Single spike solutions for strings on S-2 and S-3

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Single spike solutions for strings on S^2 and S^3

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We study solutions for rigidly rotating strings on a two-sphere. Among them we find two limiting cases that have a particular interest, one is the already known giant magnon and the other we call the single spike solution. The limiting behavior of this last solution is a string infinitely wrapped around the equator. It differs from that solution by the existence of a single spike of height $\bar{\theta}$ that points toward the north pole. We study its properties and compute its energy E and angular momentum J as a function of $\bar{\theta}$. We further generalize the solution by adding one angular momentum to obtain a solution on S^3. We find a spin chain interpretation of these results in terms of free fermions and the Hubbard model but the exact relation with the same models derived from the field theory is not clear.

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I. INTRODUCTION

The idea of a large-N duality [1] is a promising one for understanding the strong coupling limit of gauge theories. The first example of a concrete duality in 3 + 1 dimensions was provided by the AdS/CFT correspondence [2–4] which conjectures that $\mathcal{N} = 4$ super Yang-Mills theory in 3 + 1 dimensions is dual to IIB string theory on $AdS_5 \times S^5$. Strings states appear in the field theory [5,6] as long gauge-invariant operators. At this time, the string part of the correspondence is mostly understood at the classical level and for that reason classical solutions play an important role in testing and unraveling the correspondence. Multispin solutions [7,8] can be compared with particular field theory operators that can be represented as spin chains [9]. The energy of the string agrees precisely with the conformal dimension of the corresponding operators [10,11]. In fact one finds that spin chains are also connected to string theory [12–14] by the fact that the classical action of a spin chain can be interpreted as the action of a string. In fact, these spin chains are exactly the same as the ones arising from the field theory and can be used to derive the string sigma model, in the limit of a fast moving string, directly from the field theory [12].

A different type of solution are those rotating in AdS_5 one of which is the spiky string [15] which generalizes the rotating string of [6] and describes higher twist operators from the field theory point of view. These spiky strings where generalized to the sphere in [16]. A particular limit of this solution, known as the giant magnon [17], was identified with spin waves of short wavelength [17] opening new possibilities and giving rise to various interesting results [18–40].

In a particular sector with $SU(2)$ symmetry the field theory description of the operators, at one loop in perturbation theory, is in terms of the ferromagnetic spin $\frac{1}{2}$ Heisenberg model. In [41] it was conjectured that the Hubbard model was the appropriate generalization at all couplings. In this conjecture an important role is played by the antiferromagnetic state [41,42]. Later, in [43] it was proposed that the antiferromagnetic state is described by a string wound around the equator a large number of times. One important point is that, when going from small to large coupling, the Hubbard model interpolates between the Heisenberg model and free fermions. In this paper we find string solutions that look like an infinitely wound string with a spike pointing toward the north pole of the sphere. They should correspond to the free fermion states predicted by the Hubbard model which, at first sight, seems to be what we find. However, although the coefficients of the hopping term matches what the Hubbard model proposed from the field theory side, we require an extra term, proportional to the total fermion number, not present in the field theory side. Furthermore, analysis of the two angular momentum solution reveals a surprising dependence in the extra angular momentum which suggest an interpretation in terms of elementary excitations of fixed energy and not fixed angular momentum. In fact we can find a spin chain interpretation of the solutions in terms of the Hubbard model if we match the energy of the Hubbard model with the angular momentum of the string and the momentum in the Hubbard chain with the difference between energy and winding number in the string. Although these reproduce the string results, such identification is not the one resulting from the field theory calculation. The solutions therefore display the same rich behavior of the giant magnon but its direct relation to the field theory is unclear.

This paper is organized as follows. In the next section we describe briefly the spiky string and its T-dual in flat space. In Sec. III we propose an ansatz for a rigid string rotating on S^2 in static gauge and discuss the different possibilities according to the values of the constants of motion. In Sec. IV we consider two limiting solutions: one is the giant magnon and the other is the single spike. In Sec. V we repeat the calculation in conformal gauge and discuss its relation to the sine-Gordon model. After that, in Sec. VI, we generalize the solution to a string rotating on S^3 with

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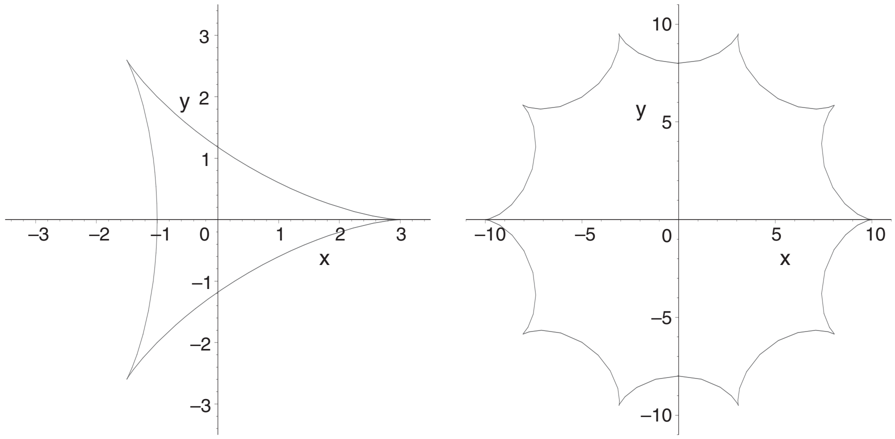

FIG. 1. Spiky strings in flat space. We show two examples with three and ten spikes, respectively.

two angular momenta. In Sec. VII we discuss the interpretation of the results in terms of spin chains. We find a description in terms of free fermions and in terms of the Hubbard model which, however, do not seem to be directly related to the field theory. Finally we give our conclusions in Sec. VIII.

II. SPIKY STRINGS IN FLAT SPACE AND THE T-DUAL SOLUTION

In [15] the spiky string classical solution was introduced.[1] It is a rigidly rotating string with spikes as shown in Fig. 1. It was generalized to the sphere in [16]. It can be described also in conformal gauge as shown in [46] where a further generalization to S^5 was constructed in terms of solutions of the Neumann-Rosochatius system [47–51]. A limit of the spiky string, known as the giant magnon, is of special importance [17].

In this section we consider a T-dual solution to the flat space spiky string which, as we see, can be generalized in precisely the same way. We start by the usual spiky string in flat space which, in conformal gauge is given by

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos(\sigma_-), \tag{2.1}$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin(\sigma_-), \tag{2.2}$$

$$t = 2A(n-1)\tau = A(n-1)(\sigma_+ + \sigma_-), \tag{2.3}$$

where $\sigma_+ = \tau + \sigma$, $\sigma_- = \tau - \sigma$, n is an integer and A is a constant which determines the size of the string. It satisfies the equations of motion:

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = \partial_{\sigma_+}\partial_{\sigma_-}X^i = 0, \tag{2.4}$$

and the constraints

$$(\partial_{\sigma_+}X)^2 = (\partial_{\sigma_-}X)^2 = 0. \tag{2.5}$$

The cases of $n = 3$ and $n = 10$ are depicted in Fig. 1.

Given a solution in flat space one can always construct a T-dual solution by changing the sign of the left movers in one of the coordinates. Doing that we get a new solution:

$$x = A\cos[(n-1)\sigma_+] - A(n-1)\cos(\sigma_-), \tag{2.6}$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin(\sigma_-), \tag{2.7}$$

$$t = 2A(n-1)\tau = A(n-1)(\sigma_+ + \sigma_-), \tag{2.8}$$

that satisfies the same equations (2.4) and constraints (2.5). The cases $n = 3$ and $n = 10$ are depicted in Fig. 2. We intend to generalize these solutions to the sphere and consider a limit similar to the one that leads to the giant magnon.

III. RIGIDLY ROTATING STRINGS ON S^2

The Nambu-Goto action is given by

$$\begin{aligned} S &= T\int d\sigma d\tau \mathcal{L} \\ &= T\int d\sigma d\tau \sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2}, \end{aligned} \tag{3.1}$$

where $T = \sqrt{\lambda}/2\pi$ is the string tension. The space time metric is set as:

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\phi^2 \tag{3.2}$$

$$= G_{\mu\nu}(X)dX^\mu dX^\nu, \tag{3.3}$$

We choose the parametrization

$$t = \kappa\tau, \tag{3.4}$$

$$\phi = \omega\tau + \sigma, \tag{3.5}$$

$$\theta = \theta(\sigma). \tag{3.6}$$

[1] In flat space similar solutions were known in the context of cosmic strings [44,45]

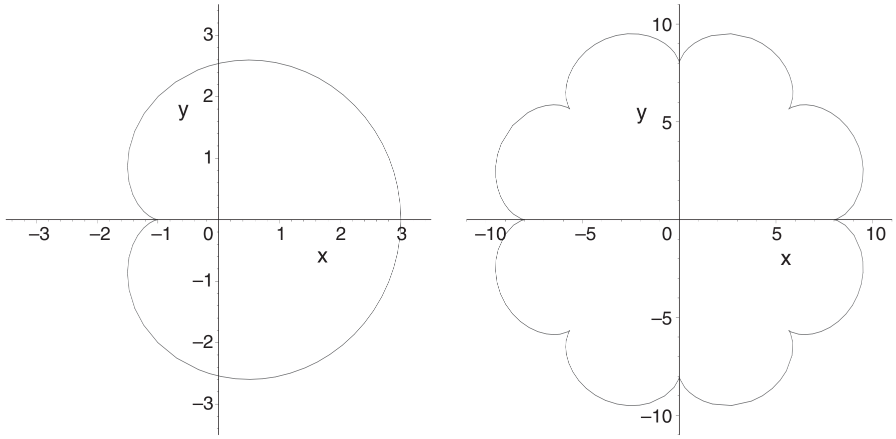

FIG. 2. T-duals of the solutions depicted in Fig. 1. The number of spikes changes by two when doing T-duality.

Then, we can obtain the following equations of motion for (3.1):

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial t'} + \partial_\tau \frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\partial \mathcal{L}}{\partial t}, \tag{3.7}$$

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial \phi'} + \partial_\tau \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{3.8}$$

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial \theta'} + \partial_\tau \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}. \tag{3.9}$$

Solving (3.7), we obtain,

$$\theta' = \frac{\kappa \sin\theta}{C}\sqrt{\frac{\omega^2\sin^2\theta - C^2}{\kappa^2 - \omega^2\sin^2\theta}}, \tag{3.10}$$

where C is a constant. We also obtain the same solution by solving (3.8), and, furthermore, the solution satisfies (3.9). Next, we compute the energy, angular momentum, and the difference in angle between two spikes.

TABLE I. Four cases for the motion of a rigid string on S^2 depending on the values of the constants. Notice that the cases $|\omega| < |C| < \kappa$ and $|\omega| < \kappa < |C|$ are forbidden. The limiting cases $|\omega| = |C|$ or $|\omega| = \kappa$ are studied separately.

Case	Range	Conditions
(i)	$(\frac{C}{\omega})^2 < \sin^2\theta < 1$	$\lvert C\rvert < \lvert\omega\rvert < \kappa$
(ii)	$(\frac{C}{\omega})^2 < \sin^2\theta < (\frac{\kappa}{\omega})^2$	$\lvert C\rvert < \kappa < \lvert\omega\rvert$
(iii)	$(\frac{\kappa}{\omega})^2 < \sin^2\theta < 1$	$\kappa < \lvert\omega\rvert < \lvert C\rvert$
(iv)	$(\frac{\kappa}{\omega})^2 < \sin^2\theta < (\frac{C}{\omega})^2$	$\kappa < \lvert C\rvert < \lvert\omega\rvert$

The energy is

$$\begin{aligned} E &= 2T\int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'}\frac{\partial \mathcal{L}}{\partial \dot{t}} \\ &= 2T\int_{\theta_0}^{\theta_1} d\theta \frac{\omega(C^2-\kappa^2)\sin\theta}{\kappa\sqrt{(\kappa^2-\omega^2\sin^2\theta)(\omega^2\sin^2\theta - C^2)}}, \end{aligned} \tag{3.11}$$

the angular momentum is

$$J = 2T\int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2T\int_{\theta_0}^{\theta_1} d\theta \sin\theta\sqrt{\frac{\omega^2\sin^2\theta - C^2}{\kappa^2-\omega^2\sin^2\theta}}, \tag{3.12}$$

and the difference in angle between two spikes is

$$\Delta\phi = 2\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'} = 2\int_{\theta_0}^{\theta_1} d\theta\frac{C}{\kappa\sin\theta}\sqrt{\frac{\kappa^2-\omega^2\sin^2\theta}{\omega^2\sin^2\theta - C^2}}. \tag{3.13}$$

Requiring that the argument of the square root in (3.10) be positive, we find that the range of θ can be, $C^2/\omega^2 < \sin^2\theta < \kappa^2/\omega^2$, or instead $\kappa^2/\omega^2 < \sin^2\theta < C^2/\omega^2$. Furthermore, in the first case we can have (i) $\kappa^2/\omega^2 < 1$ or (ii) $\kappa^2/\omega^2 > 1$. In the second case we can have (iii) $C^2/\omega^2 < 1$ or (iv) $C^2/\omega^2 > 1$. The different possibilities are summarized in Table I.

In cases (i) and (ii) we can take the limit $|\omega| \to \kappa$ which gives rise to the giant magnon. In cases (iii) and (iv) we can take the limit $|\omega| \to |C|$ which gives rise to a solution we call the single spike and that we investigate in the rest of this paper.

IV. LIMITING CASES, GIANT MAGNON AND SINGLE SPIKE SOLUTION

In the two limiting cases, $|\omega| \to \kappa$ and $|\omega| \to |C|$ the equations simplify considerably and we can compute the solution in terms of standard functions. We analyze both limits independently.

A. First limiting case, giant magnon

Consider first case (ii) and define two angles:

$$\theta_0 = \arcsin\frac{C}{\omega}, \qquad \theta_1 = \arcsin\frac{\kappa}{\omega}, \tag{4.1}$$

such that $\theta_0 \leq \theta \leq \theta_1$. The limit $|\omega| \to \kappa$ corresponds to $\theta_1 \to \frac{\pi}{2}$. In that case we can integrate Eq. (3.10):

$$\int \frac{\sin\theta_0 \cos\theta d\theta}{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_0}} = \pm\sigma, \tag{4.2}$$

and obtain:

$$\pm\sigma = -\arcsin\left(\frac{\sin\theta_0}{\sin\theta}\right), \tag{4.3}$$

or

$$\sin\theta = \mp\frac{\sin\theta_0}{\sin\sigma}. \tag{4.4}$$

Now, we have to evaluate Eqs. (3.11), (3.12), and (3.13) in the limit, $\kappa \to \omega$. From (3.13), we obtain

$$\Delta\phi = 2\arccos\left(\frac{C}{\omega}\right) \Rightarrow \frac{\Delta\phi}{2} = \frac{\pi}{2} - \theta_0. \tag{4.5}$$

Equations (3.11) and (3.12), give a divergent value for the energy and angular momentum in this limit. The difference, however, is finite:

$$E - J = 2T\sin\frac{\Delta\phi}{2} = \frac{\sqrt{\lambda}}{\pi}\sin\frac{\Delta\phi}{2}, \tag{4.6}$$

which are the known relations for the giant magnon.

B. Second limiting case, single spike

Consider now case (iv) in the limit $|\omega| \to C$. Again we define

$$\theta_0 = \arcsin\frac{\kappa}{\omega}, \qquad \theta_1 = \arcsin\frac{C}{\omega}, \tag{4.7}$$

such that $\theta_0 \leq \theta \leq \theta_1$. Notice that the definition is different from (4.1) because the limits are interchanged. The limit $|\omega| \to C$ now also corresponds to $\theta_1 \to \frac{\pi}{2}$. Integrating Eq. (3.10) we get now

$$\frac{\cos^2\theta_0}{\sin\theta_0}\int \frac{\sin\theta d\theta}{\cos\theta\sqrt{\cos^2\theta_0 - \cos^2\theta_0}} - \sin\theta_0\int\frac{\cos\theta d\theta}{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_0}} = \pm\sigma, \tag{4.8}$$

which gives

$$\pm\sigma = \frac{\cos\theta_0}{\sin\theta_0}\operatorname{arccosh}\left(\frac{\cos\theta_0}{\cos\theta}\right) - \arccos\left(\frac{\sin\theta_0}{\sin\theta}\right). \tag{4.9}$$

In this case, we use (3.12) first to obtain

$$J = 2T\frac{\kappa}{\omega}\sqrt{\left(\frac{\omega}{\kappa}\right)^2 - 1} = 2T\cos\theta_0. \tag{4.10}$$

On the other hand, Eqs. (3.11) and (3.13) diverge. By combining them, we obtain the finite result

$$E - T\Delta\phi = 2T\left(\frac{\pi}{2} - \theta_0\right). \tag{4.11}$$

The solution is plotted schematically in Fig. 3. We see that it is a string wrapped around the equator an infinite number of times and with a single spike whose tip is at $\theta = \theta_0$. We define the height of the spike $\bar{\theta}$ as

$$\bar{\theta} = \frac{\pi}{2} - \theta_0. \tag{4.12}$$

In terms of the height we can rewrite

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi}\bar{\theta}, \tag{4.13}$$

$$J = \frac{\sqrt{\lambda}}{\pi}\sin\bar{\theta}, \tag{4.14}$$

implying

$$\Delta = (E - T\Delta\phi) - J = \frac{\sqrt{\lambda}}{\pi}(\bar{\theta} - \sin\bar{\theta}), \tag{4.15}$$

which is our final expression for the energy of the single spike solution on S^2. We define Δ by subtracting the "free" anomalous dimension J from the energy. We also need to subtract the winding number $T\Delta\phi$ to get a finite result.

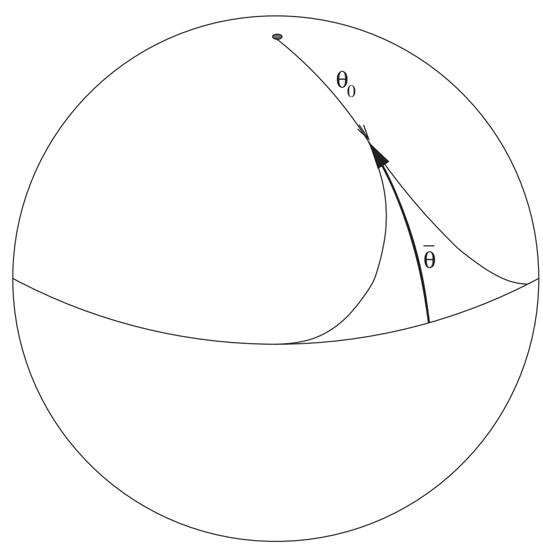

FIG. 3. Single spike solution.

V. CONFORMAL GAUGE AND SINE-GORDON MODEL

It is useful to obtain also the solution in conformal gauge. One simple way to do that is to use the results of [46] where the spiky string was studied in conformal gauge. In fact, in that paper, after Eq. (3.3) it is noted that two possibilities arise, which are described as $\kappa = \omega_1$ and $\kappa = C_1$. In [46] the first is shown to lead to the giant magnon. The second possibility actually leads to the single spike solution that we explore here. Let us summarize the points of [46] that we need to use. A string moving on S^5 is described in terms of three complex coordinates $X_{a=1\ldots3}$ with metric:

$$ds^2 = -dt^2 + \sum_a dX_a d\bar{X}_a, \qquad \sum_a |X_a|^2 = 1. \tag{5.1}$$

Then the ansatz

$$X_a = x_a(\xi)e^{i\omega_a\tau}, \qquad \xi \equiv \alpha\sigma + \beta\tau, \tag{5.2}$$

is proposed. The complex functions $x_a(\xi)$ are further parametrized as

$$x_a(\xi) = r_a(\xi)e^{i\mu_a(\xi)}, \tag{5.3}$$

where r_a, μ_a are real. The equations for the phases μ_a can be integrated giving

$$\mu'_a = \frac{1}{\alpha^2 - \beta^2}\left[\frac{C_a}{r_a^2} + \beta\omega_a\right], \tag{5.4}$$

where C_a are constants of motion. The equation of motion for the r_a can then be derived from the Lagrangian

$$\mathcal{L} = \sum_a\left[(\alpha^2 - \beta^2)r_a'^2 - \frac{1}{\alpha^2 - \beta^2}\frac{C_a^2}{r_a^2} - \frac{\alpha^2}{\alpha^2 - \beta^2}\omega_a^2 r_a^2\right] + \Lambda\Big(\sum_a r_a^2 - a\Big), \tag{5.5}$$

where Λ is a Lagrange multiplier. The Hamiltonian is

$$H = \sum_a\left[(\alpha^2 - \beta^2)r_a'^2 + \frac{1}{\alpha^2 - \beta^2}\frac{C_a^2}{r_a^2} + \frac{\alpha^2}{\alpha^2 - \beta^2}\omega_a^2 r_a^2\right], \tag{5.6}$$

and the constraints are satisfied if

$$\sum_a \omega_a C_a + \beta\kappa^2 = 0, \qquad H = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2}\kappa^2. \tag{5.7}$$

This is quite generic. We are interested in motion on an S^2 so we need to consider only the case $X_3 = 0$, X_2 real which implies $\mu_2 = 0$ and $\omega_2 = 0$. Since we need $\mu'_2 = 0$, this also implies $C_2 = 0$. So the Hamiltonian reduces to

$$H = (\alpha^2 - \beta^2)(r_1'^2 + r_2'^2) + \frac{1}{\alpha^2 - \beta^2}\frac{C_1^2}{r_1^2} + \frac{\alpha^2}{\alpha^2 - \beta^2}\omega_1^2 r_1^2. \tag{5.8}$$

Since there is a constraint $r_1^2 + r_2^2 = 1$ we can parametrize $r_{1,2}$ as

$$r_1 = \sin\theta, \qquad r_2 = \cos\theta, \tag{5.9}$$

which gives

$$H = (\alpha^2 - \beta^2)\theta'^2 + \frac{C_1^2}{\alpha^2 + \beta^2}\frac{1}{\sin^2\theta} + \frac{1}{\alpha^2 - \beta^2}\omega_1^2\sin^2\theta. \tag{5.10}$$

Using conservation of energy and the constraints (5.7) we get

$$\theta' = \pm\frac{\omega_1}{(\alpha^2 - \beta^2)\sin\theta} \times\sqrt{(\sin^2\theta_0 - \sin^2\theta)(\sin^2\theta - \sin^2\theta_1)}, \tag{5.11}$$

where

$$\sin^2\theta_0 = \frac{\beta^2\kappa^2}{\omega^2}, \qquad \sin^2\theta_1 = \frac{\kappa^2}{\omega^2}. \tag{5.12}$$

We see that two possibilities arise: $\theta_0 < \theta < \theta_1$ or $\theta_1 < \theta < \theta_0$. In the limit $\omega \to \kappa$ we have $\theta_1 = \frac{\pi}{2}$, and in the limit $|\omega| \to |C_1|$ we have $\theta_0 = \frac{\pi}{2}$. Both cases can be integrated as before. We get for the giant magnon

$$\cos\theta = \frac{\cos\theta_0}{\cosh\xi}, \tag{5.13}$$

$$\mu_1 = -\frac{\cos\theta_0}{\sin\theta_0}\arctan\left(\frac{\cos\theta_0}{\sin\theta_0}\tanh\xi\right), \tag{5.14}$$

$$\phi = \mu_1 + \omega_1\tau, \tag{5.15}$$

$$\xi = \sigma + \sin\theta_0\tau, \qquad \omega = \kappa = \cos\theta_0, \tag{5.16}$$

where we chose α and β in an appropriate way to simplify the solution and θ, ϕ are the angles parametrizing the S^2. For the single spike solution we have

$$\cos\theta = \frac{\cos\theta_1}{\cosh\xi}, \tag{5.17}$$

$$\mu_1 = \frac{\cos\theta_1}{\sin\theta_1}\xi - \arctan\left(\frac{\cos\theta_1}{\sin\theta_1}\tanh\xi\right), \tag{5.18}$$

$$\phi = \mu_1 + \omega_1\tau, \tag{5.19}$$

$$\xi = \sigma\sin\theta_1 + \tau, \qquad \omega = -\mathrm{cotan}\theta_1, \qquad \kappa = \cos\theta_1, \tag{5.20}$$

where we now chose $\beta = 1$. Notice that for the giant magnon $\beta < \alpha$ whereas for the single spike $\alpha < \beta$. If we compute the energy, angular momentum and $\Delta\phi$ we get the same result as before.

As in [17] we can see a relation to the sine-Gordon model. In conformal gauge it is interesting to compute the determinant of the world-sheet metric given by

$$h = (-\kappa^2 + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)(\theta'^2 + \sin^2\theta\phi'^2) - (\dot{\theta}\theta' + \sin^2\theta\dot{\phi}\phi')^2 \tag{5.21}$$

$$= -(\theta'^2 + \sin^2\theta\phi'^2)^2, \tag{5.22}$$

where we used the conformal constraints. Replacing the solutions we obtain:

$$\frac{1}{\kappa^2}\sqrt{-h} = \frac{1}{\cosh^2\xi}, \qquad \text{giant magnon,} \tag{5.23}$$

$$\frac{1}{\kappa^2}\sqrt{-h} = 1 - \frac{1}{\cosh^2\xi}, \qquad \text{single spike.} \tag{5.24}$$

If we now define an angle through

$$\sin^2\Phi = \frac{1}{\kappa^2}\sqrt{-h} \tag{5.25}$$

we get

$$\Phi = \arcsin\left(\frac{1}{\cosh\xi}\right), \qquad \text{giant magnon,} \tag{5.26}$$

$$\Phi = \arcsin(\tanh\xi), \qquad \text{single spike.} \tag{5.27}$$

It is now easy to check that both angles satisfy the sine-Gordon equation:

$$(\partial_\tau^2 - \partial_\sigma^2)\Phi + \tfrac{1}{2}\kappa^2\sin(2\Phi) = 0. \tag{5.28}$$

Note however that they correspond to different potentials, since $\Phi(\pm\infty) = k\pi$ for the giant magnon and $\Phi(\pm\infty) = \frac{\pi}{2} + k\phi$ for the single spike. Another way of saying it is that σ and τ are interchanged when going from one to the other solution [because interchanging σ and τ is equivalent to changing the sign of the potential in (5.28)]. In any case, the relation to sine-Gordon should be useful to compute scattering of single spikes as in [17] although we do not pursue this here. Similarly, scattering of spikes in the spiky string [15] is determined by the sinh-Gordon model.

VI. STRINGS ON S^3, TWO ANGULAR MOMENTA

In a previous section, we analyzed the motion of the string on a two-sphere. In this section we add one more dimension to the sphere, and investigate the motion with an extra angular momentum.

We can use the same action (3.1) as in the previous section. The space time metric is now

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \tag{6.1}$$

$$= G_{\mu\nu}(X)dX^\mu dX^\nu. \tag{6.2}$$

This time, we choose the parametrization as

$$t = \kappa\tau, \tag{6.3}$$

$$\theta = \theta(\sigma), \tag{6.4}$$

$$\phi_1 = \omega_1\tau + \sigma, \tag{6.5}$$

$$\phi_2 = \phi_2(\sigma) + \omega_2\tau. \tag{6.6}$$

Then, we can obtain the following equations of motion for (3.1),

$$\partial_\sigma\frac{\partial\mathcal{L}}{\partial t'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{\partial\mathcal{L}}{\partial t}, \tag{6.7}$$

$$\partial_\sigma\frac{\partial\mathcal{L}}{\partial\theta'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta}, \tag{6.8}$$

$$\partial_\sigma\frac{\partial\mathcal{L}}{\partial\phi_1'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \frac{\partial\mathcal{L}}{\partial\phi_1}, \tag{6.9}$$

$$\partial_\sigma\frac{\partial\mathcal{L}}{\partial\phi_2'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = \frac{\partial\mathcal{L}}{\partial\phi_2}. \tag{6.10}$$

Solving (6.7) and (6.9) for ϕ_2', we obtain

$$\phi_2' = \frac{\sin^2\theta\{\kappa(\kappa C_1 - \omega_1 C_2) - \omega_2^2 C_1\cos^2\theta\}}{\omega_2\cos^2\theta(\kappa C_2 - \omega_1 C_1\sin^2\theta)}, \tag{6.11}$$

where C_1 and C_2 are constants. The equation for θ' is rather complicated so, before considering it we choose the constants of motion appropriately such that

$$\theta' \to 0 \quad \text{as } \theta \to \frac{\pi}{2}. \tag{6.12}$$

We obtain the following values of the constants:

$$C_1 = \omega_1 \quad \text{and} \quad C_2 = \kappa. \tag{6.13}$$

Now, with these values we obtain simpler equations:

$$\phi_2' = \frac{\omega_1\omega_2\sin^2\theta}{\omega_1^2\sin^2\theta - \kappa^2}, \tag{6.14}$$

$$\theta' = \frac{\kappa\sin\theta\cos\theta}{\omega_1^2\sin^2\theta - \kappa^2}\sqrt{(\omega_1^2 - \omega_2^2)\sin^2\theta - \kappa^2}. \tag{6.15}$$

As in the previous section, we compute the energy, two angular momenta, and the difference in angle between the two end points of the string using Eqs. (6.14) and (6.15).

The energy is

$$E = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{t}} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{(\omega_1^2-\kappa^2)\sin\theta}{\kappa\cos\theta\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2}}, \tag{6.16}$$

the first angular momentum is

$$J_1 = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_1\sin\theta\cos\theta}{\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2}}, \tag{6.17}$$

the second angular momentum is

$$J_2 = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_2\sin\theta\cos\theta}{\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2}}, \tag{6.18}$$

and the difference in angle between the two end points of the strings is

$$\Delta\phi = 2\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'} = 2\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_1^2\sin^2\theta-\kappa^2}{\kappa\sin\theta\cos\theta\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2}}, \tag{6.19}$$

where $\theta_0 = \arcsin(\kappa/\sqrt{\omega_1^2-\omega_2^2})$ with $\omega_1^2 > \omega_2^2$, and $\theta_1 = \pi/2$. Here, θ_0 is chosen such that the inside square root of (6.15) is positive. Since $\arcsin(\kappa/\omega_1) < \arcsin(\kappa/\sqrt{\omega_1^2-\omega_2^2}) < \pi/2$, θ can never reach a value such that $\sin\theta = \kappa/\omega_1$. Thus, in this case, θ' does not go to infinity at any point. Also, we remind the reader that the tension T is $T = \frac{\sqrt{\lambda}}{2\pi}$.

Doing the integrals we find the following results:

$$E - T\Delta\phi = 2T\bar{\theta}, \tag{6.20}$$

$$J_1 = 2T\frac{1}{\cos\gamma}\sin\bar{\theta}, \tag{6.21}$$

$$J_2 = 2T\frac{\sin\gamma}{\cos\gamma}\sin\bar{\theta}, \tag{6.22}$$

where we defined

$$\bar{\theta} = \frac{\pi}{2}-\theta_0, \qquad \sin\theta_0 = \frac{\kappa}{\sqrt{\omega_1^2-\omega_2^2}}, \qquad \sin\gamma = \frac{\omega_1}{\omega_2}. \tag{6.23}$$

The result can also be written as

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi}\bar{\theta}, \tag{6.24}$$

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\bar{\theta}}. \tag{6.25}$$

In the case of $J_2 = 0$ we recover the expressions (4.14) for the energy and angular momentum of the single spike on S^2.

VII. SPIN CHAIN INTERPRETATION

It has become standard that, from the string solutions, only the parts of the string that move almost at the speed of light have a simple interpretation in the field theory. This was valid in the Berenstein-Maldacena-Nastase case [5] and also when mapping the string and spin chain actions [12] or in the spiky strings [15]. The case of the giant magnon [17] might seem different but in fact, in such a solution, the sigma coordinate spans an infinite range, most of which corresponds to the string close to the equator moving at the speed of light. In our case, the only part of the string moving at the speed of light is the spike which one might think could be interpreted as in the case of the spiky string. In principle, one might hope for more because the infinitely wound string was associated with the antiferromagnetic state in [43] and our solution is a perturbation of that. However it is not completely clear to us how the relation to the antiferromagnetic state works so, in this paper, we do not pursue this avenue any further.

Summarizing, given the fact that the string does not move at the speed of light, we anticipate a difficulty in mapping the solutions to the field theory. On the other hand, it turns out to be rather simple to find a spin chain interpretation of the results, if, at this stage, we do not require a field theory derivation of such spin chain. In fact, if we look at the string with two angular momenta whose energy and angular momentum are given by (6.25):

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi}\bar{\theta}, \tag{7.1}$$

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\bar{\theta}}, \tag{7.2}$$

we find that it is very similar to the giant magnon result [17–19] if we interpret $\bar{\theta}$ as half the momentum of the magnon $\bar{\theta} = \frac{1}{2}p$. The main difference is that we should interpret the energy of the magnon as J_1 and not E which is rather surprising. Also we have an extra quantity, namely $E - T\Delta\phi$ that we computed and should be interpreted as the momentum of the magnon. Therefore, a spin chain interpretation of the result is to take a Hubbard chain and identify the Hamiltonian of the chain with the angular momentum of the string and the momentum in the chain with the difference $E - T\Delta\phi$ in the string. This can be loosely understood as a σ, τ interchange which is what we also saw in the relation to the sine-Gordon model. Although this allows us a spin chain interpretation of the solutions and shows that their dynamics is as rich as the one of the rotating strings, the problem is that we did not derive the spin chain from the field theory. We leave the important problem of mapping these solutions to operators in $\mathcal{N} = 4$ SYM theory for the future. An important clue might be the ideas of [43].

Fermi sea.—We now point to a curious fact about the solution on S^2 that suggests a different but related spin chain interpretation. Since it does not generalize to the S^3 case we believe it to be an interesting but very particular interpretation.

In [41] the Hubbard model was proposed as a way to determine the conformal dimension of various operators in the field theory. The Hamiltonian is given by

$$H = -t \sum_{i=-\infty\ldots\infty,\alpha=\uparrow,\downarrow} (c^{\dagger}_{i,\alpha}c_{i+1,\alpha} + c^{\dagger}_{i+1,\alpha}c_{i,\alpha}) + U \sum_{i=-\infty\ldots\infty} c^{\dagger}_{i\uparrow}c_{i\uparrow}c^{\dagger}_{i\downarrow}c_{i\downarrow}, \tag{7.3}$$

where

$$U = -\frac{1}{2}, t = -\frac{\sqrt{\lambda}}{8\pi}. \tag{7.4}$$

The Hamiltonian corresponds to a system of electrons living in a one dimensional lattice whose sites are labeled by the index i. There is a hopping term proportional to t and an on-site repulsion modeling the Coulomb repulsion. The electron has two states, spin up and spin down labeled by the index $\alpha = \uparrow, \downarrow$. In the limit of small coupling $\lambda \ll 1$ the second term can be ignored and we get a system of free fermions. The energy of a single fermion is given in terms of its momentum by

$$\epsilon(k) = -2t\cos k. \tag{7.5}$$

We see that there are fermionic states with negative energy so the ground state is half-filled. It turns out that for our purpose it is necessary to add a chemical potential μ such that the ground state is empty. We take then the Hamiltonian to be

$$\tilde{H} = -t \sum_{i=-\infty\ldots\infty,\alpha=\uparrow,\downarrow} (c^{\dagger}_{i,\alpha}c_{i+1,\alpha} + c^{\dagger}_{i+1,\alpha}c_{i,\alpha} + 2c^{\dagger}_{i,\alpha}c_{i,\alpha}) + U \sum_{i=-\infty\ldots\infty} c^{\dagger}_{i\uparrow}c_{i\uparrow}c^{\dagger}_{i\downarrow}c_{i\downarrow}, \tag{7.6}$$

namely, $\mu = 2t$ if we take $\tilde{H} = H - \mu N$ with N the number of fermions. The new single particle energy is now

$$\epsilon(k) = -2t(1 + \cos k), \tag{7.7}$$

which is always positive (since $t < 0$). It is also convenient to define $\kappa = \pi - k$ since the lowest energy state has $k = \pi$. We get

$$\epsilon(\kappa) = -2t(1 - \cos\kappa). \tag{7.8}$$

If we now fill the states up to some Fermi momentum κ_F the total energy is given by

$$\mathcal{E} = -8t \int_0^{\kappa_F} (1 - \cos\kappa) d\kappa = -8t(\kappa_F - \sin\kappa_F) = \frac{\sqrt{\lambda}}{\pi}(\kappa_F - \sin\kappa_F), \tag{7.9}$$

where we introduced a degeneracy factor of 4, two because the states with momentum κ and $-\kappa$ have the same energy and another two because of the spin degeneracy. If we identify $\bar{\theta} = \kappa_F$ and $\Delta = \mathcal{E}$, then we get precisely Eq. (4.15). Tantalizingly it does so up to the coefficient. Notice that, in the previous subsection, we identified $\theta = \frac{1}{2}p$ where p is the momentum of the magnon. However, in this model a magnon is a bound state of two fermions, each with momentum $\frac{1}{2}p$. It seems then that, in both cases, $\bar{\theta}$ is related to some underlying fermionic momentum. Filling the Fermi sea corresponds to wrapping the string once more around the equator since the energy changes by $2\pi T$ when $\kappa_F = \pi$. Unfortunately, as mentioned before the interpretation in terms of a Fermi sea does not appear to generalize to the solution with two angular momenta and therefore we regard this interpretation as a curiosity. We mentioned it here because it might be useful for other purposes and also was the first interpretation we found.

VIII. CONCLUSIONS

In this paper we find and study new solutions for rigid strings moving on a sphere. They asymptote to a solution infinitely wrapped around the equator and at rest. Therefore they are fundamentally different from the rotating strings where the string moves close to the speed of light. They belong to the category of slowly moving strings described in [43]. Nevertheless we find that they have an interesting and rich dynamics that seems to be described by the same spin chains as their rotating strings counterparts. For example, the Hubbard model proposed in [41] that describes the two angular momentum giant magnon also appears useful to describe these solutions. The main difference is that we should map the angular momentum of the string to the energy of the spin chain and the difference between energy and wrapping number to the momentum of the spin chain. This makes difficult the interpretation of the spin chain in the field theory. We leave this last part for future and presumably difficult work. However other types of work seem more directly accessible, for example, it would be interesting to study the scattering of single spikes and compare them with the scattering of magnons in the spin chain. This would strengthen the map we propose between these solutions and the spin chains. We hope to report on this in the near future. Also, related ideas can be found in the more recent work [52–56].

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