\begin{aligned} &\int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz) \\ &\quad = \int_{[0,T]\times\mathbb{R}} E\left[\sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} (X)\, e_{k_i}(t)\pi_{m_i}(z) \middle| \mathcal{F}_{t-} \right] \diamond \dot{M}(t,z)\eta(dz)dt \\ &\quad = \int_{[0,T]\times\mathbb{R}} \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) e_{k_i}(t)\pi_{m_i}(z) \diamond \dot{M}(t,z)\eta(dz)dt \\ &\quad = \int_{[0,T]} \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond \int_{\mathbb{R}} \pi_{m_i}(z)\dot{M}(t,z)\eta(dz)e_{k_i}(t)dt \\ &\quad = \int_{[0,T]} \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond e_{k_i}(t) \int_{\mathbb{R}} \pi_{m_i}(z)\dot{M}(t,z)\eta(dz)dt. \end{aligned} \tag{3.325}$$

Now, since

$$e_{k_i}(t) \int_{\mathbb{R}} \pi_{m_i}(z)\dot{M}(t,z)\eta(dz)dt = e_{k_i}(t)L_{m_i}(t) = \frac{d}{dt} X^{(t)}_{k_i,m_i}. \tag{3.326}$$

Then, from the Wick chain rule and since F is be $\mathcal{F}_T-$ measurable, we finally obtain

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz)\\
&=\int_{[0,T]}\sum_{i=1}^{n}\left(\frac{\partial P}{\partial x_i}\right)^{\diamond}\left(X^{(t)}\right)\diamond\frac{d}{dt}X^{(t)}_{k_i,m_i}dt\\
&=\int_{[0,T]}\frac{d}{dt}P^{\diamond}\left(X^{(t)}\right)\\
&=P^{\diamond}\left(X^{(T)}\right)-P^{\diamond}\left(X^{(0)}\right)\\
&=E[F|\mathcal{F}_T]-E[F|\mathcal{F}_0]\\
&=F-E[F]. \qquad (3.327)
\end{aligned}$$

■

We need the following auxiliary lemma in establishing a Clark-Ocone in $L^2(P)$.

Theorem 3.10.4 *Let $F\in\mathcal{G}^*$, then we have the following:*

(i) $D_{t,z}F\in\mathcal{G}^*$, $\quad\mathcal{G}^*,\mu$ *a.e.,*

(ii) Let $F_n\in\mathcal{G}^*$, $\forall\ n\in\mathbb{N}$ such that $F_n\to F$ in $\mathcal{G}^*$ as $n\to\infty$, then there exists a sub-sequence F_{n_k}, $k\in\mathbb{N}$ such that $D_{t,z}F_{n_k}\to D_{t,z}F\in\mathcal{G}^*$ as $k\to\infty$, $\mathcal{G}^*,\mu$ a.e.

Proof The proof is similar to the proof of Okur [66] in the Wiener case.

(i) Since $F\in\mathcal{G}^*$, then it has a formal expansion

$$F=\sum_{\alpha\in\mathcal{I}}c_\alpha\mathbb{K}_\alpha \qquad (3.328)$$

and there exists $q\in\mathbb{R}$ such that

$$||F||^2_{\mathcal{G}_{-q}}=\sum_{\alpha\in\mathcal{I}}\alpha!c_\alpha^2e^{-2q|\alpha|}. \qquad (3.329)$$

Then

$$\begin{aligned}
D_{t,z}F&=\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}c_{\beta+\varepsilon_{\kappa(k,m)}}(\beta_{\kappa(k,m)}+1)e_k(t)\pi_m(z)\mathbb{K}_\beta\\
&=\sum_{\beta\in\mathcal{I}}g_\beta(t,z)\mathbb{K}_\beta \qquad (3.330)
\end{aligned}$$

where

$$g_\beta(t,z) = \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_{\beta+\varepsilon_{\kappa(k,m)}}(\beta_{\kappa(k,m)}+1)e_k(t)\pi_m(z). \tag{3.331}$$

Since $e_k p_m$ is an orthonormal basis with respect to μ, then

$$\int_{\mathbb{R}_+\times\mathbb{R}} |g_\beta(t,z)|^2\eta(dz)dt = \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c^2_{\beta+\varepsilon_{\kappa(k,m)}}(\beta_{\kappa(k,m)}+1)^2. \tag{3.332}$$

Also, since

$$\| D_{t,z}F \|^2_{\mathcal{G}_{-(q+1)}} = \sum_{\beta\in\mathcal{I}_N} g_\beta(t,z)\beta! e^{-2(q+1)|\beta|}. \tag{3.333}$$

From the identity $(z+1)e^{-z} \leq 1$ for all $z \geq 0$, then we obtain the following expression

$$\begin{aligned}
&\int_{\mathbb{R}_+\times\mathbb{R}} \| D_{t,z}F \|^2_{\mathcal{G}_{-(q+1)}} \eta(dx)dt \\
&= \sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c^2_{\beta+\varepsilon_{\kappa(k,m)}}(\beta_{\kappa(k,m)}+1)^2\beta! e^{-2(q+1)|\beta|} \\
&= \sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} (\beta_{\kappa(k,m)}+1)e^{-2(q+1)|\beta|}c^2_{\beta+\varepsilon_{\kappa(k,m)}}(\beta+\varepsilon_{\kappa(k,m)})! \\
&\leq \sum_{\beta\in\mathcal{I}}(|\beta|+1)e^{-2(q+1)|\beta|}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c^2_{\beta+\varepsilon_{\kappa(k,m)}}(\beta+\varepsilon_{\kappa(k,m)})! \\
&= \sum_{\beta\in\mathcal{I}}(|\beta|+1)e^{-2(q+1)|\beta|}\sum_{|\alpha|=|\beta|+1} c^2_\alpha \alpha! \\
&= \sum_{n=0}^{\infty}\sum_{|\alpha|=n+1} \alpha! c^2_\alpha e^{-2(q+1)n} \\
&\leq \sum_{n=0}^{\infty}\sum_{|\alpha|=n} \alpha! c^2_\alpha e^{-2qn} \\
&= \sum_{\alpha\in\mathcal{I}} \alpha! c^2_\alpha e^{-2q|\alpha|} \\
&= ||F||^2_{\mathcal{G}_{-q}}.
\end{aligned} \tag{3.334}$$

Hence, $D_{t,z}F \in \mathcal{G}_{-(q+1)} \subset \mathcal{G}^*, \quad \mathcal{G}^*, \mu$ *a.e.*

(ii) Since $F_n \to F$ in $\mathcal{G}^*$ as $n \to \infty$, then $\exists q \in \mathbb{N}_0$ such that $\| F_n - F \|_{\mathcal{G}_{-q}} \to 0$ as $n \to \infty$. Let $G_n = F_n - F$, then it is suffice to show that there exists a sub-sequence G_{n_k} such that $D_{t,x}G_{n_k} \to 0$. From our previous result, we obtain

$$\int_{\mathbb{R}_+\times\mathbb{R}} \| D_{t,z}G_n \|^2_{\mathcal{G}_{-(q+1)}} \eta(dx)dt \leq \| G_n \|^2_{\mathcal{G}_{-q}} \to 0. \tag{3.335}$$

Hence, $\| D_{t,z}G_n \|_{\mathcal{G}_{-(q+1)}} \to 0$ in $L^2(\eta \times \lambda)$. Thus, there exists a sub-sequence $\| D_{t,z}G_{n_k} \|_{\mathcal{G}_{-(q+1)}}$ for $k \in \mathbb{N}$ such that as $k \to \infty$, $D_{t,z}G_{n_k} \to 0$ in $\mathcal{G}^*$, $\mathcal{G}^*$, μ *a.e.*

■

Theorem 3.10.5 *Clark-Ocone Theorem in $L^2(P)$*

Let $F \in L^2(P)$ be $\mathcal{F}_T$measurable, then

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz) \tag{3.336}$$

where $E[D_{t,z}F|\mathcal{F}_{t-}] \in L^2(P\times\mu)$, $(t,z) \in [0,T]\times\mathbb{R}$.

Proof Since F is $\mathcal{F}_T$-measurable, then it has chaos expansion of the form

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{3.337}$$

Let F_n be the truncation of F such that

$$F_n = \sum_{\alpha\in\mathcal{I}_n} c_\alpha \mathbb{K}_\alpha \tag{3.338}$$

where $\mathcal{I}_n = \{\alpha \in \mathcal{I} : |\alpha| \leq n, Index(\alpha) \leq n\}$. Then, from the Clark-Ocone theorem for polynomials, for at $n \in \mathbb{N}$,

$$F_n = E[F_n] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz). \tag{3.339}$$

From Itô's representation theorem, there exists a unique predictable process $u(t,z)$, $(t,z) \in [0,T]\times\mathbb{R}$ such that

$$E\left[\int_{[0,T]\times\mathbb{R}} u^2(t,z)\mu(t,z)\right] < \infty \tag{3.340}$$

and

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} u(t,z)M(dt,dz). \tag{3.341}$$

From the isometry relation (Theorem 2.8.1), we obtain

$$\begin{aligned}
&E\left[|(F_n - E[F_n]) - (F - E[F])|^2\right] \\
=&E\left[\left|\int_{[0,T]\times\mathbb{R}} (E[D_{t,z}F_n|\mathcal{F}_{t-}] - u(t,z))M(dt,dz)\right|^2\right] \\
=&E\left[\int_{[0,T]\times\mathbb{R}} |E[D_{t,z}F_n|\mathcal{F}_{t-}] - u(t,z)|^2\mu(dt,dz)\right].
\end{aligned} \tag{3.342}$$

Then, since $F_n \to F$ in $L^2(P)$, then the right hand side approaches zero as $n \to \infty$. Thus, we have the following convergence:

$$E[D_{t,z}F_n|\mathcal{F}_{t\text{-}}] \to u(t,z), \qquad L^2(P\times\mu). \tag{3.343}$$

Now since $F_n \to F \in L^2(P) \subset \mathcal{G}^*$, then from Lemma 3.10.4 then there exists a sub-sequence F_{n_k}, $k \in \mathbb{N}$ such that

$$E[D_{t,x}F_{n_k}|\mathcal{F}_{t-}] \to E[D_{t,x}F|\mathcal{F}_{t-}] \in \mathcal{G}^*, \qquad k\to\infty \quad \mathcal{G}^*, \mu \quad a.e. \tag{3.344}$$

Taking a further sub-sequence, we have

$$E[D_{t,x}F_{n_k}|\mathcal{F}_{t-}] \to u(t,z), \qquad k\to\infty \qquad L^2(P), \mu \quad a.e. \tag{3.345}$$

Thus, it follows that

$$u(t,z) = E[D_{t,x}F|\mathcal{F}_{t-}], \qquad L^2(P), \mu \quad a.e. \tag{3.346}$$

■

3.11 Multivariate Extension

In this section, we provide an overview of extending the white noise frame for the Canonical Lévy process in the multivariate setting. We follow a similar framework of [1] and [64] which combines the Gaussian white noise process and pure jump Lévy white noise process as a product σ-field of these processes.

Since the arguments of the theorems in the multivariate case is similar to the univariate case, then we shall state the theorems without proof.

3.11.1 Notations

Let, $(\Omega^{(j)}, \mathcal{F}^{(j)}, \{\mathcal{F}_t^{(j)}\}_{t\geq 0}, P^{(j)})$, $j \in \{1, \cdots, N\}$ be an independent probability space for the white noise Canonical Lévy process. Its independent measure M_j is given by

$$M_j(E) = \sigma_j \int_{E_0} dW_j(t) + \int_{E'} z d\tilde{N}_j(dt, dz) \tag{3.347}$$

where $E_0 = \{t \in \mathbb{R}_+ : (t, 0) \in E\}$ and $E' = E \setminus E_0$. Then for $E_1, E_2 \in \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mu_j(E_1) < \infty, \mu_j(E_2) < \infty$

$$E[M_j(E_1)M_j(E_2)] = \mu_j(E_1 \cap E_2) \tag{3.348}$$

where μ_j is a measure on $([0, T] \times \mathbb{R}, \mathfrak{B}([0, T] \times \mathbb{R})$, where

$$\mu_j(E) = \sigma_j^2 \int_{E_0} dt + \int_{E'} z^2 d\nu_j(z) dt, \qquad E \in \mathfrak{B}([0, T] \times \mathbb{R}). \tag{3.349}$$

In differential form, we have

$$\mu_j(dt, dz) = \sigma_j^2 d\delta_0(z) dt + z^2(1 - \delta_0(z)) d\nu_j(z) dt = \lambda_j(dt)\eta_j(dx) \tag{3.350}$$

where $\lambda_j(dt) = dt$ is the Lebesgue measure and

$$\eta_j(dz) = \sigma_j^2 d\delta_0(z) dt + z^2(1 - \delta_0(z)) d\nu_j(z). \tag{3.351}$$

Denote $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space of the multivariate white noise Canonical Lévy process which is a product space of $(\Omega^{(j)}, \mathcal{F}^{(j)}, \{\mathcal{F}_t^{(j)}\}_{t\geq 0}, P^{(j)})$, $j \in \{1, \cdots, N\}$ where

$$\begin{aligned} \Omega =& \Omega^{(1)} \times \cdots \times \Omega^{(N)}, \\ \mathcal{F} =& \mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(N)}, \\ \mathcal{F}_t =& \mathcal{F}_t^{(1)} \otimes \cdots \otimes \mathcal{F}_t^{(N)}, \quad t \geq 0 \\ P =& P^{(1)} \times \cdots \times P^{(N)}. \end{aligned} \tag{3.352}$$

We let the index $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(N)})$ where $\alpha_j \in \mathcal{I}$ and the index set $\mathcal{I}_N = \mathcal{I}^{(1)} \times \cdots \times \mathcal{I}^{(N)}$ where $\mathcal{I}^{(j)} = \mathcal{I}$ where $j \in \{1, \cdots, N\}$. The white noise processes X_j, $\dot{X}_j$, M_j, and $\dot{M}_j$ are defined naturally from X, $\dot{X}$, M, and $\dot{M}$ respectively. Likewise, we have the following Radon-Nikodym relation in $(S)^*$

$$M_j(dt, dz) = \dot{M}_j(t, z)\mu_j(dt, dz). \tag{3.353}$$

3.11.2 Chaos Expansion

Denote the following notations:

$$|\alpha| = \sum_{j=1}^{N} |\alpha^{(j)}|, \qquad \alpha! = \prod_{j=1}^{N} \alpha^{(j)}!. \tag{3.354}$$

Consider the product of the form

$$\mathbb{K}_\alpha(\omega) = \prod_{j=1}^{N} \mathbb{K}_{\alpha^{(j)}}(\omega^{(j)}), \qquad \omega = (\omega^{(1)}, \cdots, \omega^{(N)}) \tag{3.355}$$

$\{\mathbb{K}_\alpha\}_{\alpha\in\mathcal{I}_N}$ forms an orthogonal basis is $L^2(P)$ with the following relation:

$$E\left[\mathbb{K}_\alpha \mathbb{K}_\beta\right] = \alpha! \mathbf{1}_{\{\alpha=\beta\}}. \tag{3.356}$$

For $F \in L^2(P)$, $\mathcal{F}_T$-measurable can be uniquely written of the form

$$F = \sum_{\alpha\in\mathcal{I}_N} c_\alpha \mathbb{K}_\alpha. \tag{3.357}$$

for some constants $c_\alpha \in \mathbb{R}$. In terms of the iterated integral, F can be written as follows:

$$F = \sum_{n \in \mathbb{N}_0^N} \prod_{j=1}^{N} I_{n_j}\left(f_{j,n_j}\right). \tag{3.358}$$

where $n = (n_1, \cdots, n_N)^T$,$n_j \in \mathbb{N}_0$ and $f_{j,n_j} \in L_s^2(\mu^{n_j})$, $j \in \{1, \cdots, N\}$. From isometry and independence, we have following relation:

$$||F||_{L^2(P)}^2 = \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \prod_{j=1}^{N} \alpha^{(j)}! = \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \alpha!. \tag{3.359}$$

Alternatively, in terms of the iterated integral:

$$||F||_{L^2(P)}^2 = \sum_{n \in \mathbb{N}_0^N} \prod_{j=1}^{N} n_j! ||f_{j,n_j}||_{L^2(\mu^{n_j})}^2 = \sum_{n \in \mathbb{N}_0^N} n! \prod_{j=1}^{N} ||f_{j,n_j}||_{L^2(\mu^{n_j})}^2. \tag{3.360}$$

3.11.3 Stochastic Test and Distribution Functions

The space $\mathcal{G}$ and $\mathcal{G}^*$

Suppose that F has a formal expansion of the form (3.357). Then, F belongs to the space $\mathcal{G}_q$, $q \in \mathbb{R}$ if

$$\begin{aligned} ||F||_{\mathcal{G}_q}^2 &= \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \prod_{j=1}^{N} \alpha^{(j)}! \exp(2q\alpha^{(j)}) \\ &= \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \alpha! \exp(2q|\alpha|) < \infty. \end{aligned} \tag{3.361}$$

Alternatively, in terms of the iterated integral:

$$\begin{aligned} ||F||_{\mathcal{G}_q}^2 &= \sum_{n \in \mathbb{N}_0^N} \prod_{j=1}^{N} n_j! ||f_{j,n_j}||_{L^2(\mu^{n_j})}^2 \exp(2qn_j) \\ &= \sum_{n \in \mathbb{N}_0^N} n! \prod_{j=1}^{N} ||f_{j,n_j}||_{L^2(\mu^{n_j})}^2 \cdot \exp(2q|n|) < \infty. \end{aligned} \tag{3.362}$$

Define the stochastic test function $\mathcal{G}$ given by

$$\mathcal{G} = \bigcap_{q>0} \mathcal{G}_q \tag{3.363}$$

endowed with inductive topology. The stochastic test function $\mathcal{G}^*$ is defined as

$$\mathcal{G} = \bigcup_{q>0} \mathcal{G}_{-q} \tag{3.364}$$

endowed with projective topology. Note that the $\mathcal{G}^*$ is the dual of $\mathcal{G}$. Let $G \in \mathcal{G}$ and $F \in \mathcal{G}^*$ with the following formal expansion:

$$F = \sum_{\alpha \in \mathcal{I}_N} c_\alpha \mathbb{K}_\alpha, \qquad G = \sum_{\alpha \in \mathcal{I}_N} d_\alpha \mathbb{K}_\alpha. \tag{3.365}$$

Then the action of F on G is given by

$$< G, F >_{\mathcal{G},\mathcal{G}^*} = \sum_{\alpha \in \mathcal{I}_N} \alpha! c_\alpha d_\alpha. \tag{3.366}$$

Kontratiev Spaces and Hida Spaces

Let $p \in [0, 1]$. Suppose that F has a formal expansion of the form (3.357). Then, F belongs to the space $(\mathcal{S})_q$, $q \in \mathbb{R}$ if

$$\begin{aligned} \|F\|_{p,q}^2 &= \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \prod_{j=1}^{N} (\alpha^{(j)}!)^{1+p} (2\mathbb{N})^{\alpha^{(j)} q} \\ &= \sum_{\alpha \in \mathcal{I}_N} c_\alpha^2 \prod_{j=1}^{N} \alpha^{(j)}! (2\mathbb{N})^{\alpha^{(j)} q} < \infty. \end{aligned} \tag{3.367}$$

Define the Kondratiev test function $(\mathcal{S})_p$ as

$$(\mathcal{S})_p = \bigcap_{q>0} (\mathcal{S})_{p,q} \tag{3.368}$$

endowed with the projective topology. The Kondratiev distribution function $(\mathcal{S})_{-p}$ as

$$(\mathcal{S})^* = \bigcup_{q>0} (\mathcal{S})_{-p,-q} \tag{3.369}$$

endowed with the inductive topology. Note that $(\mathcal{S})^*$ is a dual of $(\mathcal{S})$. Let $G \in (\mathcal{S})$ and $F \in \mathcal{S}^*$ with the following formal expansion:

$$F = \sum_{\alpha \in \mathcal{I}_N} c_\alpha \mathbb{K}_\alpha, \qquad G = \sum_{\alpha \in \mathcal{I}_N} d_\alpha \mathbb{K}_\alpha. \tag{3.370}$$

Note that $(\mathcal{S})_{-p}$ is a dual of $(\mathcal{S})_p$. The action of $G \in (\mathcal{S})_{-p}$ on $F \in (\mathcal{S})_p$, with the formal expansion of F and G of the form (3.121) is given by

$$< G, F >= \sum_{\alpha \in \mathcal{I}_N} \alpha! c_\alpha d_\alpha. \tag{3.371}$$

The Hida spaces are the special cases of the Kondratiev spaces. The Hida test function $(\mathcal{S})$ and Hida distribution function $(\mathcal{S})^*$ is given by $(\mathcal{S}) = (\mathcal{S})_0$ and $(\mathcal{S})^* = (\mathcal{S})_{-0}$ respectively. From the above definitions, we have the following inclusions for $p \in [0, 1]$:

$$(\mathcal{S})_1 \subset (\mathcal{S})_p \subset (\mathcal{S})_0 \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-p} \subset (\mathcal{S})_{-1}. \tag{3.372}$$

3.11.4 Wick Product

Definition 3.11.1 *Wick Product*
Let $F = \sum_{\alpha \in \mathcal{I}_N} a_\alpha \mathbb{K}_\alpha \in (\mathcal{S})_{-1}$ *and* $G = \sum_{\beta \in \mathcal{I}_N} b_\beta \mathbb{K}_\beta \in (\mathcal{S})_{-1}$*, then the Wick Product of* X *and* Y *denoted by* $X \diamond Y$ *is defined as*

$$X \diamond Y = \sum_{\alpha \in \mathcal{I}_N} \sum_{\beta \in \mathcal{I}_N} a_\alpha b_\beta \mathbb{K}_{\alpha+\beta} = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \mathbb{K}_\gamma. \tag{3.373}$$

3.11.5 Stochastic Derivatives

We extend the stochastic derivative in the multivariate case as follows:

$$D_{j,t,z} F = \sum_{\alpha \in \mathcal{I}_N} \sum_{i \in \mathbb{N}} c_\alpha \alpha_i \mathbb{K}_{\alpha - \varepsilon_i^{(j)}} \delta^{\hat{\otimes} \varepsilon_i^{(j)}}(t, z). \tag{3.374}$$

where $\varepsilon_i^{(j)} = (\mathbf{0}, \cdots, \varepsilon_i, \cdots, \mathbf{0})^T$ such that ε_i is the j^{th} subvector of $\varepsilon_i^{(j)}$ and a zero vector otherwise. Likewise, we can also express $D_{j,t,z}F$ as follows:

$$D_{j,t,z}F = \sum_{\alpha\in\mathcal{I}_N}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_\alpha \alpha^{(j)}_{\kappa(k,m)} \mathbb{K}_{\alpha-\varepsilon^{(j)}_{\kappa(k,m)}} \delta^{\hat{\otimes}\varepsilon^{(j)}_{\kappa(k,m)}}(t,z) \tag{3.375}$$

$$= \sum_{\alpha\in\mathcal{I}_N}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_\alpha \alpha^{(j)}_{\kappa(k,m)} \mathbb{K}_{\alpha-\varepsilon^{(j)}_{\kappa(k,m)}} e_k(t)\pi_m(z) \tag{3.376}$$

$$= \sum_{\beta\in\mathcal{I}_N}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_{\beta+\varepsilon^{(j)}_{\kappa(k,m)}} (\beta^{(j)}_{\kappa(k,m)}+1)\mathbb{K}_\beta e_k(t)\pi_m(z). \tag{3.377}$$

Theorem 3.11.1 *Let*

$$F = \sum_{n\in\mathbb{N}_0^N}\prod_{i=1}^N I_{n_i}\left(f_{i,n_i}\right) \in L^2(P). \tag{3.378}$$

where $n = (n_1,\cdots,n_N)^T$,$n_j \in \mathbb{N}_0$ *and* $f_{i,n_i} \in L^2_s(\mu^{n_i})$, $i \in \{1,\cdots,N\}$. *Then,* $D_{j,t,z}F \in \mathcal{G}^*$, μ *a.e. given by*

$$D_{j,t,z}F = \sum_{n_j=1}^\infty n_j I_{n_j}\left(f_{j,n_j}\right) \sum_{n/n_j\in\mathbb{N}_0^{N-1}} \prod_{i=1,i\neq j}^N I_{n_i}\left(f_{i,n_i}\right) \in \mathcal{G}^*. \tag{3.379}$$

Theorem 3.11.2 *Closability of Stochastic Derivatives*

Let $F_m, F \in \mathcal{G}^*$ *such that as* $m \to \infty$

(i) $F_m \to F$ *in* $\mathcal{G}^*$,

(ii) $D_{j,t,z}F_m$ *converges in* $\mathcal{G}^*$ *for* $j \in \{1,\cdots,N\}$.

Then, $D_{j,t,z}F_m \to D_{j,t,z}F$ *in* $\mathcal{G}^*$, $j \in \{1,\cdots,N\}$.

3.11.6 Generalized Conditional Expectation

Let F has a formal expansion of the form (3.358). The conditional expectation $E[F|\mathcal{F}_A]$, $A \in \mathfrak{B}([0,T])$ is given as

$$E[F|\mathcal{F}_A] = \sum_{n\in\mathbb{N}_0^N}^\infty \prod_{j=1}^N I_{n_j}\left(f_{j,n_j}\mathbf{1}_A^{\otimes n_j}\right).$$

If $F \in \mathcal{G}^*$, we can easily show, by writing the chaos expansion in terms of the iterated integral, the following properties conditional expectation in $\mathcal{G}^*$ holds in the multivariate case.

3.11.7 Skorohod Integration on $\mathcal{G}^*$

Definition 3.11.2 *Skorohod Integral in $\mathcal{G}^*$*

Let $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{G}^*$ be a random field with the formal expansion given by

$$u(t,z) = \sum_{n/n_j \in \mathbb{N}_0^{N-1}} \prod_{i=1, i\neq j}^{N} I_{n_i}(f_{i,n_i}) \sum_{n_j=0}^{\infty} I_{n_j}\left(f_{j,n_j}(\cdot,(t,z))\right) \in \mathcal{G}^*. \tag{3.380}$$

where $n = (n_1, \cdots, n_N)^T$,$n_j \in \mathbb{N}_0$, $f_{j,n_j} \in L_s^2(\mu^{n_j})$, $j \in \{1, \cdots, N\}$ such that for some $q > 0$,

$$\sum_{n/n_j \in \mathbb{N}_0^N} (n+\epsilon_j)! e^{-2q(|n|+1)} \prod_{i=0, i\neq j}^{N} ||f_{i,n_i}||^2_{L^2(\mu^{n_i})} ||\tilde{f}_{j,n_j}||^2_{L^2(\mu^{n_j+1})} < \infty \tag{3.381}$$

Define the Skorohod integral of u with respect to M_j as follows:

$$\begin{aligned} \delta_j(u) &= \int_{\mathbb{R}_+ \times \mathbb{R}} u(t,x) M_j(\delta t, dx) \\ &= \sum_{n/n_j \in \mathbb{N}_0^{N-1}} \prod_{i=0, i\neq j}^{N} I_{n_i}(f_{i,n_i}) \sum_{n_j=1}^{\infty} I_{n_{j+1}}\left(\tilde{f}_{j,n_j}\right) \end{aligned} \tag{3.382}$$

We say that u is Skorohod integrable if $\delta(u) \in \mathcal{G}^*$, that is, there exists some $q > 0$ such that $||\delta(u)||_{\mathcal{G}_{-q}} < \infty$. Moreover, we have the following

$$\begin{aligned} &||\delta(u)||^2_{\mathcal{G}_{-q}} \\ &= \sum_{n/n_j \in \mathbb{N}_0^{N-1}} \prod_{i=0, i\neq j}^{N} n_i! ||f_{i,n_i}||^2_{L^2(\mu^{n_i})} e^{-2qn_i} \sum_{n_j=1}^{\infty} n_{j+1}! ||\tilde{f}_{j,n_j}||^2_{L^2(\mu^{n_j+1})} e^{-2q(n_j+1)} \\ &= \sum_{n/n_j \in \mathbb{N}_0^N} (n+\epsilon_j)! e^{-2q(|n|+1)} \prod_{i=0, i\neq j}^{N} ||f_{i,n_i}||^2_{L^2(\mu^{n_i})} ||\tilde{f}_{j,n_j}||^2_{L^2(\mu^{n_j+1})} < \infty \end{aligned} \tag{3.383}$$

where $\epsilon_j = (0, \cdot, 1, \cdots, 0)^T$ is a unit vector of length n with one at the jth-component and zero otherwise.

Theorem 3.11.3 *Fundamental Theorem of Stochastic Calculus*

Let $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{G}^*$ be a random field satisfying the following conditions:

(i) $u \in L^2(P \times \mu)$,

(ii) $D_{j,t,z}u$ *is Skorohod integrable for all* $(t,z) \in [0,T] \times \mathbb{R}$,

(iii) $D_{j,t,z}\delta(u) \in \mathcal{G}^*$ *and* $\delta(D_{t,z}u) \in \mathcal{G}^*$, *for all* $(t,z) \in [0,T] \times \mathbb{R}$ *and there exists* $q > 0$ *such that*

$$\int_{[0,T]\times\mathbb{R}} ||D_{j,t,z}\delta(u)||^2_{\mathcal{G}_{-q}} \mu_j(dt,dz) < \infty, \tag{3.384}$$

$$\int_{[0,T]\times\mathbb{R}} ||\delta(D_{j,t,z}u)||^2_{\mathcal{G}_{-q}} \mu_j(dt,dz) < \infty. \tag{3.385}$$

Then,

$$D_{j,t,z}\left(\delta(u)\right) = u(t,z) + \delta(D_{j,t,z}u), \tag{3.386}$$

that is,

$$D_{j,t,z}\int_{[0,T]\times\mathbb{R}} u(s,x)M_j(\delta s,dx) = u(t,z) + \int_{[0,T]\times\mathbb{R}} D_{j,t,z}u(s,x)M_j(\delta s,dx). \tag{3.387}$$

Corollary 3.11.4 *Let* u *satisfy the conditions of the preceding theorem and in addition, suppose that it is also predictable, then we have the following identity:*

$$D_{j,t,z}\int_{[0,T]\times\mathbb{R}} u(s,x)M_j(ds,dx) = u(t,z) + \int_{[t,T]\times\mathbb{R}} D_{j,t,z}u(s,x)M_j(ds,dx). \tag{3.388}$$

Corollary 3.11.5 *Let* u *satisfy the conditions of the preceding corollary, then*

$$D_{j,t,z}\int_{[0,T]} u(s,0)dW_j(s) = \sigma_j^{-1}u(t,0)\mathbf{1}_{\{z=0\}} + \int_{[t,T]} D_{j,t,z}u(s,0)dW(s), \tag{3.389}$$

$$D_{j,t,z}\int_{[0,T]\times\mathbb{R}_0} u(s,x)x\tilde{N}_j(ds,dx) = \sigma_j^{-1}u(t,z)\mathbf{1}_{\{z\neq 0\}} + \int_{[t,T]\times\mathbb{R}_0} D_{j,t,z}u(s,x)\tilde{N}_j(ds,dx). \tag{3.390}$$

To conclude this section, we state the Wick-Skorohod theorem in the multivariate case.

Theorem 3.11.6 *Wick-Skorohod Theorem*

Let u *be Skorohod integrable with respect to* M_j, *then* $u(t,z)\diamond\dot{M}_j(t,z)$, *for all* $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$ *is* $(\mathcal{S})^*$ *integrable and*

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)M_j(\delta t,dz) = \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)\diamond\dot{M}_j(t,z)\mu_j(dt,dz). \tag{3.391}$$

3.11.8 Clark Ocone Theorem in $L^2(P)$

We state the Clark-Ocone theorem in the multivariate case in $L^2(P)$, as follows.

Theorem 3.11.7 *Clark-Ocone Theorem in $L^2(P)$*
Let $F \in L^2(P)$ be $\mathcal{F}_T$ measurable, then

$$F = E[F] + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}} E[D_{j,t,z}F|\mathcal{F}_{t-}] M_j(dt, dz) \tag{3.392}$$

where $E[D_{j,t,z}F|\mathcal{F}_{t-}] \in L^2(P \times \mu)$, $(t, z) \in [0, T] \times \mathbb{R}$ for $j \in \{1, \cdots, N\}$.

4. CLARK-OCONE THEOREM UNDER THE CHANGE OF MEASURE AND MEAN-VARIANCE HEDGING

4.1 Girsanov Theorem for Lévy Processes

To prove the Clark-Ocone theorem under the change in measure, we shall state the Girsanov theorem to be able to define the equivalent measure $Q \sim P$. We state the Girsanov theorem for Lévy processes.

Theorem 4.1.1 *[27] [65] Girsanov Theorem for Lévy Processes*
Suppose that there exists a predictable processes $u_j(s)$, $j \in \{1, \cdots, d\}$ and $\theta_j(s,x) < 1$, $j \in \{1, \cdots, l\}$ where $(s,x) \in [0,T] \times \mathbb{R}_0$ such that

$$\int_0^T u_j^2(s)ds < \infty, \qquad a.s., \tag{4.1}$$

$$\int_{[0,T]\times\mathbb{R}_0} (|\log(1-\theta_j(s,x))| + \theta_j^2(s,x))\nu_j(dx)ds < \infty, \qquad a.s., \tag{4.2}$$

for all $j \in \{1, \cdots, N\}$. Denote the Doleans-Dade exponential $Z(t)$ for $t \in [0,T]$ by

$$Z(t) = \exp\left(-\sum_{j=1}^{d}\left(\int_0^t u_j(s)dW_j(s) + \frac{1}{2}\int_0^t u_j^2(s)ds\right) + \right. \tag{4.3}$$

$$\left.\sum_{j=1}^{l}\int_{[0,T]\times\mathbb{R}_0}\Big(\log(1-\theta_j(s,x))\tilde{N}_j(ds,dx) + (\log(1-\theta_j(s,x)) + \theta_j(s,x))\nu_j(dx)ds\Big)\right)$$

$$= \mathcal{E}\left(-\left(\sum_{j=1}^{d}\int_0^t u_j(s)dW_j(s) + \sum_{j=1}^{l}\int_{[0,t]\times\mathbb{R}_0}\theta_j(s,x)\tilde{N}_j(ds,dx)\right)\right) \tag{4.4}$$

where $\mathcal{E}$ is the stochastic exponential operator. Define a measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = Z(T,\omega)dP(\omega). \tag{4.5}$$

Suppose that a Novikov-type condition is satisfied (to be discussed later), then $E[Z(T)] = 1$ *and* W^Q *and* $\tilde{N}^Q$ *is a Brownian motion and compensated Poisson random measure under* Q *respectively where*

$$dW_j^Q(t) = dW_j(t) + u_j(t)dt, \tag{4.6}$$
$$\tilde{N_j}^Q(dt, dz) = \tilde{N}_j(dt, dz) + \theta_j(t, z)\nu_j(dz)dt \tag{4.7}$$

for all $j \in \{1, \cdots, N\}$.

Remark 4.1.2 *We can write* (4.6) *and* (4.7) *in matrix-vector form as follows:*

$$dW^Q(t) = dW(t) + u(t)dt, \tag{4.8}$$
$$\tilde{N}^Q(dt, dz) = \tilde{N}(dt, dz) + \theta(t, z)\nu(dz)dt \tag{4.9}$$

where

$$\begin{aligned} W(t) &= [W_1(t), \cdots, W_d(t)]^T, & \tilde{N}(dt, dz) &= [\tilde{N}_1(dt, dz), \cdots, \tilde{N}_l(dt, dz)]^T, \\ W^Q(t) &= [W_1^Q(t), \cdots, W_d^Q(t)]^T, & \tilde{N}^Q(dt, dz) &= [\tilde{N}_1^Q(dt, dz), \cdots, \tilde{N}_l^Q(dt, dz)]^T, \\ u(t) &= [u_1(t), \cdots, u_d(t)]^T, & \theta(t, z) &= diag[\theta_1(t, z), \cdots, \theta_l(t, z)], \\ & & \nu(dz) &= [\nu_1(dz), \cdots, \nu_l(dz)]^T. \end{aligned} \tag{4.10}$$

Applying the Novikov-type conditions to $Z(t)$ to become a martingale implies the following for all $t \in [0, T]$, the dynamics of Z is given as

$$\begin{aligned} dZ(t) &= Z(t^-)dL(t), \\ Z(0) &= 1 \end{aligned} \tag{4.11}$$

where L is a local martingale given by

$$dL(t) = -\sum_{j=1}^{d} u_j(t)dW_j(t) - \sum_{j=1}^{l} \int_{\mathbb{R}_0} \theta_j(t, z_j)\tilde{N}_j(dt, dz). \tag{4.12}$$

Hence, $Z(t) = \mathcal{E}(M(t))$ with the following continuous and discrete parts

$$dL^c(t) = -\sum_{j=1}^{N} u_j(t)dW_j(t), \tag{4.13}$$
$$dL^d(t) = -\sum_{j=1}^{N} \int_{\mathbb{R}_0} \theta_j(t, z)\tilde{N}_j(dt, dz). \tag{4.14}$$

The corresponding angle bracket process is given as

$$< L^c >_t = \sum_{j=1}^{d} \int_0^t u_j^2(s)ds, \tag{4.15}$$

$$< L^d >_t = \sum_{j=1}^{l} \int_{[0,T]\times\mathbb{R}_0} \theta_j^2(s,z)N_j(ds,dz). \tag{4.16}$$

We state two Novikov-type theorems below. The first theorem is attributed to Lepingle and Memin [55] and the second theorem is attributed to Protter and Shimbo [70].

Theorem 4.1.3 *[55], [65] Let L be a local martingale such that $\Delta L > -1$ and*

$$A(t) = \frac{1}{2} < L^c >_t + \sum_{s\in(0,t]} [(1+\Delta L(s)))\log(1+\Delta L(s))) - \Delta L(s)] \tag{4.17}$$

which has a compensator $B = \{B(t)\}_{t\geq 0}$ such that

$$E[\exp(B(T))] < \infty. \tag{4.18}$$

Then, $\mathcal{E}(M)$ is a u.i. martingale and $\mathcal{E}(M) > 0$ almost surely.

Applying (4.12) to Theorem 4.1.3, we have as follows:

$$A(t) = \frac{1}{2}\sum_{j=1}^{d} \int_0^t u_j^2(s)ds + \sum_{j=1}^{l} \int_{[0,t]\times\mathbb{R}_0} ((1-\theta_j(s,x))\log\theta_j(s,x) + \theta_j(s,z))N_j(ds,dx) \tag{4.19}$$

then its compensator is given as follows:

$$B(t) = \frac{1}{2}\sum_{j=1}^{d} \int_0^t u_j^2(s)ds + \sum_{j=1}^{l} \int_{[0,t]\times\mathbb{R}_0} ((1-\theta_j(s,x))\log\theta_j(s,x) + \theta_j(s,x))\nu_j(dx)ds. \tag{4.20}$$

Theorem 4.1.4 *[70], [65] Let L be a square integrable martingale local such that $\Delta L > -1$. If*

$$E\left[\exp\left(\frac{1}{2} < L^c >_T + < L^d >_T\right)\right] < \infty \tag{4.21}$$

then $\mathcal{E}(L)$ is a uniformly integrable martingale.

From the Theorem 4.1.4 for $L = Z$ and from (4.15) − (4.16) we have the following Novikov-type condition

$$E\left[\exp\left(\frac{1}{2}\sum_{j=1}^{d}\int_0^T u_j^2(s)ds + \sum_{j=1}^{l}\int_{[0,T]\times\mathbb{R}_0}\theta_j^2(s,x)\nu_j(dx)ds\right)\right] < \infty. \tag{4.22}$$

Definition 4.1.1 *[27] Generalized Bayes Formula*
We let $Q(d\omega)Z(T)P(d\omega)$, where Z is the Doleans-Dade exponential. Let $F, Z(T)F \in (\mathcal{G})^*$, then we define the Generalized Bayes Formula as follows:

$$E^Q[F|\mathcal{F}_A] = \frac{E[Z(T)F|\mathcal{F}_A]}{Z(t)}, \qquad A \in \mathfrak{B}([0,T]). \tag{4.23}$$

Remark 4.1.5 *If $F, Z(T)F \in L^2(P)$ such that the Novikov condition is satisfied, then Z is a martingale and thus satisfies* (4.23) *which corresponds to the abstract Bayes rule.*

4.2 Clark-Ocone Theorem in $L^2(P) \cap L^2(Q)$

Before we present Clark-Ocone Theorem in $L^2(P) \cap L^2(Q)$, we shall present an important lemma. For simplicity of presentation, we shall assume that $N = d = l$. The summation can be adjusted accordingly if $d \neq l$.

Lemma 4.2.1 *Stochastic derivative of $Z(T)$.*
Suppose that the assumptions of the Girsanov theorem for Lévy processes and the assumptions of fundamental theorem of stochastic calculus for $u_i(t)$ and $\log(1-\theta_i(t,z))$ for $i \in \{1, \cdots, N\}$ are satisfied. Then we have following stochastic derivative for $Z(T)$.

(i) If $z = 0$, then

$$D_{j,t,0}Z(T) = Z(T)\left[-\sigma_j^{-1}u_j(t) - \sum_{i=1}^{N}\left(\int_{[t,T]} D_{j,t,0}u_i(s)dW_i^Q(s) + \int_{[t,T]\times\mathbb{R}_0}\frac{D_{j,t,0}\theta_i(s,x)}{1-\theta_i(s,x)}\tilde{N}_i^Q(ds,dx)\right)\right]. \tag{4.24}$$

(ii) If $z \neq 0$, then

$$D_{j,t,z}Z(T) = z^{-1}Z(T)\left(\exp(zD_{j,t,z}\log Z(T)) - 1\right) \tag{4.25}$$

where

$$\begin{aligned}
&D_{j,t,z}\log Z(T) = z^{-1}\log(1-\theta_j(t,z)) \\
&+\sum_{i=1}^{N}\Bigg[-\int_0^T D_{j,t,z}u_i(s)dW_i^Q(s) - \frac{1}{2}\int_t^T z(D_{j,t,z}u_i(s))^2 ds \\
&+\int_{[t,T]\times\mathbb{R}_0} z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)\tilde{N}_i^Q(ds,dx) \\
&+\int_{[t,T]\times\mathbb{R}_0}\left(z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)(1-\theta_i(s,x)) + D_{j,t,z}\theta_i(s,x)\right)\nu_i(dx)ds\Bigg].
\end{aligned} \tag{4.26}$$

Proof (i) Consider the process (4.3). We let

$$\begin{aligned}
&Y(t) = \log Z(t) = -\sum_{i=1}^{N}\left(\int_0^t u_i(s)dW_i(s) + \frac{1}{2}\int_0^t u_i^2(s)ds\right) + \\
&\sum_{i=1}^{N}\int_{[0,t]\times\mathbb{R}_0}\left(\log(1-\theta_i(s,x)\tilde{N}_i(ds,dx) + (\log(1-\theta_i(s,x)) + \theta_i(s,x))\nu_i(dx)ds\right).
\end{aligned} \tag{4.27}$$

Then for all $z \in \mathbb{R}$

$$\begin{aligned}
D_{j,t,z}Y(T) = -\sum_{j=1}^{N}\Bigg[&-\int_0^T D_{j,t,z}u_i(s)dW_i(s) - \frac{1}{2}D_{j,t,z}\int_0^T u_i^2(s)ds \\
&+ D_{j,t,z}\int_{[0,T]\times\mathbb{R}_0}\log(1-\theta_i(s,x)\tilde{N}_i(ds,dx) \\
&+D_{j,t,z}\int_{[0,T]\times\mathbb{R}_0}(\log(1-\theta_i(s,x)) + \theta_i(s,x))\nu_i(dx)ds\Bigg].
\end{aligned} \tag{4.28}$$

From the chain rule,

$$D_{j,t,0}Z(T) = Z(T)D_{j,t,0}Y(T). \tag{4.29}$$

Then, we have the following derivatives:

$$D_{j,t,0}\int_0^T u_i(s)dW_i(s) = \sigma_j^{-1}(s)u_j(t)\mathbf{1}_{\{i=j\}} + \int_t^T D_{j,t,0}u_i(s)dW_i(s), \tag{4.30}$$

$$D_{j,t,0}\int_0^T u_i^2(s)ds = \int_t^T D_{j,t,0}u_i^2(s)ds = \int_t^T 2u_i(s)D_{j,t,0}u_i(s)ds, \tag{4.31}$$

$$\begin{aligned} &D_{j,t,0}\int_{[0,T]\times\mathbb{R}_0} \log(1-\theta_i(s,x)\tilde{N}_i(ds,dx) \\ &= \int_{[t,T]\times\mathbb{R}_0} D_{j,t,0}(x^{-1}\log(1-\theta_i(s,x))x\tilde{N}_i(ds,dx) \\ &= \int_{[t,T]\times\mathbb{R}_0} \frac{D_{j,t,0}\theta_i(s,z_j)}{1-\theta_i(s,z_j)}\tilde{N}_i(ds,dz_j), \end{aligned} \tag{4.32}$$

$$\begin{aligned} &D_{j,t,0}\int_{[0,T]\times\mathbb{R}_0} (\log(1-\theta_i(s,x)) + \theta_i(s,x))\nu_i(dx)ds \\ &= \int_{[t,T]\times\mathbb{R}_0} (D_{j,t,0}\log(1-\theta_i(s,x)) + D_{j,t,0}\theta_i(s,x))\nu_i(dx)ds \\ &= \int_{[t,T]\times\mathbb{R}_0} \left(-\frac{D_{j,t,0}\theta_i(s,x)}{1-\theta_i(s,x)} + D_{j,t,0}\theta_i(s,x)\right)\nu_i(dx)ds. \end{aligned} \tag{4.33}$$

Collecting terms yields

$$\begin{aligned} D_{j,t,0}Z(T) =& Z(T)\Bigg[-\sigma_j^{-1}u_j(t) - \sum_{i=1}^N \bigg(\int_{[t,T]} D_{j,t,0}u_i(s)(dW_i(s)+u_i(s)) \\ &+ \int_{[t,T]\times\mathbb{R}_0} \frac{D_{j,t,0}\theta_i(s,x)}{1-\theta_i(s,x)}(\tilde{N}_i(ds,dx)+\nu_i(dx)ds)\bigg)\Bigg] \\ =& Z(T)\Bigg[-\sigma_j^{-1}u_j(t)\mathbf{1}_{\{\sigma_j\neq 0\}} - \sum_{i=1}^N \bigg(\int_{[t,T]} D_{j,t,0}u_i(s)dW_i^Q(s) \\ &+ \int_{[t,T]\times\mathbb{R}_0} \frac{D_{j,t,0}\theta_i(s,x)}{1-\theta_i(s,x)}\tilde{N}_i^Q(ds,dx)\bigg)\Bigg]. \end{aligned} \tag{4.34}$$

(ii) From the chain rule, we have the following derivatives for $z \neq 0$,

$$\begin{aligned} D_{j,t,z}Z(T) =& D_{j,t,z}\exp(Y(T)) \\ =& z^{-1}[\exp(Y(T)+zD_{j,t,z}Y(T)) - \exp(Y(T))] \\ =& z^{-1}Z(T)[\exp(zD_{j,t,z}Y(T)) - 1], \end{aligned} \tag{4.35}$$

$$\begin{aligned} D_{j,t,z}u_i^2(s) =& z^{-1}[(u_i(s)+zD_{j,t,z}u_i(s))^2 - u_i^2(s)] \\ =& 2u_i(s)D_{j,t,z}u_i(s) + z(D_{j,t,z}u_i(s))^2, \end{aligned} \tag{4.36}$$

$$\begin{aligned} & D_{j,t,z}(x^{-1}\log(1-\theta_i(s,x)) \\ =& x^{-1}z^{-1}[\log(1-\theta_i(s,x)) - zD_{j,t,z}\theta_i(s,x)) - \log(1-\theta_i(s,x))] \\ =& z^{-1}\log\left(1 - \frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right). \end{aligned} \tag{4.37}$$

Hence, we have following derivatives:

$$D_{j,t,z}\int_0^T u_i(s)dW_i(s) = \int_t^T D_{j,t,z}u_i(s)dW_i(s), \tag{4.38}$$

$$\begin{aligned} D_{j,t,z}\int_0^T u_i^2(s)ds =& \int_t^T D_{j,t,z}u_i^2(s)ds \\ =& \int_t^T 2u_i(s)D_{j,t,z}u_i(s) + z(D_{j,t,z}u_i(s))^2, \end{aligned} \tag{4.39}$$

$$\begin{aligned} & D_{j,t,z}\int_{[0,T]\times\mathbb{R}_0} \log(1-\theta_i(s,x))\tilde{N}_i(ds,dz_i) \\ =& z^{-1}\log(1-\theta_j(s,z))\mathbf{1}_{\{i=j\}} + \int_{[t,T]\times\mathbb{R}_0} D_{j,t,z}(x^{-1}\log(1-\theta_i(s,x))x\tilde{N}_i(ds,dx) \\ =& z^{-1}\log(1-\theta_j(s,z))\mathbf{1}_{\{i=j\}} + \int_{[t,T]\times\mathbb{R}_0} z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)\tilde{N}_i(ds,dx), \end{aligned} \tag{4.40}$$

$$\begin{aligned} & D_{j,t,z}\int_{[0,T]\times\mathbb{R}_0} (\log(1-\theta_i(s,x)) + \theta_i(s,x))\nu_i(dx)ds \\ =& \int_{[t,T]\times\mathbb{R}_0} (D_{j,t,z}\log(1-\theta_i(s,x)) + D_{j,t,z}\theta_i(s,x))\nu_i(dx)ds \\ =& \int_{[t,T]\times\mathbb{R}_0} \left(z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right) + D_{j,t,z}\theta_i(s,x)\right)\nu_i(dz)ds. \end{aligned} \tag{4.41}$$

Finally, collecting terms yield

$$
\begin{aligned}
D_{j,t,z}Y(T) &= z^{-1}\log(1-\theta_j(t,z))+ \\
&+\sum_{i=1}^{N}\Bigg[-\int_t^T D_{j,t,z}u_i(s)\,(dW_i(s)+u_i(s)ds)-\frac{1}{2}\int_t^T z(D_{j,t,z}u_i(s))^2 ds \\
&+\int_{[t,T]\times\mathbb{R}_0} z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)\left(\tilde{N}_i(ds,dx)+\nu_i(dx)ds\right) \\
&+\int_{[t,T]\times\mathbb{R}_0}\left(z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)(1-\theta_i(s,x))+D_{j,t,z}\theta_i(s,x)\right)\nu_i(dx)ds\Bigg] \\
=& z^{-1}\log(1-\theta_j(t,z))+\sum_{i=1}^{N}\Bigg[-\int_0^T D_{j,t,z}u_i(s)dW_i^Q(s)-\frac{1}{2}\int_t^T z(D_{j,t,z}u_i(s))^2 ds \\
&+\int_{[t,T]\times\mathbb{R}_0} z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)\tilde{N}_i^Q(ds,dx) \\
&+\int_{[t,T]\times\mathbb{R}_0}\left(z^{-1}\log\left(1-\frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)(1-\theta_i(s,x))+D_{j,t,z}\theta_i(s,x)\right)\nu_i(dx)ds\Bigg].
\end{aligned}
\tag{4.42}
$$

■

Theorem 4.2.2 *Clark-Ocone theorem under the change of measure*

Let $F \in L^2(P) \cap L^2(Q)$ *be* $\mathcal{F}_T$*-measurable and* $FZ(T) \in L^2(P)$. *Suppose that the assumptions of Lemma* 4.2.1 *are satisfied, then*

$$F = E^Q[F] + \sum_{j=1}^{N} \int_0^T \sigma_j E^Q[D_{j,t,0}F - FK_j(t)|\mathcal{F}_{t^-}]dW_j^Q(t) + \tag{4.43}$$

$$\sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} E^Q[F(H_j(t,z)-1) + zH_j(t,z)D_{j,t,z}F|\mathcal{F}_{t^-}]\tilde{N}_j^Q(dt,dz) \tag{4.44}$$

where

$$K_j(t) = \sum_{i=1}^{N} \left(\int_t^T D_{j,t,0}u_i(s)dW_i^Q(s) + \int_{[t,T]\times\mathbb{R}_0} \frac{D_{j,t,0}\theta_i(s,x)}{1-\theta_i(s,x)} \tilde{N}_i^Q(ds,dx) \right), \tag{4.45}$$

$$\begin{aligned} &H_j(t,z) \\ &= \exp\Bigg(\sum_{i=1}^{N} \Bigg[-\int_t^T zD_{j,t,z}u_i(s)dW_j^Q(s) - \frac{1}{2}\int_t^T (zD_{j,t,z}u_i(s))^2 ds \\ &+ \int_{[t,T]\times\mathbb{R}_0} \log\left(1 - \frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right) \tilde{N}_i^Q(ds,dx) \\ &+ \int_{[t,T]\times\mathbb{R}_0} \left(\log\left(1 - \frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)(1-\theta_i(s,x)) + zD_{j,t,z}\theta_i(s,x) \right) \nu_i(dx)ds \Bigg] \Bigg) \end{aligned} \tag{4.46}$$

for all $j \in \{1, \cdots, N\}$.

Proof We let

$$\Lambda(t) = Z(t)^{-1} \tag{4.47}$$

where $Z(t)$ is given by (4.3). From Itô's lemma,

$$
\begin{aligned}
& d\Lambda(t) \\
=& -\frac{1}{Z^2(t^-)}\sum_{i=1}^{N}(-u_i(t)Z(t^-))dW_i(t) + \frac{1}{2}\frac{2}{Z^3(t^-)} + \sum_{i=1}^{N}(-u_i(t)Z(t^-))^2 dt \\
& \sum_{i=1}^{N}\int_{\mathbb{R}_0}\left(\frac{1}{Z(t^-)+(-\theta_i(t,z)Z(t^-))} - \frac{1}{Z(t^-)}\right)\tilde{N}_i(dt,dz) + \\
& \sum_{i=1}^{N}\int_{\mathbb{R}_0}\left(\frac{1}{Z(t^-)+(-\theta_i(t,z)Z(t^-))} - \frac{1}{Z(t^-)} - \frac{-\theta_i(t,z)Z(t^-)}{Z(t^-)^2}\right)\nu_i(dz)dt \\
=& \Lambda(t^-)\left[\sum_{i=1}^{N}u_i(t)dW_i(t) + \sum_{i=1}^{N}u_i^2(t)dt\right. \\
& + \sum_{i=1}^{N}\int_{\mathbb{R}_0}\frac{1}{Z(t^-)}\left(\frac{1}{1-\theta_i(t,z)} - 1\right)\tilde{N}_i(dt,dz) + \\
& \left.\sum_{i=1}^{N}\int_{\mathbb{R}_0}\frac{1}{Z(t^-)}\left(\frac{1}{1-\theta_i(t,z)} - 1 - \theta_i(t,z)\right)\nu_i(dz)dt\right] \\
=& \Lambda(t^-)\left[\sum_{i=1}^{N}u_i(t)\left(dW_i^Q(t) - u_i(t)dt\right) + u_i^2(t)dt + \right. \\
& \sum_{i=1}^{N}\int_{\mathbb{R}_0}\frac{1}{Z(t^-)}\left(\frac{1}{1-\theta_i(t,z)} - 1\right)\left(\tilde{N}_i^Q(dt,dz) - \theta_i(t,z)\nu_i(dz)dt\right) + \\
& \left.\sum_{i=1}^{N}\int_{\mathbb{R}_0}\frac{1}{Z(t^-)}\left(\frac{1}{1-\theta_i(t,z)} - 1 - \theta_i(t,z)\right)\nu_i(dz)dt\right] \\
=& \Lambda(t^-)\left[\sum_{i=1}^{N}u_i(t)dW_i^Q(t) + \sum_{i=1}^{N}\int_{\mathbb{R}_0}\frac{\theta_i(t,z)}{1-\theta_i(t,z)}\tilde{N}_i^Q(dt,dz)\right]. \qquad (4.48)
\end{aligned}
$$

We let

$$Y(t) = E^Q[F|\mathcal{F}_t] \qquad (4.49)$$

Then assuming that a Novikov-type condition to $Z(t)$ is satisfied, from the abstract Bayes rule,

$$Y(t) = \frac{E[FZ(T)|\mathcal{F}_t]}{E[Z(T)|\mathcal{F}_t]} = \Lambda(t)V(t). \qquad (4.50)$$

Since $FZ(T) \in L^2(P)$, then $V(t) \equiv E^P[FZ(T)|\mathcal{F}_t] \in L^2(P)$. From the Clark-Ocone theorem in $L^2(P)$, we have the following:

$$\begin{aligned} E[FZ(T)|\mathcal{F}_t] =& E[E[FZ(T)|\mathcal{F}_t]] + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}} E[D_{j,s,z}E[FZ(T)|\mathcal{F}_t]|\mathcal{F}_{s^-}]M_j(ds,dx) \\ =& E[FZ(T)] + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}} E[D_{j,s,z}(FZ(T))|\mathcal{F}_{s^-}]M_j(ds,dx) \qquad (4.51) \\ =& E[FZ(T)] + \sum_{j=1}^{N} \int_0^T \sigma_j E[D_{j,t,0}(FZ(T))|\mathcal{F}_{t^-}]dW_j(t) \\ &+ \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} E[D_{j,t,z}(FZ(T))|\mathcal{F}_{t^-}]x\tilde{N}_j(dt,dx). \qquad (4.52) \end{aligned}$$

The first term is by the tower property of conditional expectation

$$E[E[FZ(T)|\mathcal{F}_t]] = E[FZ(T)]. \qquad (4.53)$$

By applying the tower property of conditional expectation yields

$$E[D_{j,s,z}E[FZ(T)|\mathcal{F}_t]\mathcal{F}_{s^-}]] = E[D_{j,s,z}(FZ(T))|\mathcal{F}_{s^-}]\mathbf{1}\{s<t\}. \qquad (4.54)$$

From the product rule

$$dY(t) = \Lambda(t^-)dV(t) + V(t^-)d\Lambda(t) + d[\Lambda, V]_t \qquad (4.55)$$

The quadratic variation is evaluated as follows:

$$\begin{aligned} d[\Lambda, V]_t =& \Lambda(t^-) \Bigg[\sum_{j=1}^{N} u_j(t)\sigma_j E[D_{j,t,0}(FZ(T))|\mathcal{F}_{t^-}]dt + \\ =& \sum_{j=1}^{N} \int_{\mathbb{R}_0} \frac{\theta_j(t,z)}{1-\theta_j(t,z)} E[D_{j,t,z}(FZ(T))|\mathcal{F}_{t^-}]zN_j(dt,dz) \Bigg]. \qquad (4.56) \end{aligned}$$

Hence,

$$
\begin{aligned}
&dY(t)\\
=&\Lambda(t^-)\left[\sum_{j=1}^{N}\sigma_j E[D_{j,t,0}(FZ(T))|\mathcal{F}_{t^-}]dW_j(t)+\right.\\
&\left.\sum_{j=1}^{N}\int_{\mathbb{R}_0} E[D_{j,t,z}(FZ(T))|\mathcal{F}_{t^-}]z\tilde{N}_j(dt,dz)\right]+\\
&E[FZ(T)|\mathcal{F}_{t^-}]\Lambda(t^-)\left[\sum_{j=1}^{N}u_j(t)dW_j^Q(t)+\sum_{j=1}^{N}\int_{\mathbb{R}_0}\frac{\theta_j(t,z)}{1-\theta_j(t,z)}\tilde{N}_j^Q(dt,dz)\right]\\
&\Lambda(t^-)\left[\sum_{j=1}^{N}\sigma_j u_j(t)E[D_{j,t,0}(FZ(T))|\mathcal{F}_{t^-}]dt+\right.\\
&\left.\sum_{j=1}^{N}\int_{\mathbb{R}_0}\frac{\theta_j(t,z)}{1-\theta_j(t,z)}E[D_{j,t,z}(FZ(T))|\mathcal{F}_{t^-}]z(\tilde{N}_j(dt,dz)+\nu_j(dz)dt)\right]\\
=&\Lambda(t^-)\left[\sum_{j=1}^{N}\sigma_j E[D_{j,t,0}(FZ(T))|\mathcal{F}_{t^-}](dW_j^Q(t)-u_j(t)dt)+\right.\\
&\left.\sum_{j=1}^{N}\int_{\mathbb{R}_0} E[D_{j,t,z}(FZ(T))|\mathcal{F}_{t^-}]z(\tilde{N}_j^Q(dt,dz)-\theta_j(t,z)\nu(dz)dt)\right]+\\
&Y(t)\left[\sum_{j=1}^{N}u_j(t)dW_j^Q(t)+\sum_{j=1}^{N}\int_{\mathbb{R}_0}\frac{\theta_j(t,z)}{1-\theta_j(t,z)}\tilde{N}_j^Q(dt,dz)\right]\\
&\Lambda(t^-)\left[\sum_{j=1}^{N}u_j(t)\sigma_j E[D_{j,t,0}(Z(T)F)|\mathcal{F}_{t^-}]dt+\right.\\
&\left.\sum_{j=1}^{N}\int_{\mathbb{R}_0}\frac{\theta_j(t,z)}{1-\theta_j(t,z)}E[D_{j,t,z}(Z(T)F)|\mathcal{F}_{t^-}]z(\tilde{N}_j(dt,dz)+(1-\theta_j(t,z))\nu_j(dz)dt)\right]\\
=&\Lambda(t^-)\left[\sum_{j=1}^{N}(\sigma_j E[D_{j,t,0}(Z(T)F)|\mathcal{F}_{t^-}]+E[Z(T)Fu_j(t)|\mathcal{F}_{t^-}])dW_j^Q(t)+\right.\\
&\left.\int_{\mathbb{R}_0}\left(\frac{E[D_{j,t,z}(Z(T)F)|\mathcal{F}_{t^-}]}{1-\theta_j(t,z)}z+\frac{\theta_j(t,z)}{1-\theta_j(t,z)}E[Z(T)F|\mathcal{F}_{t^-}]\right)\tilde{N}_j^Q(dt,dz)\right] \qquad (4.57)
\end{aligned}
$$

Since $Z(T)F \in L^2(P)$ then from product rule

$$D_{j,t,z}(Z(T)F) = FD_{j,t,z}Z(T) + Z(T)D_{j,t,z}F + zD_{j,t,z}Z(T)D_{j,t,z}F. \qquad (4.58)$$

Consider the case $z = 0$. Note that $K_j(t)$ in (4.45) can be written in terms of $D_{j,t,z}Z(T)$ in (4.24) as follows:

$$D_{j,t,0}Z(T) = Z(T)\left(F\sigma_j^{-1}u_j(t) - K_j(t)\right). \tag{4.59}$$

From the product rule (4.58), we obtain

$$\begin{aligned} D_{j,t,0}(Z(T)F) =& FZ(T)\left(F\sigma_j^{-1}u_j(t) - K_j(t)\right) + Z(T)D_{j,t,0}F \\ & Z(T)\left[D_{j,t,0}F - F\left(\sigma_j^{-1}u_j(t) + K_j(t)\right)\right]. \end{aligned} \tag{4.60}$$

On the other hand, consider the case $z \neq 0$. Note that $H_j(t,z)$ in (4.46) can be written in terms of $D_{j,t,z}\log Z(T)$ in (4.26) as follows:

$$H_j(t,z) = \exp(zD_{j,t,z}\log Z(T) - \log(1-\theta_j(t,z)). \tag{4.61}$$

Then we can express $D_{j,t,z}Z(T)$ in (4.25) as follows:

$$\begin{aligned} D_{j,t,z}Z(T) =& z^{-1}Z(T)[\exp(zD_{j,t,z}\log Z(T)) - 1] \\ =& z^{-1}Z(T)[(1-\theta_j(t,z))H_j(t,z) - 1]. \end{aligned} \tag{4.62}$$

Then, from the product rule (4.58), we obtain

$$\begin{aligned} & D_{j,t,z}(Z(T)F) \\ & \quad = z^{-1}Z(T)[(1-\theta_j(s,z))H_j(t,z) - 1]F + Z(T)D_{j,t,z}F \\ & \quad + Z(T)[(1-\theta_j(s,z))H_j(t,z) - 1]D_{j,t,z}F \\ & \quad = Z(T)[z^{-1}((1-\theta_j(s,z))H_j(t,z) - 1)F + (1-\theta_j(s,z))H_j(t,z)D_{j,t,z}F]. \end{aligned} \tag{4.63}$$

Substituting (4.60) and (4.63) into (4.57) gives us

$$
\begin{aligned}
&dY(t)\\
&= \Lambda(t^-)\left[\sum_{j=1}^{N}(\sigma_j E[Z(T)(D_{j,t,0} - F\left(\sigma_j^{-1}u_j(t) + K_j(t)\right))|\mathcal{F}_{t^-}]\right. &(4.64)\\
&+ E[Z(T)Fu_j(t)|\mathcal{F}_{t^-}])dW_j^Q(t)\\
&+ \int_{\mathbb{R}_0}\left(\frac{1}{1-\theta_j(t,z)}E[Z(T)[z^{-1}((1-\theta_j(s,z))H_j(t,z)-1)\right.\\
&\left.\left.+ \ (1-\theta_j(s,z))H_j(t,z)D_{j,t,z}F]z + \frac{\theta_j(t,z)}{1-\theta_j(t,z)}E[Z(T)F|\mathcal{F}_{t^-}]\right)\tilde{N}_j^Q(dt,dz)\right]\\
=&\Lambda(t^-)\left[\sum_{j=1}^{N}\sigma_j E[D_{j,t,0}F - FK_j(t)|\mathcal{F}_{t^-}]dW_j^Q(t)\right. &(4.65)\\
&\left.+\sum_{j=1}^{N}\int_{\mathbb{R}_0} E[F(H_j(t,z)-1) + zH_j(t,z)D_{j,t,z}F|\mathcal{F}_{t^-}]\tilde{N}_j^Q(dt,dz)\right]. &(4.66)
\end{aligned}
$$

Since F is $\mathcal{F}_T$ measurable, then

$$
Y(T) = E^Q[F|\mathcal{F}_T] = F \tag{4.67}
$$

and also

$$
Y(0) = E^Q[F|\mathcal{F}_0] = E^Q[F]. \tag{4.68}
$$

Then from the abstract Bayes rule, and from above boundary condition (4.67) and (4.68) , we finally obtain

$$
\begin{aligned}
F =&E^Q[F] + \sum_{j=1}^{d}\int_0^T \sigma_j E^Q[D_{j,t,0}F - FK_j(t)|\mathcal{F}_{t^-}]dW_j^Q(t)+ &(4.69)\\
&\sum_{j=1}^{l}\int_{[t,T]\times\mathbb{R}_0} E^Q[F(H_j(t,z)-1) + zH_j(t,z)D_{j,t,z}F|\mathcal{F}_{t^-}]\tilde{N}_j^Q(dt,dz). &(4.70)
\end{aligned}
$$

■

From the theorem, we have the following representation of $F \in L^2(P)\cap L^2(Q)$ for both continuous and pure jump case.

(i) Continuous Case

$$F = E^Q[F] + \sum_{j=1}^{N} \int_0^T \sigma_j E^Q[D_{j,t,0}F - FK_j(t)|\mathcal{F}_{t^-}]dW_j^Q(t) \tag{4.71}$$

where

$$K_j(t) = \sum_{i=1}^{N} \int_t^T D_{j,t,0}u_i(s)dW_i^Q(s). \tag{4.72}$$

(ii) Pure Jump Case

$$F = E^Q[F] + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} E^Q[F(H_j(t,z)-1) + zH_j(t,z)D_{j,t,z}F|\mathcal{F}_{t^-}]\tilde{N}_j^Q(dt,dz) \tag{4.73}$$

where

$$H_j(t,z) = \exp\left(\sum_{i=1}^{N}\left[\int_{[t,T]\times\mathbb{R}_0} \log\left(1 - \frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)\tilde{N}_i^Q(ds,dx)\right.\right.$$
$$\left.\left. + \int_{[t,T]\times\mathbb{R}_0}\left(\log\left(1 - \frac{zD_{j,t,z}\theta_i(s,x)}{1-\theta_i(s,x)}\right)(1-\theta_i(s,x)) + zD_{j,t,z}\theta_i(s,x)\right)\nu_i(dx)ds\right]\right). \tag{4.74}$$

For the deterministic case, we have the following representation for $F \in L^2(P) \cap L^2(Q)$.

Corollary 4.2.3 *Deterministic Drift*

Suppose that the assumptions of Theorem 4.2.2 are satisfied and in addition, u_j *and* θ_j *are deterministic, for all* $j \in \{1, \cdots, N\}$ *then for* $F \in L^2(P) \cap L^2(Q)$

$$\begin{aligned} F =& E^Q[F] + \sum_{j=1}^{N} \int_0^T \sigma_j E^Q[D_{j,t,0}F|\mathcal{F}_{t^-}]dW_j^Q(t) \\ &+ \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} E^Q[D_{j,t,z}F|\mathcal{F}_{t^-}]z\tilde{N}_j^Q(dt,dz) \\ =& E^Q[F] + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}} E^Q[D_{j,t,z}F|\mathcal{F}_{t^-}]M_j^Q(dt,dz). \end{aligned} \tag{4.75}$$

where M^Q *is an independent random measure on* $([0,T]\times\mathbb{R},\mathfrak{B}([0,T]\times\mathbb{R}))$ *such that*

$$M^Q(E)=\sigma\int_{E_0} dW^Q(t)+\int_{E'} zd\tilde{N}^Q(dt,dz) \tag{4.76}$$

where $E\in\mathfrak{B}(\mathbb{R}_+\times\mathbb{R})$, $E_0=\{t\in\mathbb{R}_+:(t,0)\in E\}$ *and* $E'=E\setminus E_0$.

4.3 Mean Variance Hedging

4.3.1 Financial Modeling Under a Lévy Market

In this section we give a brief overview of the financial market driven by Lévy processes. We will closely follow the discussions of Di Nunno [27], Øksendal and Sulem [65].

Asset Dynamics

We let $(\Omega,\mathcal{F},P)$ be the probability space under the usual hypothesis. For $t\in[0,T]$, we denote the following filtration:

(i) $\mathcal{F}_t$ - full information $\{\mathcal{F}\}_{t\in[0,T]}\subset\mathcal{F}$,

(ii) $\mathcal{H}_t$ - partial information $\mathcal{H}_t\subset\mathcal{F}_t$ for all $t\in[0,T]$.

We model our portfolio as follows. Let $S_0(t)$ be the risk-free asset process and $S_i(t)$, $i\in\{1,\cdots,N\}$ be the risky asset processes where $S_1(t),\cdots,S_M(t)$ are tradable $M\leq N$ and $S_{M+1}(t),\cdots,S_N(t)$ are non-tradable.

Under the objective P measure, we model the risky assets and risk-free asset with the following dynamics:

(i) risky asset P dynamics

$$\begin{aligned}
dS_i(t)=&\mu_i(t)dt+\sigma_i(t)dW(t)+\int_{\mathbb{R}_0}\gamma_i(t,z)\tilde{N}(dt,dz)\\
=&\mu_i(t)dt+\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)+\int_{\mathbb{R}_0}\sum_{j=1}^{l}\gamma_{ij}(t,z)\tilde{N}_j(dt,dz),\\
S_i(0)=&x_i>0,\qquad i\in\{1,\cdots,N\}
\end{aligned} \tag{4.77}$$

(ii) risk-free P dynamics

$$\begin{aligned} dS_0(t) =& r(t)S_0(t)dt, \\ S_0(0) =& 1 \end{aligned} \tag{4.78}$$

where the risk-free rate r is deterministic and the coefficients μ_i, σ_i, γ_i are predictable and satisfies the Lipschitz and growth conditions and

$$\begin{aligned} \sigma_i(t) =& [\sigma_{i1}(t), \cdots, \sigma_{id}(t)], \quad i \in \{1, \cdots d\}, \\ \gamma_i(t,z) =& [\gamma_{i1}(t,z), \cdots, \gamma_{il}(t,z)], \quad i \in \{1, \cdots l\}. \end{aligned} \tag{4.79}$$

In matrix-vector form, we can write (4.77) the dynamics of the risky asset of of the form

$$dS(t) = \mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz) \tag{4.80}$$

where

$$\begin{aligned} S(t) =& [S_1(t), \cdots, S_N(t)]^T, \\ W(t) =& [W_1(t), \cdots, W_d(t)]^T, \\ \tilde{N}(dt,dz) =& [\tilde{N}_1(dt,dz), \cdots, \tilde{N}_l(dt,dz)]^T, \\ \mu(t) =& [\mu_1(t), \cdots, \mu_N(t)]^T, \\ \sigma(t) =& \{\sigma_{ij}(t)\}_{1\le i\le N, 1\le j\le d}, \\ \gamma(t,z) =& \{\gamma_{ij}(t,z)\}_{1\le i\le N, 1\le j\le l}. \end{aligned} \tag{4.81}$$

Suppose that the the drift terms $u_i(t)$ and $\theta_i(t,z)$ and the Doleans-Dade exponential (4.3) satisfies the Girsanov theorem for Lévy processes (Theorem 4.1.1). Then, under the change of measure $Q \sim P$,

$$\begin{aligned} dS_i(t) =& \mu_i(t)dt + \sigma_i(t)dW(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}(dt,dz) \\ =& \mu_i(t)dt + \sigma_i(t)(dW^Q(t) - u(t)dt) + \int_{\mathbb{R}_0} \gamma_i(t,z)(\tilde{N}^Q(dt,dz) - \theta(t,z)\nu(dz)dt) \\ =& \mu_i(t)dt + \sigma_i(t)(dW^Q(t) - u(t)dt) + \int_{\mathbb{R}_0} \gamma_i(t,z)(\tilde{N}^Q(dt,dz) - \theta(t,z)\nu(dz)dt) \\ =& \alpha_i(t)dt + \sigma_i(t)dW^Q(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}^Q(dt,dz) \end{aligned} \tag{4.82}$$

where

$$\alpha_i(t) = \mu_i(t) - \sigma_i(t)u(t) - \int_{\mathbb{R}_0} \gamma_i(t,z)\theta(t,z)\nu(dz). \tag{4.83}$$

We let the discounted value process $\tilde{S}(t)$ given by

$$\tilde{S}_i(t) = \frac{S_i(t)}{S_0(t)}, \qquad t \in [0,T]. \tag{4.84}$$

then,

$$\tilde{S}_0(t) = 1. \tag{4.85}$$

From Itô's lemma, we get

$$\begin{aligned} d\tilde{S}_i(t) =& \frac{1}{S_0(t)} dS_i(t) + S_i(t) d\left(\frac{1}{S_0(t)}\right) + d\left[S_i, \frac{1}{S_0}\right]_t \\ =& \frac{1}{S_0(t)}\left(\alpha_i(t)dt + \sigma_i(t)dW^Q(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}^Q(dt,dz)\right) + S_i(t^-)\left(\frac{-r(t)dt}{S_0(t)}\right) \\ =& \frac{1}{S_0(t)}\left((\alpha_i(t) - r(t)S_i(t^-))dt + \sigma_i(t)dW^Q(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}^Q(dt,dz)\right). \end{aligned} \tag{4.86}$$

We also denote the discounted factor at the interval $[t,T]$ as follows:

$$D(t,T) = \exp\left(-\int_t^T r(s)ds\right). \tag{4.87}$$

Arbitrage-Free Condition

We assume an arbitrage-free portfolio. From the fundamental theorem of asset pricing, there exists an equivalent measure $Q \sim P$ such that $\tilde{S}_i$ is a Q-local martingale [65]. Then, there exists predictable processes u and θ such that

$$\sigma_i(t)u(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\theta(t,z)\nu(dz) = \mu_i(t) - r(t)S_i(t). \tag{4.88}$$

Denote the following processes normalized by the numeraire S_0

$$\begin{aligned} \tilde{\sigma}_{ij}(t) &= \frac{1}{S_0(t)}\sigma_{ij}(t), \qquad i \in \{1,\cdots,N\}, \quad j \in \{1,\cdots,d\}, \\ \tilde{\gamma}_{ij}(t,z) &= \frac{1}{S_0(t)}\gamma_{ij}(t,z), \qquad i \in \{1,\cdots,N\}, \quad j \in \{1,\cdots,l\}. \end{aligned} \tag{4.89}$$

Similarly, $\tilde{\sigma_i}$, $\tilde{\sigma}$, $\tilde{\gamma_i}$, $\tilde{\sigma_i}$ are defined as the normalized processes of σ_i, σ, γ_i, σ_i respectively by S_0. Hence, the discounted price dynamics under the risk-neutral measure Q is given by

$$\begin{aligned} d\tilde{S}_i(t) =& \tilde{\sigma}_i(t)dW^Q(t) + \int_{\mathbb{R}_0} \tilde{\gamma}_i(t,z)\tilde{N}^Q(dt,dz) \\ =& \sum_{j=1}^{d} \tilde{\sigma}_{ij}(t)dW_j^Q(t) + \int_{\mathbb{R}_0} \sum_{j=1}^{l} \tilde{\gamma}_{ij}(t,z)\tilde{N}_j^Q(dt,dz_j). \end{aligned} \tag{4.90}$$

Example Geometric Lévy processes for tradable risky assets

$$\begin{aligned} dS_i(t) =& S_i(t^-)\left(a_i(t)dt + \sum_{i=1}^{d} b_{ij}(t)dW_j(t) + \int_{\mathbb{R}_0} \sum_{j=1}^{l} c_{ij}(t,z_j)\tilde{N}_j(dt,dz)\right), \\ S_i(0) =& x_i > 0 \end{aligned} \tag{4.91}$$

where $i \in \{1, \cdots, M\}$ and all coefficients are predictable with $c_{ij} > -1$. In this model, we have

$$\begin{aligned} \mu_i(t) &= a_i(t)S_i(t), && i \in \{1,\cdots,M\}, \\ \sigma_{ij}(t) &= b_{ij}(t)S_i(t), && i \in \{1,\cdots,M\}, \quad j \in \{1,\cdots,d\} \\ \gamma_{ij}(t,z) &= c_{ij}(t)S_i(t), && i \in \{1,\cdots,M\}, \quad j \in \{1,\cdots,l\}. \end{aligned} \tag{4.92}$$

The solution of the SDE is given by

$$\begin{aligned} S_i(T) = \exp\Bigg(&\int_0^T \left(a_i(t) - \frac{1}{2}\sum_{j=1}^{d} u_j^2(s)\right)dt + \sum_{j=1}^{d}\int_0^T u_j(s)dW_j(t) + \\ &\sum_{j=1}^{l}\int_{[0,t]\times\mathbb{R}_0}\Big(\log(1+c_{ij}(t,z))\tilde{N}_j(dt,dz) + (\log(1+c_{ij}(t,z)) - c_{ij}(t,z))\nu_j(dz)dt\Big)\Bigg). \end{aligned} \tag{4.93}$$

The arbitrage-free condition is given by

$$a_i(t) - r(t) = \sum_{j=1}^{d} b_{ij}(t)u_j(t) + \int_{\mathbb{R}_0}\sum_{j=1}^{l} c_{ij}(t,z)\theta_j(t,z)\nu_j(dz). \tag{4.94}$$

Thus, the risk-neutral dynamics of the discounted process under the arbitrage-free condition is given by

$$d\tilde{S}_i(t) = \tilde{S}_i(t^-)\left(\sum_{j=1}^{d} b_{ij}(t)dW_j^Q(t) + \int_{\mathbb{R}_0}\sum_{j=1}^{l} c_{ij}(t,z)\tilde{N}_j^Q(dt,dz)\right). \tag{4.95}$$

Self-Financing Condition

We denote the portfolio/trading strategy $\varphi_i : [0, T] \to \mathbb{R}^{N+1}$ as an $\mathcal{F}_t$-predictable process which corresponds to the number of units the investor possess for the asset S_i at time t for all $i \in \{0, 1, \cdots, M\}$. The value/wealth process corresponding to the portfolio φ starting at x is given by

$$V_x^{\varphi}(t) \equiv V(t) = x + \varphi_0(t)S_0(t) + \sum_{i=1}^{M} \varphi_i(t)S_i(t). \tag{4.96}$$

Assume, the process is value process is self-financing, then

$$dV(t) = \varphi_0(t)dS_0(t) + \sum_{i=1}^{M} \varphi_i(t)dS_i(t). \tag{4.97}$$

We let the discounted value process $\tilde{V}(t)$ given by

$$\tilde{V}(t) = \frac{V(t)}{S_0(t)}. \tag{4.98}$$

From Itô's lemma, we get

$$d\tilde{V}(t) = \sum_{i=1}^{M} \varphi_i(t)d\tilde{S}_i(t) = \varphi(t) \cdot d\tilde{S}(t). \tag{4.99}$$

where

$$\varphi(t) = [\varphi_1(t), \cdots, \varphi_M(t)]^T, \qquad \tilde{S}(t) = [\tilde{S}_1(t), \cdots, \tilde{S}_M(t)]^T \tag{4.100}$$

Hence, the discounted value process is given by

$$\begin{aligned}\tilde{V}(t) =& x + \sum_{i=1}^{M} \int_0^t \varphi_i(s)d\tilde{S}_i(s)\\ =& x + \sum_{i=1}^{M} \int_0^T \varphi_i(t) \left(\tilde{\sigma}_i(s)dW^Q(s) + \int_{\mathbb{R}_0} \tilde{\gamma}_{ij}(s, z)\tilde{N}^Q(ds, dz) \right)\\ =& x + \sum_{i=1}^{M} \sum_{j=1}^{d} \int_0^t \varphi_i(s)\tilde{\sigma}_{ij}(s)dW_j^Q(s) + \sum_{i=1}^{M} \sum_{j=1}^{l} \int_{[0,t]\times\mathbb{R}_0} \varphi_i(s)\tilde{\gamma}_{ij}(s, z)\tilde{N}_j^Q(ds, dz).\end{aligned} \tag{4.101}$$

4.3.2 Quadratic Hedging

Motivation

A market is said to be complete if it can be replicated by a self-financing portfolio [7]. Under the Black-Scholes model, the market is complete. However, the market modeled under a Lévy process is in general incomplete and thus, the equivalent martingale measure Q is not unique.

A natural way to find a hedging portfolio is by minimizing the expected quadratic hedging error $(F - V_x^{\varphi}(T))$ for a contingent claim $F \in L^2(P)$ for all $x \in \mathbb{R}$ and φ belongs to some admissible set $\mathcal{A}^P$ with respect to the objective measure P, that is, we minimize the functional

$$J_{x,\varphi}^P = E^P[|F - V_x^{\varphi}(T)|^2], \qquad x \in \mathbb{R}, \quad \varphi \in \mathcal{A}^P. \tag{4.102}$$

This represents the mean square hedging error at maturity. The solution for this problems incorporates variance optimal martingale measure $Q^{MV} \sim P$ [8], [39], and [71] and explicit solutions are difficult to obtain in the the presence of jumps [8], [21].

A tractable way of doing quadratic hedging is when we work on the risk-neutral martingale measure Q where the discounted asset process $\tilde{S}$ is a Q-martingale. First, we define the concept of an admissible portfolio under partial information $\mathcal{H}$.

Definition 4.3.1 *[21], [27] The predictable process $\varphi(t)$ for $t \in [0,T]$ is an $\mathcal{H}$-admissible portfolio if the following conditions are satisfied:*

(i) $\varphi(t)$ is $\mathcal{H}$-caglad predictable,

(ii) $E^Q\left[\left|\int_0^T \varphi(t) \cdot d\tilde{S}(t)\right|^2\right] < \infty$.

The set of all $\mathcal{H}$-admissible portfolios is denoted by $\mathcal{A}_{\mathcal{H}}$.

To find a quadratic hedging portfolio in Q is to take the discounted $(F - V_x^{\varphi}(T))$ hedging error for $F \in L^2(Q)$ for all $x \in \mathbb{R}$ and φ belongs to some admissible set $\mathcal{A}_{\mathcal{H}}$. That is, we minimize the functional

$$J_{x,\varphi}^Q = E^Q[|\hat{F} - \hat{V}_x^{\varphi}(T)|^2], \qquad x \in \mathbb{R}, \quad \varphi \in \mathcal{A}_{\mathcal{H}}. \tag{4.103}$$

where $\hat{H}$ is the discounted claim H from maturity. Denote the set of admissible payoffs of the form

$$\mathcal{A} = \left\{ V_0 + \int_0^T \varphi(t) \cdot d\tilde{S}(t) : V_0 \in \mathbb{R}, \varphi \in \mathcal{A}_{\mathcal{H}} \right\}. \tag{4.104}$$

Then, $\mathcal{A}_{\mathcal{H}}$ is a closed subspace in $L^2(Q)$. The quadratic hedging problem under Q can be stated as follows:

$$\inf_{x \in \mathbb{R}, \varphi \in \mathcal{A}_{\mathcal{H}}} E^Q[|\hat{F} - \hat{V}_x^{\varphi}(T)|^2] = \inf_{A \in \mathcal{A}} ||\hat{F} - A||^2_{L^2(Q)}. \tag{4.105}$$

Under the assumption that $\tilde{S}$ is a square-intergrable Q martingale and $F \in L^2(Q)$ which implies $\hat{F} \in L^2(Q)$ from the dominated convergence theorem, then $\hat{F}$ admits a Galtchouk-Kunita-Watanabe (GKW) decomposition of the form

$$\hat{F} = E^Q[\hat{F}] + \int_0^T \varphi^*(t) \cdot d\tilde{S}(t) + N, \qquad a.s. \tag{4.106}$$

where $\{\varphi^*(t)\}_{t \in [0,T]}$ is a square integrable predictable process, N is orthogonal to all stochastic integrals with respect to $\tilde{S}$ and the martingale $N_t = E^Q[N|\mathcal{F}_t]$ is strongly orthogonal to $\mathcal{A}$.

From the GKW decomposition, (4.106), the stochastic integral $\int_0^T \varphi^*(t) \cdot d\tilde{S}(t)$ is the orthogonal projection of $\hat{F}$ in $L^2(Q)$ which corresponds to the hedgable component. On the other hand, N is the is the orthogonal complement of $\tilde{F}$ in $L^2(Q)$ which corresponds to the non-hedgable component or the residual risk. Using Malliavin calculus, our aim is to find $\varphi^*(t)$ by applying the Clark-Ocone reprenentation theorem.

Likewise, an alternative solution in solving the quadratic hedging error in P is to choose a risk-neutral measure $Q \sim P$ such that $\tilde{S}$ is a Q-martingale. Performing the GKW decomposition would yield different hedging strategies and since orthogonality is not invariant under the change of measure so does the the residual risk which is not desirable. However, by employing a Föllmer-Schweizer [30] minimal maringale measure $Q^{FS} \sim P$ and perfroming a GKW decomposition, the residual risk N preserves its its orthogonality under P.

Mean Variance Hedging Under the Martingale Measure

Given the contingent claim $F \in L^2(P) \cap L^2(Q)$, we want to find the hedging portfolio $\varphi \in \mathcal{A}_{\mathcal{H}}$ that minimizes the discounted residual error $(\tilde{F} - \tilde{V}_x^{\varphi}(T))$ at maturity in the mean-square sense under risk-neutral measure. That is, we want to minimize the functional $J^Q_{x,\varphi}$ in (4.103).

Theorem 4.3.1 *Quadratic hedging under the martingale measure*
Suppose that $F \in L^2(P) \cap L^2(Q)$ *has a Clark-Ocone representation of the form*

$$F = E^Q[F] + \sum_{j=1}^{N} \int_0^T \beta_j(t) dW_j^Q(t) + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} \xi_j(t,z) \tilde{N}_j^Q(dt,dz) \tag{4.107}$$

for some predictable process $\{\beta_j\}_{1\le j\le N}$ *and* $\{\xi_j\}_{1\le j\le N}$*, then the mean variance hedging portfolio problem in* (4.103) *under a martingale measure* Q *with discounted risky-asset dynamics given by* (4.90) *with partial information is given by* $\varphi(t) \in \mathbb{R}^N$ *which is a solution of the linear equation*

$$Q(t)\varphi(t) = D(t,T)R(t)$$

where

$$Q(t) = \left\{E^Q\left[N_{ik}(t)|\,\mathcal{H}_{t^-}\right]\right\} \in \mathbb{R}^{M\times M}, \qquad R(t) = \left\{E^Q\left[M_i(t)|\,\mathcal{H}_{t^-}\right]\right\} \in \mathbb{R}^M, \tag{4.108}$$

$$\begin{aligned} M_k(t) &= \sum_{j=1}^{N} \sigma_{kj}(t)\beta_j(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \gamma_{kj}(t,z)\xi_j(t,z)\nu_j(dz), \\ N_{ik}(t) &= \sum_{j=1}^{N} \sigma_{ij}(t)\sigma_{kj}(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \gamma_{ij}(t,z)\gamma_{kj}(t,z)\nu_j(dz). \end{aligned} \tag{4.109}$$

Proof From the GKW decomposition, it suffice to show that the residual $(\hat{F} - \hat{V}_x^{\varphi}(T))$ is orthogonal to all $G \in L^2(Q)$, that is,

$$E^Q[(\hat{F} - \hat{V}_x^{\varphi}(T))G] = 0 \tag{4.110}$$

where $\mathcal{F}_T$-measurable of the form

$$G = \sum_{i=1}^{M} \int_0^T \psi_i(t) d\tilde{S}_i(t) \tag{4.111}$$

and $\psi \in \mathcal{A}_{\mathcal{H}}$. Then, from (4.90), we obtain

$$\begin{aligned} G =& \sum_{i=1}^{M} \sum_{j=1}^{N} \int_0^T \psi_i(t)\tilde{\sigma}_{ij}(t) dW_j^Q(t) + \sum_{i=1}^{M} \sum_{j=1}^{l} \int_{[0,T]\times\mathbb{R}_0} \psi_i(t)\tilde{\gamma_{ij}}(t,z)\tilde{N}_j^Q(dt,dz) \\ =& \sum_{j=1}^{d} \int_0^T U_j(t) dW_j^Q(t) + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} V_j(t,z)\tilde{N}_j^Q(dt,dz). \end{aligned} \tag{4.112}$$

where

$$U_j(t) = \sum_{i=1}^{N} \psi_i(t)\tilde{\sigma}_{ij}(t), \tag{4.113}$$

$$V_j(t,z) = \sum_{i=1}^{N} \psi_i(t)\tilde{\gamma}_{ij}(t,z). \tag{4.114}$$

From the Clark-Ocone representation of $F \in L^2(P) \cap L^2(Q)$ in (4.107) since r is deterministic, then its $\hat{F}$ can be represented as

$$\hat{F} = E^Q[\hat{F}] + \sum_{j=1}^{N} \int_0^T D(0,T)\beta_j(t) dW_j^Q(t) + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} D(0,T)\xi_j(t,z)\tilde{N}_j^Q(dt,dz). \tag{4.115}$$

From dominated convergence theorem, the discounted process $\hat{F} \in L^2(P) \cap L^2(Q)$. Likewise, it can be shown that the optimal initial capital x under the mean-variance hedging [21] is given by

$$x = E^Q[F]. \tag{4.116}$$

Hence, form (4.101), we have

$$\begin{aligned} \hat{V}(T) =& E^Q[\hat{F}] + \sum_{i=1}^{M} \sum_{j=1}^{N} \int_0^T \varphi_i(t)\tilde{\sigma}_{ij}(t) dW_j^Q(t) \\ &+ \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} \varphi_i(t)\tilde{\gamma}_{ij}(t,z)\tilde{N}_j^Q(dt,dz). \end{aligned} \tag{4.117}$$

Then

$$\hat{F} - \hat{V}(T) = \sum_{j=1}^{N} \int_0^T A_j(t) dW_j^Q(t) + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} B_j(t,z) \tilde{N}_j^Q(dt, dz_j) \tag{4.118}$$

where

$$\begin{aligned} A_j(t) =& \beta_j(t) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\sigma}_{ij}(t), \\ B_j(t,z) =& \xi_j(t,z) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\gamma}_{ij}(t,z). \end{aligned} \tag{4.119}$$

Hence, the expression in (4.110) becomes

$$\begin{aligned} 0 =& E^Q[(\hat{F} - \hat{V})G] \\ =& E^Q\left[\sum_{j=1}^{N} \int_0^T A_j(t) U_j(t) dt + \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} B_j(t) V_j(t) \nu_j(dz) dt\right] \\ =& E^Q\left[\sum_{j=1}^{N} \int_0^T \left(\beta_j(t) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\sigma}_{ij}(t)\right) \sum_{i=1}^{M} \psi_i(t) \tilde{\sigma}_{ij}(t) dt + \right. \\ & \left. \sum_{j=1}^{N} \int_{[0,T]\times\mathbb{R}_0} \left(\xi_j(t,z) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\gamma}_{ij}(t,z)\right) \sum_{i=1}^{M} \psi_i(t) \tilde{\gamma}_{ij}(t,z) \nu_j(dz) dt\right] \\ & E^Q\left[\sum_{k=1}^{N} \int_0^T \psi_k(t) l_k(t) dt\right] \end{aligned} \tag{4.120}$$

where

$$\begin{aligned} l_k(t) =& \sum_{j=1}^{N} \tilde{\sigma}_{kj}(t) \left(\beta_j(t) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\sigma}_{ij}(t)\right) + \\ & + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \tilde{\gamma}_{kj}(t,z) \left(\xi_j(t,z) D(0,T) - \sum_{i=1}^{M} \varphi_i(t) \tilde{\gamma}_{ij}(t,z)\right) \nu_j(dz) dt \\ =& \left(\sum_{j=1}^{N} \tilde{\sigma}_{kj}(t) \beta_j(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \tilde{\gamma}_{kj}(t,z) \xi_j(t,z) \nu_j(dz)\right) D(0,T) \\ & - \sum_{i=1}^{M} \varphi_i(t) \left(\sum_{j=1}^{N} \tilde{\sigma}_{ij}(t) \tilde{\sigma}_{kj}(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \tilde{\gamma}_{ij}(t,z) \tilde{\gamma}_{kj}(t,z) \nu_j(dz)\right) \\ =& m_k(t) - \sum_{i=1}^{M} \varphi_i(t) n_{ik}(t), \end{aligned} \tag{4.121}$$

$$m_k(t) = \left(\sum_{j=1}^{N} \tilde{\sigma}_{kj}(t)\beta_j(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \tilde{\gamma}_{kj}(t,z)\xi_j(t,z)\nu_j(dz) \right) D(0,T) \tag{4.122}$$

$$n_{ik}(t) = \sum_{j=1}^{N} \tilde{\sigma}_{ij}(t)\tilde{\sigma}_{kj}(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \tilde{\gamma}_{ij}(t,z)\tilde{\gamma}_{kj}(t,z)\nu_j(dz). \tag{4.123}$$

Then, (4.120) holds for all $\psi \in \mathcal{A}_{\mathcal{H}}$ if and only if

$$E^Q \left[L_k(t) \middle| \mathcal{H}_{t^-} \right] = 0. \tag{4.124}$$

Since $\mathcal{H}_t \subset \mathcal{F}_t$ and $\varphi \in \mathcal{A}_{\mathcal{H}}$, and removing the tildes in $\{\tilde{\sigma}_{ij}\}$ and $\{\tilde{\gamma}_{ij}\}$ following system of linear equation for $k \in \{1, \cdots, M\}$

$$\sum_{i=1}^{M} E^Q \left[N_{ik}(t) \middle| \mathcal{H}_{t^-} \right] \varphi_i(t) = D(t,T) E^Q \left[M_k(t) \middle| \mathcal{H}_{t^-} \right] \tag{4.125}$$

where

$$M_k(t) = \sum_{j=1}^{N} \sigma_{kj}(t)\beta_j(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \gamma_{kj}(t,z)\xi_j(t,z)\nu_j(dz),$$

$$N_{ik}(t) = \sum_{j=1}^{N} \sigma_{ij}(t)\sigma_{kj}(t) + \sum_{j=1}^{N} \int_{\mathbb{R}_0} \gamma_{ij}(t,z)\gamma_{kj}(t,z)\nu_j(dz). \tag{4.126}$$

Then, $\varphi(t)$ can be solved as a linear equation of the form

$$Q(t)\varphi(t) = D(t,T)R(t) \tag{4.127}$$

where $\varphi(t) \in \mathbb{R}^M$ and

$$Q(t) = \{Q_{ik}(t)\} \in \mathbb{R}^{M\times M}, \quad Q_{ik}(t) = E^Q \left[N_{ik}(t) \middle| \mathcal{H}_{t^-} \right],$$
$$R(t) = \{R_i(t)\} \in \mathbb{R}^{M}, \qquad R_i(t) = E^Q \left[M_i(t) \middle| \mathcal{H}_{t^-} \right].$$

■

These spacial cases would yield some simplifications in the computation of the parameters.

(i) Drift parameters $u_{ij}(t), \theta_{ij}(t,z)$ are deterministic. From Corollary 4.2.3, we obtain

$$\beta_j(t) = \sigma_j E^Q[D_{j,t,0}F|\mathcal{F}_{t^-}],$$
$$\xi_j(t,z) = z E^Q[D_{j,t,z}F|\mathcal{F}_{t^-}], \tag{4.128}$$

and $\varphi(t)$ can be solved using Theorem 4.3.1.

(ii) Univariate case (single tradable risky asset)

$$\varphi(t) = D(t,T)\frac{E^Q\left[\sum_{j=1}^N \sigma_{1j}(t)\beta_j(t) + \sum_{j=1}^N \int_{\mathbb{R}_0} \gamma_{1j}(t,z)\xi_j(t,z)\nu_j(dz)\Big| \mathcal{H}_{t-}\right]}{E^Q\left[\sum_{j=1}^N \sigma_{1j}^2(t) + \sum_{j=1}^N \int_{\mathbb{R}_0} \gamma_{1j}^2(t,z)\nu_j(dz)\Big| \mathcal{H}_{t-}\right]} \tag{4.129}$$

(iii) Univariate, and the coefficients $\sigma_{ij}(t), \gamma_{ij}(t,z)$ are $\mathcal{H}$-predictable

$$\varphi(t) = D(t,T)\frac{\sum_{j=1}^N \sigma_{1j}(t)E^Q\left[\beta_j(t)|\mathcal{H}_{t-}\right] + \sum_{j=1}^N \int_{\mathbb{R}_0} \gamma_{1j}(t,z)E^Q\left[\xi_j(t,z)|\mathcal{H}_{t-}\right]\nu_j(dz)}{\sum_{j=1}^N \sigma_{1j}^2(t) + \sum_{j=1}^N \int_{\mathbb{R}_0} \gamma_{1j}^2(t,z)\nu_j(dz)} \tag{4.130}$$

(iv) Univariate, and the coefficients σ_{ij}, γ_{ij} are $\mathcal{F}$-predictable (full model)

$$\varphi(t) = D(t,T)\frac{\sum_{j=1}^d \sigma_{1j}(t)\beta_j(t) + \sum_{j=1}^l \int_{\mathbb{R}_0} \gamma_{1j}(t,z)\xi_j(t,z)\nu_j(dz)}{\sum_{j=1}^d \sigma_{1j}^2(t) + \sum_{j=1}^l \int_{\mathbb{R}_0} \gamma_{1j}^2(t,z)\nu_j(dz)} \tag{4.131}$$

4.3.3 Geometric Lévy Processes

We will discuss some characterization of quadratic hedging for a geometric Lévy process. We consider the P dynamics of a geometric Lévy process is given as

$$\begin{aligned} dS_0(t) =& r(t)S_0(t)dt, \qquad S(0) = 1 \\ dS_1(t) =& S_1(t)\left(a(t)dt + b(t)dW(t) + \int_{\mathbb{R}_0} c(t,x)\tilde{N}(dt,dx)\right), \qquad S_1(0) = x_1 > 0. \end{aligned} \tag{4.132}$$

We will discuss hedging strategies for the change of measure where the drift parameters are deterministic. Then, we will discuss quadratic hedging under minimum martingale measure.

Deterministic Coefficients

We will assume that the coefficients $a(t)$, $r(t)$, $b(t)$, and $c(t,z) > -1$ are deterministic such that

$$\int_0^T \left(|a(t)| + |r(t)| + b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right) dt < \infty. \tag{4.133}$$

For an arbitrage-free portfolio, we require the drift parameters $u(t)$ and $\theta(t,z)$ to satisfy the following:

$$a(t) - r(t) = b(t)u(t) + \int_{\mathbb{R}_0} c(t,z)\theta(t,z)\nu(dx). \tag{4.134}$$

Then, under the risk-neutral dynamics, the discounted price process is given by

$$d\tilde{S}_1(t) = \tilde{S}_1(t^-)\left(b(t)dW^Q(t) + \int_{\mathbb{R}_0} c(t,z)\tilde{N}^Q(dt,dz)\right). \tag{4.135}$$

If $u(t)$ and $\theta(t,z)$ are deterministic, then, we obtain the following mean-variance hedging portfolio for $F \in L^2(P)$ under a full model as follows:

$$\varphi(t) = D(t,T)\frac{b(t)\sigma E^Q\left[D_{t,0}F|\mathcal{F}_{t^-}\right] + \int_{\mathbb{R}_0} c(t,z)zE^Q\left[D_{t,z}F|\mathcal{F}_{t^-}\right]\nu(dz)}{S_1(t)\left(b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right)}. \tag{4.136}$$

Models where the drift coefficients are deterministic include the Merton model [59] which we will discuss in the next section and the minimum measure under the exp-Lévy model [33].

It can be shown [10], if we have the additional condition

$$\int_{\mathbb{R}_0} \left(c^4(t,z) + |\log(1 + c(t,z))|^2\right)\nu(dz) < \infty \tag{4.137}$$

then $S_1(T) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}S_1(T) = \frac{S_1(T)b(t)}{\sigma}\mathbf{1}_{\{z=0\}} + \frac{S_1(T)c(t,z)}{z}\mathbf{1}_{\{z\neq 0\}}. \tag{4.138}$$

Furthermore, consider the contingent claim $F = \Phi(S_1(T))$, where $\Phi \in C^1(\mathbb{R}_+, \mathbb{R})$ such that

(i) $\Phi(S_1(T)) \in L^2(P)$,

(ii) $\Phi'(S_1(T))D_{t,0}\Phi(S(T)) = \Phi'(S_1(T))\frac{S_1(T)b(t)}{\sigma} \in L^2(P\times\lambda)$,

(iii) $\frac{\Phi(S_1(T)+zD_{t,z}S_1(T))-\Phi(S_1(T))}{z} = \frac{\Phi(S_1(T)(1+c(t,z))-\Phi(S_1(T))}{z}. \in L^2(P\times z^2\nu(dz)dt)$, $z \neq 0$

Then, from the chain rule, $F \in \mathbb{D}^{1,2}$ and

$$D_{t,z}F = \Phi^{'}(S_1(T))\frac{S_1(T)b(t)}{\sigma}\mathbf{1}_{\{z=0\}} + \frac{\Phi(S_1(T)(1+c(t,z))) - \Phi(S_1(T))}{z}\mathbf{1}_{\{z\neq 0\}}. \tag{4.139}$$

Hence, the mean-variance hedging portfolio under a deterministic drift coefficients becomes

$$\begin{aligned}\varphi(t) =& D(t,T)\frac{1}{S_1(t)\left(b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right)}\left[b^2(t)E^Q\left[\Phi^{'}(S_1(T))S_1(T)|\mathcal{F}_{t-}\right]\right. \\ &\left.+\int_{\mathbb{R}_0} c(t,z)E^Q\left[\Phi(S_1(T)(1+c(t,z))) - \Phi(S_1(T))|\mathcal{F}_{t-}\right]\nu(dz)\right]. \end{aligned} \tag{4.140}$$

Merton Model

We examine the Merton model which is the first jump-diffusion model in option pricing [59], [21]. The P dynamics of this model is given as

$$S_1(t) = S_0 \exp\left(\mu dt + \sigma dW(t) + dJ(t)\right) \tag{4.141}$$

where μ is the rate of return, b is the diffusion are assumed to be constant. The jump process $J(t)$ is a compound Poisson process of the form

$$J(t) = \sum_{i=1}^{N(t)} Y_i \tag{4.142}$$

where $N(t) \sim Poisson(\lambda t)$, $Y_i \overset{iid}{\sim} N(m, \delta^2)$. Then, $J(t)$ is a pure jump Lévy process with Lévy measure $\nu(dz) = \lambda F(dz)$ where F is the distribution function of Y_1. Then $J(t)$ can be represented in terms of Poisson random measure as follows:

$$\begin{aligned} J(t) &= \int_{[0,t]\times\mathbb{R}_0} zN(ds,dz) \\ &= \int_{[0,t]\times\mathbb{R}_0} z\left(\tilde{N}(ds,dz) + \nu(dz)ds\right). \end{aligned} \tag{4.143}$$

We assume all of these processes are mutually independent. Then, we can write the risky-asset process process as follows:

$$S_1(t) = S_0 \exp\left[\left(\mu + \int_{\mathbb{R}_0} z\nu(dz)\right)t + \sigma dW(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(ds,dz)\right]. \tag{4.144}$$

Then the Merton model is a special case of geometric Lévy process with deterministic coefficients with

$$b(t) = \sigma, \qquad c(t, z) = e^z - 1. \tag{4.145}$$

From Itô's lemma,

$$dS_1(t) = S_1(t^-)\left[\left(\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^z - 1)\nu(dz)\right) dt + \sigma dW(t) + \int_{\mathbb{R}_0} (e^z - 1)\tilde{N}(dt, dz)\right]. \tag{4.146}$$

Under the change of measure $Q \sim P$, we have the following Brownian motions under the Q measure:

$$dW^Q(t) = dW(t) + u(t)dt, \tag{4.147}$$

$$\tilde{N}^Q(dt, dz) = \tilde{N}(dt, dz) + \theta(t, z)\nu(dz)dt \tag{4.148}$$

For an arbitrage-free condition, we have the following constraint:

$$\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^z - 1)\nu(dz) - r(t) = \sigma u(t) + \int_{\mathbb{R}_0} (e^z - 1)\theta(t, z)\nu(dz). \tag{4.149}$$

Under the risk-neutral measure Q, Merton proposed changing the drift term of the Wiener process but leaving the jump part unchanged. The economic justification of this proposal is that the "jump risk" can be diversified and there is no market risk premium attached to it. Then, we have the following drift coefficients:

$$u(t) = \frac{\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^z - 1)\nu(dz) - r(t)}{\sigma}, \qquad \theta(t, z) = 0. \tag{4.150}$$

From the distribution of Y_1, we have the following:

$$\begin{aligned} \int_{\mathbb{R}_0} (e^z - 1)\nu(dz) &= \lambda \int_{\mathbb{R}_0} (e^z - 1)F(dz) \\ &= \lambda\left(E[Y(\cdot)] - 1\right) \\ &= \lambda\left(\exp\left(m + \frac{\delta^2}{2}\right) - 1\right). \end{aligned} \tag{4.151}$$

Then, we have the following dynamics under the Q measure

$$dS_1(t) = S_1(t^-)\left(r(t)dt + \sigma dW^Q(t) + \int_{\mathbb{R}_0} (e^z - 1)\tilde{N}^Q(dt, dz)\right). \tag{4.152}$$

We claim that (4.137) is satisfied. From the second moment and the moment generating function of a normal distribution, we see that

$$\begin{aligned}&\int_{\mathbb{R}_0}\left(c^4(t,z)+|\log(1+c(t,z))|^2\right)\nu(dz)\\&=\lambda\int_{\mathbb{R}_0}\left((e^2-1)^4+z^2\right)F(dz)\\&=\lambda\left[E\left[(\exp(Y(\cdot))-1)^4\right]+E\left[Y(\cdot)^2\right]\right]<\infty\end{aligned} \tag{4.153}$$

and thus $S_1(t)\in\mathbb{D}^{1,2}$. The mean-variance hedging portfolio is given by

$$\varphi(t)=D(t,T)\frac{\sigma^2E^Q\left[D_{t,0}F|\mathcal{F}_{t-}\right]+\int_{\mathbb{R}_0}(e^z-1)zE^Q\left[D_{t,z}F|\mathcal{F}_{t-}\right]\nu(dz)}{S_1(t)\left(\sigma^2+\int_{\mathbb{R}_0}(e^z-1)^2\nu(dz)\right)}. \tag{4.154}$$

From the moment generating function of a normal distribution, we have the following integral:

$$\begin{aligned}\int_{\mathbb{R}_0}(e^z-1)^2\nu(dz)=&\lambda\int_{\mathbb{R}_0}(e^{2z}-2e^z+1)F(dz)\\=&\lambda E\left[\exp(2Y(\cdot))-\exp(Y(\cdot))+1\right]\\=&\lambda\left[\exp(2m+2\delta^2)-2\exp(m+\frac{\delta^2}{2})+1\right].\end{aligned} \tag{4.155}$$

Exponential Lévy Process

A special case of the geometric Lévy processes if the risky asset price is modeled as an exponential Lévy (exp-Lévy) model [21] given by the following Q measure

$$S_1(t)=S_0\exp(rt+L(t)) \tag{4.156}$$

where $L(t)$ is a Lévy process with characteristic triplet (a,σ^2,ν^Q). Thus, from the Lévy-Itô decomposition theorem,

$$\begin{aligned}dL(t)=&adt+\sigma dW^Q(t)+\int_{|z|\geq 1}zN^Q(dt,dz)+\int_{|z|<1}z\tilde{N}^Q(dt,dz)\\=&bdt+\sigma W(t)+\int_{\mathbb{R}_0}x\tilde{N}^Q(ds,dx).\end{aligned} \tag{4.157}$$

where

$$b = a + \int_{|z|\geq 1} z\nu^Q(dz). \tag{4.158}$$

and a, r, σ are constants. Then its normalized asset process is given as

$$\tilde{S}_1(t) = S_0 \exp(L(t)). \tag{4.159}$$

From Itô's lemma, we obtain

$$\begin{aligned}
&d\tilde{S}_1(t) \\
=&\tilde{S}_1(t^-)\left[\left(b + \frac{\sigma^2}{2}\right)dt + \sigma dW^Q(t) + \int_{\mathbb{R}_0}(e^z - 1)\tilde{N}^Q(dt, dz)\right. \\
&\left. + \int_{\mathbb{R}_0}(e^z - z - 1)\nu^Q(dz)dt\right] \\
=&\tilde{S}_1(t^-)\left[\left(a + \frac{\sigma^2}{2}\right)dt + \sigma dW^Q(t) + \int_{\mathbb{R}_0}(e^z - 1)\tilde{N}^Q(dt, dz)\right. \\
&\left. + \int_{\mathbb{R}_0}\left(e^z - z\mathbf{1}_{\{z<1\}} - 1\right)\nu^Q(dz)dt\right].
\end{aligned} \tag{4.160}$$

Moreover, under the following restrictions [21]:

$$\int_{|z|\geq 1} e^z \nu^Q(dz) < \infty, \tag{4.161}$$

$$\left(a + \frac{\sigma^2}{2}\right) + \int_{\mathbb{R}_0}\left(e^z - z\mathbf{1}_{\{z<1\}} - 1\right)\nu^Q(dz) = 0. \tag{4.162}$$

$\tilde{S}_1(t)$ is a square-integrable Q-martingale and $E^Q[\exp(L(T))] = 1$. Then under the martingale measure Q, we have

$$d\tilde{S}_1(t) = \tilde{S}_1(t^-)\left[\sigma dW^Q(t) + \int_{\mathbb{R}_0}(e^z - 1)\tilde{N}^Q(dt, dz)\right]. \tag{4.163}$$

Then, the exp-Lêvy model is the geometric Lévy process with

$$b(t) = \sigma, \qquad c(t, z) = e^z - 1. \tag{4.164}$$

Since we have already started using the martingale measure Q, then we can set the drift coefficients to zero then we have the following mean-variance hedging portfolio

$$\varphi(t) = D(t,T)\frac{\sigma^2 E^Q\left[D_{t,0}F|\mathcal{F}_{t^-}\right] + \int_{\mathbb{R}_0}(e^z - 1)zE^Q\left[D_{t,z}F|\mathcal{F}_{t^-}\right]\nu(dz)}{S_1(t)\left(\sigma^2 + \int_{\mathbb{R}_0}(e^z - 1)^2\nu(dz)\right)}. \tag{4.165}$$

Furthermore, if we have the additional condition

$$\int_{\mathbb{R}_0} \left[(e^z - 1)^4 + z^2\right] \nu(dz) < \infty \tag{4.166}$$

then $S(T) \in \mathbb{D}^{1,2}$ and

$$D_{t,z} S_1(T) = S_1(T) \mathbf{1}_{\{z=0\}} + \frac{S_1(T)(e^z - 1)}{z} \mathbf{1}_{\{z \neq 0\}}. \tag{4.167}$$

For the contingent claim $F \in \Phi(S(T))$, where $\Phi \in C^1(\mathbb{R}_+, \mathbb{R})$ and suppose that the assumptions of the chain rule holds, then

$$D_{t,z} F = \Phi'(S_1(T)) S_1(T) \mathbf{1}_{\{z=0\}} + \frac{\Phi(S)_1(T)e^z) - \Phi(S_1(T))}{z} \mathbf{1}_{\{z \neq 0\}}. \tag{4.168}$$

Hence, the mean-variance hedging portfolio is given by

$$\begin{aligned} \varphi(t) =& D(t,T) \frac{1}{S_1(t)\left(\sigma^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz)\right)} \left[\sigma^2 E^Q \left[\Phi'(S_1(T)) S_1(T) | S_1(t^-)\right]\right. \\ & \left. + \int_{\mathbb{R}_0} (e^z - 1) E^Q \left[\Phi(S_1(T)e^z) - \Phi(S_1(T)) | \mathcal{F}_{t^-}\right] \nu(dz)\right]. \end{aligned} \tag{4.169}$$

Example [10] Consider the European call option with strike price K

$$\Phi(S_1(T)) = (S_1(T) - K)^+ \tag{4.170}$$

Then, from the chain rule under the geometric Lévy process,

$$\begin{aligned} D_{t,z} F =& \frac{S_1(T) b(t)}{\sigma} \mathbf{1}_{\{S_1(T) > K\}} \mathbf{1}_{\{z=0\}} \\ & + \frac{(S_1(T)(1 + c(t,z)) - K)^+ - (S_1(T) - K)^+}{z} \mathbf{1}_{\{z \neq 0\}}. \end{aligned} \tag{4.171}$$

On the other hand, for the exp-Lévy model we have the following:

$$D_{t,z} F = S(T) \mathbf{1}_{\{S_1(T) > K\}} \mathbf{1}_{\{z=0\}} + \frac{(S_1(T)e^z - K)^+ - (S_1(T) - K)^+}{z} \mathbf{1}_{\{z \neq 0\}}. \tag{4.172}$$

4.3.4 Minimal Martingale Measure

We examine the mean-variance hedging under the minimal martingale measure (MMM). In addition, we shall present results with the Barndorff-Nielsen and Shepard (BNS) stochastic volatility model in light of results of Arai and Suzuki [11].

(i) **Deterministic Coefficients**

Consider the asset dynamics (4.132) in Section (4.3.3) with deterministic coefficients under the objective P-measure satisfying the integrability condition (4.133). Then S_1 is a special semi-martingale [69] with a unique canonical decomposition

$$S_1(t) = S_1(0) + A(t) + M(t) \tag{4.173}$$

where $M(t)$ is a local martingale with $M(0) = 0$ and $A(t)$ is a finite variation process with $A(0) = 0$. Hence

$$dM(t) = S_1(t^-)\left(b(t)dW(t) + \int_{\mathbb{R}_0} c(t,x)\tilde{N}(dt,dz)\right), \tag{4.174}$$

$$dA(t) = S_1(t^-)a(t)dt. \tag{4.175}$$

The minimal martingale measure Q is given by the following Radon-Nikodym derivative [75]

$$Z(t) = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\int_{[0,t]} \Lambda(s)dM(s)\right) \in L^2(P) \tag{4.176}$$

that is, $Z(t)$ satisfies the SDE

$$dZ(t) = -Z(t^-)\Lambda(t)dM(t), \qquad Z(0) = 1 \tag{4.177}$$

for some predictable process $\Lambda(t)$. If we let

$$u(t) = \Lambda(t)b(t)S_1(t^-), \qquad \theta(t,z) = \Lambda(t)c(t,z)S_1(t^-) \tag{4.178}$$

then

$$\Lambda(t)dM(t) = u(t)dW(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(dt,dz). \tag{4.179}$$

Assume that Z satisfies the Novikov-type conditions, then from the Girsanov theorem for Lévy processes, and W^Q and $\tilde{N}^Q$ is a Brownian motion and compensated Poisson random measure under Q respectively where

$$dW^Q(t) = dW(t) + u(t)dt, \tag{4.180}$$

$$\tilde{N}^Q(dt,dz) = \tilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt. \tag{4.181}$$

From the arbitrage-free condition, we have the following,

$$a(t) - r(t) = b(t) \cdot \Lambda(t) b(t) S_1(t^-) + \int_{\mathbb{R}_0} c(t,z) \cdot \Lambda(t) c(t,z) S_1(t^-) \nu(dz). \quad (4.182)$$

Solving for $\Lambda(t)$ yields

$$\Lambda(t) = \frac{a(t) - r(t)}{S_1(t^-)\left(b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right)}. \quad (4.183)$$

Thus, the drift parameters are given as follows:

$$u(t) = \frac{(a(t) - r(t))b(t)}{\left(b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right)}, \qquad \theta(t,z) = \frac{(a(t) - r(t))c(t,z)}{\left(b^2(t) + \int_{\mathbb{R}_0} c^2(t,z)\nu(dz)\right)}. \quad (4.184)$$

which are deterministic. Thus, the mean-variance hedging portfolio is given by (4.136).

(ii) **Barndorff-Nielsen and Shephard Model**

The Barndorff-Nielsen and Shephard (BNS) model is a stochastic volatility model driven by a positive Lévy Orstein-Uhlenbeck (OU) process. The risky asset under the historical P dynamics is given by the following model [12], [21]

$$\begin{aligned} S_1(t) =& S_0 \exp(X(t)), \\ dX(t) =& (\mu + \beta v(t))dt + \sqrt{v(t)} dW(t) + \rho dL(\lambda t), \\ dv(t) =& -\lambda v(t) dt + L(\lambda t), \quad v(0) > 0 \end{aligned} \quad (4.185)$$

where $L(t)$ is the driving Lévy process which is a subordinator (an increasing Lévy process almost surely) without drift with Lévy measure ν^Z, $\beta v(t)$ is the volatility risk premium, $\rho dZ(t)$ is the leverage effect, $\rho \leq 0$, and $\lambda > 0$ is the mean-reversion parameter.

We let $J(t) = L(\lambda t)$ be the jump process with Poisson random measure $N(dt, dz)$, and Lévy measure $\nu(dz)$. Then $\nu(dz) = \lambda \nu^L(dz)$ and

$$J(t) = L(\lambda t) = \int_{[0,t]\times\mathbb{R}_+} x N(ds, dx) \quad (4.186)$$

from Itô's lemma, we obtain

$$dS_1(t) = S_1(t^-)\left(a(t)dt + \sqrt{v(t)}dW(t) + \int_{\mathbb{R}_+}(e^{\rho x} - 1)\tilde{N}(dt, dz)\right) \tag{4.187}$$

where

$$a(t) = \mu + \left(\beta + \frac{1}{2}\right)v(t) + \int_0^\infty (e^{\rho z} - 1)\nu(dz). \tag{4.188}$$

The difference in the preceding case is that the coefficient $\sigma(t)$ is random. If we set $\beta = -1/2$, then

$$a(t) = \mu + \int_0^\infty (e^{\rho z} - 1)\nu(dz). \tag{4.189}$$

which is deterministic. For now, we will be dealing the case $\beta = -1/2$ since the boundedness of the drift parameters $u(t)$ and $\theta(t, z)$ no longer holds [9], [11]. In addition, the solution for the Lévy OU process is given by

$$v(t) = v(0)e^{-\lambda t} + \int_{[0,t]\times\mathbb{R}_+} e^{\lambda(s-t)} zN(ds, dz). \tag{4.190}$$

Assume that S_1 is a special semi-martingale then repeating the same calculations in the preceding case, we obtain the MMM drift parameters. Solving for $\Lambda(t)$ yields

$$\Lambda(t) = \frac{a(t) - r(t)}{S_1(t^-)\left(v(t) + \int_0^\infty (e^{\rho z} - 1)^2\nu(dz)\right)}. \tag{4.191}$$

Thus, the drift parameters are given as follows:

$$u(t) = \frac{(a(t) - r(t))b(t)}{\left(v(t) + \int_0^\infty (e^{\rho z} - 1)^2\nu(dz)\right)}, \qquad \theta(t, z) = \frac{(a(t) - r(t))(e^{\rho z} - 1)}{\left(v(t) + \int_0^\infty (e^{\rho z} - 1)^2\nu(dz)\right)}. \tag{4.192}$$

which is random.

Then, we have the following Malliavin derivatives [11] for $t \leq s$,

$$D_{t,z}v(s) = e^{-\lambda(s-t)}\mathbf{1}_{\{z>0\}}, \tag{4.193}$$

$$D_{t,z}\sqrt{v(s)} = \frac{\sqrt{v(s)+ze^{-\lambda(s-t)}}-\sqrt{v(s)}}{z}\mathbf{1}_{\{z>0\}}, \tag{4.194}$$

$$\begin{aligned} D_{t,z}u(s) =& \frac{f_u(\sqrt{v(s)}+zD_{t,z}\sqrt{v(s)})-f_u(\sqrt{v(s)})}{z}\mathbf{1}_{\{z>0\}} \\ =& \frac{f_u(\sqrt{\sigma^2(s)+ze^{-\lambda(s-t)}})-f_u(\sqrt{v(s)})}{z}\mathbf{1}_{\{z>0\}}, \end{aligned} \tag{4.195}$$

$$\begin{aligned} D_{t,z}\theta(s,x) =& \frac{f_\theta(\sqrt{v(s)}+zD_{t,z}\sqrt{v(s)})}{z}\mathbf{1}_{\{z>0\}} \\ =& \frac{f_u(\sqrt{v(s)+ze^{-\lambda(s-t)}})-f_u(\sqrt{v(s)})}{z}\mathbf{1}_{\{z>0\}}, \end{aligned} \tag{4.196}$$

where

$$f_u(y) = \frac{(a(t)-r(t))y}{y^2+\int_0^\infty (e^{\rho z}-1)^2\nu(dz)}, \qquad y\in\mathbb{R} \tag{4.197}$$

and

$$f_\theta(y) = \frac{(a(t)-r(t))}{y^2+\int_0^\infty (e^{\rho z}-1)^2\nu(dz)}, \qquad y\in\mathbb{R}. \tag{4.198}$$

Hence, the mean-variance hedging portfolio is given by

$$\varphi(t) = D(t,T)\frac{\sigma\sqrt{v(t)}\beta(t)+\int_0^\infty (e^z-1)z\xi(t,z)\nu(dz)}{S_1(t^-)\left(v(t)+\int_0^\infty (e^z-1)^2\nu(dz)\right)}. \tag{4.199}$$

The from the Clark-Ocone theorem under the change of measure, $\beta(t)$ and $\xi(t,z)$ are determined as follows:

$$\beta(t) = \sigma E^Q[D_{t,0}F - FK(t)|\mathcal{F}_{t^-}], \tag{4.200}$$

$$\xi(t,z) = E^Q[F(H(t,z)-1)+zH(t,z)D_{t,z}F|\mathcal{F}_{t^-}] \tag{4.201}$$

where

$$K(t) = \int_t^T D_{t,0}u(s)dW^Q(s) + \int_{[t,T]\times\mathbb{R}_+}\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}^Q(ds,dx) = 0, \tag{4.202}$$

$$\begin{aligned} H(t,z) = \exp\Bigg(&-\int_t^T zD_{t,z}u(s)dW^Q(s) - \frac{1}{2}\int_t^T (zD_{t,z}u(s))^2ds \\ &+ \int_{[t,T]\times\mathbb{R}_+}\log\left(1-\frac{zD_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)\tilde{N}^Q(ds,dx) \\ &+ \int_{[t,T]\times\mathbb{R}_+}\left(\log\left(1-\frac{zD_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x))+zD_{t,z}\theta(s,x)\right)\nu(dx)ds\Bigg). \end{aligned} \tag{4.203}$$

4.3.5 The Bates Model

Preliminaries

The Bates model is an extension of the Heston stochastic volatility [42] by adding a proportional log-normal jumps in the risky asset process [13], [21]. Its P dynamics

$$\begin{aligned} dS(t) &= S(t^-)\left(\mu dt + \sqrt{v(t)}dB(t) + dJ(t)\right), \qquad S(0) = S_0 > 0, \\ dv(t) &= \kappa(\theta - v(t))dt + \beta\sqrt{v(t)}dW(t), \qquad v(0) = v_0 > 0 \end{aligned} \tag{4.204}$$

where B and W are correlated Brownian motion with $d[B, W] = \rho dt$ and μ is the rate of return, κ is the mean-reversion rate, θ is the long-run variance, β is the volatility of the volatility (vol-vol). The jump process $J(t)$ is a Compound Poisson Process of the form

$$J(t) = \sum_{i=1}^{N(t)} Y_i \tag{4.205}$$

where $N(t) \sim Poisson(\lambda t)$,

$$\log(1 + Y_i) \overset{iid}{\sim} N\left(\log(1 + \bar{k}) - \frac{1}{2}\delta^2, \delta^2\right), \tag{4.206}$$

and $N(t)$, and Y_i's are mutually independent. Then, $J(t)$ is a pure jump Lévy process with Lévy measure $\nu(dz) = \lambda F(dz)$ where F is the distribution function of Y_1. Then $J(t)$ can be represented in terms of Poisson random measure as follows:

$$\begin{aligned} J(t) &= \int_{[0,t]\times\mathbb{R}_0} zN(ds, dz) \\ &= \int_{[0,t]\times\mathbb{R}_0} z\left(\tilde{N}(ds, dz) + \nu(dz)ds\right). \end{aligned} \tag{4.207}$$

The variance process $v(t)$ is a mean-reverting square root Cox-Ingersoll-Ross (CIR) model. We shall denote this process by $CIR(\kappa, \theta, \beta)$. We assume that all parameters are constant and it satisfies Feller's condition

$$2\kappa\theta > \beta^2. \tag{4.208}$$

The Feller's condition [22] assures that $v(t) > 0$, a.s. Hence, the P dynamics of the Bates model can be written as follows:

$$dS(t) = S(t^-)\left(\alpha dt + \sqrt{v(t)}(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)) + \int_{\mathbb{R}_0} z\tilde{N}_3(dt,dz)\right)$$
$$dv(t) = \kappa(\theta - v(t))dt + \beta\sqrt{v(t)}dW_1(t). \tag{4.209}$$

where W_1 and W_2 are Wiener processes and $\tilde{N}_3$ is a compensated Poisson random process and are all mutually independent and

$$\alpha = \mu + \int_{\mathbb{R}_0} z\nu(dz). \tag{4.210}$$

Characterization of the CIR and 3/2 model

We need to characterize some of the important properties of the CIR model and its reciprocal, which is the 3/2 model which is needed in deriving the mean-variance hedging portfolio. From the CIR model of the variance process $v(t)$, the conditional distribution of $v(t)$ given $v(s)$ [22] is given by

$$v(t)|v(s) \sim \frac{\beta^2(1-e^{-\kappa(t-s)})}{4\kappa}\chi_\nu'^2(\lambda) \tag{4.211}$$

where $\chi_\nu'^2(\lambda)$ is non-central chi-square distribution with ν degrees of freedom and non-centrality parameter λ where

$$\nu = \frac{4\kappa\theta}{\beta^2}, \qquad \lambda = \frac{4\kappa e^{-\kappa(t-s)}}{\sigma^2(1-e^{-\kappa(t-s)})}v(s). \tag{4.212}$$

The pdf of $X \sim \chi_\nu'^2(\lambda)$ is given by

$$\begin{aligned} f_X(x) =& \frac{1}{2}e^{-(x+\lambda)/2}\left(\frac{x}{\lambda}\right)^{\nu/4-1} I_{\nu/2-1}(\sqrt{\lambda x})\mathbf{1}_{\{x\geq 0\}} \\ =& \frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}e^{-(\lambda+x)/2}x^{\nu/2-1}{}_0F_1\left(;,\frac{\nu}{2},\frac{\lambda x}{4}\right)\mathbf{1}_{\{x\geq 0\}} \end{aligned} \tag{4.213}$$

where

$$I_\alpha(z) = \sum_{\alpha=0}^{\infty}\frac{1}{n!\Gamma(n+\alpha+1)}\left(\frac{z}{2}\right)^{2n+\alpha} \tag{4.214}$$

is the modified Bessel function of the first kind,

$$ {}_0F_1\left(;, b, z\right) = \sum_{n=0}^{\infty} \frac{z^n}{(b)_n n!} \tag{4.215} $$

is the confluent hypergeometric limit function, and

$$ (a)_n = \frac{\Gamma(n+\alpha)}{\Gamma(a)}, \qquad n \in \mathbb{N}_0 \tag{4.216} $$

is the Pochhammer symbol [3]. The moments of X are given as

$$ E[X^\alpha] = 2^\alpha e^{-\lambda/2} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} {}_1F_1\left(\alpha + \frac{\nu}{2}, \frac{\nu}{2}, \frac{\lambda x}{4}\right) \tag{4.217} $$

where

$$ {}_1F_1\left(a, b, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \tag{4.218} $$

is the confluent hypergeometric function. Hence, its conditional moments are given by

$$ \begin{aligned} E[v(t)^\alpha | v(s)] = & \left(\frac{\sigma^2(1-e^{-\kappa(t-s)})}{2\kappa e^{-\kappa(t-s)}}\right)^\alpha \exp\left(-\frac{2\kappa e^{-\kappa(t-s)}}{\sigma^2(1-e^{-\kappa(t-s)})} v(s)\right) \cdot \\ & \frac{\Gamma\left(\alpha + \frac{2\kappa\theta}{\beta^2}\right)}{\Gamma\left(\frac{2\kappa\theta}{\beta^2}\right)} {}_1F_1\left(\alpha + \frac{2\kappa\theta}{\beta^2}, \frac{2\kappa\theta}{\beta^2}, \frac{\lambda x}{4}\right). \end{aligned} \tag{4.219} $$

The 3/2 stochastic volatility model [43] is given by

$$ d\bar{v}(t) = \bar{\kappa} v(t)(\bar{\theta} - v(t))dt + \bar{\beta}\bar{v}^{3/2}(t)dW(t), \qquad \bar{v}(0) = \bar{v}_0 > 0. \tag{4.220} $$

This model is a non-affine stochastic volatility model with a 3/2 power law and nonlinear reversion rate $\bar{\kappa}v(t)$ [28], [37]. The Fourier-Laplace transform of the log spot price and the integrated variance of $\bar{v}$ is given in closed form [19], [40]. Hence, the Laplace integrated variance of $\bar{v}(s)$ can be derived [52] and is given as follows:

$$ E\left[\exp\left(-z\int_0^T \bar{v}(s)ds\right)\right] = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\alpha)} \left(\frac{2}{\bar{\beta}} y(0, v_0)\right)^\alpha {}_1F_1\left(\alpha, \gamma, \frac{-2}{\bar{\beta}^2 y(0, \bar{v}_0)}\right) \tag{4.221} $$

where $z \in \mathbb{C}$ and

$$\alpha = \left(\frac{1}{2} + \frac{\bar{\kappa}}{\bar{\beta}^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{\bar{\kappa}}{\bar{\beta}^2}\right)^2 + \frac{2z}{\bar{\beta}^2}}, \tag{4.222}$$

$$\gamma = 1 + 2\sqrt{\left(\frac{1}{2} + \frac{\bar{\kappa}}{\bar{\beta}^2}\right)^2 + \frac{2z}{\bar{\beta}^2}}, \tag{4.223}$$

$$y(t,v) = v\int_0^T \exp\left(\int_t^u \bar{\kappa}\bar{\theta}(s)ds\right) du. \tag{4.224}$$

Since we assume $\theta(s) = \bar{\theta}$ is a constant, then

$$y(0,\bar{v}_0) = \frac{\bar{v}_0\left(\exp(\bar{\kappa}\bar{\theta}T) - 1\right)}{\bar{\kappa}\bar{\theta}}. \tag{4.225}$$

Since the square-root term in α and γ should be non-negative, then the range of z is given as follows:

$$z \geq -\left(\frac{\bar{\beta}^2 + 2\bar{\kappa}}{2\sqrt{2}\bar{\beta}}\right)^2. \tag{4.226}$$

The relationship between the Heston model and the 3/2 model is as follows. We let $\bar{v}(t) = 1/v(t)$, that is, $\bar{v}(t)$ reciprocal Heston process. From Itô's lemma, we obtain

$$d\bar{v}(t) = \left(\kappa\bar{v}(t) - (\kappa\theta - \beta^2)\bar{v}^2\right)dt + \beta\bar{v}^{3/2}(t)dW(t), \qquad \bar{v}(0) = 1/v_0 > 0. \tag{4.227}$$

Hence, $\bar{v}(t)$ is a 3/2 model [40], [43] with the following parameters

$$\bar{\kappa} = \kappa\theta - \beta^2, \qquad \bar{\theta} = \frac{\kappa}{\kappa\theta - \beta^2}, \qquad \bar{\beta} = -\beta. \tag{4.228}$$

Its conditional moments which can be obtained by computing the negative moments of the non-central chi-square distribution

$$E[\bar{v}(t)^\alpha|\bar{v}(s)] = \left(\frac{\sigma^2(1 - e^{-\kappa(t-s)})}{2\kappa e^{-\kappa(t-s)}}\right)^{-\alpha} \exp\left(-\frac{2\kappa e^{-\kappa(t-s)}}{\sigma^2(1 - e^{-\kappa(t-s)})}\frac{1}{y(s)}\right)\cdot$$
$$\frac{\Gamma\left(-\alpha + \frac{2\kappa\theta}{\beta^2}\right)}{\Gamma\left(\frac{2\kappa\theta}{\beta^2}\right)} {}_1F_1\left(-\alpha + \frac{2\kappa\theta}{\beta^2}, \frac{2\kappa\theta}{\beta^2}, \frac{\lambda x}{4}\right). \tag{4.229}$$

This conditional moment is finite if

$$\alpha < \frac{2\kappa\theta}{\beta^2}. \tag{4.230}$$

Conversely, taking the dynamics of reciprocal of $\bar{v}(t)$, From Itô's lemma, we obtain [28]

$$d\left(\frac{1}{\bar{v}(t)}\right) = \bar{\kappa}\bar{\theta}\left(\frac{\bar{\kappa}+\bar{\beta}^2}{\bar{\kappa}\bar{\theta}} - \frac{1}{\bar{v}(t)}\right)dt - \bar{\beta}\left(\frac{1}{\bar{v}(t)}\right)^{1/2} dW(t) \tag{4.231}$$

that is, $\frac{1}{\bar{v}(t)}$ is a $CIR(\bar{\kappa}\bar{\theta}, \frac{\bar{\kappa}+\bar{\beta}^2}{\bar{\kappa}\bar{\theta}}, -\bar{\beta})$ process. Applying Feller's condition for $1/\hat{v}(t) > 0$ which implies the non-explosion of $\bar{v}(t)$ [28] gives us

$$\bar{\kappa} \geq -\frac{\bar{\beta}^2}{2}. \tag{4.232}$$

Change of Measure

Under the change of measure $Q \sim P$, we have the following Brownian motions under the Q measure:

$$dW_i^Q(t) = dW_i(t) + u_i(t)dt, \qquad i \in \{1,2\}, \tag{4.233}$$

$$\tilde{N}_i^Q(dt,dz) = \tilde{N}_i(dt,dz) + \theta(t,z)\nu(dz)dt, \qquad i \in \{3\}, \tag{4.234}$$

For an arbitrage-free condition, we have the following constraints:

$$\mu + \int_{\mathbb{R}_0} z\nu(dz) - r(t) = \rho\sqrt{v(t)}u_1(t) + \sqrt{1-\rho^2}\sqrt{v(t)}u_2(t) + \int_{\mathbb{R}_0} z\theta_3(t,z)\nu(dz). \tag{4.235}$$

In this case, we have an incomplete model. We can choose the drift parameters such that the P and Q dynamics of $v(t)$ are the same and in the same spirit as the Merton model, where there is no drift in the jump term. In addition,

$$\int_{\mathbb{R}_0} z\nu(dz) = \lambda\int_{\mathbb{R}_0} zF(dz) = E[Y_1] = \lambda\bar{k}. \tag{4.236}$$

Hence, we have the following drift terms:

$$u_1(t) = 0, \qquad u_2(t) = \frac{\mu + \lambda\bar{k} - r(t)}{\sqrt{(1-\rho^2)v(t)}}, \qquad \theta(t,z) = 0. \tag{4.237}$$

Then, we have the following dynamics under the Q measure

$$
\begin{aligned}
dS(t) =& S(t^-)\left(r(t)dt + \sqrt{v(t)}S(\rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t)) + \int_{\mathbb{R}_0} z\tilde{N}_3^Q(dt,dz)\right), \\
S(0) =& S_0 > 0, \\
dv(t) =& \kappa(\theta - v(t))dt + \beta\sqrt{v(t)}dW_1^Q(s), \\
v(0) =& v_0 > 0.
\end{aligned} \tag{4.238}
$$

From Itô's lemma, we get

$$
\begin{aligned}
&S(T) \\
&= S_0 \exp\left(\int_0^T \left(r(t) - \frac{v(t)}{2}\right) ds + \rho\int_0^T \sqrt{v(t)}dW_1^Q(s) + \sqrt{1-\rho^2}\int_0^T \sqrt{v(t)}dW_2^Q(t)\right. \\
&\qquad \left. + \int_{[0,T]\times\mathbb{R}_0} (\log(1+z) - z)\nu(dz)dt + \int_{[0,T]\times\mathbb{R}_0} \log(1+z)\tilde{N}_3^Q(dt,dz)\right).
\end{aligned} \tag{4.239}
$$

We denote the normalized log stock process as follows:

$$
Y(t) = \log\left(\frac{S(t)}{S(0)}\right). \tag{4.240}
$$

Then, we have

$$
\begin{aligned}
Y(T) =& \int_0^T \left(r(t) - \frac{v(t)}{2}\right) ds + \rho\int_0^T \sqrt{v(t)}dW_1^Q(s) + \sqrt{1-\rho^2}\int_0^T \sqrt{v(t)}dW_2^Q(t) \\
&+ \int_{[0,T]\times\mathbb{R}_0} (\log(1+z) - z)\nu(dz)dt + \int_{[0,T]\times\mathbb{R}_0} \log(1+z)\tilde{N}_3^Q(dt,dz).
\end{aligned} \tag{4.241}
$$

Under the Heston model, $Y(T)$ has the same form as (4.241) except for the absence of the last two terms. It was shown that in the Heston case, $Y(T) \in \mathbb{D}^{1,2}$ and by chain rule, $S(T) \in \mathbb{D}^{1,2}$ [29]. The final two terms of (4.241) consist of a deterministic term and from (4.206), a compensated Poisson random process with normally distributed jumps. Hence, under the Bates model, $Y(T) \in \mathbb{D}^{1,2}$ and by chain rule, $S(T) \in \mathbb{D}^{1,2}$.

Since $\bar{v}(t) = 1/v(t)$ is a 3/2 model and from the Laplace transform of the integrated variance of $\bar{v}(t)$, we obtain the Novikov condition

$$
E\left[\exp\left(\frac{1}{2}\int_0^T u_2^2(t)dt\right)\right] = E\left[\exp\left(-\lambda\int_0^T \bar{v}(t)dt\right)\right]. \tag{4.242}
$$

where the expression at the right is given by (4.221) which finite under certain conditions where

$$z = -\frac{(\mu + \lambda\bar{k} - r(t))^2}{1-\rho^2} \tag{4.243}$$

and the 3/2 model parameters is given by (4.228). Under the restriction for z in (4.226), we get

$$z \geq -\left(\frac{\bar{\beta}^2 + 2\bar{\kappa}}{2\sqrt{2}\bar{\beta}}\right)^2 = -\left(\frac{2\kappa\theta - \beta^2}{2\sqrt{2}\beta}\right)^2. \tag{4.244}$$

From (4.243) and (4.244), we obtain the following restriction for ρ^2 which will satisfy the Novikov condition:

$$\rho^2 \leq 1 - \left(\frac{2(\mu + \lambda\bar{k} - r(t))\beta}{2\kappa\theta - \beta^2}\right)^2. \tag{4.245}$$

Mean-Variance Hedging

We assume to have a full model, then the mean-variance portfolio is given as

$$\begin{aligned}\varphi(t) =& D(t,T)\frac{\rho\sqrt{v(t)}S(t)\beta_1(t) + \sqrt{1-\rho^2}\sqrt{v(t)}S(t)\beta_2(t) + \int_{\mathbb{R}_0} zS(t)\xi_3(t,z)\nu(dz)}{\left(\rho\sqrt{v(t)}S(t)\right)^2 + \left(\sqrt{1-\rho^2}\sqrt{v(t)}S(t)\right)^2 + + \int_{\mathbb{R}_0}(zS(t))^2\nu(dz)} \\ =& \frac{D(t,T)}{S(t)}\frac{\rho\sqrt{v(t)}\beta_1(t) + \sqrt{1-\rho^2}\sqrt{v(t)}\beta_2(t) + \int_{\mathbb{R}_0} z\xi_3(t,z)\nu(dz)}{v(t) + \int_{\mathbb{R}_0} z^2\nu(dz)}.\end{aligned} \tag{4.246}$$

where $\beta_j(t)$ are computed from the Clark-Ocone theorem under the change of measure as follows:

$$\begin{aligned}\beta_j(t) =& E^Q[D_{j,t,0}F - FK_j(t)|\mathcal{F}_{t^-}], \\ K_j(t) =& \sum_{i=1}^{2}\int_t^T D_{j,t,0}u_i(s)dW_i^Q(s) \qquad j \in \{1,2\},\end{aligned} \tag{4.247}$$

and

$$\xi_3(t,z) = E^Q[D_{3,t,z}F|\mathcal{F}_{t^-}]. \tag{4.248}$$

From the moments of the log-normal distribution, we have the following:

$$\begin{aligned}\int_{\mathbb{R}_0} z^2\nu(dz) =&\lambda\int_{\mathbb{R}_0} z^2F(dz) = \lambda E[Y(\cdot)^2]\\ =&\lambda\left[e^{\delta^2}(1+\bar{k})^2 + -2(1+\bar{k})+1\right]. \end{aligned} \tag{4.249}$$

We assume that our contingent claim $F \in \mathbb{D}^{1,2}$. Without loss of generality, we assume that $\sigma_j = 1$ in (3.347). Hence, the stochastic derivative $D_{j,t,0}$ is the Malliavin derivative operator with respect to W_j^Q.

Although the Heston model has no closed form, Alós and Ewald [4] were able to derive the following characterizations of the variance process $v(t)$ and volatility process $\sigma(t) = \sqrt{v(t)}$ as follows:

(i) $v(s) \in \mathbb{L}^{1,2}$ and for $t \leq s$,

$$D_{1,t,z}v(s) =\beta\sqrt{v(s)}\exp\left(\int_t^s\left(-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\beta^2}{8}\right)\frac{1}{v(u)}\right)du\right)\mathbf{1}_{\{z=0\}}, \tag{4.250}$$

(ii) $\sqrt{v(s)} \in \mathbb{L}^{1,2}$ and for $t \leq s$,

$$D_{1,t,z}\sqrt{v(s)} = \frac{\beta}{2}\exp\left(\int_t^s\left(-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\beta^2}{8}\right)\frac{1}{v(u)}\right)du\right)\mathbf{1}_{\{z=0\}}. \tag{4.251}$$

Hence, under Feller's condition we have following upper bounds:

$$\begin{aligned}|D_{1,t,z}v(s)| \leq&\beta\sqrt{v(t)}\exp\left(-\frac{\kappa}{2}(s-t)\right),\\ |D_{1,t,z}\sqrt{v(s)}| \leq&\frac{\beta}{2}\exp\left(-\frac{\kappa}{2}(s-t)\right). \end{aligned} \tag{4.252}$$

Since $v(s)$ and $\sigma(s)$ do not depend on W_2^Q, then

$$D_{2,t,0}v(s) = D_{2,t,0}\sigma(s) = 0. \tag{4.253}$$

In addition, since $u_1(s) = 0$, then

$$D_{j,s,0}u_1(t) = 0, \qquad j \in \{1,2\}. \tag{4.254}$$

If the Feller condition holds, then from (4.229) - (4.230) implies

$$\frac{1}{\sqrt{v(s)}} \in L^2(Q). \tag{4.255}$$

On the other hand, since

$$|v^{-3/2}(s)D_{1,t,0}v(s)| \leq \frac{1}{v(s)}\exp\left(-\frac{\kappa}{2}(s-t)\right). \tag{4.256}$$

Then, from (4.229) - (4.230) and from dominated convergence theorem

$$2 < \frac{2\kappa\theta}{\beta^2} \Rightarrow v^{-3/2}(s)D_{1,t,0}v(s) \in L^2(Q \times \lambda). \tag{4.257}$$

From the chain rule, we obtain

$$D_{1,t,0}\frac{1}{\sqrt{v(s)}} = v^{-3/2}(s)D_{1,t,0}v(s) \tag{4.258}$$

and thus,

$$D_{1,t,0}u_2(s) = -\frac{\mu - r(t)}{2\sqrt{1-\rho^2}}v^{-3/2}(s)D_{1,t,0}v(s). \tag{4.259}$$

So therefore, we have the following:

$$K_1(t) = -\frac{\mu - r(t)}{2\sqrt{1-\rho^2}}\int_t^T v^{-3/2}(s)D_{1,t,0}v(s)ds, \qquad K_2(t) = 0. \tag{4.260}$$

Collecting terms, we obtain the following mean-variance hedging portfolio

$$\begin{aligned}\varphi(t) = &\frac{D(t,T)}{v(t) + \int_{\mathbb{R}_0} z^2\nu(dz)}\cdot \\ &\left(\rho E^Q\left[D_{1,t,0}F + F\frac{\mu - r(t)}{2\sqrt{1-\rho^2}}\int_t^T v^{-3/2}(s)D_{1,t,0}v(s)ds\middle|\mathcal{F}_{t^-}\right]\right. \\ &\left.+\sqrt{1-\rho^2}E^Q\left[D_{2,t,0}F|\mathcal{F}_{t^-}\right] + \int_{\mathbb{R}_0} zE^Q[D_{3,t,z}F|\mathcal{F}_{t^-}]\nu(dz)\right).\end{aligned} \tag{4.261}$$

To solve the mean-variance hedging portfolio explicitly, we need to compute for $D_{j,t,z}F$. We consider the contingent claim of the form $F = \Phi(S(T))$, where $\Phi \in C^1(\mathbb{R}_+, \mathbb{R})$, such that $\Phi(S(T)) \in L^2(Q)$, $\Phi'(S(T))D_{j,t,0}S(T) \in L^2(Q \times \lambda)$, and $\frac{\Phi(S(T)+zD_{j,t,z}S(T))-\Phi(S(T))}{z} \in L^2(Q \times z^2\nu(dz)dt)$ for $z \neq 0$, then from the chain rule, $F \in \mathbb{D}^{1,2}$ and

$$D_{j,t,z}F = \Phi'(S(T))D_{j,t,0}S(T)\mathbf{1}_{\{z=0\}} + \frac{\Phi(S(T)+zD_{j,t,z}S(T)) - \Phi(S(T))}{z}\mathbf{1}_{\{z\neq 0\}} \tag{4.262}$$

where

$$
\begin{aligned}
&D_{j,t,z}S(T)\\
=&D_{j,t,z}(S_0\exp(Y(T)))\\
=&S_0\exp(Y(T))D_{j,t,z}Y(T)\mathbf{1}_{\{z=0\}}+S_0\frac{\exp(Y(T)+zD_{j,t,z}Y(T))-\exp(Y(T))}{z}\mathbf{1}_{\{z\neq 0\}}\\
=&S(T)D_{j,t,z}Y(T)\mathbf{1}_{\{z=0\}}+S(T)\frac{\exp(zD_{j,t,z}Y(T))-1}{z}\mathbf{1}_{\{z\neq 0\}}. \qquad (4.263)
\end{aligned}
$$

Since $Y(t)\in\mathbb{D}^{1,2}$ and $v(t)$, $\sqrt{v(t)}\in\mathbb{L}^{1,2}$, then from chain rule $D_{j,t,z}Y(T)$ is computed as follows:

$$
\begin{aligned}
D_{1,t,z}Y(T)=&\left(-\frac{1}{2}\int_t^T D_{1,t,0}v(s)ds+\rho\left(\sqrt{v(t)}+\int_t^T D_{1,t,0}\sqrt{v(s)}dW_1^Q(s)\right)\right.\\
&\left.+\sqrt{1-\rho^2}\int_t^T D_{1,t,0}\sqrt{v(s)}dW_2^Q(s)\right)\mathbf{1}_{\{z=0\}}\\
=&\left(-\frac{1}{2}\int_t^T D_{1,t,0}v(s)ds+\rho\sqrt{v(t)}+\int_t^T D_{1,t,0}\sqrt{v(s)}dB^Q(s)\right)\mathbf{1}_{\{z=0\}}
\end{aligned}
\tag{4.264}
$$

where

$$dB^Q(s)=\rho dW_1^Q(s)+\sqrt{1-\rho^2}dW_2^Q(s) \tag{4.265}$$

is a Q Brownian motion,

$$
\begin{aligned}
D_{2,t,z}Y(T)=&\sqrt{1-\rho^2}\sqrt{v(t)}\mathbf{1}_{\{z=0\}}, \qquad (4.266)\\
D_{3,t,z}Y(T)=&\frac{\log(1+z)}{z}\mathbf{1}_{\{z\neq 0\}}. \qquad (4.267)
\end{aligned}
$$

So therefore, we have the following stochastic derivatives:

$$
\begin{aligned}
D_{1,t,z}F=&\Phi'(S(T))S(T)\cdot\\
&\left(-\frac{1}{2}\int_t^T D_{1,t,0}v(s)ds+\rho\sqrt{v(t)}+\int_t^T D_{1,t,0}\sqrt{v(s)}dB^Q(s)\right)\mathbf{1}_{\{z=0\}}, \qquad (4.268)\\
D_{2,t,z}F=&\Phi'(S(T))S(T)\sqrt{1-\rho^2}\sqrt{v(t)}\mathbf{1}_{\{z=0\}}, \qquad (4.269)\\
D_{3,t,z}F=&\frac{\Phi(S(T)(1+z))-\Phi(S(T))}{z}\mathbf{1}_{\{z\neq 0\}}. \qquad (4.270)
\end{aligned}
$$

Remark 4.3.2 *With the recent results of Alós and Rheinländer on the Malliavin differentiability of the* 3/2 *model [6] and the reciprocal relation of the the Heston and the* 3/2 *model, then we find mean-variance portfolio with the similar fashion for the Bates model under the* 3/2 *stochastic volatility model, that is,*

$$dS(t) = S(t^-)\left(\mu dt + \sqrt{v(t)}dB(t) + dJ(t)\right), \qquad S(0) = S_0 > 0,$$
$$dv(t) = \kappa(\theta - v(t))dt + \beta v(t)^{3/2}dW(t), \qquad v(0) = v_0 > 0. \tag{4.271}$$

5. DONSKER DELTA AND ITS APPLICATIONS TO FINANCE

The Donsker Delta is an approach to compute the generalized conditional expectation $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$ without explicitly evaluating the Malliavin derivative [2], [27], [66] is first studied in the Wiener case. We shall present some applications in finding the mean-variance hedging portfolio.

5.1 Donsker Delta

Definition 5.1.1 *Let $Y : \Omega \to \mathbb{R}$ be a random variable, belongs to $(\mathcal{S})^*$. Then the mapping $\delta_Y : \mathbb{R} \to (\mathcal{S})^*$ is called the Donsker Delta function Y if for all measurable $g : \mathbb{R} \to \mathbb{R}$*

$$g(Y) = \int_{\mathbb{R}} g(u)\delta_Y(u)dy \quad , a.e. \tag{5.1}$$

such that the integral on the RHS conveges [27].

We assume that the Lévy process $X(t)$ satisfies the following condition to ensure the convergence of the Donsker Delta in $(\mathcal{S})^*$.

Assumption There exists $\epsilon \in (0, 1)$ such that

$$\lim_{|u|\to\infty} |u|^{-(1+\epsilon)} Re\Psi(u) = \infty. \tag{5.2}$$

- This entails strong Feller property of the semigroup which implies absolute continuity of its distribution with respect to the Lebesgue measure [17].
- A Lévy process with a Brownian term satisfies this condition.

This assumption was stated [58] to assure convergence of the Donsker Delta in $(\mathcal{S})^*$ in the pure jump Lévy case. However, this assumption still holds with a Lévy process with a Wiener and pure jump components to assure convergence of the Donsker Delta in $(\mathcal{S})^*$. We state the following theorem for the Donsker Delta of the Itô-Lévy process

Theorem 5.1.1 *[26] The Donsker Delta of the Itô-Lévy process*

$$\begin{aligned} dY(s) =& \alpha(s)ds + \beta(s)dW(s) + \int_{\mathbb{R}_0} \gamma(s,x)\tilde{N}(ds,dx), \quad s \in [t,T] \\ Y(t) =& y \end{aligned} \tag{5.3}$$

where y is a constant, $\alpha(s)$, $\beta(s)$, and $\gamma(s,x) > -1$, deterministic $(s,x) \in [0,T] \times \mathbb{R}_0$ such that

$$\int_0^T \left(|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,x)\nu(dx) \right) ds < \infty \tag{5.4}$$

is given as follows

$$\begin{aligned} \delta_{Y(T)}(u) =& \frac{1}{2\pi} \int_{\mathbb{R}} \exp^{\diamond} \left(-i\lambda u + i\lambda y + \int_t^T \left(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s) \right) ds \right. \\ &+ \int_t^T i\lambda\beta(s)dW(s) + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds \\ &\left. + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx) \right) d\lambda. \end{aligned} \tag{5.5}$$

whenever it converges in $(\mathcal{S})^*$.

Let $g : \mathbb{R} \to \mathbb{R}$ be measurable, then from the Donsker Delta function of the Itô-Lévy process, we obtain

$$\begin{aligned} g(Y(T)) = \int_{\mathbb{R}} g(u) \Bigg[\frac{1}{2\pi} \int_{\mathbb{R}} \exp^{\diamond} &\left(-i\lambda u + i\lambda y + \int_t^T \left(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s) \right) ds \right. \\ &+ \int_t^T i\lambda\beta(s)dW(s) + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds \\ &\left. + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx) \right) d\lambda \Bigg] du. \end{aligned} \tag{5.6}$$

5.2 Evaluation of $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$

We evaluate $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$ by considering 2 cases. Case I: $z = 0$ and Case II: $z \neq 0$

5.2.1 Case I: $E[D_{t,0}g(Y(T))|\mathcal{F}_t]$

Taking the stochastic derivative $D_{t,0}$ in (5.1) yields

$$D_{t,0}g(Y(T)) = \int_{\mathbb{R}} g(u)D_{t,0}\delta_{Y(T)}(u)du. \tag{5.7}$$

Then, taking the generalized conditional expectation, we obtain

$$E[D_{t,0}g(Y(T))|\mathcal{F}_t] = \int_{\mathbb{R}} g(u)E[D_{t,0}\delta_{Y(T)}(u)|\mathcal{F}_t]du. \tag{5.8}$$

From the Wick chain rule and since $i\lambda\beta(t)$ is deterministic, then

$$\begin{aligned} D_{t,0}\delta_{Y(T)}(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\Big(-i\lambda u + i\lambda y + \int_t^T \Big(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s)\Big)ds \\ + \int_t^T i\lambda\beta(s)dW(s) + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds \\ + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\Big) i\lambda\frac{\beta(t)}{\sigma}d\lambda. \end{aligned} \tag{5.9}$$

Taking the generalized conditional expectation, we obtain

$$\begin{aligned} E[D_{t,0}\delta_{Y(T)}(u)|\mathcal{F}_t] = \frac{\beta(t)}{2\pi\sigma}\int_{\mathbb{R}} i\lambda\exp(-i\lambda u + i\lambda y)E\left[\exp^{\diamond}\left(\int_t^T i\lambda\beta(s)dW(s)\right)\middle|\mathcal{F}_t\right]\diamond \\ E\left[\exp^{\diamond}\left(\int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\right)\middle|\mathcal{F}_t\right]\cdot \\ \exp\Big(\int_t^T \Big(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s)\Big)ds \\ + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds\Big)d\lambda. \end{aligned} \tag{5.10}$$

Now since

$$\begin{aligned} E\left[\exp^{\diamond}\left(\int_t^T i\lambda\beta(s)dW(s)\right)\middle|\mathcal{F}_t\right] &= \exp^{\diamond}\left(E\left[\int_t^T i\lambda\beta(s)dW(s)\middle|\mathcal{F}_t\right]\right) \\ &= \exp^{\diamond}(0) = 1 \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} &E\left[\exp^{\diamond}\left(\int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\right)\middle|\mathcal{F}_t\right] \\ &= \exp^{\diamond}\left(E\left[\int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\middle|\mathcal{F}_t\right]\right) = \exp^{\diamond}(0) = 1. \end{aligned} \tag{5.12}$$

Hence, using the above results and from previous theorem yield

$$
\begin{aligned}
E[D_{t,0}\delta_{Y(T)}(u)|\mathcal{F}_t] =& \frac{\beta(t)}{2\pi\sigma}\int_{\mathbb{R}} i\lambda e^{-i\lambda u}\exp\left(i\lambda y + \int_t^T \left(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s)\right) ds + \right.\\
&\left.\int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds\right) d\lambda \\
=& \frac{\beta(t)}{2\pi\sigma}\int_{\mathbb{R}} i\lambda e^{-i\lambda u} E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda. \qquad (5.13)
\end{aligned}
$$

If the characteristic function is known but not the pdf, the last expression is sufficient to compute the expression $E[D_{t,0}g(Y(T))|\mathcal{F}_t]$. Since $|\exp(i\lambda Y(T))| \in L^1$, the conditional expectation on the right-hand side is an ordinary conditional expectation. On the other hand, if the pdf of $Y(T)$ conditional to $\mathcal{F}_t$ denoted by $f_{Y(T)}(\cdot|\mathcal{F}_t)$ is differentiable, then, from the differentiation property of the Fourier transform, we obtain

$$
E[D_{t,0}\delta_{Y(T)}(u)|\mathcal{F}_t] = -\frac{\beta(t)}{\sigma}\frac{d}{du}f_{Y(T)}(u|\mathcal{F}_t). \qquad (5.14)
$$

Hence, $E[D_{t,0}g(Y(T))|\mathcal{F}_t]$ can be expressed of the form

$$
E[D_{t,0}g(Y(T))|\mathcal{F}_t] = \int_{\mathbb{R}} g(u)\frac{\beta(t)}{2\pi\sigma}\int_{\mathbb{R}} i\lambda e^{-i\lambda u}E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda du. \qquad (5.15)
$$

This can be simulated by Fourier transform techniques [18], [20] together with Monte Carlo simulation. Alternatively, if $f_{Y(T)}(\cdot|\mathcal{F}_t)$ is differentiable, then,

$$
\begin{aligned}
E[D_{t,0}g(Y(T))|\mathcal{F}_t] &= -\frac{\beta(t)}{\sigma}\int_{\mathbb{R}} g(u)\frac{d}{du}f_{Y(T)}(u|\mathcal{F}_t)du \\
&= -\frac{\beta(t)}{\sigma}\int_{\mathbb{R}} g(u)\frac{d}{du}\log f_{Y(T)}(u|\mathcal{F}_t)f_{Y(T)}(u|\mathcal{F}_t)du \\
&= -\frac{\beta(t)}{\sigma}E\left[g(Y(T))\log f_{Y(T)}(Y(T)|\mathcal{F}_t)\middle|\,\mathcal{F}_t\right]. \qquad (5.16)
\end{aligned}
$$

The conditional expectation in the right-hand is an ordinary conditional expectation and its form resembles a likelihood ratio method estimator [34]. This can be simulated by Monte Carlo simulation.

5.2.2 Case II: $E[D_{t,z}g(Y(T))|\mathcal{F}_t], z \neq 0$

Taking the stochastic derivative $D_{t,z}$, where $z \neq 0$, in (5.1) yields

$$D_{t,z}g(Y(T)) = \int_{\mathbb{R}} g(u) D_{t,z}\delta_{Y(T)}(u)du. \tag{5.17}$$

Then, taking the generalized conditional expectation, we obtain

$$E[D_{t,z}g(Y(T))|\mathcal{F}_t] = \int_{\mathbb{R}} g(u) E[D_{t,z}\delta_{Y(T)}(u)|\mathcal{F}_t]du. \tag{5.18}$$

From the Wick chain rule and since $\exp(i\lambda\gamma(t,z)) - 1$ is deterministic, then

$$\begin{aligned} D_{t,z}\delta_{Y(T)}(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\Big(-i\lambda u + i\lambda y + \int_t^T \Big(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s)\Big) ds \\ + \int_t^T i\lambda\beta(s)dW(s) + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds \\ + \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\Big) \frac{(\exp(i\lambda\gamma(t,z)) - 1)}{z} d\lambda. \end{aligned} \tag{5.19}$$

Taking the generalized conditional expectation, we obtain

$$\begin{aligned} &E[D_{t,z}\delta_{Y(T)}(u)|\mathcal{F}_t] \\ =&\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\exp(i\lambda\gamma(t,z)) - 1)}{z} \exp(-i\lambda u + i\lambda y) E\left[\exp^{\diamond}\left(\int_t^T i\lambda\beta(s)dW(s)\right)\Big|\, \mathcal{F}_t\right] \diamond \\ &E\left[\exp^{\diamond}\left(\int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1\right)\tilde{N}(ds,dx)\right)\Big|\, \mathcal{F}_t\right] \cdot \\ &\exp\Big(\int_t^T \Big(i\lambda\alpha(s) - \frac{1}{2}\lambda^2\beta^2(s)\Big) ds \\ &+ \int_{[t,T]\times\mathbb{R}_0} \left(\exp(i\lambda\gamma(s,x)) - 1 - i\lambda\gamma(s,x)\right)\nu(dx)ds\Big) d\lambda. \end{aligned} \tag{5.20}$$

Performing similar calculations from the Brownian case, we obtain

$$E[D_{t,z}\delta_{Y(T)}(u)|\mathcal{F}_t] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda u} E[\exp(i\lambda Y(T))|\mathcal{F}_t] \frac{(\exp(i\lambda\gamma(t,z)) - 1)}{z} d\lambda. \tag{5.21}$$

If the characteristic function is known but not the pdf, the last expression is sufficient to compute the expression $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$. Since $|\exp(i\lambda Y(T))| \in L^1$, the conditional expectation on the right-hand side is an ordinary conditional expectation. On the other hand, provided that the pdf of $Y(T)$ conditional to $\mathcal{F}_t$ denoted by

$f_{Y(T)}(\cdot|\mathcal{F}_t)$ is known, then, from the translation property of the Fourier Transform, we obtain

$$\begin{aligned}
&E[D_{t,z}\delta_{Y(T)}(u)|\mathcal{F}_t] \\
=&\frac{1}{2\pi z}\int_{\mathbb{R}} e^{-i\lambda(u-\gamma(t,z))}E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda - \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda u}E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda \\
=&\frac{f_{Y(T)}(u-\gamma(t,z)|\mathcal{F}_t) - f_{Y(T)}(u|\mathcal{F}_t)}{z}. \qquad (5.22)
\end{aligned}$$

Hence, $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$ can be expressed as follows:

$$\begin{aligned}
E[D_{t,z}g(Y(T))|\mathcal{F}_t] = \int_{\mathbb{R}} \frac{g(u)}{z}\bigg(&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda(u-\gamma(t,z))}E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda \\
&-\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda u}E[\exp(i\lambda Y(T))|\mathcal{F}_t]d\lambda\bigg)\, du. \qquad (5.23)
\end{aligned}$$

This can be simulated by Fourier transform techniques [18], [20] together with Monte Carlo simulation. Likewise, if $f_{Y(T)}(\cdot|\mathcal{F}_t)$ is known, then,

$$\begin{aligned}
E[D_{t,z}g(Y(T))|\mathcal{F}_t] &= \int_{\mathbb{R}} g(u)\left(\frac{f_{Y(T)}(u-\gamma(t,z)|\mathcal{F}_t) - f_{Y(T)}(u|\mathcal{F}_t)}{z}\right) du \\
&= \int_{\mathbb{R}} \left(\frac{g(u+\gamma(t,z)) - g(u)}{z}\right) f_{Y(T)}(u|\mathcal{F}_t)du \\
&= \frac{E[g(Y(T)+\gamma(t,z))|\mathcal{F}_t] - E[g(Y(T))|\mathcal{F}_t]}{z}. \qquad (5.24)
\end{aligned}$$

Since

$$Y(T) = y + \int_t^T \alpha(s)ds + \int_t^T \beta(s)dW(s) + \int_{[t,T]\times\mathbb{R}_0} \gamma(t,x)\tilde{N}(ds,dx), \quad s \in [t,T] \qquad (5.25)$$

then, for $(t,z) \in [0,T]\times\mathbb{R}_0$,

$$D_{t,z}Y(T) = \frac{\gamma(t,z)}{z}. \qquad (5.26)$$

Hence, we obtain

$$E[D_{t,z}g(Y(T))|\mathcal{F}_t] = \frac{E[g(Y(T)+zD_{t,z}Y(T))|\mathcal{F}_t] - E[g(Y(T))|\mathcal{F}_t]}{z}. \qquad (5.27)$$

The result shows this corresponds to the conditional expectation of a increment operator. This is the same result in the $\mathbb{D}^{1,2}$ case. Moreover, this can be simulated by Monte Carlo simulation.

5.3 Examples

5.3.1 Merton Model

We take a look again at the Merton model in Chapter 4.3.3. Consider the jump-diffusion of the form

$$X(t) = \mu t + \sigma W(t) + \sum_{i=0}^{N(t)} Y_i \qquad Y_i \overset{iid}{\sim} N(m, \delta^2). \tag{5.28}$$

Then its pdf and cdf is given respectively as follows:

$$f_{X(t)}(x) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \phi(x, \mu t + km, \sigma^2 t + k\delta^2), \tag{5.29}$$

$$F_{X(t)}(x) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} N(x, \mu t + km, \sigma^2 t + k\delta^2) \tag{5.30}$$

where $\phi(\cdot, a, b^2)$ and $N(\cdot, a, b^2)$ is the pdf and cdf of the normal distribution $N(a, b^2)$ respectively.

From the risk-neutral model (4.152) of the Merton model, we have the following solution of the geometric Lévy process,

$$\begin{aligned} S_1(t) =& S_1(0) \exp \left(\int_0^t \left(r(s) - \frac{\sigma^2}{2} \right) ds + \int_0^t \sigma dW^Q(s) \right. \\ & \left. + \int_{[0,t]\times\mathbb{R}_0} (z - (e^x - 1))\nu(dx)ds + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(ds, dx) \right). \end{aligned} \tag{5.31}$$

Then $Y(t)$ defined as

$$Y(t) = \log \left(\frac{S_1(t)}{S_1(0)} \right) \tag{5.32}$$

is an Itô-Lévy process of the form

$$Y(t) = \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_{[0,t]\times\mathbb{R}_0} \gamma(s, x)\tilde{N}(ds, dx) \tag{5.33}$$

where

$$\alpha(s) = r(s) - \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (x - (e^x - 1))\nu(dz), \qquad \beta(s) = \sigma, \qquad \gamma(s, z) = z. \tag{5.34}$$

Let F be the distribution function of Y_1, then have the following integrals:

$$\begin{aligned}\int_{\mathbb{R}_0}(e^x - x - 1)\nu(dx) =& \lambda \int_{\mathbb{R}_0}(e^x - x - 1)F(dx), \\ =& \lambda \left(E[e^{Y_1}] - E[Y_1] - 1\right) \\ =& \lambda \left(\exp\left(m + \frac{\delta^2}{2}\right) - m - 1\right) \end{aligned} \tag{5.35}$$

$$\begin{aligned}\int_{\mathbb{R}_0}\gamma(s,x)\nu(dx) =& \lambda \int_{\mathbb{R}_0} z^2 F(dx) \\ =& \lambda E[Y_1^2] \\ =& \lambda \left(m + \delta^2\right). \end{aligned} \tag{5.36}$$

If $r(t)$ is deterministic and belongs to $L^1[0,T]$, then (2.31) is satisfied.

Consider the contingent claim $F = \Phi(S_1(T))$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(u) = \Phi(S_0 e^u), \qquad u \in \mathbb{R} \tag{5.37}$$

Then, we have the following relationship:

$$F = \Phi(S_1(T)) = g(Y(T)). \tag{5.38}$$

For the Merton model, the mean-variance hedging portfolio is given by (4.154).

The Wick product is invariant under the change of measure in the Wiener case [48]. This identity also holds in the Lévy case by an can be proven using the similar approach of showing this identity in the Wiener case by showing the identity using the Doleans-Dade exponential then extend using a density argument [48]. Hence, we can use the results from the previous section to evaluate $E^Q[D_{t,z} g(Y(T))|\mathcal{F}_t]$

Example Binary Option

We consider the Binary option

$$F = \Phi(S(T)) = \mathbf{1}_{[K,L]}(S_1(T)) \tag{5.39}$$

then,

$$g(u) = \mathbf{1}_{[\log \frac{K}{S_0}, \log \frac{L}{S_0}]}(u), \qquad u \in \mathbb{R}. \tag{5.40}$$

Although, $F \notin \mathbb{D}^{1,2}$, nevertheless, we can compute $E[D_{t,z}g(Y(T))|\mathcal{F}_t]$ using the Donsker Delta approach as follows:

$$\begin{aligned} E^Q[D_{t,0}F|\mathcal{F}_t] &= -\frac{\beta(t)}{\sigma}\int_{\mathbb{R}} g(u)\frac{d}{du}f_{Y(T)}(u|\mathcal{F}_t)du \\ &= -\frac{b}{\sigma}\int_{\mathbb{R}} \mathbf{1}_{[\log \frac{K}{S_0}, \frac{L}{S_0}]}(u)\frac{d}{du}\log f_{Y(T)}(u|\mathcal{F}_t)du \\ &= -\frac{b}{\sigma}\int_{\log \frac{K}{S_0}}^{\log \frac{L}{S_0}} \frac{d}{du}f_{Y(T)}(u|\mathcal{F}_t)du \\ &= -\frac{b}{\sigma}\left[f_{Y(T)}(\log K|\mathcal{F}_t) - f_{Y(T)}(\log L|\mathcal{F}_t)\right]. \end{aligned} \tag{5.41}$$

and for $z \neq 0$,

$$\begin{aligned} & E^Q[D_{t,z}F|\mathcal{F}_t] \\ =& \frac{E^Q[g(Y(T)+zD_{t,z}Y(T))|\mathcal{F}_t] - E^Q[g(Y(T))|\mathcal{F}_t]}{z} \\ =& \frac{E^Q[\mathbf{1}_{[\log \frac{K}{S_0}, \log \frac{L}{S_0}]}(Y(T)+z)|\mathcal{F}_t] - E^Q[\mathbf{1}_{[\log \frac{K}{S_0}, \log \frac{L}{S_0}]}Y(T)|\mathcal{F}_t]}{z} \\ =& \frac{Q\left(Y(T)+z \in \left[\log \frac{K}{S_0}, \log \frac{L}{S_0}\right] \middle| \mathcal{F}_t\right) - Q\left(Y(T) \in \left[\log \frac{K}{S_0}, \log \frac{L}{S_0}\right] \middle| \mathcal{F}_t\right)}{z} \\ =& \frac{1}{z}\left[F_{Y(T)}\left(\log \frac{L}{S_0} - z \middle| \mathcal{F}_t\right) - F_{Y(T)}\left(\log \frac{K}{S_0} - z \middle| \mathcal{F}_t\right)\right. \\ & \left. - F_{Y(T)}\left(\log \frac{L}{S_0} \middle| \mathcal{F}_t\right) + F_{Y(T)}\left(\log \frac{K}{S_0} \middle| \mathcal{F}_t\right)\right] \end{aligned} \tag{5.42}$$

where its conditional pdf and cdf of the jump diffusion process $Y(T)$ under the risk-neutral measure is given as

$$f_{Y(T)}(x|\mathcal{F}_t) = \sum_{k=0}^{\infty} e^{-\lambda(T-t)}\frac{(\lambda(T-t))^k}{k!}\phi(x, \alpha(T-t)+km, \sigma^2(T-t)+k\delta^2), \tag{5.43}$$

$$F_{Y(T)}(x|\mathcal{F}_t) = \sum_{k=0}^{\infty} e^{-\lambda(T-t)}\frac{(\lambda(T-t))^k}{k!}N(x, \alpha(T-t)+km, \sigma^2(T-t)+k\delta^2). \tag{5.44}$$

5.3.2 Continuous Case

We demonstrate some special results in the continuous case. Using generalized conditional expectation, $E^Q[D_{t,0}g(Y(T))|\mathcal{F}_t]$, we are able to obtain a Delta of an option coinciding with the Delta obtained from the likelihood method.

From the risk-neutral model (4.152) of the geometric Brownian motion

$$dS_1(t) = S_1(t)\left(rdt + \sigma dW^Q(t)\right), \qquad S_1(0) = S_0. \tag{5.45}$$

The solution for this SDE is as follows:

$$S_1(t) = S_0 \exp\left(\int_0^t \left(r(s) - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dW^Q(s)\right). \tag{5.46}$$

Then $Y(t)$ defined as

$$Y(t) = \log\left(\frac{S_1(t)}{S_1(0)}\right) \tag{5.47}$$

is a an Itô-Lévy process of the form

$$Y(t) = \int_0^t \alpha(s)ds + \int_0^t \beta(s)d^QW(s) \sim N\left(\int_0^t \alpha(s)ds, \int_0^t \beta^2(s)dW(s)\right). \tag{5.48}$$

where

$$\alpha(s) = r(s) - \frac{\sigma^2}{2}, \qquad \beta(s) = \sigma, \qquad \gamma(s,x) = 0. \tag{5.49}$$

Hence,

$$Y(T) = Y(t) + \int_t^T \alpha(s)ds + \int_t^T \beta(s)dW^Q(s). \tag{5.50}$$

If $r(t)$ is deterministic and belongs to $L^1[0,T]$, then (2.31) is satisfied. The mean-variance hedging portfolio, from (4.136) for the continuous case is given as

$$\varphi(t) = D(t,T)\frac{E^Q\left[D_{t,0}F|\mathcal{F}_{t^-}\right]}{S_1(t)}. \tag{5.51}$$

Consider the contingent claim $F = \Phi(S_T)$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(u) = \Phi(S_0 e^u), \qquad u \in \mathbb{R} \tag{5.52}$$

We can compute for $E^Q\left[D_{t,0}F|\mathcal{F}_t\right]$ as follows:

$$E^Q[D_{t,0}g(Y(T))|\mathcal{F}_t] = -\frac{\beta(t)}{\sigma}E\left[g(Y(T))\log f_{Y(T)}(Y(T)|\mathcal{F}_t)\middle|\,\mathcal{F}_t\right]. \quad (5.53)$$

We can express $Y(T)$ conditional to $\mathcal{F}_t$ as follows,

$$Y(T)|\mathcal{F}_t = Y(t) + \int_t^T \alpha(s)ds + \int_t^T \beta(s)dW(s) \sim N\left(Y(t) + m_{[t,T]}, \Sigma_{[t,T]}\right) \quad (5.54)$$

where

$$m_{[t,T]} = \int_t^T \alpha(s)ds, \qquad \Sigma_{[t,T]} = \int_t^T \beta^2(s)ds. \quad (5.55)$$

Then density of $Y(T)$ conditional to $\mathcal{F}_t$

$$f_{Y(T)}(u|\mathcal{F}_t) = \frac{1}{\sqrt{2\pi\Sigma_{[t,T]}}}\exp\left(-\frac{\left(u-(Y(t)+m_{[t,T]})\right)^2}{2\Sigma_{[t,T]}}\right). \quad (5.56)$$

Hence, taking the logarithm gives us

$$\frac{d}{du}\log f_{Y(T)}(Y(T)|\mathcal{F}_t) = -\frac{Y(T)-(Y(t)+m_{[t,T]})}{\Sigma_{[t,T]}} = -\frac{\int_t^T \beta(s)dW^Q(s)}{\int_t^T \beta^2(s)ds}. \quad (5.57)$$

So therefore,

$$E^Q[D_{t,0}F|\mathcal{F}_t] = E\left[\Phi(S_1(T))\frac{\int_t^T \sigma dW^Q(s)}{\int_t^T \sigma^2 ds}\middle|\,\mathcal{F}_t\right]. \quad (5.58)$$

Hence, the mean-variance hedging portfolio is given by

$$\varphi(t) = D(t,T)E^Q\left[\frac{\Phi(S_1(T))}{S_1(t)}\frac{\int_t^T \sigma dW^Q(s)}{\int_t^T \sigma^2 ds}\middle|\,\mathcal{F}_t\right]. \quad (5.59)$$

If we let $r(s) = r$ an constant and $t = 0$, we obtain the familiar Delta of the option using the likelihood method as well as from the technique using the first variation process by Fournie, et al., [31] process given by

$$\Delta = e^{-rT}E^Q\left[\Phi(S_1(T))\frac{W^Q(T)}{\sigma S_0 T}\right]. \quad (5.60)$$

Example We consider the Binary option

$$F = \Phi(S(T)) = \mathbf{1}_{[K,L]}(S_1(T)). \tag{5.61}$$

Then, the mean-variance hedging portfolio is given by

$$\varphi(t)\frac{D(t,T)}{\sigma S_1(t)\sqrt{T-t}}\left[\phi(d_2(K)) - \phi(d_2(L))\right] \tag{5.62}$$

where $\phi(\cdot)$ is the density function of the standard normal distribution and

$$d_2(H) = \frac{\log\frac{S_1(t)}{H} + m_{[t,T]}}{\sqrt{\Sigma_{[t,T]}}} = \frac{\log\frac{S(t)}{H} + \int_t^T \left(r(s) - \frac{\sigma^2}{2}\right) ds}{\sqrt{\int_t^T \sigma^2 ds}}. \tag{5.63}$$

6. EVALUATING GREEKS IN EXOTIC OPTIONS

6.1 Preliminaries

We investigate the Greeks for exotic options using Malliavin calculus which involves the running supremum and running infimum of an asset process such as the barrier and lookback options. To be able to use Malliavin calculus on these options, one should show Malliavin differentiability of the running supremum. Nualart and Vives were able to prove in the continuous case [62]. Moreover, a recent paper from Arai and Suzuki proved the Malliavin differentiability in the Lévy case [10]. Througout the rest of this chapter, we shall present the assumptions taken from [38] and [16].

Without loss of generality, we consider a single risky asset S. Consider the risky-asset price under the Q dynamics modeled as an exponential Lévy process

$$S(t) = S_0 \exp(rt + L(t)) \tag{6.1}$$

where $r \geq 0$ is the risk-free interest rate and $L(t)$ is a square-integrable Lévy process with characteristic triplet (b, σ^2, ν), that is, $L(t)$ can be written as follows:

$$L(t) = bt + \sigma W^Q(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(dt, dz). \tag{6.2}$$

In addition, we assume that $S \in L^2(Q)$ and $e^{-rt}S(t)$ is a Q-martingale. Hence, we the following restrictions [21]:

$$\begin{aligned} \int_{|z|\geq 1} e^{2z}\nu(dz) &< \infty, \\ b + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^z - z - 1)\nu(dz) &= 0. \end{aligned} \tag{6.3}$$

From Itô's lemma, the SDE of the exp-Lévy process is given by

$$dS(t) = \alpha(t, S(t))dt + \beta(t, S(t))dW^Q(t) + \int_{\mathbb{R}_0} \gamma(t, S(t), z)\tilde{N}^Q(dt, dz) \tag{6.4}$$

where

$$
\begin{aligned}
\alpha(t,x) =& \left(\left(r + \frac{\sigma^2}{2} \right) + \int_{\mathbb{R}_0} (e^z - z - 1)\nu(dz) \right) x, \\
\beta(t,x) =& \sigma x, \\
\gamma(t,x,z) =& (e^z - 1)x.
\end{aligned} \tag{6.5}
$$

Then, by choosing

$$
c(t) = \left| \left(r + \frac{\sigma^2}{2} \right) + \int_{\mathbb{R}_0} (e^z - z - 1)\nu(dz) \right| + \sigma^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz) \tag{6.6}
$$

in Chapter 2.4, then the SDE of the exp-Lévy process has a strong solution provided that

$$
\begin{aligned}
\int_{\mathbb{R}_0} (e^z - z - 1)\nu(dz) <& \infty, \\
\int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz) <& \infty.
\end{aligned} \tag{6.7}
$$

From Chapter 2.4 implies $\sup_{t\in[0.T]} S(t) \in L^2(Q)$.

Denote the following running supremum and infimum processes as follows:

$$
\begin{aligned}
M^S(t) &= \sup_{u \in I \cap [0,t]} S(u), \qquad m^S(t) = \inf_{u \in I \cap [0,t]} S(u), \\
M^L(t) &= \sup_{u \in I \cap [0,t]} L(u), \qquad m^L(t) = \inf_{u \in I \cap [0,t]} L(u)
\end{aligned} \tag{6.8}
$$

There are two monitoring schemes of interest, namely:

- continuous-time monitoring: $I = [0, T]$
- discrete-time monitoring: $I = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}$.

Without loss of generality, we denote the following shorthand notation:

$$
M^S = M^S(T), \qquad m^S = m^S(T), \qquad S = S(T). \tag{6.9}
$$

Consider the payoff of the form $\Phi = \Phi(M^S, m^S, S)$ be square integrable under the following assumptions.

Assumption (S) There exists $a > 0$ such that the following condition holds:

- $\Phi(M^S, m^S, S)$ does not depend on M^S if $M^S < S_0 \exp(a)$,
- $\Phi(M^S, m^S, S)$ does not depend on m^S if $m^S > S_0 \exp(-a)$.

Remark 6.1.1 *This assumption is not so restrictive and it includes a large class of barrier and lookback options. Some examples are as follows: Let φ be a vanilla option payoff and $a > 0$.*

- *Single barrier options*
  - *Up and out barrier option:*
$$\Phi(M^S, m^S, S) = \mathbf{1}_{M^S < U}\varphi(S), \quad a = \log(U/S_0)$$
  - *Down and in barrier option:*
$$\Phi(M^S, m^S, S) = \mathbf{1}_{m^S \leq D}\varphi(S), \quad a = \log(S_0/D)$$
- *Double barrier options*
  - *Double in barrier option:*
$$\Phi(M^S, m^S, S) = \mathbf{1}_{M^S \geq U}\mathbf{1}_{m^S \leq D}\varphi(S), \quad a = \min(\log(U/S_0), \log(S_0/D))$$
  - *Double out barrier option:*
$$\Phi(M^S, m^S, S) = \mathbf{1}_{M^S < U}\mathbf{1}_{m^S > D}\varphi(S), \quad a = \min(\log(U/S_0), \log(S_0/D))$$
  - *Mixed in/out out barrier option:*
$$\Phi(M^S, m^S, S) = \mathbf{1}_{M^S \geq U}\mathbf{1}_{m^S \leq D}\varphi(z)$$

This option doesn't directly satisfy (S). However, we can write the payoff as follows:
$$\Phi(M^S, m^S, S) = \mathbf{1}_{m^S \leq D}\varphi(S) - \mathbf{1}_{m^S \geq U}\mathbf{1}_{m^S \leq D}\varphi(S)$$
which is a linear combination of payoff verifying (S).

- *Backward start lookback put:*

$$\Phi(M^S, m^S, S) = \max(M_0, M^S) - S, \qquad M_0 > S_0, \quad a = \log(M_0/S_0)$$

- *Out of the money put on minimum:*

$$\Phi(M^S, m^S, S) = (K - m^S)^+, \qquad K < S_0, \quad a = \log(S_0/K)$$

Let $\Psi : [0, \infty) \to [0, 1]$ be a localizing process belongs to C_b^∞ such that

$$\Psi(x) = \begin{cases} 1, & x \leq \frac{a}{2}, \\ 0, & x \geq a \end{cases} \tag{6.10}$$

where $a > 0$ is given by Assumption (S).

Assumption (NS)
We assume that the Lévy measure ν satisfies the so-called Nualart-Schoutens assumption (3.23).

Definition 6.1.1 *X-dominating process*
An increasing, predictable, cadlag process $Y = \{Y(t) : t \in [0, T]\}$ is a dominating process for $X = \{X(t) : t \in [0, T]\}$ or an X-dominating process if the following condition holds: For all $t \in I$,

$$|X(t)| \leq Y(t). \tag{6.11}$$

Moreover, we assume that the dominating process has the following moment property.

Assumption (H)
There exists a positive function $\alpha : \mathbb{N} \to [0, \infty)$ with $\lim_{q\to\infty} \alpha(q) = \infty$, such that $\forall q \in \mathbb{N}$

$$E[Y^q(t)] < C_q t^{\alpha(q)}, \qquad \forall [0, T]. \tag{6.12}$$

In particular, $Y(0) = 0$.

Moreover, we have the following Malliavin differentiability assumption.

Assumption $R(q)$

$\Psi(Y(t)) \in \mathbb{D}^{q,\infty}$ *for all* $t \in [0,T]$*. Moreover, for* $j \in \{1, \cdots, q\}$*,*

$$\sup_{r_1,\cdots,r_j\in[0,T]} E\left[\sup_{r_1\vee,\cdots,\vee r_j\leq t\leq T} |D_{(r_1,0)\cdots,(r_j,0)}\Psi(Y(\cdot))|^p\right] \leq C_p. \tag{6.13}$$

6.2 Markovian Property of the Payoff

Let $\Phi \in L^2$ be Borel measurable and $s \leq t$, then

$$\begin{aligned}&\Phi\left(M^S(t), m^S(t), S(t)\right)\\&=\Phi\left(\sup_{u\in I\cap[0,t]} S(u), \inf_{u\in I\cap[0,t]} S(u), S(t)\right)\\&=\Phi\left(S_0e^{rt}\exp\left(\sup_{u\in I\cap[0,t]} L(u)\right), S_0e^{rt}\exp\left(\inf_{u\in I\cap[0,t]} L(u)\right), S_0e^{rt}\exp(L(t))\right).\end{aligned} \tag{6.14}$$

From the independent increments property of Lévy processes, the last term of the following expressions are independent of $X(s)$:

$$\begin{aligned}\sup_{u\in I\cap[0,t]} L(u) &= \max\left(\sup_{u\in I\cap[0,s]} L(u), \sup_{u\in I\cap[s,t]} L(u)\right)\\&= \max\left(M^L(s), L(s) + \sup_{u\in I\cap[s,t]}(L(u) - L(s))\right),\end{aligned} \tag{6.15}$$

$$\begin{aligned}\inf_{u\in I\cap[0,t]} L(u) &= \min\left(\inf_{u\in I\cap[0,s]} L(u) \min_{u\in I\cap[s,t]} L(u)\right),\\&= \min\left(m^L(s), L(s) + \min_{u\in I\cap[s,t]}(L(u) - L(s))\right),\end{aligned} \tag{6.16}$$

$$L(t) = L(s) + (L(t) - L(s)). \tag{6.17}$$

Hence, we have the following conditional expectation with respect to $\mathcal{F}_s$:

$$\begin{aligned}&E[\Phi\left(M^S(t), m^S(t), S(t)\right)|\mathcal{F}_s]\\&=E[\Phi\left(M^S(t), m^S(t), S(t)\right)|(M^L(s), m^L(s), L(s))]\\&=E[\Phi\left(M^S(t), m^S(t), S(t)\right)|(M^S(s), m^S(s), S(s))].\end{aligned} \tag{6.18}$$

6.3 Malliavin Derivatives of the Supremum and Infimum

Before we proceed in deriving the Greeks, we need to characterize the Malliavin derivative of the infimum and supremum for both discrete and continuous monitoring. Most of the derivation for the Malliavin derivatives for the supremum process were done by Arai and Suzuki [10]. For completeness, we shall give a infiumum process.

Let $x \in \mathbb{R}$, then the positive and negative part of x are given as follows:

$$x^+ = max(x, 0) = x \cdot \mathbf{1}\{x \geq 0\}, \tag{6.19}$$

$$x^- = -\, min(x, 0) = -x \cdot \mathbf{1}\{x < 0\}. \tag{6.20}$$

Furthermore, let $y \in \mathbb{R}$, then we can write the maximum and minimum of x and y as follows:

$$max(x, y) = (x - y)^+ + y, \tag{6.21}$$

$$min(x, y) = -(x - y)^- + y. \tag{6.22}$$

Lemma 6.3.1 *Let $F \in \mathbb{D}^{1,2}, K \in \mathbb{R}$, then $(F - K)^+ \in \mathbb{D}^{1,2}$ with*

(i)

$$D_{t,z}(F - K)^+ = \mathbf{1}_{\{F>K\}} D_{t,0} F \mathbf{1}_{\{z=0\}} + \frac{(F + zD_{t,z}F - K)^+ - (F - K)^+}{z} \mathbf{1}_{\{z \neq 0\}}, \tag{6.23}$$

(ii)

$$D_{t,z}(F - K)^- = -\mathbf{1}_{\{F \leq K\}} D_{t,0} F \mathbf{1}_{\{z=0\}} + \frac{(F + zD_{t,z}F - K)^- - (F - K)^-}{z} \mathbf{1}_{\{z \neq 0\}}. \tag{6.24}$$

Proof The proof of (6.23) is given by [10]. To complete the lemma, it suffice to show (6.24). Since

$$(F - K)^- = (F - K)^+ - (F - K) \tag{6.25}$$

then

$$
\begin{aligned}
&D_{t,z}(F-K)^- \\
=&D_{t,z}(F-K)^+ - D_{t,z}(F-K) \\
=&[\mathbf{1}_{\{F>K\}}D_{t,0}F - D_{t,0}F]\mathbf{1}_{\{z=0\}} + \left[\frac{(F+zD_{t,z}F-K)^+ - (F-K)^+}{z} - D_{t,z}F\right]\mathbf{1}_{\{z\neq 0\}} \\
=&\mathbf{1}_{\{F>K\}}D_{t,0}F \cdot \mathbf{1}_{\{z=0\}} + \frac{(F+zD_{t,z}F-K)^+ - (F-K)^+}{z}\mathbf{1}_{\{z\neq 0\}}. \qquad (6.26)
\end{aligned}
$$

■

Corollary 6.3.2 *Let $F_1, F_2 \in \mathbb{D}^{1,2}$, then*

(i)

$$
\begin{aligned}
D_{t,z}(F_2-F_1)^+ =&\mathbf{1}_{\{F_2>F_1\}}D_{t,0}(F_2-F_1)\cdot\mathbf{1}_{\{z=0\}} \\
&+\frac{((F_2-F_1)+zD_{t,z}(F_2-F_1))^+ - (F_2-F_1)^+}{z}\mathbf{1}_{\{z\neq 0\}}, \qquad (6.27)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
D_{t,z}(F_2-F_1)^- =&-\mathbf{1}_{\{F\leq K\}}D_{t,0}(F_2-F_1)\cdot\mathbf{1}_{\{z=0\}} \\
&+\frac{((F_2-F_1)+zD_{t,z}(F_2-F_1))^- - (F_2-F_1)^-}{z}\mathbf{1}_{\{z\neq 0\}}. \qquad (6.28)
\end{aligned}
$$

Proof From the previous lemma, we take $F = F_2 - F_1$ and $K = 0$. ■

Theorem 6.3.3 *Malliavin Derivatives of the Maximum and Minimum*
Let $F_k \in \mathbb{D}^{1,2}, k \in \mathbb{N}, 1 \leq k \leq n$ for all $n \in \mathbb{N}$, then

(i) $M_n \equiv \max_{1\leq k\leq n} F_k \in \mathbb{D}^{1,2}$ *and*

$$
D_{t,z}M_n = \sum_{k=1}^{n}\mathbf{1}_{A_{n,k}}D_{t,0}F_k\mathbf{1}_{\{z=0\}} + \frac{\max_{1\leq k\leq n}(F_k+zD_{t,z}F_k) - M_n}{z}\mathbf{1}_{\{z\neq 0\}}, \qquad (6.29)
$$

(ii) $m_n \equiv \min_{1\leq k\leq n} F_k \in \mathbb{D}^{1,2}$ *and*

$$
D_{t,z}m_n = \sum_{k=1}^{n}\mathbf{1}_{a_{n,k}}D_{t,0}F_k\mathbf{1}_{\{z=0\}} + \frac{\min_{1\leq k\leq n}(F_k+zD_{t,z}F_k) - M_n}{z}\mathbf{1}_{\{z\neq 0\}} \qquad (6.30)
$$

where

$$A_{n,1} = \{M_n = F_1\}, \quad A_{n,k} = \{M_n \neq F_1, \cdots, M_n \neq F_{k-1}, M_n = F_k\}, \quad 2 \leq k \leq n$$

$$a_{n,1} = \{m_n = F_1\}, \quad a_{n,k} = \{m_n \neq F_1, \cdots m_n \neq F_{k-1}, m_n = F_k\}, \quad 2 \leq k \leq n. \tag{6.31}$$

Proof The proof of (6.29) is given by [10]. To complete the lemma, it suffice to show (6.30).

Note that $m_1 = F_1, m_2 = F_2 \wedge F_1 = -(F_2 - F_1)^- + F_1 \in \mathbb{D}^{1,2}$ and by induction, it follows that $m_n = F_n \wedge m_{n-1} = -(F_n - m_{n-1})^- + m_{n-1} \in \mathbb{D}^{1,2}$. For $z = 0$,

$$\begin{aligned} D_{t,0}m_n =& - D_{t,0}(F_n - m_{n-1})^- + D_{t,0}m_{n-1} \\ =& \mathbf{1}_{\{F_n < m_{n-1}\}} D_{t,0}(F_n - m_{n-1}) + D_{t,0}m_{n-1} \\ =& \mathbf{1}_{\{F_n < m_{n-1}\}} D_{t,0}F_n + (1 - \mathbf{1}_{\{F_n < m_{n-1}\}}) D_{t,0}m_{n-1} \\ =& \mathbf{1}_{\{F_n < m_{n-1}\}} D_{t,0}F_n + \mathbf{1}_{\{F_n \geq m_{n-1}\}} D_{t,0}m_{n-1} \\ =& \mathbf{1}_{a_{n,n}} D_{t,0}F_n + \mathbf{1}_{\{m_n = m_{n-1}\}} D_{t,0}m_{n-1}. \end{aligned} \tag{6.32}$$

Recursively, we obtain

$$D_{t,0}m_n = \sum_{k=1}^{n} \mathbf{1}_{a_{n,k}} D_{t,0}F_k. \tag{6.33}$$

For $z \neq 0$,

$$\begin{aligned} D_{t,z}m_n =& - D_{t,z}(F_n - m_{n-1})^- + D_{t,z}m_{n-1} \\ =& - z^{-1}[((F_n - m_{n-1}) + zD_{t,z}(F_n - m_{n-1}))^- - (F_n - m_{n-1})^-] + D_{t,z}m_{n-1} \\ =& - z^{-1}[((F_n + zD_{t,z}F_n) - (m_{n-1} + zD_{t,z}m_{n-1}))^- \\ & - (m_{n-1} + zD_{t,z}m_{n-1}) - ((F_n - m_{n-1})^- - m_{n-1})] \\ =& z^{-1}[(F_n + zD_{t,z}F_n) \wedge (m_{n-1} + zD_{t,z}m_{n-1}) - m_n]. \end{aligned} \tag{6.34}$$

Now since

$$D_{t,z}m_n = z^{-1}[(m_n + zD_{t,z}m_n) - m_n] \tag{6.35}$$

so therefore,

$$m_n + zD_{t,z}m_n = (F_n + zD_{t,z}F_n) \wedge (m_{n-1} + zD_{t,z}m_{n-1}). \tag{6.36}$$

From the recursion, (6.36), we obtain

$$m_n + zD_{t,z}m_n = \min_{1\le k\le n}(F_k + zD_{t,z}F_k) \tag{6.37}$$

so therefore,

$$D_{t,z}m_n = \frac{\min_{1\le k\le n}(F_k + zD_{t,z}F_k) - m_n}{z}. \tag{6.38}$$

■

Consider the risky-asset process $S(t) = S_0\exp(rt + L(t))$. Denote the following:

$$\begin{aligned} M_n^S &= \sup_{1\le k\le n, t_k\le t} S(t_k), \qquad M_n^L = \sup_{1\le k\le n, t_k\le t} L(t_k), \\ m_n^S &= \inf_{1\le k\le n, t_k\le t} S(t_k), \qquad m_n^L = \inf_{1\le k\le n, t_k\le t} L(t_k). \end{aligned} \tag{6.39}$$

Since $L(t) \in \mathbb{D}^{1,2}$ for all $t \in [0,T]$, then we have the following corollary.

Corollary 6.3.4 *Malliavin Derivative of Supremum and Infimum (Discrete Monitoring)*

(i)

$$\begin{aligned} D_{t,z}m_n^L =& \sum_{k=1}^{n} \mathbf{1}_{\{A_{n,k}\}} D_{t,0}L(t_k)\mathbf{1}_{\{z\in\mathbb{R}_0\}} \\ &+ \frac{\sup_{1\le k\le n, t_k\le t}(L(t_k) + zD_{t,z}L(t_k)) - M_n^L}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}}, \end{aligned} \tag{6.40}$$

(ii)

$$\begin{aligned} D_{t,z}M_n^L =& \sum_{k=1}^{n} \mathbf{1}_{\{a_{n,k}\}} D_{t,0}L(t_k)\mathbf{1}_{\{z\in\mathbb{R}_0\}}, \\ &+ \frac{\inf_{1\le k\le n, t_k\le t}(L(t_k) + zD_{t,z}L(t_k)) - m_n^L}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}} \end{aligned} \tag{6.41}$$

where

$$\begin{aligned}
A_{n,1} =&\{M_n^L = L(t_1)\},\\
A_{n,k} =&\{M_n^L \neq L(t_1), \cdots M_n^L \neq L(t_{k-1}), M_n^L = L(t_k)\}, \quad 2 \leq k \leq n,\\
a_{n,1} =&\{m_n^L = L(t_1)\},\\
a_{n,k} =&\{m_n^L \neq L(t_1), \cdots m_n^L \neq L(t_{k-1}), m_n^L = L(t_k)\}, \quad 2 \leq k \leq n. \qquad (6.42)
\end{aligned}$$

Corollary 6.3.5 *Malliavin Derivative of Supremum and Infimum (Discrete Monitoring)*

(i)

$$\begin{aligned}
D_{t,z}M_n^S =&M_n^S \sum_{k=1}^{n} \mathbf{1}_{\{A_{n,k}\}} D_{t,0}L(t_k)\mathbf{1}_{\{z=0\}}\\
&+ \frac{\sup_{1\leq n, t_k \leq t}(S(t_k)e^{zD_{t,z}L(t_k)}) - M_n^S}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}}
\end{aligned}$$

(ii)

$$\begin{aligned}
D_{t,z}m_n^S =&m_n^S \sum_{k=1}^{n} \mathbf{1}_{\{a_{n,k}\}} D_{t,0}L(t_k)\mathbf{1}_{\{z=0\}}\\
&+ \frac{\inf_{1\leq n, t_k \leq t}(S(t_k)e^{zD_{t,z}L(t_k)}) - m_n^S}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}} \qquad (6.43)
\end{aligned}$$

where

$$\begin{aligned}
A_{n,1} =&\{M_n^S = S(t_1)\},\\
A_{n,k} =&\{M_n^S \neq S(t_1), \cdots M_n^S \neq S(t_{k-1}), M_n^S = S(t_k)\}, \quad 2 \leq k \leq n,\\
a_{n,1} =&\{m_n^S = S(t_1)\},\\
a_{n,k} =&\{m_n^S \neq S(t_1), \cdots m_n^S \neq S(t_{k-1}), m_n^S = S(t_k)\}, \quad 2 \leq k \leq n. \qquad (6.44)
\end{aligned}$$

Proof Since the exponential is strictly increasing, then $A_{n,k}$ and $a_{n,k}$ holds for $1 \leq k \leq n$ for all $n \in \mathbb{N}$.

(i) From the chain rule

$$\begin{aligned} D_{t,z}M_n^S =& D_{t,0}M_n^S \mathbf{1}_{\{z=0\}} + D_{t,z}M_n^S \mathbf{1}_{\{z\in\mathbb{R}_0\}} \\ =& D_{t,0}S_0e^{rt}e^{M_n^L}\mathbf{1}_{\{z=0\}} + \frac{S_0e^{rt}e^{M_n^L+zD_{t,z}M_n^L} - S_0e^{rt}e^{M_n^L}}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}}. \quad (6.45) \end{aligned}$$

From the preceding corollary and the chain rule, we have the following: For $z = 0$,

$$D_{t,0}S_0e^{rt}e^{M_n^L} = S_0e^{rt}e^{M_n^L}D_{t,0}M_n^L = M_n^S \sum_{k=1}^{n} \mathbf{1}_{\{A_{n,k}\}} D_{t,0}L(t_k), \quad (6.46)$$

and for $z \neq 0$,

$$\begin{aligned} & S_0e^{rt}\exp(M_n^L + zD_{t,z}M_n^L) \\ =& S_0e^{rt}\exp\left(M_n^L + \sup_{\{1\le k\le n, t_k\le t\}} (L(t_k) + zD_{t,z}L(t_k)) - M_n^L\right) \\ =& \sup_{\{1\le k\le n, t_k\le t\}} S_0e^{rt}\exp(L(t_k) + zD_{t,z}L(t_k)) \\ =& \sup_{\{1\le k\le n, t_k\le t\}} S_0e^{rt}\exp(zD_{t,z}L(t_k)). \quad (6.47) \end{aligned}$$

Hence, from the last two equations gives us the desired expression for $D_{t,z}M_n^S$.

(ii) From the chain rule,

$$\begin{aligned} D_{t,z}m_n^S =& D_{t,0}m_n^S \mathbf{1}_{\{z=0\}} + D_{t,z}m_n^S \mathbf{1}_{\{z\in\mathbb{R}_0\}} \\ =& D_{t,0}S_0e^{rt}e^{m_n^L}\mathbf{1}_{\{z=0\}} + \frac{S_0e^{rt}e^{m_n^L+zD_{t,z}m_n^L} - S_0e^{rt}e^{m_n^L}}{z}\mathbf{1}_{\{z\in\mathbb{R}_0\}} \quad (6.48) \end{aligned}$$

From the preceding corollary and the chain rule, we have the following: For $z = 0$,

$$D_{t,0}S_0e^{rt}e^{m_n^L} = S_0e^{rt}e^{m_n^L}D_{t,0}m_n^L = m_n^S \sum_{k=1}^{n} \mathbf{1}_{\{a_{n,k}\}} D_{t,0}L(t_k) \quad (6.49)$$

and for $z \neq 0$,

$$
\begin{aligned}
& S_0 e^{rt} \exp(m_n^L + zD_{t,z} m_n^L) \\
&= S_0 e^{rt} \exp\left(m_n^L + \inf_{\{1 \le k \le n, t_k \le t\}} (L(t_k) + zD_{t,z} L(t_k)) - m_n^L \right) \\
&= \inf_{\{1 \le k \le n, t_k \le t\}} S_0 e^{rt} \exp(L(t_k) + zD_{t,z} L(t_k)) \\
&= \inf_{\{1 \le k \le n, t_k \le t\}} S_0 e^{rt} \exp(zD_{t,z} L(t_k)). \qquad (6.50)
\end{aligned}
$$

Hence, from the last two equations gives us the desired expression for $D_{t,z} m_n^S$.

■

We let

$$
\begin{aligned}
\tau^M &= \inf\{t \in [0, T] : L(t) \vee L(t^-) = M^L\}, \\
\tau^m &= \inf\{t \in [0, T] : L(t) \wedge L(t^-) = m^L\}. \qquad (6.51)
\end{aligned}
$$

Note that

$$
\begin{aligned}
M^L &= \sup_{t \in [0,T]} L(t) = L(\tau^M) \vee L(\tau^{M-}), \\
m^L &= \inf_{t \in [0,T]} L(t) = L(\tau^m) \wedge L(\tau^{m-}). \qquad (6.52)
\end{aligned}
$$

Assumption The countable dense subset $\mathcal{U} \equiv \{u_k : k \in \mathbb{N}\}$ where $0, T \in \mathcal{U}$ exists such that

$$
M^L = \sup_{t \in \mathcal{U}} L(t), \qquad m^L = \inf_{t \in \mathcal{U}} L(t) \qquad (6.53)
$$

and $P(L(s) = L(t)) = 0$ for all $s \neq t$, where $s, t \in \mathcal{U}$.

Remark 6.3.6 *If L is a Lévy process that is not a compound Possion process, the above assumption holds [73].*

Let $\mathcal{U} \equiv \{u_k : k \in \mathbb{N}\}$ be a countable dense where $0, T \in \mathcal{U}$. Then, we have the following identities.

Lemma 6.3.7 *[10] Let $Y = \{Y(t) : t \in [0,T]\}$ be a cadlag and let*

$$M_n^Y = \max_{1\le k\le n} Y(u_k), \qquad M^Y = \sup_{t\in[0,T]} Y(t) \tag{6.54}$$

Then, $M_n^Y \overset{a.s.}{\to} M^Y$, as $n \to \infty$.

Corollary 6.3.8 *Let $Y = \{Y(t) : t \in [0,T]\}$ be a cadlag and let*

$$m_n^Y = \min_{1\le k\le n} Y(u_k), \qquad m^Y = \inf_{t\in[0,T]} Y(t) \tag{6.55}$$

Then, $m_n^Y \overset{a.s.}{\to} m^Y$, as $n \to \infty$.

Proof Applying the previous lemma, we obtain

$$m_n^Y = -\max_{1\le k\le n}(-Y(u_k)) \overset{a.s.}{\to} -\sup_{t\in k[0,T]}(-Y(t)) = \inf_{t\in[0,T]} Y(t) = m^Y \tag{6.56}$$

■

Definition 6.3.1 *[72] Uniform Integrablility*

A collection of random variables $\{A_i : i \in \mathcal{I}\}$ where $\mathcal{I}$ is an index set is said to integrable if $\sup_{i\in\mathcal{I}} E[|L_i|] < \infty$. Further, $\{A_i : i \in \mathcal{I}\}$ is said to be uniformly integrable (u.i.) if

$$\sup_{i\in\mathcal{I}} E[|A_i| \cdot \mathbf{1}\{|A_i| \ge \lambda\}] \to 0, \qquad \lambda \to \infty. \tag{6.57}$$

Lemma 6.3.9 *[72] Sufficiency condition for Uniform Integrability*

If $|A_i| \le B$ for all $i \in \mathcal{I}$ for some $B \in L^1(P)$, then then the $A_i's$ are uniformly integrable.

Claim As $n \to \infty$,

(i) $M_n^L \overset{L^2(Q)}{\to} M^L$,

(ii) $m_n^L \overset{L^2(Q)}{\to} m^L$.

Remark 6.3.10 *Arai and Suzuki [10] has just proceeded after they have shown that $M_n^L \overset{a.s.}{\to} M^L$, then they have just mentioned that $M_n^L \overset{L^2(Q)}{\to} M^L$. Moreover, in addition to almost sure convergence, uniform integrability is also required to justify the $L^2(P)$ convergence.*

Proof (i) From the previous lemma, $M_n^L \overset{a.s.}{\to} M^L$, then, $(M_n^L)^2 \overset{a.s.}{\to} (M^L)^2$. It suffice to show that $(M_n^L)^2$ is u.i.. Note that

$$M_n^L \leq \max((M^L)^2, (m^L)^2) \leq (M^L)^2 + (m^L)^2 \tag{6.58}$$

Now since

$$M^L(t)^2 = \left(\sup_{t\in[0,T]} L(t)\right)^2 \leq \sup_{t\in[0,T]} |L(t)|^2, \tag{6.59}$$

$$m^L(t)^2 = \left(\inf_{t\in[0,T]} L(t)\right)^2 \leq \sup_{t\in[0,T]} |L(t)|^2. \tag{6.60}$$

So therefore,

$$M_n^L \leq 2 \sup_{t\in[0,T]} |L(t)|^2. \tag{6.61}$$

From the Lévy process $L(t)$ we have the following upper bound:

$$E^Q\left[\sup_{t\in[0,T]} |L(t)|^2\right] \leq$$
$$3^2\left((\mu T)^2 + E^Q\left[\sup_{t\in[0,T]} \left|\sigma W^Q(t)\right|^2\right] + E^Q\left[\sup_{t\in[0,T]} \left|\int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(ds,dz)\right|^2\right]\right). \tag{6.62}$$

From the Burkholder's Inequality [7], there exists $C_p > 0$ such that

$$E^Q\left[\sup_{t\in[0,T]} \left|\sigma W^Q(t)\right|^2\right] \leq C_p[\sigma W]_T = C_p\sigma^2 T. \tag{6.63}$$

where $[\cdot]$ is the quadratic variation process. Also, from Kunita's maximal inequality for a pure jump process [7], there exists $D_p > 0$ such that

$$E^Q\left[\sup_{t\in[0,T]}\left|\int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(ds,dz)\right|^2\right] \leq 2D_pE^Q\left[\int_{[0,T]\times\mathbb{R}_0} z^2 ds\nu(dz)\right]$$
$$=2D_pTE\left[\int_{\mathbb{R}_0} z^2\nu(dz)\right]. \tag{6.64}$$

Hence, we obtain,

$$E^Q\left[\sup_{t\in[0,T]}|L(t)|^2\right] < \infty. \tag{6.65}$$

By letting,

$$A_n = M_n^L, \qquad B = \sup_{t\in[0,T]}|L(t)|^2, \qquad \mathcal{I}\in\mathbb{N} \tag{6.66}$$

then from the previous lemma, implies M_n^L is u.i.

(ii) Similarly, from the inequality

$$m_n^L \leq \max((M^L)^2,(m^L)^2) \leq 2\sup_{t\in[0,T]}|L(t)|^2 \tag{6.67}$$

by letting,

$$A_n = M_n^L, \qquad B = \sup_{t\in[0,T]}|L(t)|^2, \qquad \mathcal{I}\in\mathbb{N} \tag{6.68}$$

then from the previous lemma, implies m_n^L is u.i.

■

Theorem 6.3.11 *Malliavin Derivatives of the Maximum and Minimum (Continuous Monitoring)*

(i) $M^L \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}M^L = \mathbf{1}_{\{\tau^M\leq t\}}\mathbf{1}_{\{z=0\}} + \frac{\sup_{t\in[0,T]}(L(s)+z\mathbf{1}_{\{t\leq s\}}) - M^L}{z}\mathbf{1}_{\{z\neq 0\}}, \tag{6.69}$$

(ii) $m^L \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}m^L = \mathbf{1}_{\{\tau^m\leq t\}}\mathbf{1}_{\{z=0\}} + \frac{\inf_{t\in[0,T]}(L(s)+z\mathbf{1}_{\{t\leq s\}}) - m^L}{z}\mathbf{1}_{\{z\neq 0\}}. \tag{6.70}$$

Proof The proof of (6.69) is given by [10]. To complete the theorem, it suffice to show (6.70). Consider the normalized log-returns for the exp-Lévy process (6.2), then,

$$D_{t,z}L(s) = \mathbf{1}_{\{t\leq s\}}. \tag{6.71}$$

Since we have already shown that $m_n^L \to m^L$ in $L^2(Q)$, then it suffice to show that $D_{t,z}m_n^L$ converges to $L^2(Q\times\mu)$. First, consider the case $z \neq 0$. Since

$$D_{t,z}m_n^L = \frac{\min_{1\leq k\leq n}\left(L(u_k) + zD_{t,z}L(u_k)\right) - m_n^L}{z} \tag{6.72}$$

and is a cadlag with respect to s, then

$$D_{t,z}m_n^L \overset{a.s.}{\to} \frac{\inf_{s\in[0,T]}(L(u) + zD_{t,z}L(u)) - m^L}{z}. \tag{6.73}$$

From the identity

$$\left|\inf_{i\in\mathcal{I}}(a_i + b_i) - \inf_{i\in\mathcal{I}}(a_i)\right| \leq \sup_{i\in\mathcal{I}}|b_i| \tag{6.74}$$

where $\mathcal{I}$ is some index set, then

$$\begin{aligned}
&\left|D_{t,z}m_n^L - \frac{\inf_{u\in[0,T]}(L(u) + zD_{t,z}L(u)) - m^L)}{z}\right| \\
&\leq 2\left[\left|D_{t,z}m_n^L\right|^2 + \frac{\left|\inf_{u\in[0,T]}(L(u) + zD_{t,z}L(u)) - m^L\right|^2}{|z|^2}\right] \\
&\leq \frac{2}{|z|^2}\left[\left|\min_{1\leq k\leq n}\left(L(u_k) + zD_{t,z}L(u_k)\right) - m_n^L\right| + \left|\inf_{u\in[0,T]}(L(u) + zD_{t,z}L(u)) - m^L\right|^2\right] \\
&\leq 2\left[\max_{1\leq k\leq n}\left|D_{t,z}L(u_k)\right|^2 + \sup_{u\in[0,T]}\left|D_{t,z}L(u)\right|^2\right] \\
&\leq 4\sup_{u\in[0,T]}\left|D_{t,z}L(u)\right|^2 = 4.
\end{aligned} \tag{6.75}$$

The convergence in $L^2(Q\times\mu)$ follows from the dominated convergence theorem.

Next, we consider the case $z = 0$. Denote the following:

$$\begin{aligned}
a_{n,1}^L =& \{m_n^L = L(u_1)\}, \\
a_{n,k}^L =& \{m_n^L \neq L(u_1), \cdots, m_n^L \neq L(u_{k-1}), m_n^L = L(u_k)\}, \qquad 2\leq k\leq n
\end{aligned} \tag{6.76}$$

$$\tau_n = \sum_{k=1}^{n} u_k \mathbf{1}_{\{a_{n,k}^L\}}. \tag{6.77}$$

Then, from the above assumption, $\tau_n = u_k$, whenever $m_n^L = L(u_k)$. Moreover, since $\mathcal{U}$ contains the index set that maximizes L, then

$$D_{t,0} m_n^L = \sum_{k=1}^{n} \mathbf{1}_{\{a_{n,k}^L\}} D_{t,0} L(u_k) = \sum_{k=1}^{n} \mathbf{1}_{\{a_{n,k}^L\}} \mathbf{1}_{\{t \le u_k\}} \stackrel{a.s.}{\to} \mathbf{1}_{\{t \le \tau^a\}}. \tag{6.78}$$

■

6.4 Some Important Identities

Lemma 6.4.1 *[60] Let $\Sigma = \{\Sigma_{ij} \in \mathbb{R}^{m\times m}\}$ positive definite symmetric matrix such that each of the entries Σ_{ij} have all orders and that for any $p \ge 2$, there exists $\epsilon_0(p)$ such that for all $\epsilon \le \epsilon_0(p)$*

$$\sup_{\|v\|_2 = 1} P(v^T \Sigma v \le \epsilon) \le \epsilon^p. \tag{6.79}$$

Then $\det(\Sigma)^{-1} \in L^p$ *for all p.*

Claim $\left(\int_0^T \Psi(Y(s))ds\right)^{-1} \in L^p$, for all $p \ge 1$.

Note that

$$\begin{aligned} \int_0^T \Psi(Y(t))dt = &\int_0^T \mathbf{1}_{\{Y(t)\in[0,a/2]\}} \Psi(Y(t))dt + \int_0^T \mathbf{1}_{\{Y(t)\in(a/2,a)\}} \Psi(Y(t))dt \\ &+ \int_0^T \mathbf{1}_{\{Y(t)\in[a,\infty)\}} \Psi(Y(t))dt \\ \ge & T \wedge Y^{(-1)}(a/2) \end{aligned} \tag{6.80}$$

where $Y^{(-1)}$ is the generalized inverse function

$$Y^{(-1)}(u) = \inf\{x \in \mathbb{R} : Y(x) \ge u\}. \tag{6.81}$$

Then for $\epsilon < 0$ small enough,

$$P(T \wedge Y^{(-1)}(a/2) < \epsilon) \le P(Y^{(-1)}(a/2) < \epsilon). \tag{6.82}$$

From the event $\{Y^{(-1)}(a/2) < \epsilon\}$, then we have

$$a/2 \leq Y(Y^{(-1)}(a/2)) \leq Y(\epsilon). \tag{6.83}$$

Hence, (6.83) together with Markov Inequality and (H), we obtain

$$P(Y^{(-1)}(a/2) < \epsilon) \leq P(Y(\epsilon \geq a) \leq \frac{E[Y^q(\epsilon)]}{(a/2)^q} \leq C_q \frac{\epsilon^{\alpha(q)}}{(a/2)^q}. \tag{6.84}$$

So that we can choose a positive function α such that

$$P\left(\int_0^T \Psi(Y(s))ds < \epsilon\right) = O(\epsilon^p) \to 0 \tag{6.85}$$

as $\epsilon \to 0$. Then, from the preceding lemma, $\left(\int_0^T \Psi(Y(s))ds\right)^{-1} \in L^p$, for all $p \geq 1$.

6.5 Delta

Theorem 6.5.1 *The Delta of the contingent claim $\Phi = \Phi(M^S, m^S, S)$ from an exp-Lévy process provided that the Assumptions (S), (NS), and the dominating process Y satisfies Assumptions (H) and $R(1)$ holds is given by the following:*

$$\Delta = \frac{\partial}{\partial S_0} E^Q[e^{-rT}\Phi] = e^{-rT} E^Q[\Pi_\Delta \Phi] \tag{6.86}$$

where

$$\Pi_\Delta = \frac{1}{S_0}\delta^W \left(\frac{\Psi(Y(\cdot))}{\int_0^T \Psi(Y(t))dt}\sigma^{-1}\right), \tag{6.87}$$

δ^W is the Skorohod integral with respect to the Wiener process W^Q.

Proof From the density argument, it suffices to show that the identity holds for a smooth function $\Phi \in C_b^\infty(\mathbb{R})$ [31]. From the chain rule, we have

$$\Delta = \frac{\partial}{\partial S_0} E\left[e^{-rT}\frac{\partial \Phi}{\partial S_0}\right] = \frac{\partial}{\partial S_0} E\left[e^{-rT}\left(\Phi_1' \frac{\partial M^S}{\partial S_0} + \Phi_2' \frac{\partial M^S}{\partial S_0} + \Phi_3' \frac{\partial S(T)}{\partial S_0}\right)\right] \tag{6.88}$$

where Φ_i' is the partial derivative of Φ with respect to the i^{th} argument. For an exponential Lévy process, we have the following:

$$\frac{\partial M^S}{\partial S_0} = \frac{M^S}{S_0}, \qquad \frac{\partial m^S}{\partial S_0} = \frac{m^S}{S_0}, \qquad \frac{\partial S(T)}{\partial S_0} = \frac{S(T)}{S_0}. \tag{6.89}$$

Since

$$D_{t,z}\Phi = \left(\Phi_1' D_{t,0}M^S + m\Phi_2' D_{t,0}m^S + \Phi_3{}' D_{t,0}S(T)\right)\mathbf{1}_{\{z=0\}} + \frac{\Phi(M^S + zD_{t,z}M^S, m^S + zD_{t,z}m^S, S(T) + zD_{t,z}S(T)) - \Phi(M, m, S(T))}{z}\mathbf{1}_{\{z\neq 0\}} \tag{6.90}$$

and from the chain rule

$$D_{t,z}M^S = M^S\mathbf{1}_{\{t\leq\tau_M\}}\mathbf{1}_{\{z=0\}} + \frac{\sup_{u\in[0,t]} S(u)\exp\left(z\mathbf{1}_{\{t\leq u\}}\right) - M^S}{z}\mathbf{1}_{\{z\neq 0\}}, \tag{6.91}$$

$$D_{t,z}m^S = m^S\mathbf{1}_{\{t\leq\tau_m\}}\mathbf{1}_{\{z=0\}} + \frac{\inf_{u\in[0,t]} S(u)\exp\left(z\mathbf{1}_{\{t\leq u\}}\right) - m^S}{z}\mathbf{1}_{\{z\neq 0\}}, \tag{6.92}$$

$$D_{t,z}S(T) = S(T)\mathbf{1}_{\{z=0\}} + S(T)\frac{\exp\left(z\mathbf{1}_{\{t\leq u\}}\right) - 1}{z}\mathbf{1}_{\{z\neq 0\}}. \tag{6.93}$$

Hence,

$$D_{t,0}\Phi = \frac{1}{\sigma}D_t^W\Phi = \Phi_1' M^S\mathbf{1}_{\{t\leq\tau_M\}} + \Phi_2' m^S\mathbf{1}_{\{t\leq\tau_m\}} + \Phi_3{}' S(T) \tag{6.94}$$

By localization, we have the following:

$$\Phi_1'\mathbf{1}_{\{t\leq\tau_M\}}\Psi(Y(t)) = \Phi_1'\Psi(Y(t)), \tag{6.95}$$

$$\Phi_2'\mathbf{1}_{\{t\leq\tau_m\}}\Psi(Y(t)) = \Phi_2'\Psi(Y(t)). \tag{6.96}$$

We extend the reasoning of the above localization from the Wiener case [38] to the Lévy case. For (6.95), consider the following cases. If $M^S < S_0\exp(a)$, from Assumption (S), then Φ doesn't depend on M^L so $\Phi_1' = 0$ and thus both sides of (6.95) becomes $0 = 0$. Conversely, if $M^S \geq S_0\exp(a)$, that is,

$$\sup_{s\in I\cap[0,T]} S(s) \geq S_0\exp(a). \tag{6.97}$$

Then, suppose there exists $t \in [0,T]$ such that $\Psi(Y(t)) \neq 0$, then $Y(t) < \exp(a)$. From the dominating process (6.11) implies

$$\sup_{s\in I\cap[0,t]} S(s) < S_0\exp(a). \tag{6.98}$$

Combining the above inequalities gives us,

$$\sup_{s\in I\cap[0,t]} S(s) < S_0 \exp(a) \leq \sup_{s\in I\cap[0,T]} S(s) \tag{6.99}$$

and thus, $t \leq \tau_M$. On the other hand, for (6.96), consider the following cases. If $m^S > S_0 \exp(-a)$, from Assumption (S), then Φ doesn't depend on m^S so $\Phi_2' = 0$ and thus both sides of (6.95) becomes $0 = 0$. Conversely, If $m^S \leq S_0 \exp(-a)$, that is,

$$\inf_{s\in I\cap[0,T]} L(s) \geq S_0 \exp(-a). \tag{6.100}$$

Again, suppose there exists $t \in [0, T]$ such that $\Psi(Y(t)) \neq 0$, then $Y(t) < \exp(a)$. From the dominating process (6.11) implies

$$S_0 \exp(-a) < \inf_{s\in I\cap[0,t]} S(s). \tag{6.101}$$

Combining the above inequalities gives us,

$$\inf_{s\in I\cap[0,T]} S(s) \leq S_0 \exp(-a) < \inf_{s\in I\cap[0,t]} S(s) \tag{6.102}$$

and thus, $t \leq \tau_m$. Plugging (6.95) and (6.96) into 6.94 then multiplying by $\Psi(Y(t))$ and finally, integrating both sides yields

$$\int_0^T D_{t,0}\Phi \cdot \Psi(Y(t))dt = (\Phi_1' M^S + m\Phi_2' m^S + \Phi_3{}' S(T)) \int_0^T \Psi(Y(t))dt. \tag{6.103}$$

From the isometry $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; \Omega_J)$ [77], [78] the divergence relation [60], as well as the preceding lemma, we have as follows:

$$\begin{aligned}
\Delta =& E^Q \left[\frac{e^{-rT}}{S_0} \int_0^T D_{t,0}\Phi \left(\frac{\Psi(Y(t))}{\int_0^T \Psi(Y(u))du} \right) dt \right] \\
=& E^Q \left[\frac{e^{-rT}}{S_0} \int_0^T D_t^W \Phi \left(\frac{\Psi(Y(t))\sigma^{-1}}{\int_0^T \Psi(Y(u))du} \right) dt \right] \\
=& E^Q \left[\frac{e^{-rT}}{S_0} \delta^W \left(\frac{\Psi(Y(\cdot))\sigma^{-1}}{\int_0^T \Psi(Y(u))du} \right) \Phi \right].
\end{aligned} \tag{6.104}$$

■

To evaluate the Skorohod integral, we need to recall the following proposition [60].

Proposition 6.5.1 *[60] Let $F \in \mathbb{D}^{1,2}$ and $u \in Dom(\delta^W)$, then*

$$\delta^W(Fu) = F\delta^W(u) - < D^W F, u >_{L^2[0,T]} . \tag{6.105}$$

In addition, if u is adapted, then

$$\delta^W(Fu) = F\int_0^T u(t)dW(t) - \int_0^T D_t^W F \cdot u(t)dt. \tag{6.106}$$

We let

$$F = \left(\int_0^T \Psi(Y(s))ds\right)^{-1}, \qquad u(t) = \Psi(Y(t))\sigma^{-1} \tag{6.107}$$

From the condition $R(q)$ and since $F \in L^2$ from the above claim, then $F \in \mathbb{D}^{1,2}$. Moreover, since Y is adapted, then $u \in Dom(\Delta)$ and is adapted. From chain rule,

$$D_t^W F = \frac{-1}{\left(\int_0^T \Psi(Y(s))ds\right)^2}\int_t^T \Psi'(Y(s))D_t^W Y(s)ds. \tag{6.108}$$

Then

$$\begin{aligned}
&\int_0^T D_t^W F \cdot u(s)ds \\
=&\frac{-1}{\left(\int_0^T \Psi(Y(s))ds\right)^2}\int_0^T\int_t^T \Psi'(Y(s))D_t^W Y(s)ds \cdot \Psi(Y(t))\sigma^{-1}dt \\
=&\frac{-1}{\left(\int_0^T \Psi(Y(s))ds\right)^2}\int_0^T \Psi'(Y(s))\sigma^{-1} \cdot \int_0^s \Psi(Y(t))D_t^W Y(s)dtds
\end{aligned} \tag{6.109}$$

where the last expression is done using Fubini's theorem. Hence, performing integration by parts (6.106), we obtain

$$\begin{aligned}
S_0\Pi_\Delta =&\Delta\left(\frac{\Psi(Y(\cdot))}{\int_0^T \Psi(Y(t))dt}\sigma^{-1}\right) \\
=&\frac{\int_0^T \Psi(Y(t))\sigma^{-1}dW(t)}{\int_0^T \Psi(Y(t))dt} + \frac{\int_0^T \Psi'(Y(s))\sigma^{-1} \cdot \int_0^s \Psi(Y(t))D_t^W Y(s)dtds}{\left(\int_0^T \Psi(Y(s))ds\right)^2}.
\end{aligned} \tag{6.110}$$

6.6 Gamma

Theorem 6.6.1 *The Gamma of the contingent claim $\Phi = \Phi(M^S, m^S, S)$ from an exp-Lévy process provided that the Assumptions (S), (NS), and the dominating process Y satisfies Assumptions (H) and $R(2)$ holds is given by the following:*

$$\Gamma = \frac{\partial^2}{\partial S_0^2} E^Q[e^{-rT}\Phi] = e^{-rT} E^Q[\Pi_\Gamma \Phi] \tag{6.111}$$

where

$$\Pi_\Gamma = \frac{1}{S_0^2}\delta^W\left(\delta^W\left(\frac{\Psi(Y(\cdot))\cdot\sigma^{-1}}{\int_0^T \Psi(Y(t))dt}\right)\frac{\Psi(Y(\cdot))\cdot\sigma^{-1}}{\int_0^T \Psi(Y(t))dt}\right) - \frac{1}{S_0^2}\delta^W\left(\frac{\Psi(Y(\cdot))\cdot\sigma^{-1}}{\int_0^T \Psi(Y(t))dt}\right). \tag{6.112}$$

Proof From our previous result,

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{\partial}{\partial S_0} E^Q\left[\frac{e^{-rT}}{S_0}\delta^W(u)\,\Phi\right] \tag{6.113}$$

where

$$u = \frac{\Psi(Y(\cdot))\sigma^{-1}}{\int_0^T \Psi(Y(u))du}. \tag{6.114}$$

By density argument, it suffice to show the identity for $\Phi \in \mathcal{C}_b^\infty(\mathbb{R})$, [31]. Note that,

$$\Gamma = e^{-rT}\left(\frac{1}{S_0}E^Q\left[\delta^W(u)\frac{\partial\Phi}{\partial S_0}\right] - \frac{1}{S_0^2}E^Q\left[\delta^W(u)\,\Phi\right]\right). \tag{6.115}$$

From the derivation of the Delta,

$$\frac{\partial \Phi}{\partial S_0} = \frac{1}{S_0}\left(M\Phi_1' + m\Phi_2' + S(T)\Phi_3{}'\right) = \frac{1}{S_0}\int_0^T D_t^W u \cdot dt. \tag{6.116}$$

Also, from the divergence relation,

$$E^Q\left[\delta^W(u)\frac{\partial\Phi}{\partial S_0}\right] = E^Q\left[\int_0^T D_t^W u \cdot \delta^W(u)\,dt\right] = E^Q\left[\delta^W\left(u\delta^W(u)\right)\right]. \tag{6.117}$$

Hence,

$$\Gamma = \frac{e^{-rT}}{S_0^2}E^Q\left[\left(\delta^W\left(u\delta^W(u)\right) - \delta^W(u)\right)\Phi\right]. \tag{6.118}$$

■

Note that we can write

$$\Gamma = \frac{e^{-rT}}{S_0} E^Q \left[\left(\delta^W \left(\Pi_\Delta u \right) - \Pi_\Delta \right) \Phi \right] \tag{6.119}$$

then, its weight is given by

$$\Pi_\Gamma = \frac{1}{S_0} \left[\left(\delta^W \left(\Pi_\Delta u \right) - \Pi_\Delta \right) \Phi \right]. \tag{6.120}$$

To explicitly evaluate Γ, we need to evaluate the Skorohod integral using integration by parts as well as the Malliavin derivative D^W that will be involved in the derivation. Since Π_Δ has been computed explicitly, then it suffice to compute for the Skorohod integral $\delta^W \left(\Pi_\Delta u \right)$. We assume a suitable differentiability and Skorohod integrability conditions hold. Consider the expression

$$\Pi_\Delta u = \left[\frac{\int_0^T \Psi(Y(t))\sigma^{-1} dW(t)}{\left(\int_0^T \Psi(Y(t))dt \right)^2} + \frac{\int_0^T \Psi'(Y(s))\sigma^{-1} \cdot \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds}{\left(\int_0^T \Psi(Y(s))ds \right)^3} \right] \Psi(Y(\cdot))\sigma^{-1}. \tag{6.121}$$

The Skorohod integral of the first term of $\Pi_\Delta u$ is evaluated as follows. We let

$$F = \left(\int_0^T \Psi(Y(s))ds \right)^{-2}, \qquad u = \int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \Psi(Y(\cdot))\sigma^{-1}. \tag{6.122}$$

From integration by parts,

$$\begin{aligned} &\delta^W \left(\frac{\int_0^T \Psi(Y(t))\sigma^{-1} dW(t)}{\left(\int_0^T \Psi(Y(t))dt \right)^2} \Psi(Y(\cdot))\sigma^{-1} \right) \\ =& \frac{1}{\left(\int_0^T \Psi(Y(t))dt \right)^2} \delta^W \left(\int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \Psi(Y(\cdot))\sigma^{-1} \right) \\ &+ \frac{2}{\left(\int_0^T \Psi(Y(t))dt \right)^3} \int_0^T \int_t^T \Psi'(Y(s)) D_t^W Y(s) ds \Psi(Y(t))\sigma^{-1} dt \end{aligned} \tag{6.123}$$

We let

$$F = \int_0^T \Psi(Y(t)) dW(t) \sigma^{-1} dW(t), \qquad u = \Psi(Y(\cdot))\sigma^{-1} \tag{6.124}$$

Again, applying by integration by parts,

$$\begin{aligned}
&\delta^W \left(\int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \Psi(Y_\cdot)\sigma^{-1} \right) \\
=& \int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \cdot \int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \\
&\qquad - \int_0^T \int_t^T \Psi'(Y(s)) D_t^W Y(s) ds \cdot \Psi(Y(t))\sigma^{-1} dt \\
=& \left(\int_0^T \Psi(Y(t))\sigma^{-1} dW(t) \right)^2 - \int_0^T \Psi'(Y(s))\sigma^{-1} \cdot \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \quad (6.125)
\end{aligned}$$

On the other hand, the Skorohod integral of the second term of $\Pi_\Delta u$ is evaluated as follows. We let

$$F = \left(\int_0^T \Psi(Y(s)) ds \right)^{-3}, u = \int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \cdot \Psi(Y(\cdot))\sigma^{-1}. \quad (6.126)$$

From integration by parts,

$$\begin{aligned}
&\delta^W \left(\frac{\int_0^T \Psi'(Y(s))\sigma^{-1} \cdot \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds}{\left(\int_0^T \Psi(Y(s)) ds \right)^3} \Psi(Y(\cdot))\sigma^{-1} \right) \\
=& \frac{1}{\left(\int_0^T \Psi(Y(s)) ds \right)^3} \delta^W \left(\int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \cdot \Psi(Y(\cdot))\sigma^{-1} \right) \\
&+ \frac{3}{\left(\int_0^T \Psi(Y(s)) ds \right)^4} \int_0^T \int_t^T \Psi'(Y(s)) D_t^W Y(s) ds \cdot \\
&\qquad \int_0^T \Psi'(Y(u))\sigma^{-1} \int_0^u \Psi(Y(v)) D_v^W Y(u) dv du \cdot \Psi(Y(\cdot))\sigma^{-1} dt.
\end{aligned} \quad (6.127)$$

The last term is simplified as follows:

$$\frac{3}{\left(\int_0^T \Psi(Y(s)) ds \right)^4} \left(\int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \cdot \right)^2. \quad (6.128)$$

We let

$$F = \int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds, \qquad u = \Psi(Y(\cdot))\sigma^{-1}. \quad (6.129)$$

Again, by integration by parts,

$$\begin{aligned}
&\delta^W \left(\int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \cdot \Psi(Y(\cdot))\sigma^{-1} \right) \\
=& \int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \cdot \int_0^T \Psi(Y(s))\sigma^{-1} ds \\
&- \int_0^T D_u^W \left(\int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \right) \Psi(Y(u))\sigma^{-1} du.
\end{aligned} \tag{6.130}$$

Performing differentiation, we have the following:

$$\begin{aligned}
& D_u^W \left(\int_0^T \Psi'(Y(s))\sigma^{-1} \int_0^s \Psi(Y(t)) D_t^W Y(s) dt ds \right) \\
=& \quad \sigma^{-1} \int_u^T D_u^W \left(\Psi'(Y(s)) \int_0^s \Psi(Y(t)) D_t^W Y(s) dt \right) ds
\end{aligned} \tag{6.131}$$

where

$$\begin{aligned}
& D_u^W \left(\Psi'(Y(s)) \int_0^s \Psi(Y(t)) D_t^W Y(s) dt \right) \\
=& \Psi''(Y(s)) D_u^W Y(s) \int_0^s \Psi(Y(t)) D_t^W Y(s) dt + \Psi'(Y(s)) D_u^W \left(\int_0^s \Psi(Y(t)) D_t^W Y(s) dt \right)
\end{aligned} \tag{6.132}$$

and

$$\begin{aligned}
& D_u^W \left(\int_0^s \Psi(Y(t)) D_t^W Y(s) dt \right) \\
=& \int_u^s D_u^W \left(\Psi(Y(t)) D_t^W Y(s) \right) dt \\
=& \int_u^s \left[\Psi'(Y(t)) D_u^W Y(t) D_t^W Y(s) + \Psi(Y(t)) D_u^W (D_t^W Y(s)) \right] dt.
\end{aligned} \tag{6.133}$$

6.7 Construction of Dominating Processes

We extend the construction of the dominating process for the exp-Lévy process. We verify whether dominating process proposed by Bernis, Gobet, and Kohatsu-Higa, [16] for the geometric Brownian motion will be still carried over to the exp-Lévy process.

6.7.1 Continuous-Time Monitoring

- Extrema Process

$$Y(t) = \sup_{s\in[0,t]}(L(s)-L(0)) - \inf_{s\in[0,t]}(L(s)-L(0)) \tag{6.134}$$

Claim: $Y(t)$ is an L-dominating process. Note that (6.11) is clearly satisfied since

$$L(t) = (L(t)-L(0)) \leq \sup_{s\in[0,t]}(L(s)-L(0)) - \inf_{s\in[0,t]}(L(s)-L(0)) = Y(t). \tag{6.135}$$

Moreover, for $q \geq 1$, and from the identity $(a+b)^q \leq 2^q(a^q+b^q)$, we obtain

$$\begin{aligned} E^Q[Y(t)^q] =& E^Q\left[\left|\sup_{s\in[0,t]}(L(s)-L(0)) - \inf_{s\in[0,t]}(L(s)-L(0))\right|^q\right] \\ \leq& 2^q E^Q\left[\left|\sup_{s\in[0,t]}(L(s)-L(0))\right|^q + \left|\inf_{s\in[0,t]}(L(s)-L(0))\right|^q\right] \\ \leq& 2^q E^Q\left[\sup_{s\in[0,t]}|L(s)-L(0)|^q + \sup_{s\in[0,t]}|L(s)-L(0)|^q\right] \\ \leq& 2^{(q+1)} E^Q\left[\sup_{s\in[0,t]}|L(s)-L(0)|^q\right]. \end{aligned} \tag{6.136}$$

From the Lévy process $L(t)$,

$$E^Q\left[\sup_{s\in[0,t]}|L(s)-L(0)|^q\right] \leq \\ 3^q\left[(\mu t)^q + E^Q\left[\sup_{s\in[0,t]}\left|\int_0^t \sigma dW^Q(s)\right|^q\right] + E^Q\left[\sup_{s\in[0,t]}\left|\int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(ds,dz)\right|^q\right]\right]. \tag{6.137}$$

From Burkholder's Inequality and Kunita's Inequality [7], there exists $A_p > 0$ and $B_p > 0$ such that

$$E^Q\left[\sup_{s\in[0,t]}\left|\int_0^t \sigma dW^Q(s)\right|^q\right] \leq A_q\sigma^q t^{q/2}, \tag{6.138}$$

$$E^Q\left[\sup_{s\in[0,t]}\left|\int_{[0,t]\times\mathbb{R}_0} z\tilde{N}^Q(ds,dz)\right|^q\right] \leq B_q\left[t^{q/2}\left(\int_{\mathbb{R}_0} z^2\nu(dz)\right)^{q/2} + t\int_{\mathbb{R}_0}|z|^q\nu(dz)\right]. \tag{6.139}$$

Hence, to satisfy (6.12) we can pick $\alpha(q) = q$. Hence, $Y(t)$ is an L-dominating process. Moreover, $Y(t)$ it satisfies $R(1)$ since for $r \leq s$,

$$\begin{aligned} D_{r,0}Y(t) =& D_{r,0} \sup_{s\in[0,t]} (L(s) - L(0)) - D_{r,0} \inf_{s\in[0,t]} (L(s) - L(0)) \\ =& \mathbf{1}_{\{r\leq\tau^M(t)\}} - \mathbf{1}_{\{r\leq\tau^m(t)\}} \end{aligned} \tag{6.140}$$

where $\tau^M(t)$ and $\tau^m(t)$ is the running supremum and running infimum of L respectively. Hence,

$$\begin{aligned} \sup_{r\in[0,t]} E^Q \left[\sup_{t\in[r,T]} |D_{t,0}\Psi(Y(t))|^p \right] &= \sup_{r\in[0,t]} E^Q \left[\sup_{t\in[r,T]} |\Psi'(Y(t))|^p |D_{t,0}Y(t)|^p \right] \\ &\leq \left(\frac{2C_1}{a} \right)^p. \end{aligned} \tag{6.141}$$

- Average Modulus of Continuity Process

$$Y(t) = 8 \left(4 \int_{[0,t]^2} \frac{|L(s) - L(u)|^\gamma}{|s-u|^{m+2}} dsdu \right)^{1/\gamma} \frac{m+2}{m} t^{m\gamma} \tag{6.142}$$

where γ is an even integer, and $m \in \left(0, \frac{\gamma}{2} - 2\right)$. This dominating process is applicable only in the continuous case. To show that $Y(t)$ is a dominating process, we use of the Garsia, Rodemich, Rumsey (GRR) lemma [32], [35] which assumes that L is continuous. The GRR lemma is stated as follows:

Theorem 6.7.1 *Let (E, d) be a metric space, $f \in C([0,T], E)$ and Ψ, p be continuous and strictly increasing functions on $[0, \infty)$ such that $p(0) = g(0) = 0$ and $\lim_{t\uparrow\infty} g(t) = \infty$. Then*

$$\int_{[0,t]^2} g \left(\frac{d(f(s), f(u))}{p(|u-s|)} \right) dsdu \leq B \tag{6.143}$$

implies for $0 \leq s < u \leq T$,

$$d(f(s), f(u)) \leq 8 \int_0^{u-s} g\left(\frac{4B}{v^2}\right) dp(v). \tag{6.144}$$

Denote $\omega_f(\Delta) = \sup\{d(L(s), L(t)) : s, t \in [0,T], |t-s| \leq \Delta\}$ be the modulus of continuity of f, then

$$\omega_f(\Delta) \leq 8 \int_0^{\Delta} g\left(\frac{4B}{u^2}\right) dp(u). \tag{6.145}$$

6.7.2 Discrete-Time Monitoring

- Extrema Process

$$Y(t) = \sup_{0\le k\le n, t_k\le t}(L(t_k) - L(0)) - \inf_{0\le i\le n, t_k\le t}(L(t_k) - L(0)) \tag{6.146}$$

Claim: $Y(t)$ is an L-dominating process. Note that (6.11) is clearly satisfied since

$$L(t) = (L(t) - L(0)) \le \sup_{0\le k\le n, t_k\le t}(L(t_i) - L(0)) - \inf_{0\le k\le n, t_k\le t}(L(t_k) - L(0)) = Y(t). \tag{6.147}$$

Moreover, for $q \ge 1$, and from the identity $(a+b)^q \le 2^q(a^q + b^q)$, we obtain

$$\begin{aligned} E^Q[Y(t)^q] =& E^Q\left[\left|\sup_{0\le k\le n, t_k\le t}(L(t_k) - L(0)) - \inf_{0\le k\le n, t_k\le t}(L(t_k) - L(0))\right|^q\right] \\ \le& 2^{(q+1)}E^Q\left[\sup_{0\le k\le n, t_k\le t}|L(t_k) - L(0)|^q\right] \\ \le& 2^{(q+1)}E^Q\left[\sup_{s\in[0,t]}|L(s) - L(0)|^q\right]. \end{aligned} \tag{6.148}$$

This upper bound is similar to the continuous monitoring case. Hence, to satisfy (6.12) we can pick $\alpha(q) = q$. Moreover, $Y(t)$ it satisfies $R(1)$ since for $r \le s$,

$$\begin{aligned} D_{r,0}Y(t) =& D_{r,0}\sup_{0\le k\le n, t_k\le t}(L(s) - L(0)) - D_{r,0}\inf_{0\le k\le n, t_k\le t}(L(s) - L(0)) \\ =& \sum_{k=0}^{n}\mathbf{1}_{\{A_{n,k}\}}D_{r,0}L(t_k) - \sum_{k=0}^{n}\mathbf{1}_{\{a_{n,k}\}}D_{r,0}L(t_k) \\ =& \sum_{k=0}^{n}\left(\mathbf{1}_{\{A_{n,k}\}} - \mathbf{1}_{\{a_{n,k}\}}\right)\mathbf{1}_{\{t\le t_k\}} \end{aligned} \tag{6.149}$$

where

$$\begin{aligned} A_{n,1} = \{M_n = F_1\}, \quad A_{n,k} = \{M_n \ne F_1, \cdots, M_n \ne F_{k-1}, M_n = F_k\}, \quad 2 \le k \le n, \\ a_{n,1} = \{m_n = F_1\}, \quad a_{n,k} = \{m_n \ne F_1, \cdots m_n \ne F_{k-1}, m_n = F_k\}, \quad 2 \le k \le n, \end{aligned} \tag{6.150}$$

and

$$M_n = \sup_{0\le k\le n, t_k\le t}(L(s)-L(0)), \qquad m_n = \sup_{0\le k\le n, t_k\le t}(L(s)-L(0)). \tag{6.151}$$

Hence,

$$\begin{aligned}\sup_{r\in[0,t]} E^Q\left[\sup_{t\in[s,T]}|D_{t,0}\Psi(Y(t))|^p\right] &= \sup_{s\in[0,t]} E^Q\left[\sup_{t\in[s,T]}|\Psi'(Y(t))|^p|D_{t,0}Y(t)|^p\right]\\ &\le \left(\frac{2C_1}{a}\right)^p. \end{aligned}\tag{6.152}$$

.

- Averaged Quadratic Increments Process

$$Y(t) = \sqrt{n\sum_{1\le k\le n, t_k\le t}|L(t_k)-L(t_{k-1})|^2}. \tag{6.153}$$

Claim: $Y(t)$ is an L-dominating process. Note that (6.11) is satisfied by triangle inequality, followed by Cauchy-Schwartz inequality, that is,

$$\begin{aligned}|L(t)-L(0)| &\le \sum_{1\le k\le n, t_k\le t}|L(t_k)-L(t_{k-1})|\\ &\le\sqrt{n\sum_{1\le k\le n, t_k\le t}|L(t_k)-L(t_{k-1})|^2} = Y(t).\end{aligned}\tag{6.154}$$

From the stationary increments,

$$\begin{aligned}E^Q[Y^q(t)] =&E^Q\left[\left(n\sum_{1\le k\le n, t_k\le t}|L(t_k)-L(t_{k-1})|^2\right)^{q/2}\right]\\ =&E^Q\left[\left(n\sum_{1\le k\le n, t_k\le t}|L(t_k-t_{k-1})|^2\right)^{q/2}\right]\\ =&n^{q/2}E^Q\left[\left(\sum_{1\le k\le n, t_k\le t}|L(t_k-t_{k-1})|^2\right)^{q}\right]^{1/2}.\end{aligned}\tag{6.155}$$

Then by Minkowski's inequality,

$$\begin{aligned}E^Q\left[\left(\sum_{1\le k\le n, t_k\le t}|L(t_k-t_{k-1})|^2\right)^{q}\right]^{1/q} \le& \sum_{1\le k\le n, t_k\le t}E[|L(t_k-t_{k-1})|^{2q}]^{1/q}\\ \le& nE^Q\left[\sup_{s\in[0,t]}|L(s)|^{2q}\right]^{1/q}.\end{aligned}\tag{6.156}$$

Hence,

$$E^Q[Y^q(t)] \leq n^q E^Q \left[\sup_{s\in[0,t]} |L(s)|^{2q} \right]^{1/2}. \tag{6.157}$$

The upper bound for inequality on the RHS of the last equation is evaluated similar to the continuous monitoring case where q is replaced by $2q$. Hence, to satisfy (6.12) we can pick $\alpha(q) = q/2$. Hence, $Y(t)$ is an L-dominating process. Moreover, since $Y(t) \in \mathbb{D}^\infty$, then it satisfies $R(q)$.

6.8 Example: Merton Model

We take a look again at the Merton model in Chapter 4.3.3. From the risk-neutral dynamics of the Merton model, (4.152) we have the following solution

$$S_1(t) = S_0 \exp(rt + L(t)) \tag{6.158}$$

where $L(t)$ is of the form (6.2) where

$$\begin{aligned} b &= -\frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (z - (e^z - 1))\, \nu(dz) \\ &= -\frac{\sigma^2}{2} + \lambda \left[m - \left(e^{m+\delta^2/2} - 1 \right) \right]. \end{aligned} \tag{6.159}$$

The numerical computation of Delta depends on $Y(s)$. Consider the expression of the weights Π_Δ in (6.110). To make the numerical computation explicit, it suffice to compute for $D_t^W Y(s)$.

6.8.1 Continuous Monitoring

Extrema Process

$$Y(s) = \sup_{u\in[0,s]} (L(u) - L(0)) - \inf_{u\in[0,s]} (L(u) - L(0)) \tag{6.160}$$

Denote the following running supremum and infimum

$$M^L(s) = \sup_{u\in[0,s]} L(s) \qquad m^L(s) = \inf_{u\in[0,s]} L(s). \tag{6.161}$$

Then, $D_t^W Y(s)$ is computed for $s \le t$ as follows:

$$D_t^W Y(s) = \sigma \left(\mathbf{1}_{\{\tau^M(s) \le t\}} - \mathbf{1}_{\{\tau^m(s) \le t\}} \right) \tag{6.162}$$

where

$$\begin{aligned} \tau^M(s) =& \inf\{t \in [0,s] : L(t) \vee L(t^-) = M^L(s)\}, \\ \tau^m(s) =& \inf\{t \in [0,s] : L(t) \wedge L(t^-) = m^L(s)\}. \end{aligned} \tag{6.163}$$

6.8.2 Discrete Monitoring

For $s \in [0,T]$, there exists $l \in \{0, \cdots, n\}$ such that

$$0 = t_0 < t_1 < \cdots < t_l \le s < \cdots \le t_n = T. \tag{6.164}$$

In particular, for an equally spaced partition, we have

$$t_k = \frac{kT}{n}, \qquad k \in \{0, \cdots, n\}. \tag{6.165}$$

In this case, $l = \left\lfloor \frac{s}{T/n} \right\rfloor$. Also, we denote

$$M(s) = \sup_{0 \le k \le l, t_k \le s} L(t_k), \qquad m(s) = \inf_{0 \le k \le l, t_k \le s} L(t_k). \tag{6.166}$$

- Extrema Process

$$\begin{aligned} Y(s) =& \sup_{0 \le k \le l, t_k \le s} (L(t_k) - L(0)) - \inf_{0 \le k \le l, t_k \le s} (L(t_k) - L(0)) \\ =& M(s) - m(s) \end{aligned} \tag{6.167}$$

Then, $D_t^W Y(s)$ is computed for $s \le t$ as follows:

$$D_t^W Y(s) = \sigma \sum_{k=1}^{l} \left(\mathbf{1}_{\{A_{l,k}\}} - \mathbf{1}_{\{a_{l,k}\}} \right) \mathbf{1}_{\{t_k \le t\}}. \tag{6.168}$$

where

$$\begin{aligned} A_{l,1} =& \{M(s) = L(t_1)\}, \\ A_{l,k} =& \{M_l \neq L(t_1), \cdots, M(s) \neq L(t_{k-1}), M(s) = L(t)\}, \quad 2 \le k \le l, \\ a_{l,1} =& \{m(s) = L(t_1)\}, \\ a_{l,k} =& \{m(s) \neq L(t_1), \cdots m(s) \neq L(t_{k-1}), m(s) = L(t)\}, \quad 2 \le k \le l. \end{aligned} \tag{6.169}$$

- Averaged Quadratic Increments Process

$$Y(s) = \sqrt{l \sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1}))^2}. \tag{6.170}$$

For $s \in [0, t_1)$, $Y(s) = 0$ and thus, $D_t^W Y(s) = 0$. On the other hand, for $s \in [t_1, T]$, from chain rule, we obtain:

$$D_t^W Y(s) = \frac{1}{2Y(s)} D_t^W \left(l \sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1}))^2 \right). \tag{6.171}$$

Now since

$$\begin{aligned}
&D_t^W \left(\sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1}))^2 \right) \\
=&2 \sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1})) D_t^W (L(t_k) - L(t_{k-1})) \\
=&2\sigma \sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1})) \left(\mathbf{1}_{\{t\le t_k\}} - \mathbf{1}_{\{t\le t_{k-1}\}} \right) \\
=&2\sigma \sum_{1\le k\le l, t_k \le s} (L(t_k) - L(t_{k-1})) \mathbf{1}_{\{t_{k-1}<t<t_k\}} \\
=&2\sigma L(t_j)
\end{aligned} \tag{6.172}$$

where $j = \inf\{k : t < t_k\}$. For an equally-spaced partition, $j = \left\lfloor \frac{t}{T/n} \right\rfloor$. Thus, for $s \in [t_1, T]$,

$$D_t^W Y(s) = \frac{\sigma l L(t_j)}{Y(s)}. \tag{6.173}$$

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APPENDICES

A. WIENER AND POISSON CHAOS EXPANSIONS

We review some of the important concepts in white noise Malliavin Calculus in both Wiener and pure jump (compensated Poisson random measure) cases [1], [24], [27], and [64]. We state the classical and the alternative chaos expansions for both the Wiener and Poisson case.

A.1 Hermite Polynomial and Hermite Function

The Hermite polynomial $h_n(x)$ is defined as follows:

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \tag{A.1}$$

Its generating function is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = \exp\left(tx - \frac{t^2}{2} \right). \tag{A.2}$$

The first few Hermite polynomials are as follows:

$$\begin{aligned} h_0(x) =& 1, \\ h_1(x) =& x, \\ h_2(x) =& x^2 - 1, \\ h_3(x) =& x^3 - 3x, \\ h_4(x) =& x^4 - 6x^2 + 3, \\ h_5(x) =& x^5 - 10x^3 + 15x. \end{aligned} \tag{A.3}$$

The Hermite polynomial has a weighted orthogonal property given by

$$\int_{\mathbb{R}} h_n(x) h_m(x) e^{-x^2/2} dx = \sqrt{2\pi} n! \delta_{mn}. \tag{A.4}$$

The Hermite function $e_n(x)$ is defined as follows:

$$e_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}e^{-x^2/2}h_{n-1}(\sqrt{2}x), \qquad n \in \mathbb{N}. \tag{A.5}$$

The first few Hermite functions are as follows:

$$\begin{aligned} e_1(x) =& \pi^{-1/4}e^{-x^2/2}, \\ e_2(x) =& \sqrt{2}\pi^{-1/4}xe^{-x^2/2}, \\ e_3(x) =& (\sqrt{2}\pi^{1/4})^{-1}(2x^2-1)e^{-x^2/2}, \\ e_4(x) =& (\sqrt{3}\pi^{1/4})^{-1}(2x^3-3x)e^{-x^2/2}, \\ e_5(x) =& (2\sqrt{6}\pi^{1/4})^{-1}(4x^4-12x^2+3)e^{-x^2/2}. \end{aligned} \tag{A.6}$$

Some important characterization of Hermite functions:

1. $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R})$,
2. $e_n \in \mathcal{S}(\mathbb{R})$,
3. $\sup_{x\in\mathbb{R}} |e_n(x)| = O(n^{-1/12})$,
4. $e_n(x) = O(n^{-1/4}) \quad \forall x \in \mathbb{R}$.

Remark A.1.1 *The Hermite functions also play a role in quantum mechanics. The relation*

$$\psi_n(x) \propto e_{n+1}(x), \quad n \in \mathbb{N}_0 \tag{A.7}$$

is an eigenfunction of the harmonic oscillator in the Schrödinger's time-independent wave equation [41].

A.2 Wiener Chaos Expansions

We let $g \in L^2([0,T]^n)$ and denote

$$I_n^W(g) = \int_{[0,T]^n} gW(dt)^{\otimes n} \tag{A.8}$$

be the n-fold iterated Itô integral with respect to the Wiener process.

Theorem A.2.1 *Wiener Chaos Expansion.*

Let $F \in L^2(P)$ *be* $\mathcal{F}_T$*-measurable, then there exists a unique sequence* $f_n \in L^2_s([0,T]^n)$, $\forall n \in \mathbb{N}_0$ *such that*

$$F = \sum_{n=0}^{\infty} I_n^W(f_n). \tag{A.9}$$

In addition, we have the following isometry relation:

$$\| F \|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \| f_n \|^2_{([0,T])^n} . \tag{A.10}$$

We let

$$\theta_k \equiv \int_{\mathbb{R}} e_k(t) dW(t)$$

$$H_\alpha^W = \prod_{k=1}^{n} h_{\alpha_k}(\theta_k) = I_n^W(e^{\hat{\otimes}\alpha}). \tag{A.11}$$

where $\{h_k\}_{k\in\mathbb{N}_0}$ and $\{e_k\}_{k\in\mathbb{N}}$ and is the Hermite polynomial and Hermite function respectively. Let $\mathcal{I}$ be the set of mulli-indices $\alpha = (\alpha_1, \cdots \alpha_n), n \in \mathbb{N}_0, n \in \mathcal{N}$. Also, we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Then, $\{H_\alpha^W\}_{\alpha\in\mathcal{I}}$ is an orthogonal basis for $L^2(P)$ with norm

$$\| H_\alpha^W \|^2_{L^2(P)} = \alpha!. \tag{A.12}$$

We now state the alternative Wiener chaos expansion.

Theorem A.2.2 *Let* $F \in L^2(P)$, *then there exists unique* $c_\alpha \in \mathbb{R}$ *such that*

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha^W. \tag{A.13}$$

In addition, we have the following isometry condition,

$$\| F \|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{I}} \alpha! c_\alpha^2. \tag{A.14}$$

A.3 Poisson Chaos Expansions

We let $g \in L^2(([0,T] \times \mathbb{R}_0)^n)$ and denote

$$I_n^{\tilde{N}}(g) = \int_{([0,T]\times\mathbb{R}_0)^n} g \tilde{N}(dt, dx)^{\otimes n} \tag{A.15}$$

be the n-fold iterated Itô integral with respect to the compensated Poisson random measure.

Theorem A.3.1 *Poisson Chaos Expansion.*
Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable, then there exists a unique sequence $f_n \in L_s^2([0,T] \times \mathbb{R}_0)$, $\forall n \in \mathbb{N}_0$ such that

$$F = \sum_{n=0}^{\infty} I_n^{\tilde{N}}(f_n). \tag{A.16}$$

In addition, we have the following isometry relation:

$$\| F \|_{L^2(P)}^2 = \sum_{n=0}^{\infty} n! \| f_n \|_{([0,T]\times\mathbb{R}_0)^n}^2 . \tag{A.17}$$

We assume that the Lévy measure ν satisfied the so-called Nualart-Schoutens assumption (3.23). Let $\{l_m\}_{m\in\mathbb{N}_0}$ be the orthogonalization of $\{z^m\}_{m\in\mathbb{N}_0}$ with respect to $L^2(\rho)$ where $\rho(dz) = z^2\nu(dz)$ (Note: This is different from the π_m presented in the canonical Lévy case). Define

$$p_m(z) \equiv \frac{z l_{m-1}(z)}{\| l_{m-1} \|_{L^2(\rho)}} \qquad m \in \mathbb{N} \tag{A.18}$$

which consists of an orthonormal basis functions in $L^2(\rho)$. Denote the Cantor diagonalization mapping $\kappa :\rightarrow \mathbb{N} \times \mathbb{N}$ as follows:

$$\kappa(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}. \tag{A.19}$$

Let $k = \kappa(i,j)$ and

$$d_k(t,z) = e_i(t) p_j(z) \tag{A.20}$$

then $\{d_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\lambda \times \rho)$. Suppose that $m = Index(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $n = |\alpha|$, define the following tensor product as follows:

$$\begin{aligned} & d^{\otimes\alpha}((t_1,z_1)\cdots(t_n,z_n)) \\ =& d_1^{\otimes\alpha_1} \otimes \cdots \otimes d_m^{\otimes\alpha_m}((t_1,z_1)\cdots(t_n,z_n)) \\ =& d_1(t_1,z_1)\cdots d_1(t_{\alpha_1},z_{\alpha_1})\cdots \qquad d_m(t_{n-\alpha_m+1},z_{n-\alpha_m+1})\cdots d_m(t_n,z_n) \end{aligned} \tag{A.21}$$

with the convention $d_i^{\otimes 0} = 1$, $i \in \{1, \cdots, m\}$. Also, we denote the following symmetrized tensor product

$$\begin{aligned} d^{\hat{\otimes}\alpha}((t_1,z_1)\cdots(t_n,z_n)) &= \left(d^{\otimes\alpha}((t_1,z_1)\cdots(t_n,z_n))\right)^{\wedge} \\ &= d_1^{\hat{\otimes}\alpha_1}\hat{\otimes}\cdots\hat{\otimes}d_m^{\hat{\otimes}\alpha_m}((t_1,z_1)\cdots(t_n,z_n)). \end{aligned} \tag{A.22}$$

and

$$H_\alpha^{\tilde{N}} = I_{|\alpha|}\left(d^{\hat{\otimes}\alpha}\right). \tag{A.23}$$

Then, $\{H_\alpha^{\tilde{N}}\}_{\alpha\in\mathcal{I}}$ is an orthogonal basis for $L^2(P)$ with norm

$$\| H_\alpha^{\tilde{N}} \|_{L^2(P)}^2 = \alpha!. \tag{A.24}$$

We now state the alternative Poisson chaos expansion.

Theorem A.3.2 *Let $F \in L^2(P)$, then it has a unique chaos expansion of the form*

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha^{\tilde{N}}. \tag{A.25}$$

In addition, we have the following isometry condition,

$$\| F \|_{L^2(P)}^2 = \sum_{\alpha\in\mathcal{I}} \alpha! c_\alpha^2. \tag{A.26}$$

VITA

VITA

ROLANDO DANGANAN NAVARRO, JR.

Date of Birth: 3 February 1978

Place of Birth: City of Manila, Philippines

RESEARCH INTERESTS

- Lévy Processes
- Long Memory Dependence
- Machine Learning
- Malliavin Calculus
- Mathematical Finance
- Optimal Execution Using Stochastic Optimal Control
- Signal Processing
- Stochastic Analysis
- Stochastic Volatility Modeling

EDUCATION

- Ph.D. Statistics, Purdue University, West Lafayette, IN 2011-2015
  Research: *Malliavin Calculus in the Canonical Lévy Process: White Noise Theory and Financial Applications.* Adviser: Frederi G. Viens, Ph.D.
- Certificate of Applied Management Principles (mini-MBA) -Purdue University, West Lafayette, IN 2014
- M.S. Statistics - University of the Philippines, Diliman,2004-2008 Research:*Estimating the Gauss-Markov Random Field Parameters for Remote Sensing Image Textures.* Adviser: Joselito C. Magadia, Ph.D., Enrico C. Paringit, D.Eng.

- B.S. Electrical Engineering University of the Philippines, Los Baños 1996-2001
  Research: *Recognition of Tagalog Alphabets Using the Hidden Markov Model*
  Adviser: Haidee P. Rosete

TEACHING EXPERIENCE

- Laboratory Instructor at the Undergraduate Level
  - STAT 301 - Elementary Statistics (Regular and Honors)
- Teaching Assistant to the following Graduate Level Courses
  - MA 515 / STAT 540 - Mathematics of Finance
  - MA 516 / STAT 541 - Adv. Probability & Options, w/ Numerical Methods
  - MA 539 / STAT 539 - Theory of Probability II
  - STAT 520 - Time Series and Applications
  - STAT 532 - Elementary Stochastic Processes

PROFESSIONAL EXPERIENCE

- Graduate Teaching Assistant (Aug 2011 to Present)
  Department of Statistics, Purdue University
  West Lafayette, IN, USA
  Evaluate students laboratory performance in Elementary Statistics, using SPSS; and homework covering graduate-level courses in Mathematical Finance, Time Series, Measure-Theoretic Probability, and Stochastic Processes.
- *Risk Analyst, Group Model Validation* (May 2010 - Jun 2011)
  Standard Chartered Bank
  Marina Bay Financial Centre, Singapore
  Identified modelling and SAS code development issues; formulated solutions for consumer banking models for Basel II compliance; and collaborated with Head of Methodology to improve PD and LGD estimation using Survival Analysis and Kalman Filtering.

- *Data Analytics Engineer* (May 2008 to Apr 2010)
  Vision Analytics, Inc. (formerly Vinta Systems, Inc.)
  Makati City, Philippines
  Analyzed the performance of machine learning algorithms, such as Support Vector Machines and Neural Networks for application and behavioral credit scoring. Recommended statistical analysis tools to develop credit scorecard for RCBC Savings Bank.

- *Design Engineer* (Jan 2003 to Apr 2008)
  Luxembourg Electronics, Inc.
  Makati City, Philippines
  Responsible for the testing and installation of single and multimode fiber optic systems on a family-run business.

- *Science Research Specialist*, DSP Team (Apr 2002 to Dec 2002)
  Advanced Science and Technology Institute, Dept. of Science and Technology
  Quezon City, Philippines
  Investigated the performance and stability of timing synchronization of $\pi/4$-DQPSK modulation for baseband modem design

SKILLS

- Computing: C/C++, Python, SQL, Unix, Matlab, Excel-VBA, SPSS, SAS, R, Mathematica, LaTeX, Algorithmic Trading.
- Finance: Options and Fixed Income, Time Series, Market and Credit Risk, Portfolio Optimization, Macroeconomics.
- Mathematics: Numerical Analysis, Optimization, Linear Algebra, Stochastic Calculus, PDE, Stochastic Optimal Control.
- Statistics: Machine Learning, Bayesian Analysis, Signal Processing, Multivariate Analysis, Mathematical Statistics, Monte Carlo.

NOTABLE ACHIEVEMENTS

- Frederick Andrews Fellowship, Purdue University, 2011-2015.
- Best M.S. Statistics Thesis, Philippine Council for Advanced Science and Technology Research and Development, 2008.
- MS Thesis Fellowship Grant, Statistical Research and Training Center (Philippines), 2007.
- Rank 13th Philippine Registered Electrical Engineering Board Licensure Examination (REE License Number 28215), 2001.
- Philippine delegate to the 36th International Mathematical Olympiad, York University, Toronto, Canada, 1995.
- 2nd Place - Philippine Mathematical Olympiad, National Level, 1995.

JOURNAL PUBLICATIONS

- R. Navarro and F. Viens. White Noise Analysis in the Canonical Levy Process, to be submitted in *Communications in Stochastic Analysis.*
- R. Navarro, R. Tamangan, N. Natan-Guba, E. Ramos, and A. de Guzman. Identification of the Long Memory Process in the ASEAN-4 Stock Markets by Fractional and Multifractional Brownian Motion, *Philippine Statistician,* 55(2):65-83, 2006.

BOOK CHAPTER

- R. Navarro, J. Magadia, and E. Paringit. Estimation of the Separable MGMRF Parameters for Thematic Classification. In *Remote Sensing - Advanced Techniques and Platforms,* B. Escalante (ed.), 2012.

CONFERENCE PRESENTATIONS

- R. Navarro. White Noise Analysis in the Canonical Levy Process , Probability Seminar, Purdue University, 2015.
- R. Navarro. Canonical Levy Process in Finance Using White Noise Analysis, Computational Finance Seminar, Purdue University, 2015.

- R. Navarro. *Mean-Variance Hedging with Partial Information Using the Clark-Ocone Representation with the Change of Measure for Lévy Process,* Barcelona GSE Summer Forum: Statistics, Jump Processes and Malliavin Calculus, 2014.
- R. Navarro, J. Magadia, and E. Paringit. *Estimating the Gauss-Markov Random Field Parameters for Remote Sensing Image Textures,* IEEE TENCON 2009, Singapore, 2009.
- R. Navarro. *Recognition of Tagalog Alphabets Using The Hidden Markov Model,* 10th National Convention on Statistics, Manila, 2007.
- R. Navarro and J. R. Albert. *A Compound Gauss-Markov Random Field Modeling of Philippine Unemployment Data,* Physics Society of the Philippines National Congress, Ateneo de Davao University, 2006.
- R. Navarro and J. Noche. *Classification of Mixtures of Student Grade Distribution Based on the Gaussian Mixture Model Using the Expectation-Maximization Algorithm,* 4th National ECE Conference, University of San Carlos, Cebu City, Philippines, 2003.
- R. Navarro, C. G. Santos, and A. Manlapat.*Performance Analysis of Gardner Timing Error Detector Over* $\pi/4$*-DQPSK Modulation,* 3rd National ECE Conference, University of the Philippines, Diliman, Quezon City, Philippines, 2002.

PEER REVIEW ACTIVITIES

- Referee - The IET Signal Processing

FUNDED PARTICIPATION IN CONFERENCES

- AMS Mathematics Research Communities Financial Mathematics, Snowbird, UT, Jun 2015.
- High Frequency Conference, Stevens Institute of Technology, Hoboken, NJ, (Jul 2012 and Oct 2013).
- ASA Joint Statistical Meetings Diversity Workshop San Diego, CA, (Jul-Aug 2012)

LIFETIME MEMBERSHIPS OF HONOR SOCIETIES

- Golden Key International Honour Society (since March 2013)
- Phi Kappa Phi International Honor Society (since April 2007)

PROFESSIONAL AFFILIATION

- American Mathematical Society
- American Statistical Association
- Society of Industrial and Applied Mathematics

LEADERSHIP ACTIVITIES

- Student Team Leader
  2015 Rotman International Trading Competition, (Dec 2014 to Feb 2015)
  Spearheaded training of 6-member team in futures, commodities, and algorithmic trading events.
- President
  Purdue Quantitative Finance Club (PQFC), (Feb 2014 to May 2015)
  Revitalized student interest on Quantitative Finance by initiating seminars and organizing trading research group.
- Senator
  Purdue Graduate Student Government, (Jun 2013 - May 2014).
- Founding President
  UPLB Society of Electrical Engineering Students, (Jun 1997 - Nov 1998).