INCLUDING REACTION TIMES IN MULTINOMIAL PROCESSING TREES

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Xiaofang Zheng

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STATEMENT OF COMMITTEE APPROVAL

Dr. Richard Schweickert, Chair

Department of Psychological Sciences

Dr. Robert W. Proctor

Department of Psychological Sciences

Dr. Sebastien Hélie

Department of Psychological Sciences

Approved by:

Dr. David Rollock

Head of the Graduate Program

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ABSTRACT

Author: Zheng, Xiaofang. MS
Institution: Purdue University
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Major Professor: Richard Schweickert

Processing trees are widely used to model the accuracy in tasks. For the mental process of a task, people's responses can be analyzed in terms of response probabilities and reaction time. Our project is to include reaction times in processing trees. Given a processing tree, if two sets of parameter values lead to the same predictions for response probabilities and response times, then there are equations relating parameter values in one set to those in the other set. We derived these equations and calculated relevant degrees of freedom. The current study also includes results on the following topics: the restricting conditions for the nonnegativity of time parameters, mixtures of trees with factors selectively influencing processes, and conditions under which an arbitrary processing tree is equivalent to the Standard Binary Tree for Ordered Processes.

INTRODUCTION

Many models of humans performing a cognitive task are specialized models of a single process. For example, there are models of perception, decision, memory search, motor preparation and movement (e.g., Fitts, 1954; Luce, 1986). Often, the duration of the process of interest is modeled in detail. Durations of all processes except the one of special interest are considered to add and their total, called the base time, is simply added to the time for the process of interest. If researchers want to analyze the data of probability and response time to reveal underlying mental processes, the complexity of the processes may make the traditional decomposition of the observed response time distribution into unobservable times for the separate mental processes unfeasible (Hu, 2001).

A Multinomial Processing Tree (MPT) model for data of accuracy and response time, incorporating several processes, could be useful. One advantage of a Multinomial Processing Tree model is that it can include several processes, allowing for perception, decision, etc. An MPT model emphasizes the organization of the processes and the inner workings of each process are ignored. Whereas MPT models are valuable tools to disentangle cognitive processes based on categorical data, they usually do not account for response times (Heck & Erdfelder, 2016), and are used almost exclusively for the probability of responses in various classes. Here, we extend MPT models to response time to improve the usefulness of MPT models. Theorems developed here can help find the relationship of the assumed cognitive processes, for example, whether two processes are ordered or unordered.

MPT models are widely used to model the accuracy, especially response probabilities, of performance in cognitive tasks. For instance, Keppel and Underwood (1962) demonstrated that the retention of items over very short intervals (short-term memory) shares the same effects of proactive inhibition as that of items over much longer intervals (long-term memory) in three experiments. A processing tree is applicable to Experiment 2 in their study. Three consonant syllables (KQF, MHZ, and CXJ) were presented visually to the subjects as for study. Subjects were required to recall them after 3 retention intervals. During the intervals, subjects were required to count backward by threes from a three-digit number spoken immediately after the presentation of the syllable. To increase the degree of initial learning, a 2-second visual exposure of the items was used. Then the retention for all syllables over three different intervals of 3 seconds, 9 seconds, 18 seconds was measured by counting the proportion of subjects' correct responses.

A processing tree model was built (Schweickert, Fisher & Sung, 2012) and gave a good account of the data of Keppel and Underwood's (1962) Experiment 2. In the model, the probability of a correct response is

$$P\ (\text{correct}) = I + (1 - I)\ R \qquad (1.1)$$

There is a value I_i for each retention interval i and a value R_j for each trial j. I is the probability that the memory trace for an item is intact, and R is the probability a nonintact memory trace can be correctly reconstructed. In Experiment 2, estimated values were I_1 = .723, I_2 = .526, I_3 = .407, and R_1 = .959, R_2 = .494, R_3 = 296 (Table 1).

Table 1

Keppel and Underwood (1962), Experiment 2

		Frequency of Correct Recall					Frequency of Incorrect Recall		
			Trial Number					Trial Number	
Retention Interval		1	2	3	Retention Interval		1	2	3
3	obs	71.0	62.0	58.0	3	obs	1.0	10.0	14.0
	pred	71.2	61.9	57.9		pred	.8	10.1	14.1
9	obs	70.0	54.0	49.0	9	obs	2.0	18.0	23.0
	pred	70.6	54.7	48.0		pred	1.4	17.3	24.0
18	obs	71.0	51.0	41.0	18	obs	1.0	21.0	31.0
	pred	70.2	50.4	41.9		pred	1.8	21.6	30.0

Observed and predicted values are labeled obs and pred, respectively. Predicted values are from Equation (1.1). For predicted and observed correct frequencies, r^2 = .999. And for both correct and incorrect frequencies, G^2 = .84, 4 df., which indicates a good fit.

A processing tree is made of points, called vertices, and lines, called arcs, that go from one vertex to another. Each vertex represents a process. For example, a vertex in a processing tree for a memory task might represent memory retrieval. After a process starts, outcomes of it become possible. Memory retrieval might succeed or fail, for example. Each arc descending from a vertex represents a possible outcome of the process represented by the vertex. Each arc has a probability associated with it, and the probability that the outcome represented by the arc will occur after the process starts. The sum of the probabilities associated with the arcs descending from a vertex is 1.

When the task starts, processing begins in the processing tree at the source, a vertex preceded by no other vertex. Then one of the possible outcomes of the source occurs. In other words, an arc descending from the source is selected, and the process represented by the end vertex of the arc begins. Such steps continue until a vertex with no arcs descending from it, a terminal vertex, is reached. When a terminal vertex is reached, a response is made. Responses are partitioned into classes, such as correct and incorrect. More than one terminal vertex may be associated with the same response class. For simplicity, we suppose that there are two response classes. For more than two response classes, see Schweickert and Xi (2011).

On each trial of the task, a response is made via a path that starts at the source and ends at a terminal vertex. For the mental processes of a task, people's responses can be analyzed in terms of response probabilities and times. However, existing studies mostly talk about the probabilities in processing trees. As for the response times, only a few studies (e.g., Heck & Erdfelder, 2016; Hu, 2001; Link, 1982) have investigated the features of response times in processing trees. Here, we are aiming to augment processing trees by dealing with response times. We assume, as Hu (2001) did, that associated with each arc is a time. The probability of a path is the product of the probabilities on the arcs on the path. The time required for a path from the source to a terminal vertex is the sum of the times associated with the arcs on the path. There is exactly one path from the source to a particular terminal vertex. The probability of a response in a particular class is the sum of the probabilities of all the paths from the source to the terminal vertices associated with that class. The time for a response in a particular class is a mixture distribution of the times of all paths from the source to the terminal vertex associated with that class. Two trees are defined to be *equivalent* if the probabilities and response times in both trees for every response class at each level of the factors are the same.

In a processing tree study, predictions of both probabilities and response times will be much more complex than those of only probabilities. We want to simplify the predictions by employing a technique already widely used to analyze response times, selectively influencing processes. In a processing tree, we say a factor *selectively influences* a vertex if there is exactly one vertex v with arcs whose associated probabilities and response times change as the level of the experimental factor changes (Schweickert & Chen, 2008). If X is an outcome of a process not influenced by a factor, then we denote the probability of outcome X as p_X and the time to produce outcome X as t_X. If Y is an outcome of a process which is selectively influenced by a factor, and the level of the factor is *k*, then the probability of outcome *Y* is written as $p_Y(k)$ and the time to produce outcome *Y* is written as $t_Y(k)$.

Work in this thesis is part of a larger project on factors selectively influencing processes in processing trees that produce response times. The thesis proposal mentioned that some of the work intended for the thesis was included in a manuscript submitted for publication (Schweickert & Zheng, 2017a). The proposal also mentioned that other work intended for the thesis was in a manuscript in progress. That manuscript is now split into two parts. One part is now itself a manuscript submitted for publication (Schweickert & Zheng, 2017b). The other part is now a manuscript in preparation (Schweickert & Zheng, 2018). As the topics of the thesis are discussed, if the work is included in one of the manuscripts submitted for publication, it will be cited here. Work not included in those manuscripts is in the manuscript in preparation or is intended to be included.

In our study, three standard trees are discussed, which are the Standard Tree for Unordered Processes, the Standard Tree for Ordered Processes, and the Standard Binary Tree for Ordered Processes. Consider an experimental factor with levels $i = 1, 2, 3, \ldots, I$ and another experimental factor with levels $j = 1, 2, 3, \ldots, J$. The Standard Tree for Unordered Processes is illustrated in Figure 1. *A* and *B* are two outcomes not influenced by any factors, and we denote the probabilities of outcomes *A* and *B* as p_A and p_B respectively, and the corresponding times as t_A and t_B. Moreover, outcomes *C*, *D*, *E*, and *F*, which can be influenced by the two experimental factors *i* and *j*, are labeled with $p_C(i)$, $p_D(i)$, $p_E(j)$, and $p_F(j)$ indicating the probability values, and $t_C(i)$, $t_D(i)$, $t_E(j)$, and $t_F(j)$ denoting the time values. Figure 2 illustrates the Standard Tree for Ordered

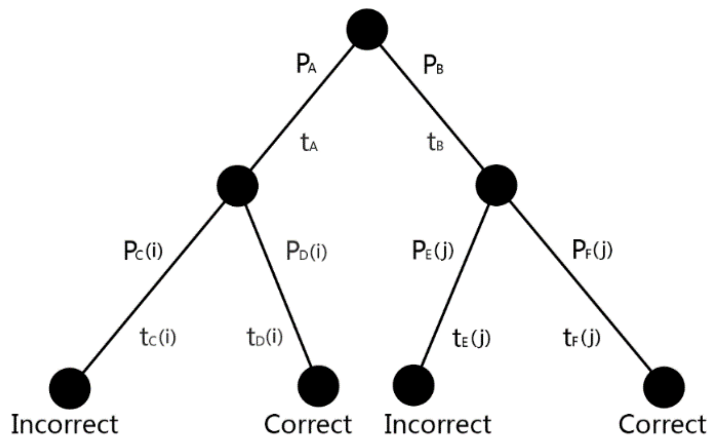

Figure 1. Standard Tree for Unordered Processes.

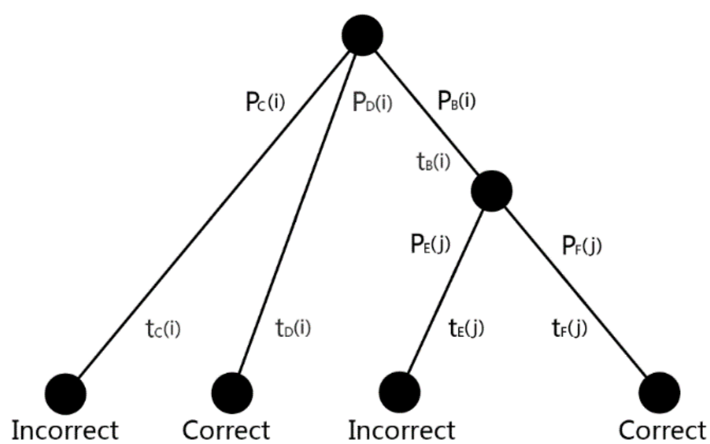

Figure 2. Standard Tree for Ordered Processes.

Processes with $p_B(i)$, $p_C(i)$, $p_D(i)$, $p_E(j)$, and $p_F(j)$ indicating the probability values and $t_B(i)$, $t_C(i)$, $t_D(i)$, $t_E(j)$, and $t_F(j)$ denoting the time values. Figure 3 illustrates the Standard Binary Tree for Ordered Processes with $p_A(i)$, $p_B(i)$, p_C, p_D, $p_E(j)$, and $p_F(j)$ indicating the probability values and $t_A(i)$, $t_B(i)$, t_C, t_D, $t_E(j)$, and $t_F(j)$ denoting the time values.

If a processing tree accounts for response times and probabilities, the parameters of both probability and response time are not unique. Consider a path with two arcs, one with probability *p* and one with probability *q*. The probability of the path is *pq*, which equals *(cp)(q/c)* with a constant *c*. So suitable parameters are $p^* = cp$ and $q^* = p/c$, provided *c* can be chosen so p^* and q^* are between 0 and 1. The same deduction can be applied to the parameters of response time.

Given a processing tree, if two sets of parameter values lead to the same predictions for response probabilities and response times, then there are equations relating parameter values in one set to those in the other set. We derived the equations, which relate parameter values in one set to those in the other set. One use of the equations is to calculate the relevant degrees of freedom for the standard trees. Works about the uniqueness of parameters are from Schweickert and Zheng (2017a), so we put the proofs for the uniqueness of parameters in the appendix.

Furthermore, suppose responses are produced by executing processes organized in any processing tree, and each factor selectively influences a different process in the tree. We showed that even if response times are included together with response probabilities, the tree is equivalent to one of the two simple trees, either the one in Figure 1 or the one in Figure 2 (Schweickert & Zheng, 2018). Following this, we derived straightforward

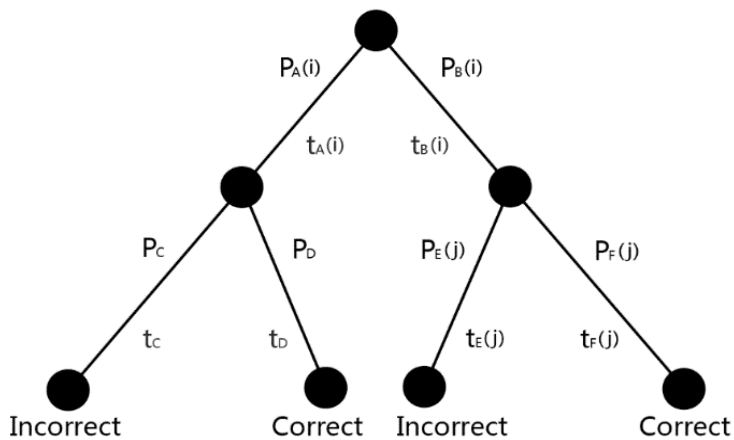

Figure 3. Standard Binary Tree for Ordered Processes.

predictions for response probabilities and response times in experiments, in which two experimental manipulations selectively influence two different vertices in a processing tree (Schweickert & Zheng, 2017b).

A participant might not use the same processing tree on every trial, and different participants might use different processing trees. If so, the response probabilities and response times may be generated by a mixture of trees. This thesis reports the restriction conditions required for a mixture of trees to be equivalent to one of the standard trees if both response probabilities and an additional measure such as response time are observed.

Before reading the following theorems, one needs to define what probability matrix **P** and time matrix **T** are. A probability matrix is a matrix whose entries are numbers between 0 and 1, inclusive (Schweickert & Chen, 2008). A time matrix is a matrix whose entries are nonnegative numbers. When an experiment has two factors, observations of probabilities and response times can be put in a probability matrix and a time matrix, respectively, with cell values p_{ij} and t_{ij} as the probability and response time of a correct response when Factor *A* is at level *i* and Factor *B* is at level *j*.

Suppose a processing tree, together with a set of parameter values of probabilities and response times for its arcs, produces the cell values in the probability matrix **P** and the time matrix **T**. An infinite number of other processing trees also produce the cell values. Then questions about the uniqueness of structure for processing trees appear. Former work (Schweickert & Chen, 2008) discussed the uniqueness of structure for processing trees, which, however, ignored a distinction. Suppose there are two factors, *A* and *B*, which selectively influence two different vertices in an MPT model. There are only two ways the vertices can be arranged: ordered or unordered. Consider a path from the source to a terminal vertex, going along each arc in the direction indicated by its arrow. Suppose on this path the vertex selectively influenced by Factor *A* is followed by an arc whose parameters change when the level of Factor *A* changes, and this arc is followed by the vertex selectively influenced by Factor *B*. (Following need not be immediately following.) Then we say the selectively influenced vertices are *ordered by function* (see examples in Figure 2 and Figure 3). If there is no such path, we say the selectively influenced vertices are *unordered by function* (see Figure 1). Two vertices ordered by form may or may not also be ordered by function. To incorporate the distinction, we have finished refining the proofs for the uniqueness of structure for the Standard Tree for Unordered Processes and the Standard Tree for Ordered Processes. The thesis refines the proofs for the uniqueness of structure for the Standard Binary Tree for Ordered Processes.

VERTEX ARRANGEMENTS

In a two-factor experiment, since the necessary and sufficient conditions are known for the Standard Tree for Unordered Processes and the Standard Tree for Ordered Processes to account for response probabilities, researchers only need consider these two trees for probabilities (Schweickert & Chen, 2008). Special cases, like the Standard Binary Tree for Ordered Processes, can be used with more detailed information. Terminal vertices of a processing tree are partitioned into two response classes. We call them here correct and incorrect. For the Standard Tree for Unordered Processes, for the subtree leading to incorrect responses, we find a subtree leading to correct responses of similar form. We now have developed equations and theorems for both the probability and the response time for a correct response, as will be shown later. Then it is straightforward to see that equations for the probability and the response time for an incorrect response are analogous. The same is true for the Standard Tree for Ordered Processes.

Suppose each of two factors selectively influences a different vertex in an arbitrary processing tree. Those two vertices are either ordered by function or they are not. As stated earlier, the arbitrary processing tree is equivalent to either the processing tree in Figure 1 or the one in Figure 2 (Schweickert & Zheng, 2018). Therefore, understanding the properties of these two processing trees is important.

The Standard Tree for Unordered Processes

We begin with properties of the processing tree in Figure 1, the Standard Tree for Unordered Processes. The first property we consider is the uniqueness of its parameters. Some of the works about the uniqueness of parameters are from Schweickert and Zheng (2017b), which are as follows.

Suppose two factors selectively influence two processes in one of the standard trees as shown above with notations in the figures. The following theorems work to figure out all possible sets of parameter values as well as the degrees of freedom and the boundary conditions. See their proofs in appendixes. In each theorem, results about probability were derived in Schweickert and Chen (2008). Results about response time and degrees of freedom were newly derived in Schweickert and Zheng (2017b).

Uniqueness of Parameter Values

Theorem 1. Suppose probability matrix **P** and time matrix **T** are produced by two factors selectively influencing two unordered processes in the Standard Tree for Unordered Processes. That is, there are probabilities p_A, p_B, $p_D(i)$, $p_F(j)$, and response times t_A, t_B, $t_D(i)$ and $t_F(j)$ such that for all *i*, $1 \le i \le I$ and all *j*, $1 \le j \le J$,

$$p(i, j) = p_A p_D(i) + p_B p_F(j)$$

and

$$p(i, j)t(i, j) = p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)].$$

Then for all *i*, $1 \le i \le I$ and all *j*, $1 \le j \le J$, $p^*_A, p^*_D(i)$, and $p^*_F(j)$ are probabilities and $t^*_A, t^*_B, t^*_D(i)$ and $t^*_F(j)$ are times such that

$$p(i, j) = p^*_A\ p^*_D(i) + p^*_B\ p^*_F(j).$$

and

$$p(i, j)t(i, j) = p^*_A\ p^*_D(i)[t^*_A + t^*_D(i)] + p^*_B\ p^*_F(j)[t^*_B + t^*_F(j)]$$

if and only if there are constants *c, d, h* and *m* such that

$$p^*_A = p_A / c\,,$$

$$p^*_D(i) = cp_D(i) + d\,,$$

$$p^*_F(j) = c(1-p_A)p_F(j)/(c-p_A) - p_A d/(c-p_A)\,.$$

$$t^*_A = t_A - m$$

$$t^*_B = t_B - h$$

$$t^*_D(i) = \frac{p_D(i)[t_D(i)+m] + \frac{d}{c}\times[t_B - t_A + m + t^*_F(j') - h] + (\frac{1}{p_A}-1)p_F(j')[t_F(j') - t^*_F(j') + h]}{p_D(i) + \frac{d}{c}}$$

$$t^*_F(j) = \frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t^*_F(j')] - \frac{p_A d}{[c(1-p_A)]}t^*_F(j')}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}$$

where *j'* can be any different value from *j*.

Further, *c, d, h,* and *m* satisfy certain inequalities

$$p_A < c,$$

$$-c\min\{p_D(i)\} \le d \le 1 - c\max\{p_D(i)\}\,,$$

$$[p_A - c + (1-p_A)c\max\{p_F(j)\}]/p_A \le d \le [(1-p_A)c\min\{p_F(j)\}]/p_A\,,$$

$$\max\{-t^*_A\} \le m \le \min\{t_A\}$$

$$\max\{-t^*_B\} \le h \le \min\{t_B\}$$

and the degrees of freedom are *3IJ-3I-3J+5*.

The proof is in the appendix. See Figure 4 and Figure 5 for a numerical example of the uniqueness of parameters in the Standard Tree for Unordered Processes.

Mixtures of Trees With Factors Selectively Influencing Unordered Processes

As we said earlier, a participant might not use the same processing tree on every trial, and different participants might use different processing trees. If so, the response probabilities and times may be generated by a mixture of trees. Notwithstanding, for response probabilities under certain conditions, a mixture of trees is equivalent to one of the standard trees. For response probabilities, mixtures of trees were considered in Theorems 4, 10 and 15 in Schweickert and Chen's work (2008). Suppose both response probabilities and an additional measure such as response time are observed. Then what conditions are required for a mixture of trees to be equivalent to one of the standard trees?

The proof of the following theorem uses necessary and sufficient conditions for two factors to selectively influence two processes in the Standard Tree for Unordered Processes. Those conditions are as follows (Schweickert & Zheng, 2017b):

> *Probability matrix* $\mathbf{P} = (p(i, j))$ *and correct-response-measure matrix* $\mathbf{T} = (t(i,j))$ *are produced by Factor A and Factor B selectively influencing two vertices unordered by form in the Standard Tree for Unordered Processes if and only if the following two conditions are true:*
>
> *1. There are levels i^* and j^* such that for all levels i of Factor A and all levels j of Factor B, such that* $p(i,j) = p(i,j^*) + p(i^*,j) - p(i^*,j^*)$;

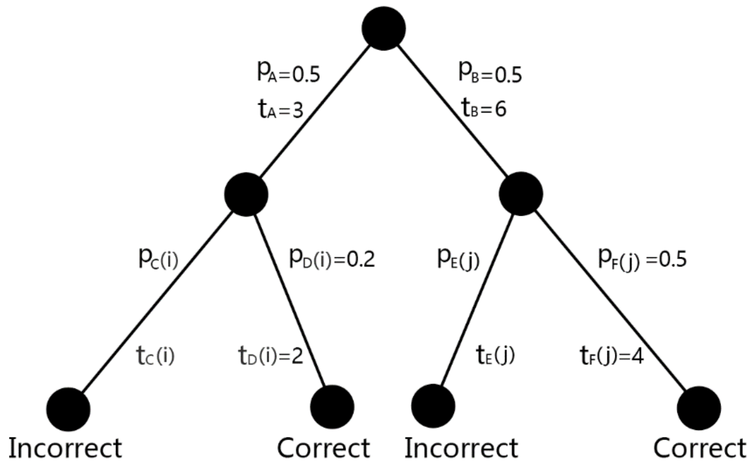

Figure 4. A numerical example of the uniqueness of parameters in the Standard Tree for Unordered Processes: Original parameters.

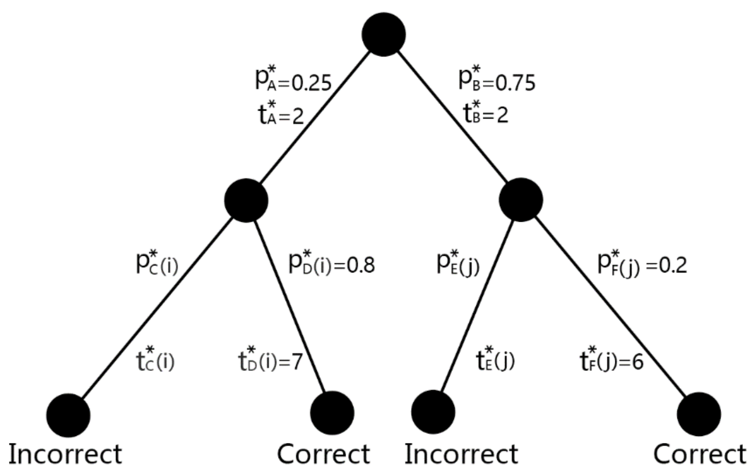

Figure 5. A numerical example of the uniqueness of parameters in the Standard Tree for Unordered Processes: Transformed parameters.

2. There are levels m and n such that for all levels i of Factor A and all levels j of Factor B, such that p(i,j)t(i,j) = p(i,n)t(i,n) + p(m,j)t(m,j) – p(m,n)t(m,n).

The following theorem answers this question for the Standard Tree for Unordered Processes.

Theorem 2. Suppose probability matrix **P** and measure matrix **T** are produced by two factors, Factor *A* and Factor *B*, selectively influencing two different vertices in each tree of a mixture of trees. Suppose in each tree in the mixture, the factors selectively influence two vertices unordered by function. Then **P** and **T** are produced by two factors selectively influencing two unordered vertices in an equivalent Standard Tree for Unordered Processes.

Proof. For each tree *z* in the mixture, there is an equivalent Standard Tree for Unordered Processes (Schweickert & Zheng, 2018). With this standard tree, when Factor *A* is at level *i* and Factor *B* is at level *j*, we can write the probability of a correct response as

$$p_z(i, j) = p_{Az} p_{Dz}(i) + (1 - p_{Az}) p_{Fz}(j)$$

and the product of probability and response time of a correct response for tree *z* as

$$p_z(i, j) t_z(i, j) = p_{Az} p_{Dz}(i)[t_{Az} + t_{Dz}(i)] + (1 - p_{Az}) p_{Fz}(j)[t_{Bz} + t_{Fz}(j)]$$

where $1 - p_{Az} = p_{B_z}$.

Suppose tree *z* is selected with probability τ_z.

For the probability of the mixture of trees, we have

$$p(i, j) = \sum \tau_z p_z(i, j) = \sum \tau_z p_{Az} p_{Dz}(i) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)$$

where all sums are over *z*, and $\sum \tau_z = 1$.

Consider any levels *i** and *j**. For all levels *i* of Factor *A* and all levels *j* of Factor *B*, we have

$$p(i, j^*) = \sum \tau_z p_z(i, j^*) = \sum \tau_z p_{Az} p_{Dz}(i) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j^*)$$

$$p(i^*, j) = \sum \tau_z p_z(i^*, j) = \sum \tau_z p_{Az} p_{Dz}(i^*) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)$$

$$p(i^*, j^*) = \sum \tau_z p_z(i^*, j^*) = \sum \tau_z p_{Az} p_{Dz}(i^*) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j^*)$$

Then,

$$p(i, j^*) + p(i^*, j) - p(i^*, j^*)$$

$$= \sum \tau_z p_{Az} p_{Dz}(i) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j^*) + \sum \tau_z p_{Az} p_{Dz}(i^*) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)$$

$$- \sum \tau_z p_{Az} p_{Dz}(i^*) - \sum \tau_z (1 - p_{Az}) p_{Fz}(j^*)$$

$$= \sum \tau_z p_{Az} p_{Dz}(i) + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)$$

$$= p(i, j)$$

This satisfies the first necessary and sufficient condition for the Standard Tree for Unordered Processes (Schweickert & Zheng, 2017b).

For the product of probabilities and response times of the mixture of trees, we have

$$p(i, j)t(i, j) = \sum \tau_z p_z(i, j)t_z(i, j)$$
$$= \sum \tau_z p_{Az} p_{Dz}(i)[t_{Az} + t_{Dz}(i)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)[t_{Bz} + t_{Fz}(j)]$$

For any levels *m* and *n*, for all levels *i* of Factor *A* and all levels *j* of Factor *B*, we have

$$p(i,n)t(i,n) = \sum \tau_z p_{Az} p_{Dz}(i)[t_{Az} + t_{Dz}(i)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(n)[t_{Bz} + t_{Fz}(n)]$$
$$p(m, j)t(m, j) = \sum \tau_z p_{Az} p_{Dz}(m)[t_{Az} + t_{Dz}(m)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)[t_{Bz} + t_{Fz}(j)]$$
$$p(m,n)t(m,n) = \sum \tau_z p_{Az} p_{Dz}(m)[t_{Az} + t_{Dz}(m)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(n)[t_{Bz} + t_{Fz}(n)]$$

Then,

$$p(i,n)t(i,n) + p(m, j)t(m, j) - p(m,n)t(m,n)$$
$$= \sum \tau_z p_{Az} p_{Dz}(i)[t_{Az} + t_{Dz}(i)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(n)[t_{Bz} + t_{Fz}(n)]$$
$$+ \sum \tau_z p_{Az} p_{Dz}(m)[t_{Az} + t_{Dz}(m)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)[t_{Bz} + t_{Fz}(j)]$$
$$- \sum \tau_z p_{Az} p_{Dz}(m)[t_{Az} + t_{Dz}(m)] - \sum \tau_z (1 - p_{Az}) p_{Fz}(n)[t_{Bz} + t_{Fz}(n)]$$
$$= \sum \tau_z p_{Az} p_{Dz}(i)[t_{Az} + t_{Dz}(i)] + \sum \tau_z (1 - p_{Az}) p_{Fz}(j)[t_{Bz} + t_{Fz}(j)]$$
$$= p(i, j)t(i, j)$$

This satisfies the second of the two necessary and sufficient conditions for the Standard Tree for Unordered Processes (Schweickert & Zheng, 2017b).

Hence, the mixture is equivalent to the Standard Tree for Unordered Processes.

□

Restricting Conditions for the Nonnegativity of Time Parameters

We developed formulas for parameter estimators for probabilities and the product of probabilities and response times, such as $p_F(j)$, $p_B(i)t_B(i)$ and so on, from the observed probabilities of correct responses and correct response times. The proofs (Schweickert & Zheng, 2018) showed that the probability estimators are between 0 and 1. It turns out that the time parameters with the Standard Tree for Unordered Process are always nonnegative. It can be straightforwardly shown as follows:

Consistent with Theorem 1 (Schweickert & Zheng, 2017b), one can renumber the levels *i* of Factor *A* and *j* of Factor *B* so for every *j*

$$p(\mathrm{m},j)\, t(\mathrm{m},j) \leq \ldots \leq p(i,j)\, t(i,j) \leq \ldots \leq p(M,j)\, t(M,j)$$

and for every *i*

$$p(\mathrm{i},n)\, t(\mathrm{i},n) \leq \ldots \leq p(i,j)\, t(i,j) \leq \ldots \leq p(i,N)\, t(i,N).$$

Let $\hat{t}_A = 0$ and $\hat{t}_B = 0$.

For every *i*, define $\hat{t}_D(i)$ as follows:

If $\hat{p}_D(i) = 0$, let $\hat{t}_D(i) = 0$.

If $\hat{p}_D(i) \neq 0$, let $\hat{t}_D(i) = \dfrac{p(i,n)t(i,n) - \frac{1}{2}p(m,n)t(m,n)}{\hat{p}_A \hat{p}_D(i)}$.

Because $p(m,n)t(m,n) \leq p(i,n)t(i,n)$, $\hat{t}_D(i) \geq 0$.

Similarly, for every j, define $\hat{t}_F(j)$ as follows:

If $\hat{p}_F(j) = 0$, let $\hat{t}_F(j) = 0$.

If $\hat{p}_F(j) \neq 0$, let $\hat{t}_F(j) = \dfrac{p(m,j)t(m,j) - \frac{1}{2}p(m,n)t(m,n)}{(1-\hat{p}_A)\hat{p}_F(j)}$.

Because $p(m,n)t(m,n) \leq p(\mathrm{m},j)t(\mathrm{m},j)$, $\hat{t}_F(j) \geq 0$.

As stated earlier, if two factors selectively influence two vertices in an arbitrary processing tree, that tree is equivalent to either the Standard Tree for Unordered Processes or the Standard Tree for Ordered Processes. The preceding theorems were about the former, we now turn to the latter.

The Standard Tree for Ordered Processes

Uniqueness of Parameter Values

The proof for the following theorem is in appendix.

Theorem 3. Suppose probability matrix **P** and time matrix **T** are produced by two factors selectively influencing two ordered processes in the Standard Tree for Ordered Processes. That is, there are parameters $p_B(i)$, $p_D(i)$, $p_F(j)$, $t_B(i)$, $t_D(i)$ and $t_F(j)$ such that for all i, $1 \leq i \leq I$ and all j, $1 \leq j \leq J$, there are

$$p(i,j) = p_D(i) + p_B(i)p_F(j).$$

and

$$p(i,j)t(i,j) = p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

Then for all i, $1 \le i \le I$ and all j, $1 \le j \le J$, $p*_B(i)$, $p^*_D(i)$ and $p*_F(j)$ are probabilities and $t*_B(i)$, $t^*_D(i)$ and $t*_F(j)$ are times such that

$$p(i,j) = p*_D(i) + p*_B(i)p*_F(j).$$

and

$$p(i,j)t(i,j) = p*_D(i)t*_D(i) + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

if and only if there are numbers *c*, *d* and *e* such that

$$p*_B(i) = cp_B(i)$$

$$p*_D(i) = p_D(i) - cdp_B(i)$$

$$p*_F(j) = p_F(j)/c + d$$

$$t*_B(i) = t_B(i) - e$$

$$t*_D(i) = \frac{-p_D(i)t_D(i) + p_B(i)[p_F(j') + cd][t*_F(j') - e] - p_B(i)p_F(j')t_F(j') + cdp_B(i)t_B(i)}{cdp_B(i) - p_D(i)}$$

$$t*_F(j) = \frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] + cdt*_F(j') + p_F(j)t_F(j)}{cd + p_F(j)}$$

where *j'* can be any different value of *j*, and *c*, *d*, and *e* should be in the boundaries as

$$0 < c \le 1/\max\{p_B(i)\}$$

$$-\min\{p_F(j)\} \le cd$$

$$\max\{p_F(j)\} \le c(1-d)$$

$$\max\{[p_D(i)-1]/p_B(i)\} \le cd \le \min\{p_D(i)/p_B(i)\}$$

$$\max\{-t*_B(i)\} \le e \le \min\{t_B(i)\}$$

And the degrees of freedom are *3IJ-5I-3J+6*.

Mixtures of Trees With Factors Selectively Influencing Ordered Processes

We now consider mixtures of processing trees. According to Schweickert and Zheng (2017b), there are three necessary and sufficient conditions for the Standard Tree for Ordered Processes to produce **P** and **T**:

There is a level n of Factor B and for every level i of Factor A there are numbers $r_i \geq 0$ *and* s_i *such that*

1. The columns of **P** *can be numbered so j > j' implies that for every i,* $p(i,j) \geq p(i,j')$.

2. There exist i and j* such that for every i and j,*

$p(i,j) - p(i,j^*) = r_i[p(i^*,j) - p(i^*,j^*)]$.

3. Let $\max\{r_i\} = r_h$. *For every j,*

$$r_h r_i s_i[p(h,j) - p(h,n)]$$

$$= r_h[p(i,j)t(i,j) - p(i,n)t(i,n)] - r_i[p(h,j)t(h,j) - p(h,n)t(h,n)].$$

The following theorem shows the conditions that are required for a mixture of trees to be equivalent to one of the standard trees for the Standard Tree for Ordered Processes.

Theorem 4. Suppose probability matrix **P, 0 < P < 1,** and time matrix **T** are produced by Factors *A* and *B* selectively influencing processes in each MPT of a mixture of MPTs. Suppose in each MPT in the mixture, the factors selectively influence two vertices ordered by function, with the vertex influenced by Factor *A* preceding the vertex influenced by Factor *B*.

Then **P** and **T** are produced by two factors selectively influencing two ordered processes ordered by function in the Standard Tree for Ordered Processes, with the process influenced by Factor *A* preceding the process influenced by Factor *B*.

Proof. For each MPT *z* in the mixture, there is an equivalent Standard Tree for Ordered Processes (Schweickert & Zheng, 2018). With this standard tree, when Factor *A* is at level *i* and Factor *B* is at level *j*, we can write the probability of a correct response as

$$p_z(i,j) = p'_{Dz}(i) + \ p'_{Bz}(i)p'_{Fz}(j)$$

and the product of probability and response time of a correct response for MPT *z* as

$$p_z(i,j)t_z(i,j) = p'_{Dz}(i)t'_{Dz}(i) + \ p'_{Bz}(i)p'_{Fz}(j)[t'_{Bz}(i) + \ t'_{Fz}(j)].$$

Parameter values temporarily written with ‘ ′ ’, can be transformed with admissible transformations. According to Theorem 3 above, for MPT *z* an admissible transformation for $p'_{Bz}(i)$ is multiplication by a positive constant (a scaling parameter) and an admissible transformation for $t'_{Bz}(i)$ is addition of a constant (a scaling parameter). Choose the scaling parameters as follows.

For some MPT z and level *i* parameter $p'_{Bz}(i)$ takes on its maximum value. Denote this maximum value as $p_B(i)$. For every MPT *z* let c_z be the number so $c_z p'_{Bz}(i) =$

$p_B(i)$. Note that if $p_B(i) = 0$, then in every MPT z, changing level j has no effect on response probability, contrary to assumption. So, $c_z > 0$. Then the bounds required in Theorem 3 for scaling parameter c_z are met, that is for MPT z, for every i, $0 < c_z \leq 1/\max\{p'_{Bz}(i)\}$.

Similarly, for some MPT z and level i parameter $t'_{Bz}(i)$ takes on its maximum value. Denote this maximum value as $t_B(i)$. For every MPT z let e_z be the number so $t'_{Bz}(i) - e_z = t_B(i)$. Now for every level i, for every MPT z transform $p'_{Bz}(i)$ as $c_z p'_{Bz}(i) = p_B(i)$ and transform $t'_{Bz}(i)$ as $t'_{Bz}(i) - e_z = t_B(i)$. Transform the remaining parameters of MPT z with their admissible transformations, using scaling parameters c_z and e_z. Denote the remaining transformed parameters as $p_{Dz}(i)$, $t_{Dz}(i)$, $p_{Fz}(j)$, and $t_{Fz}(j)$.

According to Theorem 3 above, the transformed parameters produce **P** and **T** with the Standard Tree for Ordered Processes. That is

$$p_z(i,j) = p_{Dz}(i) + p_B(i)p_{Fz}(j)$$

and

$$p_z(i,j)t_z(i,j) = p_{Dz}(i)t_{Dz}(i) + p_B(i)p_{Fz}(j)[t_B(i) + t_{Fz}(j)].$$

Suppose tree z is selected with probability τ_z.

For the probability of the mixture of trees, we have

$$p(i,j) = \sum \tau_z p_z(i,j) = \sum \tau_z p_{Dz}(i) + \sum \tau_z p_B(i)p_{Fz}(j)$$

where all sums are over z, and $\sum \tau_z = 1$.

The columns of ($p(i, j)$) can be ordered so $j > j'$ implies $\sum \tau_z\, p_{Fz}(j) \geq \sum \tau_z\, p_{Fz}(j')$, where j' is a different level of j, and then, $p(i,j) \geq p(i,j')$ for all i. Hence the first of the necessary and sufficient conditions is satisfied.

Then we move to the second condition. Choose $i^* = h$, where $p_B(h) = \max\{p_B(i)\}$. If $p_B(i) = 0$ for every level i, then Factor B is ineffective for response probabilities. Hence $p_B(i) \neq 0$. Set $r_i = \frac{p_B(i)}{p_B(h)}$. Choose any level of Factor B as j^*. Then for any level j of Factor B,

$$p(i,j) - p(i,j^*) = \sum \tau_z p_{Dz}(i) + \sum \tau_z p_B(i) p_{Fz}(j) - \sum \tau_z p_{Dz}(i) - \sum \tau_z p_B(i) p_{Fz}(j^*)$$

$$= \sum \tau_z p_B(i) p_{Fz}(j) - \sum \tau_z p_B(i) p_{Fz}(j^*)$$

$$= p_B(i) \sum \tau_z \left[p_{Fz}(j) - p_{Fz}(j^*)\right]$$

$$r_i[p(i^*,j) - p(i^*,j^*)]$$

$$= \frac{p_B(i)}{p_B(h)} \left[\sum \tau_z p_{Dz}(i^*) + \sum \tau_z p_B(i^*) p_{Fz}(j) - \sum \tau_z p_{Dz}(i^*) - \sum \tau_z p_B(i^*) p_{Fz}(j^*)\right]$$

$$= \frac{p_B(i)}{p_B(h)} \left[p_B(i^*) \sum \tau_z\, p_{Fz}(j) - p_B(i^*) \sum \tau_z\, p_{Fz}(j^*)\right]$$

$$= \frac{p_B(i)}{p_B(h)} p_B(i^*) \sum \tau_z \left[p_{Fz}(j) - p_{Fz}(j^*)\right]$$

$$= p_B(i) \sum \tau_z \left[p_{Fz}(j) - p_{Fz}(j^*)\right]$$

Hence, there exist i^* and j^*, such that for every i and j, there is

$$p(i,j) - p(i,j^*) = r_i[p(i^*,j) - p(i^*,j^*)].$$

The second of the necessary and sufficient conditions is satisfied. Suppose *n* is chosen such that $p(h,n)t(h,n) = \min\{p(h, j)t(h, j)\}$. For the product of probabilities and response times of the mixture of trees, we have

$$p(i,j)t(i,j) = \sum \tau_z p_z(i,j) t_z(i,j)$$
$$= \sum \tau_z \{p_{Dz}(i) t_{Dz}(i) + p_B(i) p_{Fz}(j)[t_B(i) + t_{Fz}(j)]\}$$
$$= \sum \tau_z p_{Dz}(i) t_{Dz}(i) + \sum \tau_z p_B(i) p_{Fz}(j)[t_B(i) + t_{Fz}(j)]$$

Similarly, we have

$$p(i,n)t(i,n) = \sum \tau_z p_{Dz}(i) t_{Dz}(i) + \sum \tau_z p_B(i) p_{Fz}(n)[t_B(i) + t_{Fz}(n)]$$
$$p(h,j)t(h,j) = \sum \tau_z p_{Dz}(h) t_{Dz}(h) + \sum \tau_z p_B(h) p_{Fz}(j)[t_B(h) + t_{Fz}(j)]$$
$$p(h,n)t(h,n) = \sum \tau_z p_{Dz}(h) t_{Dz}(h) + \sum \tau_z p_B(h) p_{Fz}(n)[t_B(h) + t_{Fz}(n)] .$$

Because $r_h = \max(\frac{p_B(i)}{p_B(h)}) = 1$, we have

$$r_h[p(i, j)t(i, j) - p(i,n)t(i,n)]$$
$$= 1 \times \{\sum \tau_z p_{Dz}(i) t_{Dz}(i) + \sum \tau_z p_B(i) p_{Fz}(j)[t_B(i) + t_{Fz}(j)] - \sum \tau_z p_{Dz}(i) t_{Dz}(i) - \sum \tau_z p_B(i) p_{Fz}(n)[t_B(i) + t_{Fz}(n)]\}$$
$$= \sum \tau_z p_B(i) p_{Fz}(j)[t_B(i) + t_{Fz}(j)] - \sum \tau_z p_B(i) p_{Fz}(n)[t_B(i) + t_{Fz}(n)]$$
$$= p_B(i)\{\sum \tau_z p_{Fz}(j)[t_B(i) + t_{Fz}(j)] - \sum \tau_z p_{Fz}(n)[t_B(i) + t_{Fz}(n)]\} \qquad (2.1)$$

$$r_i[p(h,j)t(h,j) - p(h,n)t(h,n)]$$

$$= \frac{p_B(i)}{p_B(h)}\{\sum \tau_z p_{Dz}(h)t_{Dz}(h) + \sum \tau_z p_B(h)p_{Fz}(j)[t_B(h) + t_{Fz}(j)] - \sum \tau_z p_{Dz}(h)t_{Dz}(h)$$

$$- \sum \tau_z p_B(h)p_{Fz}(n)[t_B(h) + t_{Fz}(n)]\}$$

$$= \frac{p_B(i)}{p_B(h)}\{\sum \tau_z p_B(h)p_{Fz}(j)[t_B(h) + t_{Fz}(j)] - \sum \tau_z p_B(h)p_{Fz}(n)[t_B(h) + t_{Fz}(n)]\}$$

$$= \frac{p_B(i)}{p_B(h)} p_B(h)\{\sum \tau_z p_{Fz}(j)[t_B(h) + t_{Fz}(j)] - \sum \tau_z p_{Fz}(n)[t_B(h) + t_{Fz}(n)]\}$$

$$= p_B(i)\{\sum \tau_z p_{Fz}(j)[t_B(h) + t_{Fz}(j)] - \sum \tau_z p_{Fz}(n)[t_B(h) + t_{Fz}(n)]\} \qquad (2.2)$$

The right hand side of (2.1) minus the right hand side of (2.2) is

$$p_B(i)\{\sum \tau_z p_{Fz}(j)[t_B(i) + t_{Fz}(j)] - \sum \tau_z p_{Fz}(n)[t_B(i) + t_{Fz}(n)]\}$$

$$-p_B(i)\{\sum \tau_z p_{Fz}(j)[t_B(h) + t_{Fz}(j)] - \sum \tau_z p_{Fz}(n)[t_B(h) + t_{Fz}(n)]\}$$

$$= p_B(i)\{\sum \tau_z p_{Fz}(j)t_B(i) - \sum \tau_z p_{Fz}(n)t_B(i) - \sum \tau_z\, p_{Fz}(j)t_B(h) + \sum \tau_z p_{Fz}(n)t_B(h)\}$$

$$= p_B(i)\{\sum \tau_z[p_{Fz}(j) - p_{Fz}(n)]t_B(i) - \sum \tau_z[p_{Fz}(j) - p_{Fz}(n)]t_B(h)\}$$

$$= p_B(i)[t_B(i) - t_B(h)]\sum \tau_z[p_{Fz}(j) - p_{Fz}(n)]$$

$$= \frac{p_B(i)}{p_B(h)}[t_B(i) - t_B(h)]\sum \tau_z p_B(h)[p_{Fz}(j) - p_{Fz}(n)]$$

$$= 1 * \frac{p_B(i)}{p_B(h)}[t_B(i) - t_B(h)]\sum\{\tau_z[p_B(h)p_{Fz}(j) + p_{Dz}(h) - p_B(h)p_{Fz}(n) - p_{Dz}(h)]\}$$

$$= r_h r_i[t_B(i) - t_B(h)][p(h,j) - p(h,n)]$$

$$= r_h r_i s_i[p(h,j) - p(h,n)]$$

where $s_i = t_B(i) - t_B(h)$.

Hence, for every j,

$$r_h r_i s_i[p(h,j)-p(h,n)]$$
$$=r_h[p(i,j)t(i,j)-p(i,n)t(i,n)]-r_i[p(h,j)t(h,j)-p(h,n)t(h,n)].$$

Then all the conditions for **P** and **T** to be produced by two factors selectively influencing two processes in the Standard Tree for Ordered Processes are satisfied.

□

Restricting Conditions for the Nonnegativity of Time Parameters

The following theorem gives restricting conditions for the calculated values of the time parameters in the Standard Tree for Ordered Processes to be nonnegative.

Based on Theorem 4 (Schweickert & Zheng 2017b), for every level i of Factor A, $\hat{p}_D(i) = p(i,1)$, $\hat{p}_B(i) = p(i,J) - p(i,1)$, and for every level j of Factor B,

$$\hat{p}_F(j)=\frac{p(i^{**},j)-p(i^{**},1)}{p(i^{**},J)-p(i^{**},1)},$$

where i^{**} is one level of Factor A such that $p(i^{**}, J) > p(i^{**}, 1)$.

Additionally, for any level i of Factor A and any level j of Factor B, we calculate the values of $t_B(i)$, $t_D(i)$ and $t_F(j)$ as follows:

If $\hat{p}_F(j)=0$, then $\hat{t}_F(j) = 0$; Otherwise, $\hat{t}_F(j)=\dfrac{p(h,j)t(h,j)-p(h,n)t(h,n)}{\hat{p}_B(h)\hat{p}_F(j)}$.

If $r_i = 0$, $\hat{t}_B(i) = 0$; Otherwise, $\hat{t}_B(i) = r_h s_i$.

$$\hat{t}_D(i) = \frac{p(i,n)t(i,n) - \hat{p}_B(i)\hat{p}_F(n)\hat{t}_B(i)}{\hat{p}_D(i)}.$$

Here, h is a level of factor i such that $p_B(h) = \max\{p_B(i)\}$, and n is a level of factor j such that $p(h,n)t(h,n) = \min\{p(h,j)t(h,j)\}$ and $t(i,n) - t_B(i) \geq 0$.

Theorem 5. Assume the time matrix **T**= $(t(i,j))$ produced by the Standard Tree for Ordered Processes is nonnegative. Let h be a level of factor A such that $p_B(h) = \max\{p_B(i)\}$. Suppose for any i, there is always $t_B(i) \geq t_B(h)$, and we can choose n such that $t(i,n) - t_B(i) \geq 0$. Then $\hat{t}_B(i)$, $\hat{t}_D(i)$, and $\hat{t}_F(j)$, the values of $t_B(i)$, $t_D(i)$ and $t_F(j)$ as calculated in the proof of Theorem 4 (Schweickert & Zheng, 2017b), are always nonnegative.

Proof. Obviously, $\hat{t}_F(j) = \dfrac{p(h,j)t(h,j) - p(h,n)t(h,n)}{\hat{p}_B(h)\hat{p}_F(j)} \geq 0$

From Theorem 4 (Schweickert & Zheng, 2017b), we know that $\hat{t}_B(i) = r_h s_i = r_h[t_B(i) - t_B(h)]$, because $t_B(i) \geq t_B(h)$, we get $\hat{t}_B(i) \geq 0$.

From the proof for Theorem 4 (Schweickert & Zheng, 2017b), we know that $\hat{p}_B(i)\hat{p}_F(j) = p(i,j) - p(i,1)$. Remember that $\hat{t}_B(i) = r_h s_i$ and $s_i = t_B(i) - t_B(h)$. So we get

$$\hat{t}_D(i) = \frac{p(i,n)t(i,n) - \hat{p}_B(i)\hat{p}_F(n)\hat{t}_B(i)}{\hat{p}_D(i)}$$

$$= \frac{p(i,n)t(i,n) - [p(i,n) - p(i,1)]r_h s_i}{\hat{p}_D(i)}$$

$$= \frac{p(i,n)t(i,n) - p(i,n)r_h s_i + p(i,1)r_h s_i}{\hat{p}_D(i)}$$

$$= \frac{p(i,n)[t(i,n) - r_h s_i] + p(i,1)r_h s_i}{\hat{p}_D(i)}$$

$$= \frac{p(i,n)\{t(i,n) - \max(\frac{p_B(i)}{p_B(h)})[t_B(i) - t_B(h)]\} + p(i,1)r_h s_i}{\hat{p}_D(i)}$$

because $p_B(h) = \max(p_B(i))$, we can get $\max(\frac{p_B(i)}{p_B(h)}) \leq 1$. Then,

$$\hat{t}_D(i) \geq \frac{p(i,n)\{t(i,n) - [t_B(i) - t_B(h)]\} + p(i,1)r_h s_i}{\hat{p}_D(i)}.$$

We can choose n such that $t(i,n) - t_B(i) \geq 0$, hence,

$$\hat{t}_D(i) \geq \frac{p(i,n)[t(i,n) - t_B(i)] + p(i,n)t_B(h) + p(i,1)r_h s_i}{\hat{p}_D(i)} \geq 0.$$

□

The Standard Binary Tree for Ordered Processes

Uniqueness of Parameter Values

The following theorem works to figure out all possible sets of parameter values as well as the degrees of freedom and the boundary conditions in the Standard Binary Tree for Ordered Processes.

Theorem 6. Suppose probability matrix **P** and time matrix **T** are produced by two factors selectively influencing two ordered processes in the Standard Binary Tree for Ordered Processes. That is, there are parameters s $p_A(i)$, $p_B(i)$, p_D, $p_F(j)$, and response times $t_A(i)$, $t_B(i)$, t_D and $t_F(j)$ such that for all *i*, $1 \le i \le I$ and all *j*, $1 \le j \le J$,

$$p(i,j) = p_A(i)p_D + p_B(i)p_F(j)$$

$$p(i,j)t(i,j) = p_A(i)p_D[t_A(i) + t_D] + p_B(i)p_F(j)[t_B(i) + t_F(j)].$$

Then for all *i*, $1 \le i \le I$ and all *j*, $1 \le j \le J$, $p*_A(i), p*_B(i), p*_D$ and $p*_F(j)$ are probabilities and $t*_A(i), t*_B(i), t*_D$ and $t*_F(j)$ are times such that

$$p(i,j) = p*_A(i)p*_D + p*_B(i)p*_F(j).$$

and

$$p(i,j)t(i,j) = p*_A(i)p*_D[t*_A(i) + t*_D] + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

if and only if there are numbers *c*, *e* and *f* such that

$p*_B(i) = cp_B(i)$,

$p*_F(j) = p_F(j)/c + (c-1)p_D/c$,

$p*_D = p_D$

$t*_B(i) = t_B(i) + f$,

$$t^*_F(j) =$$

$$\frac{p_F(j)t_F(j) + f\cdot[p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j) + (c-1)p_D},$$

$$t^*_D = t_D + e,$$

and

$$t^*_A(i) = \frac{p_A(i)t_A(i) + p_B(i)\{[t_B(i) - t_D](1-c) + [f + t^*_F(j')][1 - c - \frac{p_F(j')}{p_D}] + \frac{p_F(j')t_F(j')}{p_D} + ce\} - e}{1 - cp_B(i)}$$

where *j'* can be any different value of *j*, and *c*, *e*, and *f* should be in the boundaries as

$$0 < c \le 1/\max\{p_B(i)\}$$

$$p_D - \min\{p_F(j)\} \le p_D \cdot c$$

$$\max\{p_F(j)\} - p_D \le (1 - p_D)c$$

$$\max\{-t_D\} \le e \le \min\{t^*_D\}$$

$$\max\{-t_B(i)\} \le f \le \min\{t^*_B(i)\}$$

and the degrees of freedom are *3IJ-3I-3J+3*.

The proof is in the Appendix.

Uniqueness of Structure

Suppose two factors selectively influence two vertices in an arbitrary processing tree. The theorem is about conditions under which the arbitrary tree is equivalent to the Standard Binary Tree for Ordered Processes.

Results from Lemma 2 (Schweickert & Zheng, 2018) are used. The statement of Lemma 2 is as follows:

Suppose in an MPT changing the level of Factor A changes parameters on M arcs descending from a vertex u_1*, but on no other arcs. Suppose changing the level of Factor B changes parameters on N arcs descending from a vertex* u_2*, which may be the same as vertex* u_1*, but on no other arcs.*

Then there is an equivalent MPT with the following properties.

1. Changing the level of Factor A changes parameters on M arcs descending from a vertex v_1*, and on no other arcs, and no arc descending from vertex* v_1 *has parameter values invariant over all levels of Factor A. Further, for a vertex* v_2 *different from* v_1*, changing the level of Factor B changes parameters on N arcs descending from vertex* v_2*, and on no other arcs, and no arc descending from vertex* v_2 *has parameter values invariant over all levels of Factor B.*

2. If u_1 *and* u_2 *are the same vertex, then in the equivalent MPT,* v_1 *and* v_2 *are vertices unordered by form.*

3. Suppose there is no path from vertex u_1 *to vertex* u_2 *through an arc whose parameter values change when the level of Factor A changes, and there is no path from vertex* u_2 *to vertex* u_1 *through an arc whose parameter values change when the level of Factor B changes. Then in the equivalent MPT* v_1 *and* v_2 *are vertices unordered by form.*

4. Suppose there is a path from vertex u_1 *to vertex* u_2 *through an arc whose parameter values change when the level of Factor A changes. Then in the equivalent MPT, there is a path from vertex* v_1 *to vertex* v_2 *through an arc whose parameter values change when the level of Factor A changes.*

Additionally, the proof of the following theorem uses necessary and sufficient conditions for two factors to selectively influence two processes in the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b). Those conditions are as follows:

Probability matrix $\mathbf{P} = \big(p(i, j)\big)$, $\mathbf{0} < \mathbf{P} < \mathbf{1}$, *and measure matrix* $\mathbf{T} = \big(t(i, j)\big)$ *are produced by Factor A and Factor B selectively influencing two different functionally ordered vertices in the Standard Binary Tree for Ordered Processes, with the vertex selectively influenced by Factor A preceding the vertex selectively influenced by Factor B, if and only if there is a level n of Factor B and for every level i of Factor A there are numbers* $r_i \geq 0$ *and* s_i, *such that the following conditions are true. Let* $max\{r_i\} = r_h$.

1. There is a constant k, $0 \leq k \leq 1$, *such that for every* $i \neq h$ *and* $j \neq n$

$$[p(i,j)p(h,n) - p(i,n)p(h,j)] = k\,[p(i,j) - p(h,j) - p(i,n) + p(h,n)].$$

2. For every j,

$$p(i,j) - k = r_i[p(h,j) - k].$$

3. For every j,

$$r_h r_i s_i [p(h, j) - p(h, n)]$$

$$= r_h [p(i, j)t(i, j) - p(i, n)t(i, n)] - r_i [p(h, j)t(h, j) - p(h, n)t(h, n)]. \quad (2.3)$$

Theorem 7. Suppose probability matrix **P**, **0** < **P** < **1**, and time matrix **T** are produced by Factor *A* and Factor *B* each selectively influencing a different vertex in a Multinomial Processing Tree with two response classes. Suppose there is a path from the vertex selectively influenced by Factor *A*, through an arc influenced by Factor *A*, to the vertex selectively influenced by Factor *B*. Suppose when levels of Factor *A* change, parameters on exactly two arcs descending from the vertex selectively influenced by Factor *A* change.

Then **P** and **T** are produced by Factor *A* and Factor *B* each selectively influencing a different vertex in an equivalent Standard Binary Tree for Ordered Processes, with the vertex selectively influenced by Factor *A* preceding the vertex selectively influenced by Factor *B*.

Proof. Suppose the premises of the theorem are true. Construct a Multinomial Processing Tree that is equivalent to the given tree but having the properties described in Lemma 2 (Schweickert & Zheng, 2018). The proof continues with the constructed tree.

Denote the vertices selectively influenced by Factor *A* and Factor *B* as v_1 and v_2, respectively. There is one simple path α from the source to v_1. Denote the probability path α is taken as p_α and the measure for path α as t_α.

There are three ways a correct response can be made.

1. A path goes from the source through v_1, then through v_2, and then to a terminal vertex for a correct response.

2. A path goes from the source through v_1, does not go through v_2, and goes to a terminal vertex for a correct response.

3. A path goes from the source to a terminal vertex for a correct response, but does not go through v_1 or v_2.

Note that it is not possible for a path to go from the source to v_2 without going through v_1, because if such a path existed there would be two paths from the source to v_2.

A correct response is made by way 1 as follows. Vertex v_1 is reached with probability p_α and associated measure t_α. Denote the arcs descending from v_1 as $e1$ and $e2$. Exactly one of those two arcs is on a path from v_1 to v_2; denote it as $e1$. With probability $p_{e1}(i)$ arc $e1$ is selected, with associated measure $t_{e1}(i)$.

There is a path β from the end vertex of $e1$ to v_2. With probability p_β this path is taken from the end vertex of $e1$ to v_2, with associated measure t_β. Denote the arcs descending from vertex v_2 as $f1, \ldots, fN$. At vertex v_2 a descending arc fq is taken with probability $p_{fq}(j)$ and associated measure $t_{fq}(j)$.

There is a subtree consisting of the end vertex of arc fq together with every arc and vertex preceded by fq. When the end vertex of arc fq is reached, there is a probability π_q that through this subtree a terminal vertex for a correct response is reached, with associated measure τ_q.

Hence, for every i and j the probability a correct response is made by way 1 is

$$p_1(i, j) = p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q \, .$$

Given that a correct response is made by way 1, let $t_1(i, j)$ denote the resulting measure. Then

$$p_1(i, j)t_1(i, j) = p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_\alpha + t_{e1}(i) + t_\beta + t_{fq}(j) + \tau_q].$$

A correct response made by way 2 as follows. Vertex v_1 is reached with probability p_α and associated measure t_α. Descending arc $e1$ of vertex v_1 could be selected, with probability $p_{e1}(i)$ and measure $t_{e1}(i)$, and the end vertex of arc $e1$ reached. Descending arc $e2$ of vertex v_1 could be selected, with probability $p_{e2}(i)$, and measure $t_{e2}(i)$, and the end vertex of arc $e2$ reached.

On a path to a correct response made by way 2, no parameter on an arc following the end vertex of $e1$ or $e2$ depends on the level of Factor A or Factor B. When the end vertex of arc $e1$ is reached, there is a probability π_1 that through this subtree a terminal vertex for a correct response is reached, with associated measure τ_2. For the end vertex of arc $e2$, the corresponding statement is true, with similar notation.

Hence, for every i and j the probability a correct response is made by way 2 is

$$p_2(i, j) = p_\alpha p_{e1}(i)\pi_1 + p_\alpha p_{e2}(i)\pi_2 \, .$$

Given that a correct response is made by way 2, let $t_2(i, j)$ denote the resulting measure. Then

$$p_2(i, j)t_2(i, j) = p_\alpha p_{e1}(i)\pi_1[t_\alpha + t_{e1}(i) + \tau_1] + p_\alpha p_{e2}(i)\pi_2[t_\alpha + t_{e2}(i) + \tau_2].$$

In way 3 for a correct response to be made, a path is followed from the source to a terminal vertex for a correct response, and the path goes through neither v_1 nor v_2. No arcs on this path have parameters that depend on the level *i* of Factor *A* or the level *j* of Factor *B*. We can denote the probability of a correct response made by way 3 as p_3 and the measure produced as t_3.

Hence, for every *i* and *j* the probability a correct response is made by way 3 is

$$p_3(i,j) = p_3.$$

Given that a correct response is made by way 3, let $t_3(i, j)$ denote the resulting measure. Then

$$p_3(i,j)t_3(i,j) = p_3t_3.$$

The three ways of making a correct response are mutually exclusive. Hence for every *i* and *j*,

$$p(i,j) = p_1(i,j) + p_2(i,j) + p_3(i,j) = p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q + p_\alpha p_{e1}(i)\pi_1 + p_\alpha p_{e2}(i)\pi_2 + p_3.$$

Further,

$$p(i,j)t(i,j) = p_1(i,j)t_1(i,j) + p_2(i,j)t_2(i,j) + p_3(i,j)t_3(i,j)$$

$$= p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_\alpha + t_{e1}(i) + t_\beta + t_{fq}(j) + \tau_q]$$

$$+ \ p_\alpha p_{e1}(i)\pi_1[t_\alpha + t_{e1}(i) + \tau_1] + p_\alpha p_{e2}(i)\pi_2[t_\alpha + t_{e2}(i) + \tau_2] + p_3 t_3. \quad (2.4)$$

We now show that the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes in Schweickert & Zheng (2017b) are satisfied.

For simplicity, let $\sum_{q=1}^{Q} p_{fq}(j)\pi_q = p_j$ and $p_{e1}(i) = p_i$. Then $p_{e2}(i) = 1 - p_i$.

$$p(i,j) = p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q \ + \ p_\alpha p_{e1}(i)\pi_1 + p_\alpha p_{e2}(i)\pi_2 + p_3$$

$$= p_\alpha p_i p_\beta p_j \ + \ p_\alpha p_i \pi_1 + p_\alpha [1 - p_i]\pi_2 + p_3.$$

Suppose that h is chosen such that $p_h = \max(p_i)$, and n is chosen such that

$$p(h,n)t(h,n) = \min\{p(h,j)t(h,j)\}.$$

Then we have

$$p(i,j)p(h,n) - p(i,n)p(h,j) =$$

$$\{p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha[1-p_i]\pi_2 + p_3\} \times \{p_\alpha p_h p_\beta p_n + p_\alpha p_h \pi_1 + p_\alpha[1-p_h]\pi_2 + p_3\}$$

$$-\{p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha[1-p_i]\pi_2 + p_3\} \times \{p_\alpha p_h p_\beta p_j + p_\alpha p_h \pi_1 + p_\alpha[1-p_h]\pi_2 + p_3\}$$

$$= \{p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha[1-p_i]\pi_2 + p_3\} \times p_\alpha p_h p_\beta p_n$$

$$+\{ p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_\alpha p_h \pi_1$$

$$+\{ p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_\alpha [1- p_h]\pi_2$$

$$+\{ p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_3$$

$$-\{ p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_\alpha p_h p_\beta p_j$$

$$-\{ p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_\alpha p_h \pi_1$$

$$-\{ p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_\alpha [1- p_h]\pi_2$$

$$-\{ p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\}\times p_3$$

$$= p_\alpha p_\beta (p_j - p_n)\times \{-p_h p_\alpha \pi_2 + p_h p_\alpha p_i \pi_2 - p_h p_3 + p_i p_\alpha \pi_2 - p_i p_\alpha p_h \pi_2 + p_i p_3\}$$

$$= p_\alpha p_\beta (p_j - p_n)\times \{-p_h p_\alpha \pi_2 - p_h p_3 + p_i p_\alpha \pi_2 + p_i p_3\}$$

$$= p_\alpha p_\beta (p_j - p_n)\times (p_\alpha \pi_2 + p_3)(p_i - p_h)$$

Also,

$$p(i,j) - p(h,j) - p(i,n) + p(h,n)$$

$$=\{ p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\} - \{ p_\alpha p_h p_\beta p_j + p_\alpha p_h \pi_1 + p_\alpha [1- p_h]\pi_2 + p_3\}$$

$$-\{ p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 + p_\alpha [1- p_i]\pi_2 + p_3\} + \{ p_\alpha p_h p_\beta p_n + p_\alpha p_h \pi_1 + p_\alpha [1- p_h]\pi_2 + p_3\}$$

$$=[p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 - p_\alpha p_i \pi_2] - [p_\alpha p_h p_\beta p_j + p_\alpha p_h \pi_1 - p_\alpha p_h \pi_2]$$

$$-[p_\alpha p_i p_\beta p_n + p_\alpha p_i \pi_1 - p_\alpha p_i \pi_2] + [p_\alpha p_h p_\beta p_n + p_\alpha p_h \pi_1 - p_\alpha p_h \pi_2]$$

$$= p_\alpha p_i p_\beta p_j - p_\alpha p_h p_\beta p_j - p_\alpha p_i p_\beta p_n + p_\alpha p_h p_\beta p_n$$

$$= p_\alpha p_\beta (p_i p_j - p_h p_j - p_i p_n + p_h p_n)$$

$$= p_\alpha p_\beta [p_j (p_i - p_h) - p_n (p_i - p_h)]$$

$$= p_\alpha p_\beta (p_j - p_n)(p_i - p_h)$$

then

$$[p(i,j)p(h,n) - p(i,n)p(h,j)]/ [p(i,j) - p(h,j) - p(i,n) + p(h,n)]$$

$$= p_\alpha p_\beta (p_j - p_n)\ (p_\alpha \pi_2 + p_3)(p_i - p_h) / [p_\alpha p_\beta (p_j - p_n)(p_i - p_h)]$$

$$= p_\alpha \pi_2 + p_3$$

Set $k = p_\alpha \pi_2 + p_3$.

The value of k doesn't change when i or j changes. Hence, Condition 1 of the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b) is satisfied.

Define $r_i = \frac{p_i}{p_h}$. Then $\max\{r_i\} = r_h$. Here, r_h=1.

For any i and j,

$$\begin{aligned}
p(i,j) - k &= p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1 - p_i]\pi_2 + p_3 - k \\
&= p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 + p_\alpha [1 - p_i]\pi_2 + p_3 - p_\alpha \pi_2 - p_3 \\
&= p_\alpha p_i p_\beta p_j + p_\alpha p_i \pi_1 - p_\alpha p_i \pi_2 \\
&= p_\alpha p_i (p_\beta p_j + \pi_1 - \pi_2) \\
&= \frac{p_i}{p_h} p_\alpha p_h (p_\beta p_j + \pi_1 - \pi_2) \\
&= \frac{p_i}{p_h} (p_\alpha p_h p_\beta p_j + p_\alpha p_h \pi_1 + p_\alpha [1 - p_h]\pi_2 + p_3 - p_\alpha \pi_2 - p_3) \\
&= r_i [p(h,j) - k].
\end{aligned}$$

Hence, Condition 2 of the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b) is satisfied.

For Condition3,

if $r_i = 0$, let $s_i = 0$;

for *i* such that $r_i \neq 0$, let $s_i = t_{e1}(i) - t_{e1}(h)$. Clearly, s_i does not depend on *j*.

For every *j*, the left hand side of (2.3) is

$$r_h r_i s_i [p(h,j) - p(h,n)]$$

$=$

$$r_h r_i s_i \{p_\alpha p_h p_\beta p_j + p_\alpha p_h \pi_1 + p_\alpha (1-p_h)\pi_2 + p_3 - [p_\alpha p_h p_\beta p_n + p_\alpha p_h \pi_1 + p_\alpha (1-p_h)\pi_2 + p_3]\}$$

$$= r_h r_i s_i (p_\alpha p_h p_\beta p_j - p_\alpha p_h p_\beta p_n) \qquad (2.5)$$

From equation (2.4),

$$p(i,j)t(i,j) = p_\alpha p_{e1}(i) p_\beta \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_\alpha + t_{e1}(i) + t_\beta + t_{fq}(j) + \tau_q]$$

$$+ \ p_\alpha p_{e1}(i)\pi_1 [t_\alpha + t_{e1}(i) + \tau_1] + p_\alpha p_{e2}(i)\pi_2 [t_\alpha + t_{e2}(i) + \tau_2] + p_3 t_3.$$

Remember that for simplicity, we let $\sum_{q=1}^{Q} p_{fq}(j)\pi_q = p_j$, $p_{e1}(i) = p_i$, and $p_{e2}(i) = 1 - p_i$.

With the simpler notation,

$$p(i,j)t(i,j) = p_\alpha p_i p_\beta \{p_j [t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_{fq}(j) + \tau_q]\}$$

$$+ \ p_\alpha p_i \pi_1 [t_\alpha + t_{e1}(i) + \tau_1] + p_\alpha (1-p_i)\pi_2 [t_\alpha + t_{e2}(i) + \tau_2] + p_3 t_3.$$

Then

$$p(i,j)t(i,j) - p(i,n)t(i,n)$$

$$= p_\alpha p_i p_\beta \{ p_j [t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_{fq}(j) + \tau_q]\}$$

$$+ p_\alpha p_i \pi_1 [t_\alpha + t_{e1}(i) + \tau_1] + p_\alpha (1 - p_i)\pi_2 [t_\alpha + t_{e2}(i) + \tau_2] + p_3 t_3$$

$$- p_\alpha p_i p_\beta \{ p_n [t_\alpha + t_{e1}(i) + t_\beta] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q [t_{fq}(j) + \tau_q]\} - p_\alpha p_i \pi_1 [t_\alpha + t_{e1}(i) + \tau_1]$$

$$- p_\alpha (1 - p_i)\pi_2 [t_\alpha + t_{e2}(i) + \tau_2] - p_3 t_3$$

$$= p_\alpha p_i p_\beta \{ p_j [t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_{fq}(j) + \tau_q]\}$$

$$- p_\alpha p_i p_\beta \{ p_n [t_\alpha + t_{e1}(i) + t_\beta] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q [t_{fq}(n) + \tau_q]\}$$

$$= p_\alpha p_i p_\beta \{ (p_j - p_n)[t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_{fq}(j) + \tau_q]$$
$$- \sum_{q=1}^{Q} p_{fq}(n)\pi_q [t_{fq}(n) + \tau_q]\}$$

Similarly, we can get

$$p(h,j)t(h,j) - p(h,n)t(h,n) =$$

$$p_\alpha p_h p_\beta \{ (p_j - p_n)[t_\alpha + t_{e1}(h) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q [t_{fq}(j) + \tau_q]$$
$$- \sum_{q=1}^{Q} p_{fq}(n)\pi_q [t_{fq}(n) + \tau_q]\}$$

because $r_h = 1$, the right hand side of (2.3) is

$$r_h[p(i,j)t(i,j) - p(i,n)t(i,n)] - r_i[p(h,j)t(h,j) - p(h,n)t(h,n)]$$

$$= r_h\{p_\alpha p_i p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}\}$$

$$- r_i\{p_\alpha p_h p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(h) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}\}$$

$$= p_\alpha p_i p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}$$

$$- \frac{p_i}{p_h} p_\alpha p_h p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(h) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}$$

$$= p_\alpha p_i p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(i) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}$$

$$- p_\alpha p_i p_\beta\{(p_j - p_n)[t_\alpha + t_{e1}(h) + t_\beta] + \sum_{q=1}^{Q} p_{fq}(j)\pi_q[t_{fq}(j) + \tau_q] - \sum_{q=1}^{Q} p_{fq}(n)\pi_q[t_{fq}(n) + \tau_q]\}$$

$$= p_\alpha p_i p_\beta(p_j - p_n)[t_{e1}(i) - t_{e1}(h)] \tag{2.6}$$

To show (2.3) is true, we only need show the right hand sides of (2.5) and (2.6) are equal.

Suppose $p_j - p_n = 0$. Then the right hand side of (2.5) is

$$r_h r_i s_i(p_\alpha p_h p_\beta p_j - p_\alpha p_h p_\beta p_n) = r_h r_i s_i p_\alpha p_h p_\beta(p_j - p_n) = 0,$$

and the right hand side of (2.6) is

$$p_\alpha p_i p_\beta(p_j - p_n)[t_{e1}(i) - t_{e1}(h)] = 0.$$

Suppose $p_j - p_n \neq 0$. We can divide the right hand side of (2.6) by $(p_j - p_n)$. We get

$$p_\alpha p_i p_\beta [t_{e1}(i) - t_{e1}(h)] = p_\alpha r_i p_h p_\beta [t_{e1}(i) - t_{e1}(h)] = p_\alpha r_i p_h p_\beta s_i = r_i r_h p_\alpha p_h p_\beta s_i,$$

where the last line follows because $r_h = 1$.

The above expression equals the right hand side of (2.5) divided by $(p_j - p_n)$. Hence, the Condition 3 of the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b) is satisfied.

Because Conditions 1, 2, and 3 are true, **P** and **T** are produced by Factor *A* and Factor *B* each selectively influencing a different vertex in the equivalent Standard Binary Tree for Ordered Processes, with the vertex selectively influenced by Factor *A* preceding the vertex selectively influenced by Factor *B*.

□

Mixtures of Trees With Factors Selectively Influencing Ordered Processes

Suppose both response probabilities and response time are observed. The following theorem is about conditions required for a mixture of trees to be equivalent to the Standard Binary Tree for Ordered Processes. The proof for the following theorem is based on the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b)

Theorem 8. Suppose probability matrix **P**, **0 < P < 1**, and time matrix **T** are produced by Factors *A* and *B* selectively influencing processes in each tree of a mixture of trees. Suppose there is a path from the vertex selectively influenced by Factor *A*, through an arc influenced by Factor *A*, to the vertex selectively influenced by Factor *B*.

Suppose when levels of Factor *A* change, parameters on exactly two arcs descending from the vertex selectively influenced by Factor *A* change.

In tree *z* in the mixture of trees, there is a single edge that precedes the process influenced by Factor *B* and whose probability is dependent on the level of Factor *A*. When Factor *A* is at level *i*, let $e_z(i)$ be this probability in tree *z*. Suppose for every level *i* of Factor *A* there is a probability $e(i)$ and for every tree *z* there is a number c_z such that $e_z(i) = c_z e(i)$.

Then **P** and **T** are produced by two factors selectively influencing two ordered vertices in the Standard Binary Tree for Ordered Processes, with the vertex influenced by Factor *A* preceding the vertex influenced by Factor *B*.

Proof. Each tree *z* in the mixture is equivalent to the Standard Binary Tree for Ordered Processes. When Factor *A* is at level *i* and Factor *B* is at level *j*, we can write the probability of a correct response as

$$p_z(i, j) = p_{Az}(i)p_D + p_{Bz}(i)p_{Fz}(j)$$

and the product of probability and response time of a correct response for tree *z* as

$$p_z(i, j)t_z(i, j) = p_{Az}p_{Dz}(i)[t_{Az}(i) + t_{Dz}] + p_{Bz}(i)p_{Fz}(j)[t_{Bz}(i) + t_{Fz}(j)].$$

The first two conditions about probability have been proved in Theorem 10 in Schweickert and Chen (2008). To be consistent with the former work, here, we set $p_B(i) = e(i)$ and $r_i = \frac{p_B(i)}{p_B(h)}$.

We start to prove Condition 3.

Suppose tree *z* is selected with probability τ_z. For the probability of the mixture of trees, we have

$$p(i, j) = \sum \tau_z p_z(i, j) = \sum \tau_z [p_{Az}(i) p_D + p_{Bz}(i) p_{Fz}(j)] = \sum \tau_z p_{Az}(i) p_D + \sum \tau_z p_{Bz}(i) p_{Fz}(j)$$

where all sums are over *z*, and $\sum \tau_z = 1$.

From Theorem 10 in Schweickert and Chen (2008), we know that $p_{Bz}(i)$ in the standard tree for tree z can be set equal to $e_z(i)$ in the original tree z, so here we let $p_{Bz}(i)$ $= e_z(i) = c_z p_B(i)$. Then

$$p(i, j) = \sum \tau_z p_z(i, j) = \sum \tau_z p_{Az}(i) p_D + \sum \tau_z p_{Bz}(i) p_{Fz}(j) = \sum \tau_z p_{Az}(i) p_D + p_B(i) \sum \tau_z c_z p_{Fz}(j).$$

Let $p_F(j) = \sum \tau_z c_z p_{Fz}(j)$.

If $p_B(i) \sum \tau_z c_z p_{Fz}(j) = 0$, then let $t_B(i) = 0$ and $t_F(j) = 0$; Otherwise, let

$$t_B(i) = \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Bz}(i) \Big/ p_B(i) \sum \tau_z c_z p_{Fz}(j) = \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Bz}(i) \Big/ p_B(i) p_F(j),$$

$$t_F(j) = \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Fz}(j) \Big/ p_B(i) \sum \tau_z c_z p_{Fz}(j) = \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Fz}(j) \Big/ p_B(i) p_F(j).$$

Choose *h* such that $p_B(h) = \max\{p_B(i)\}$. If $p_B(h) = 0$, then for every *i*, $p_B(i) = 0$. Then factor *B* is not effective. Hence, $p_B(h) \neq 0$.

We also set $r_i = \dfrac{p_B(i)}{p_B(h)}$, $r_h = \max(\dfrac{p_B(i)}{p_B(h)}) = 1$, $s_i = t_B(i) - t_B(h)$. Besides, we chose n such that $p(h,n)t(h,n) = \min\{p(h, j)t(h, j)\}$.

We get,

$$p(h,j)-p(h,n)=\sum\tau_z p_{Az}(h)p_D+\sum\tau_z p_{Bz}(h)p_{Fz}(j)-\sum\tau_z p_{Az}(h)p_D-\sum\tau_z p_{Bz}(h)p_{Fz}(n)$$

$$=p_B(h)\sum\tau_z c_z[p_{Fz}(j)-p_{Fz}(n)].$$

Then the left side of (2.3) is

$$r_h r_i s_i[p(h,j)-p(h,n)]=r_h r_i s_i p_B(h)\sum\tau_z c_z[p_{Fz}(j)-p_{Fz}(n)]$$

$$=\frac{p_B(i)}{p_B(h)}s_i p_B(h)\sum\tau_z c_z[p_{Fz}(j)-p_{Fz}(n)]$$

$$=s_i p_B(i)\sum\tau_z c_z[p_{Fz}(j)-p_{Fz}(n)]$$

$$=[t_B(i)-t_B(h)]p_B(i)[p_F(j)-p_F(n)]$$

For the product of probabilities and response times of the mixture of trees, we have

$$p(i,j)t(i,j)=\sum\tau_z p_z(i,j)t_z(i,j)$$

$$=\sum\tau_z\{p_{Az}p_{Dz}(i)[t_{Az}(i)+t_{Dz}]+p_{Bz}(i)p_{Fz}(j)[t_{Bz}(i)+t_{Fz}(j)]\}$$

$$=\sum\tau_z p_{Az}p_{Dz}(i)[t_{Az}(i)+t_{Dz}]+\sum\tau_z p_{Bz}(i)p_{Fz}(j)[t_{Bz}(i)+t_{Fz}(j)]$$

Similarly,

$$p(i,n)t(i,n)=\sum\tau_z p_z(i,n)t_z(i,n)$$

$$=\sum\tau_z p_{Az}p_{Dz}(i)[t_{Az}(i)+t_{Dz}]+\sum\tau_z p_{Bz}(i)p_{Fz}(n)[t_{Bz}(i)+t_{Fz}(n)]$$

$$p(h,j)t(h,j)=\sum\tau_z p_z(h,j)t_z(h,j)$$

$$=\sum\tau_z p_{Az}p_{Dz}(h)[t_{Az}(h)+t_{Dz}]+\sum\tau_z p_{Bz}(h)p_{Fz}(j)[t_{Bz}(h)+t_{Fz}(j)]$$

$$p(h,n)t(h,n) = \sum \tau_z p_z(h,n) t_z(h,n)$$

$$= \sum \tau_z p_{Az} p_{Dz}(h)[t_{Az}(h) + t_{Dz}] + \sum \tau_z p_{Bz}(h) p_{Fz}(n)[t_{Bz}(h) + t_{Fz}(n)].$$

Then the right side of (2.3) is

$$r_h[p(i,j)t(i,j) - p(i,n)t(i,n)] - r_i[p(h,j)t(h,j) - p(h,n)t(h,n)]$$

$$= r_h\{\sum \tau_z p_{Az} p_{Dz}(i)[t_{Az}(i) + t_{Dz}] + \sum \tau_z p_{Bz}(i) p_{Fz}(j)[t_{Bz}(i) + t_{Fz}(j)]$$

$$- \sum \tau_z p_{Az} p_{Dz}(i)[t_{Az}(i) + t_{Dz}] - \sum \tau_z p_{Bz}(i) p_{Fz}(n)[t_{Bz}(i) + t_{Fz}(n)]\}$$

$$- \frac{p_B(i)}{p_B(h)}\{\sum \tau_z p_{Az} p_{Dz}(h)[t_{Az}(h) + t_{Dz}] + \sum \tau_z p_{Bz}(h) p_{Fz}(j)[t_{Bz}(h) + t_{Fz}(j)]$$

$$- \sum \tau_z p_{Az} p_{Dz}(h)[t_{Az}(h) + t_{Dz}] - \sum \tau_z p_{Bz}(h) p_{Fz}(n)[t_{Bz}(h) + t_{Fz}(n)]\}$$

$$= \sum \tau_z p_{Bz}(i) p_{Fz}(j)[t_{Bz}(i) + t_{Fz}(j)] - \sum \tau_z p_{Bz}(i) p_{Fz}(n)[t_{Bz}(i) + t_{Fz}(n)]$$

$$- \frac{p_B(i)}{p_B(h)}\{\sum \tau_z p_{Bz}(h) p_{Fz}(j)[t_{Bz}(h) + t_{Fz}(j)] - \sum \tau_z p_{Bz}(h) p_{Fz}(n)[t_{Bz}(h) + t_{Fz}(n)]\}$$

$$= p_B(i)\{\sum \tau_z c_z p_{Fz}(j)[t_{Bz}(i) + t_{Fz}(j)] - \sum \tau_z c_z p_{Fz}(n)[t_{Bz}(i) + t_{Fz}(n)]\}$$

$$- \frac{p_B(i)}{p_B(h)} p_B(h)\{\sum \tau_z c_z p_{Fz}(j)[t_{Bz}(h) + t_{Fz}(j)] - \sum \tau_z c_z p_{Fz}(n)[t_{Bz}(h) + t_{Fz}(n)]\}$$

$$= p_B(i)\{\sum \tau_z c_z p_{Fz}(j)[t_{Bz}(i) + t_{Fz}(j)] - \sum \tau_z c_z p_{Fz}(n)[t_{Bz}(i) + t_{Fz}(n)]$$

$$- \sum \tau_z c_z p_{Fz}(j)[t_{Bz}(h) + t_{Fz}(j)] + \sum \tau_z c_z p_{Fz}(n)[t_{Bz}(h) + t_{Fz}(n)]\}$$

$$= \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Bz}(i) - \sum \tau_z p_{Bz}(i) p_{Fz}(n) t_{Bz}(i)$$

$$- \sum \tau_z p_{Bz}(i) p_{Fz}(j) t_{Bz}(h) + \sum \tau_z p_{Bz}(i) p_{Fz}(n) t_{Bz}(h)$$

$$= p_B(i)p_F(j)t_B(i) - p_B(i)p_F(n)t_B(i) - p_B(i)p_F(j)t_B(h) + p_B(i)p_F(n)t_B(h)$$

$$= [t_B(i) - t_B(h)]p_B(i)[p_F(j) - p_F(n)],$$

which is the left side of (2.3). Hence, the Condition 3 is satisfied.

Then Conditions 1, 2, and 3 of the necessary and sufficient conditions for the Standard Binary Tree for Ordered Processes (Schweickert & Zheng, 2017b) are satisfied, so **P** and **T** are produced by Factors *A* and *B* selectively influencing two processes in the Standard Binary Tree for Ordered Processes, with the vertex influenced by Factor *A* preceding the vertex influenced by Factor *B*.

□

CONCLUSION

Since MPT models are widely used in various domains, conditions for the nonnegativity of time parameters, conditions for the mixture of trees, and the uniqueness of trees in terms of both parameters and structure are worthwhile detecting. Our work augmented MPT models by dealing with response times, which can be greatly helpful in several ways. First, MPT models with only probability may be not enough for describing some phenomena. MPT models with both probability and time can make models more widely used and firmly built. Second, most testing of MPT models is based on goodness of fit, and the technique of selective influence provides an alternative way to test the MPT models. Including hypotheses about response time makes the tests more stringent. Last, our theorems about the uniqueness of parameters, including response time, allow researchers to calculate the degrees of freedom of MPT models. A main disadvantage is that the possibility of feedback is ignored, which could be one future direction based on current study. But MPT models do not differ in this respect from most other prominent models for response time, including the accumulator model (e.g., Luce, 1986) and the diffusion model (Ratcliff, 1978). Extending MPT models to response time and other measures makes MPT models more useful and more widely applicable.

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APPENDIX A

Proof for Theorem 1

(a) Suppose $p^*_A, p^*_B, p^*_D(i), p^*_F(j)$, $t^*_A, t^*_B, t^*_D(i)$, and $t^*_F(j)$ exist with

$$p(i,j)t(i,j) = p^*_A\, p^*_D(i)[t^*_A + t^*_D(i)] + p^*_B\, p^*_F(j)[t^*_B + t^*_F(j)]$$

Also, for any *i* and *j*, there exists

$$p(i,j)t(i,j) = p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)]$$

Let *j* and *j'* be two different values of *j*. Then we have

$$p(i,j)t(i,j) - p(i,j')t(i,j')$$

$$= p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)]$$

$$- p_A p_D(i)[t_A + t_D(i)] - p_B p_F(j')[t_B + t_F(j')]$$

$$= p_B p_F(j)[t_B + t_F(j)] - p_B p_F(j')[t_B + t_F(j')] \qquad (2.1.1)$$

Also, because there is

$$p(i,j)t(i,j) = p^*_A\, p^*_D(i)[t^*_A + t^*_D(i)] + p^*_B\, p^*_F(j)[t^*_B + t^*_F(j)],$$

we have

$$p(i,j)t(i,j) - p(i,j')t(i,j')$$

$$= p^*_B\, p^*_F(j)[t^*_B + t^*_F(j)] - p^*_B\, p^*_F(j')[t^*_B + t^*_F(j')] \qquad (2.1.2)$$

Because Equation (2.1.1) and Equation (2.1.2) have the same left side, the right sides should be the same, which is,

$$p_B p_F(j)[t_B + t_F(j)] - p_B p_F(j')[t_B + t_F(j')]$$

$$= p^*_B\, p^*_F(j)[t^*_B + t^*_F(j)] - p^*_B\, p^*_F(j')[t^*_B + t^*_F(j')]$$

Replace p^*_A, $p^*_D(i)$, $p^*_F(j)$ with p_A, $p_D(i)$, $p_F(j)$ using the equations:

$$p^*_A = p_A / c,$$

$$p^*_D(i) = cp_D(i) + d,$$

$$p^*_F(j) = c(1 - p_A)p_F(j)/(c - p_A) - p_A d/(c - p_A),$$

(Schweickert and Chen, 2008)

There is

$$p_B p_F(j)[t_B + t_F(j)] - p_B p_F(j')[t_B + t_F(j')]$$

$$= (1 - p_A/\mathrm{c})[c(1 - p_A)p_F(j)/(c - p_A) - p_A d/(c - p_A)][t^*_B + t^*_F(j)]$$

$$-(1 - p_A/\mathrm{c})[c(1 - p_A)p_F(j')/(c - p_A) - p_A d/(c - p_A)][t^*_B + t^*_F(j')]$$

Equivalently,

$$(1 - p_A)p_F(j)[t_B + t_F(j)] - (1 - p_A)p_F(j')[t_B + t_F(j')]$$

$$= [(1 - p_A)p_F(j) - p_A d/c][t^*_B + t^*_F(j)] - [(1 - p_A)p_F(j') - p_A d/c][t^*_B + t^*_F(j')]$$

Divide $(1 - p_A)$ in both sides of the equation, there is

$$p_F(j)[t_B + t_F(j)] - p_F(j')[t_B + t_F(j')]$$

$$= \{p_F(j) - p_A d/[c(1 - p_A)]\}[t^*_B + t^*_F(j)]$$

$$-\{p_F(j') - p_A d/[c(1 - p_A)]\}[t^*_B + t^*_F(j')]$$

Then,

$$p_F(j)[t_B + t_F(j)] - p_F(j')[t_B + t_F(j')]$$

$$= p_F(j)[t^*_B + t^*_F(j)] - \frac{p_A d}{[c(1 - p_A)]}[t^*_F(j) - t^*_F(j')] - p_F(j')[t^*_B + t^*_F(j')]$$

Equally, for any *j* such that $t^*_F(j) \neq t^*_F(j')$, we have

$$\frac{p_F(j)[t_B + t_F(j) - t^*_B - t^*_F(j)] - p_F(j')[t_B + t_F(j') - t^*_B - t^*_F(j')]}{t^*_F(j) - t^*_F(j')}$$

$$= -\frac{p_A d}{[c(1-p_A)]}$$

Then,

$$t_B - t^*_B = \frac{-\frac{p_A d}{[c(1-p_A)]}[t^*_F(j) - t^*_F(j')] + p_F(j')[t_F(j') - t^*_F(j')] - p_F(j)[t_F(j) - t^*_F(j)]}{p_F(j) - p_F(j')}$$

Because the left part of the equation won't change when *j* changes, so we can set

$$t_B - t^*_B = \frac{-\frac{p_A d}{[c(1-p_A)]}[t^*_F(j) - t^*_F(j')] + p_F(j')[t_F(j') - t^*_F(j')] - p_F(j)[t_F(j) - t^*_F(j)]}{p_F(j) - p_F(j')} = h$$

where *h* is a constant.

So we get

$$t^*_B = t_B - h$$

Then,

$$t^*_F(j) = \frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t^*_F(j')] - \frac{p_A d}{[c(1-p_A)]}t^*_F(j')}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}$$

Similarly, let *i* and *i'* be two different values of *i*, we have

$$p(i,j)t(i,j) - p(i',j)t(i',j)$$

$$= p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)]$$

$$- p_A p_D(i')[t_A + t_D(i')] - p_B p_F(j)[t_B + t_F(j)]$$

$$= p_A p_D(i)[t_A + t_D(i)] - p_A p_D(i')[t_A + t_D(i')] \quad (2.1.3)$$

Also, there is

$$p(i,j)t(i,j) - p(i',j)t(i',j)$$

$$= p*_A \, p*_D(i)[t*_A + t*_D(i)] - p*_A \, p*_D(i')[t*_A + t*_D(i')] \quad (2.1.4)$$

Because Equation (2.1.3) and Equation (2.1.4) have the same left side, the right sides should be the same, which is,

$$p_A p_D(i)[t_A + t_D(i)] - p_A p_D(i')[t_A + t_D(i')]$$

$$= p*_A \, p*_D(i)[t*_A + t*_D(i)] - p*_A \, p*_D(i')[t*_A + t*_D(i')]$$

Replace $p*_A$ and $p*_D(i)$ with p_A and $p_D(i)$ using the equations:

$$p*_A = p_A / c,$$

$$p*_D(i) = cp_D(i) + d,$$

There is

$$p_A p_D(i)[t_A + t_D(i)] - p_A p_D(i')[t_A + t_D(i')]$$

$$= (p_A/c)[cp_D(i) + d][t*_A + t*_D(i)] - (p_A/c)[cp_D(i') + d][t*_A + t*_D(i')]$$

Equivalently,

$$p_A p_D(i)[t_A + t_D(i)] - p_A p_D(i')[t_A + t_D(i')]$$

$$= p_A[p_D(i) + d/c][t*_A + t*_D(i)] - p_A[p_D(i') + d/c][t*_A + t*_D(i')]$$

Divide p_A in both sides of the equation, there is

$$p_D(i)[t_A + t_D(i)] - p_D(i')[t_A + t_D(i')]$$

$$= [p_D(i) + d/c][t*_A + t*_D(i)] - [p_D(i') + d/c][t*_A + t*_D(i')]$$

Then,

$$[p_D(i) - p_D(i')][t_A - t*_A] + p_D(i)[t_D(i) - t*_D(i)] - p_D(i')[t_D(i') - t*_D(i')]$$

$$= d/c[t*_D(i) - t*_D(i')]$$

Equally, for any *i* such that $p_D(i) \neq p_D(i')$, we have

$$t_A - t*_A = \frac{d/c[t*_D(i) - t*_D(i')] - p_D(i)[t_D(i) - t*_D(i)] + p_D(i')[t_D(i') - t*_D(i')]}{p_D(i) - p_D(i')}$$

Because the left part of the equation won't change when *i* changes, we can set

$$t_A - t*_A =$$

$$\frac{d/c[t*_D(i) - t*_D(i')] - p_D(i)[t_D(i) - t*_D(i)] + p_D(i')[t_D(i') - t*_D(i')]}{p_D(i) - p_D(i')} = m$$

where m is a constant.

So $t*_A = t_A - m$

Then because

$$p(i, j)t(i, j)$$

$$= p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)]$$

$$= p*_A \, p*_D(i)[t*_A + t*_D(i)] + p*_B \, p*_F(j)[t*_B + t*_F(j)]$$

With the relationships

$$p*_A = p_A/c,$$

$$p*_D(i) = cp_D(i) + d,$$

$$p*_F(j) = c(1 - p_A)p_F(j)/(c - p_A) - p_A d/(c - p_A),$$

$$t*_A = t_A - m,$$

$$t*_B = t_B - h,$$

and

$$t*_F(j) = \frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1 - p_A)]} t*_F(j')}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)},$$

we have

$$p_A p_D(i)[t_A + t_D(i)] + (1 - p_A) p_F(j)[t_B + t_F(j)]$$

$$= (p_A/c) \cdot [cp_D(i) + d][t_A - m + t*_D(i)]$$

$$+ (1 - p_A/c)[c(1 - p_A) p_F(j)/(c - p_A) - p_A d/(c - p_A)] \times$$

$$[t_B - h + \frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1 - p_A)]} t*_F(j')}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)}]$$

$$= p_A[p_D(i) + \frac{d}{c}][t_A - m + t*_D(i)] + [(1 - p_A) p_F(j) - p_A d/c] \times$$

$$[t_B + \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1 - p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)}]$$

So,

$$p_A[p_D(i) + \frac{d}{c}][t_A - m + t*_D(i)]$$

$$= p_A p_D(i)[t_A + t_D(i)] + (1 - p_A) p_F(j)[t_B + t_F(j)] - [(1 - p_A) p_F(j) - p_A d/c] \times$$

$$[t_B + \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1 - p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)}]$$

Equivalently,

$$p_A[p_D(i) + \frac{d}{c}]t*_D(i)$$

$$= p_A p_D(i)[t_D(i) + m] + (1 - p_A) p_F(j)[t_B + t_F(j)]$$

$$- p_A \frac{d}{c}[t_A - m] - [(1 - p_A) p_F(j) - p_A d/c] \times$$

$$[t_B + \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1-p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}]$$

$$= p_A p_D(i)[t_D(i) + m] + (1-p_A)p_F(j)[t_B + t_F(j)] - p_A \frac{d}{c}[t_A - m]$$

$$-(1-p_A)p_F(j) \times$$

$$[t_B + \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1-p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}]$$

$$+ p_A \frac{d}{c} \times [t_B + \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1-p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}]$$

$$= p_A p_D(i)[t_D(i) + m] + p_A \frac{d}{c} \times [t_B - t_A + m]$$

$$+ (1-p_A)p_F(j) \frac{hp_F(j') + p_F(j')[t_F(j') - t*_F(j')] + \frac{p_A d}{[c(1-p_A)]}[t*_F(j') - h] - \frac{p_A d}{[c(1-p_A)]} t_F(j)}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}$$

$$+ p_A \frac{d}{c} \times \frac{p_F(j)t_F(j) - hp_F(j') - p_F(j')[t_F(j') - t*_F(j')] - \frac{p_A d}{[c(1-p_A)]}[t*_F(j') - h]}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)} =$$

$$p_A p_D(i)[t_D(i) + m] + p_A \frac{d}{c} \times [t_B - t_A + m] + \frac{(1-p_A)p_F(j)hp_F(j')}{-\frac{p_A d}{[c(1-p_A)]} + p_F(j)}$$

$$-\frac{p_A \frac{d}{c} h p_F(j')}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}+\frac{(1-p_A)p_F(j)p_F(j')[t_F(j')-t*_F(j')]}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}$$

$$-\frac{p_A \frac{d}{c} p_F(j')[t_F(j')-t*_F(j')]}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}+\frac{(1-p_A)p_F(j)\frac{p_A d}{[c(1-p_A)]}[t*_F(j')-h]}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}$$

$$-\frac{p_A \frac{d}{c}\frac{p_A d}{[c(1-p_A)]}[t*_F(j')-h]}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}+\frac{p_A \frac{d}{c} p_F(j) t_F(j)}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}-\frac{p_F(j)\frac{p_A d}{c} t_F(j)}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}$$

$$= p_A p_D(i)[t_D(i)+m] + p_A \frac{d}{c}\times[t_B-t_A+m]+\frac{[(1-p_A)p_F(j)-p_A\frac{d}{c}]}{-\frac{p_A d}{[c(1-p_A)]}+p_F(j)}\times$$

$$\{hp_F(j')+p_F(j')[t_F(j')-t*_F(j')]+\frac{p_A d}{[c(1-p_A)]}[t*_F(j')-h]\}$$

$$= p_A p_D(i)[t_D(i)+m]+p_A\frac{d}{c}\times[t_B-t_A+m]+(1-p_A)\{hp_F(j')$$

$$+p_F(j')[t_F(j')-t*_F(j')]+\frac{p_A d}{[c(1-p_A)]}[t*_F(j')-h]\}$$

$$= p_A p_D(i)[t_D(i)+m]+p_A\frac{d}{c}\times[t_B-t_A+m]+(1-p_A)\{hp_F(j')$$

$$+(1-p_A)p_F(j')[t_F(j')-t*_F(j')]+\frac{p_A d}{c}[t*_F(j')-h]\}$$

$$= p_A p_D(i)[t_D(i)+m] + p_A\frac{d}{c}\times[t_B-t_A+m+t*_F(j')-h]$$

$$+(1-p_A)p_F(j')[t_F(j')-t*_F(j')+h]$$

So we have here

$$p_A[p_D(i)+\frac{d}{c}]t^*_D(i)$$

$$= p_A p_D(i)[t_D(i)+m] + p_A \frac{d}{c}\times[t_B - t_A + m + t^*_F(j') - h]$$

$$+(1-p_A)p_F(j')[t_F(j') - t^*_F(j') + h]$$

So $t^*_D(i) =$

$$\frac{p_A p_D(i)[t_D(i)+m] + p_A \frac{d}{c}[t_B - t_A + m + t^*_F(j') - h] + (1-p_A)p_F(j')[t_F(j') - t^*_F(j') + h]}{p_A[p_D(i)+\frac{d}{c}]}$$

$$= \frac{p_D(i)[t_D(i)+m] + \frac{d}{c}\times[t_B - t_A + m + t^*_F(j') - h] + (\frac{1}{p_A}-1)p_F(j')[t_F(j') - t^*_F(j') + h]}{p_D(i)+\frac{d}{c}}$$

Then for the bounds of *c*, *d*, *m*, and *h*, from Schweickert and Chen (2008), we know that,

$$p_A < c$$

$$-c\min\{p_D(i)\} \le d \le 1 - c\max\{p_D(i)\}$$

$$[p_A - c + (1-p_A)c\max\{p_F(j)\}]/p_A \le d \le [(1-p_A)c\min\{p_F(j)\}]/p_A$$

Because,

$$t^*_A = t_A - m$$

$$t^*_B = t_B - h$$

$$t^*_A, t^*_B, t_A, t_B \ge 0$$

We get the bounds of constants *m* and *h* as

$$\max\{-t^*_A\} \le m \le \min\{t_A\}$$

$$\max\{-t^*_B\} \le h \le \min\{t_B\}$$

b) Conversely, suppose there exists $p^*_A, p^*_B, p^*_D(i), p^*_F(j), t^*_A, t^*_B, t^*_D(i), t^*_F(j)$

such that

$$0 \le p^*_A, p^*_B, p^*_D(i), p^*_F(j) \le 1$$

$$t^*_A, t^*_B, t^*_D(i), t^*_F(j) \ge 0$$

With all the following equations,

$$p^*_A = p_A / c,$$

$$p^*_D(i) = cp_D(i) + d,$$

$$p^*_F(j) = c(1 - p_A)p_F(j)/(c - p_A) - p_A d/(c - p_A).$$

$$t^*_A = t_A - m$$

$$t^*_B = t_B - h$$

$$t^*_D(i) = \frac{p_D(i)[t_D(i) + m] + \frac{d}{c} \times [t_B - t_A + m + t^*_F(j') - h] + (\frac{1}{p_A} - 1)p_F(j')[t_F(j') - t^*_F(j') + h]}{p_D(i) + \frac{d}{c}}$$

$$t^*_F(j) = \frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t^*_F(j')] - \frac{p_A d}{[c(1 - p_A)]} t^*_F(j')}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)}$$

as well as the bounds for *c*, *d*, *h*, and *m*,

$$p_A < c,$$

$$-c \min\{p_D(i)\} \le d \le 1 - c \max\{p_D(i)\},$$

$$[p_A - c + (1 - p_A)c \max\{p_F(j)\}]/p_A \le d \le [(1 - p_A)c \min\{p_F(j)\}]/p_A,$$

(Schweickert & Chen, 2008)

$$\max\{-t^*_A\} \le m \le \min\{t_A\}$$

$$\max\{-t*_B\} \le h \le \min\{t_B\}$$

Then,

$$p*_A\ p*_D\ (i)[t*_A + t*_D\ (i)] + p*_B\ p*_F\ (j)[t*_B + t*_F\ (j)]$$

$$= \frac{p_A}{c}[cp_D(i) + d][t_A - m +$$

$$\frac{p_D(i)[t_D(i) + m] + \frac{d}{c} \times [t_B - t_A + m + t*_F\ (j') - h] + (\frac{1}{p_A} - 1)p_F(j')[t_F(j') - t*_F\ (j') + h]}{p_D(i) + \frac{d}{c}}]$$

$$+ (1 - \frac{p_A}{c})[c(1 - p_A)p_F(j)/(c - p_A) - p_A d/(c - p_A)][t_B - h +$$

$$\frac{p_F(j)t_F(j) + h[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F\ (j')] - \frac{p_A d}{[c(1 - p_A)]}t*_F\ (j')}{-\frac{p_A d}{[c(1 - p_A)]} + p_F(j)}]$$

$$= p_A p_D(i)[t_A + t_D(i)] + \frac{d}{c}p_A[t_B + t*_F\ (j') - h] + p_B p_F(j')[t_F(j') - t*_F\ (j') + h]$$

$$+ (1 - p_A)p_F(j)[t_B + t_F(j)] - (1 - p_A)p_F(j')[t_F(j') - t*_F\ (j') + h]$$

$$- \frac{p_A d}{c}[t*_F\ (j') + t_B - h]$$

$$= p_A p_D(i)[t_A + t_D(i)] + p_B p_F(j)[t_B + t_F(j)]$$

$$= p(i, j)t(i, j)$$

Additionally, the degree of freedom is calculated as below

$$3IJ-(I+J+1)-(2+2I+2J)+4+4=3IJ-3I-3J+5$$

for the total observations are $3IJ$, the number of arc probabilities is $(I+J+1)$, the number of arc times is $(2+2I+2J)$, and there are four constants c, d, h, m in the calculation for situation of correct answer as well as 4 constants for situation of incorrect response. Here we exclude the situation where p_A is 0 or 1, for that leads to a totally different processing tree.

APPENDIX B

Proof for Theorem 3

a) Suppose $p*_A(i)$, $p*_D$, $p*_B(i)$, $p*_F(j)$, $t*_A(i), t*_B(i), t*_D$, and $t*_F(j)$ exist with

$$p(i,j)t(i,j) = p*_A(i)p*_D[t*_A(i)+t*_D] + p*_B(i)p*_F(j)[t*_B(i)+t*_F(j)]$$

Also, for any *i* and *j*, there is

$$p(i,j)t(i,j) = p_A(i)p_D[t_A(i)+t_D] + p_B(i)p_F(j)[t_B(i)+t_F(j)]$$

Let *j* and *j'* be two different values of *j*. Then we have

$$p(i,j)t(i,j) - p(i,j')t(i,j')$$

$$= p_A(i)p_D[t_A(i)+t_D] + p_B(i)p_F(j)[t_B(i)+t_F(j)] - p_A(i)p_D[t_A(i)+t_D]$$

$$- p_B(i)p_F(j')[t_B(i)+t_F(j')]$$

$$= p_B(i)p_F(j)[t_B(i)+t_F(j)] - p_B(i)p_F(j')[t_B(i)+t_F(j')] \quad (2.2.1)$$

Additionally, for the same *i* and *j*, due to the assumption of

$$p(i,j)t(i,j) = p*_A(i)p*_D[t*_A(i)+t*_D] + p*_B(i)p*_F(j)[t*_B(i)+t*_F(j)]$$

we have for any *i*, *j* and *j'*

$$p(i,j)t(i,j) - p(i,j')t(i,j')$$

$$= p*_B(i)p*_F(j)[t*_B(i)+t*_F(j)] - p*_B(i)p*_F(j')[t*_B(i)+t*_F(j')] \quad (2.2.2)$$

Because Equation (2.2.1) and Equation (2.2.2) have the same left side, the right sides should be the same, that is,

$$p_B(i)p_F(j)[t_B(i)+t_F(j)] - p_B(i)p_F(j')[t_B(i)+t_F(j')]$$

$$= p*_B(i)p*_F(j)[t*_B(i)+t*_F(j)] - p*_B(i)p*_F(j')[t*_B(i)+t*_F(j')]$$

Substitute

$$p*_B(i) = cp_B(i)$$
$$p*_F(j) = p_F(j)/c + (c-1)p_D/c. \qquad \text{(Schweickert \& Chen, 2008)}$$

There is

$$p_B(i)p_F(j)[t_B(i)+t_F(j)] - p_B(i)p_F(j')[t_B(i)+t_F(j')]$$

$$= cp_B(i)[p_F(j)/c+(c-1)p_D/c][t*_B(i)+t*_F(j)]$$

$$-cp_B(i)[p_F(j')/c+(c-1)p_D/c][t*_B(i)+t*_F(j')]$$

Equivalently,

$$p_F(j)t_B(i)+p_F(j)t_F(j)-p_F(j')t_B(i)-p_F(j')t_F(j')$$

$$= p_F(j)t*_B(i)+p_F(j)t*_F(j)+(c-1)p_D[t*_F(j)-t*_F(j')]$$

$$-p_F(j')t*_B(i)-p_F(j')t*_F(j')$$

Then we get

$$[t*_B(i)-t_B(i)][p_F(j')-p_F(j)]$$

$$= p_F(j)t*_F(j)+(c-1)p_D[t*_F(j)-t*_F(j')]-p_F(j')t*_F(j')$$

$$-p_F(j)t_F(j)+p_F(j')t_F(j')$$

As a result, for any *j* such that $p_F(j) \neq p_F(j')$

$$t*_B(i)-t_B(i) = \frac{p_F(j)t*_F(j)+(c-1)p_D[t*_F(j)-t*_F(j')]-p_F(j')t*_F(j')-p_F(j)t_F(j)+p_F(j')t_F(j')}{p_F(j')-p_F(j)}.$$

The left hand side of the above equation cannot change when j changes. So the left hand side must be a constant, denote it as *f*.

Hence, $t*_B(i) = t_B(i) + f$.

For both $t*_B(i)$ and $t_B(i)$ should be nonnegative, f has boundaries as

$$t^*_B(i) = t_B(i) + f \geq 0$$

$$t_B(i) = t^*_B(i) - f \geq 0$$

So for f, there is

$$\max\{-t_B(i)\} \leq f \leq \min\{t^*_B(i)\}$$

Further, the right hand side equals f.

$$\frac{p_F(j)t^*_F(j) + (c-1)p_D[t^*_F(j) - t^*_F(j')] - p_F(j')t^*_F(j') - p_F(j)t_F(j) + p_F(j')t_F(j')}{p_F(j') - p_F(j)} = f .$$

Hence, there is

$t^*_F(j) =$

$$\frac{p_F(j)t_F(j) + f \cdot [p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j) + (c-1)p_D}$$

Similarly, according to the assumptions,

$$p(i,j)t(i,j) = p_A(i)p_D[t_A(i) + t_D] + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$p(i,j)t(i,j) = p^*_A(i)p^*_D[t^*_A(i) + t^*_D] + p^*_B(i)p^*_F(j)[t^*_B(i) + t^*_F(j)]$$

for any i such that $p_B(i) \neq 0$ and $p^*_B(i) \neq 0$, we have

$$\frac{p(i,j)t(i,j)}{p_B(i)} = \frac{p_A(i)p_D[t_A(i) + t_D]}{p_B(i)} + p_F(j)[t_B(i) + t_F(j)] \quad (2.2.3)$$

$$= \frac{p^*_A(i)p^*_D[t^*_A(i) + t^*_D]}{p^*_B(i)} + p^*_F(j)[t^*_B(i) + t^*_F(j)] \quad (2.2.4)$$

Because $p^*_B(i) = cp_B(i)$, we have $\dfrac{p(i,j)t(i,j)}{p^*_B(i)} = \dfrac{p(i,j)t(i,j)}{cp_B(i)}$

Also there is $p^*_D = p_D$

So Equation (2.2.4) can be transformed into

$$\frac{p(i,j)t(i,j)}{p_B(i)} = c \cdot \frac{p*_A(i)p_D[t*_A(i)+t*_D]}{p*_B(i)} + c \cdot p*_F(j)[t*_B(i)+t*_F(j)] \quad (2.2.5)$$

Then according to Equation (2.2.3), for any *i* and *i'* which can be any different value of *i*, we have

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i',j)t(i',j)}{p_B(i')}$$

$$= \frac{p_A(i)p_D[t_A(i)+t_D]}{p_B(i)} + p_F(j)[t_B(i)+t_F(j)] -$$

$$\frac{p_A(i')p_D[t_A(i')+t_D]}{p_B(i')} - p_F(j)[t_B(i')+t_F(j)]$$

$$= \frac{p_A(i)p_Dt_A(i)}{p_B(i)} + \frac{p_A(i)p_Dt_D}{p_B(i)} + p_F(j)[t_B(i)-t_B(i')] - \frac{p_A(i')p_Dt_A(i')}{p_B(i')} - \frac{p_A(i')p_Dt_D}{p_B(i')}$$

$$= \frac{p_A(i)p_Dt_A(i)}{p_B(i)} - \frac{p_A(i')p_Dt_A(i')}{p_B(i')} + p_F(j)[t_B(i)-t_B(i')] + \frac{p_A(i)p_Dt_D}{p_B(i)} - \frac{p_A(i')p_Dt_D}{p_B(i')}$$

$$= \frac{p_A(i)p_Dt_A(i)}{p_B(i)} - \frac{p_A(i')p_Dt_A(i')}{p_B(i')} + p_F(j)[t_B(i)-t_B(i')] + [\frac{1-p_B(i)}{p_B(i)} - \frac{1-p_B(i')}{p_B(i')}]p_Dt_D$$

$$= \frac{p_A(i)p_Dt_A(i)}{p_B(i)} - \frac{p_A(i')p_Dt_A(i')}{p_B(i')} + p_F(j)[t_B(i)-t_B(i')] + [\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_Dt_D \quad (2.2.6)$$

Also, according to Equation (2.2.5), we have

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i',j)t(i',j)}{p_B(i')}$$

$$= c\frac{p*_A(i)p_D[t*_A(i)+t*_D]}{p*_B(i)} + cp*_F(j)[t*_B(i)+t*_F(j)]$$

$$-c\frac{p*_A(i')p_D[t*_A(i')+t*_D]}{p*_B(i')} - cp*_F(j)[t*_B(i')+t*_F(j)]$$

$$= c\{\frac{p*_A(i)p_D[t*_A(i)+t*_D]}{p*_B(i)} + p*_F(j)[t*_B(i)+t*_F(j)]$$

$$-\frac{p*_A(i')p_D[t*_A(i')+t*_D]}{p*_B(i')} - p*_F(j)[t*_B(i')+t*_F(j)]\}$$

$$= c\{\frac{p*_A(i)p_D t*_A(i)}{p*_B(i)} + \frac{p*_A(i)p_D t*_D}{p*_B(i)} + p*_F(j)[t*_B(i)-t*_B(i')]$$

$$-\frac{p*_A(i')p_D t*_A(i')}{p*_B(i')} - \frac{p*_A(i')p_D t*_D}{p*_B(i')}\}$$

$$= c\{\frac{p*_A(i)p_D t*_A(i)}{p*_B(i)} - \frac{p*_A(i')p_D t*_A(i')}{p*_B(i')} + p*_F(j)[t*_B(i)-t*_B(i')]$$

$$+\frac{p*_A(i)p_D t*_D}{p*_B(i)} - \frac{p*_A(i')p_D t*_D}{p*_B(i')}\}$$

$$= c\{\frac{p*_A(i)p_D t*_A(i)}{p*_B(i)} - \frac{p*_A(i')p_D t*_A(i')}{p*_B(i')} + p*_F(j)[t*_B(i)-t*_B(i')]$$

$$+[\frac{1-p*_B(i)}{p*_B(i)} - \frac{1-p*_B(i')}{p*_B(i')}]p_D t*_D\}$$

$$= c\{\frac{p*_A(i)p_D t*_A(i)}{p*_B(i)} - \frac{p*_A(i')p_D t*_A(i')}{p*_B(i')} + p*_F(j)[t*_B(i)-t*_B(i')]$$

$$+[\frac{1}{p*_B(i)} - \frac{1}{p*_B(i')}]p_D t*_D\} \qquad (2.2.7)$$

Substitute all the following values into Equation (2.2.7),

$$p*_B(i) = cp_B(i),$$
$$p*_F(j) = p_F(j)/c + (c-1)p_D/c,$$
$$t*_B(i) = t_B(i) + f.$$

There is,

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i',j)t(i',j)}{p_B(i')}$$

$$= c\{\frac{p^*_A(i)p_D t^*_A(i)}{cp_B(i)} - \frac{p^*_A(i')p_D t^*_A(i')}{cp_B(i')} + [p_F(j)/c + (c-1)p_D/c][t_B(i) - t_B(i')]$$

$$+[\frac{1}{cp_B(i)} - \frac{1}{cp_B(i')}]p_D t^*_D\}$$

$$= \frac{p^*_A(i)p_D t^*_A(i)}{p_B(i)} - \frac{p^*_A(i')p_D t^*_A(i')}{p_B(i')} + [p_F(j) + (c-1)p_D][t_B(i) - t_B(i')]$$

$$+[\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D t^*_D \tag{2.2.8}$$

Because Equation (2.2.6) and Equation (2.2.8) have the same left side, the right sides should be the same, that is,

$$\frac{p_A(i)p_D t_A(i)}{p_B(i)} - \frac{p_A(i')p_D t_A(i')}{p_B(i')} + p_F(j)[t_B(i) - t_B(i')] + [\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D t_D$$

$$= \frac{p^*_A(i)p_D t^*_A(i)}{p_B(i)} - \frac{p^*_A(i')p_D t^*_A(i')}{p_B(i')} + [p_F(j) + (c-1)p_D][t_B(i) - t_B(i')]$$

$$+[\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D t^*_D$$

Equivalently,

$$\frac{p_A(i)p_D t_A(i)}{p_B(i)} - \frac{p_A(i')p_D t_A(i')}{p_B(i')} + p_F(j)[t_B(i) - t_B(i')] - \frac{p^*_A(i)p_D t^*_A(i)}{p_B(i)}$$

$$+ \frac{p^*_A(i')p_D t^*_A(i')}{p_B(i')} - [p_F(j) + (c-1)p_D][t_B(i) - t_B(i')]$$

$$= [\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D t^*_D - [\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D t_D$$

The left side can be simplified into

$$\frac{p_D}{p_B(i)}\{p_A(i)t_A(i) - p*_A(i)t*_A(i)\}$$

$$-\frac{p_D}{p_B(i')}\{p_A(i')t_A(i') - p*_A(i')t*_A(i')\} + [(1-c)p_D][t_B(i) - t_B(i')]$$

Meanwhile, the right side can be simplified into

$$[\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D(t*_D - t_D)$$

Then we have,

$$\frac{p_D}{p_B(i)}\{p_A(i)t_A(i) - p*_A(i)t*_A(i)\} - \frac{p_D}{p_B(i')}\{p_A(i')t_A(i') - p*_A(i')t*_A(i')\}$$

$$+[(1-c)p_D][t_B(i) - t_B(i')]$$

$$=[\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D(t*_D - t_D)$$

So for any *i* such that $p_B(i) \neq p_B(i')$, there is,

$t*_D - t_D =$

$$\frac{\frac{p_D}{p_B(i)}\{p_A(i)t_A(i) - p*_A(i)t*_A(i)\} - \frac{p_D}{p_B(i')}\{p_A(i')t_A(i') - p*_A(i')t*_A(i')\} + [(1-c)p_D][t_B(i) - t_B(i')]}{[\frac{1}{p_B(i)} - \frac{1}{p_B(i')}]p_D}$$

The left hand side of the above equation cannot change when i changes. So the left hand side must be a constant, denote it as *e*.

Hence, $t*_D = t_D + e$

For both $t*_D$ and t_D should be nonnegative, *e* has boundaries as

$$t*_D = t_D + e \geq 0$$

$$t_D = t*_D - e \geq 0$$

So for e, there is

$$\max\{-t_D\} \le e \le \min\{t^*_D\}$$

Because

$$p(i,j)t(i,j) = p_A(i)p_D[t_A(i)+t_D] + p_B(i)p_F(j)[t_B(i)+t_F(j)]$$

$$p(i,j)t(i,j) = p^*_A(i)p^*_D[t^*_A(i)+t^*_D] + p^*_B(i)p^*_F(j)[t^*_B(i)+t^*_F(j)]$$

There is

$$p_A(i)p_D[t_A(i)+t_D] + p_B(i)p_F(j)[t_B(i)+t_F(j)]$$

$$= p^*_A(i)p^*_D[t^*_A(i)+t^*_D] + p^*_B(i)p^*_F(j)[t^*_B(i)+t^*_F(j)]$$

Substitute

$$p^*_B(i) = cp_B(i),$$

$$p^*_F(j) = p_F(j)/c + (c-1)p_D/c,$$

$$t^*_B(i) = t_B(i) + f,$$

$$t^*_F(j) =$$

$$\frac{p_F(j)t_F(j) + f\cdot[p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j)+(c-1)p_D},$$

and

$$t^*_D = t_D + e,$$

We get

$$p_A(i)p_D[t_A(i)+t_D] + p_B(i)p_F(j)[t_B(i)+t_F(j)]$$

$$= [1-cp_B(i)]p_D[t^*_A(i)+t_D+e] + cp_B(i)[p_F(j)/c + (c-1)p_D/c]\times$$

$$\{t_B(i) + f + \frac{p_F(j)t_F(j) + f\cdot[p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j)+(c-1)p_D}\}$$

Equivalently,

$$[1-cp_B(i)]p_D[t^*_A(i)+t_D+e] - [1-p_B(i)]p_D[t_A(i)+t_D]$$

$$= p_B(i)p_F(j)[t_B(i)+t_F(j)] - cp_B(i)[p_F(j)/c+(c-1)p_D/c]\times\{t_B(i)+f$$

$$+\frac{p_F(j)t_F(j)+f[p_F(j')-p_F(j)]+t^*_F(j')[p_F(j')+(c-1)p_D]-p_F(j')t_F(j')}{p_F(j)+(c-1)p_D}\}$$

The left side can be simplified into

$$p_D\{[1-cp_B(i)]t^*_A(i)-p_A(i)t_A(i)+(1-c)p_B(i)t_D-cep_B(i)+e\}$$

The right side can be simplified into

$$p_B(i)\{t_B(i)(1-c)p_D+[f+t^*_F(j')][(1-c)p_D-p_F(j')]+p_F(j')t_F(j')\}$$

Hence,

$$p_D\{[1-cp_B(i)]t^*_A(i)-p_A(i)t_A(i)+(1-c)p_B(i)t_D-cep_B(i)+e\}$$

$$= p_B(i)\{t_B(i)(1-c)p_D+[f+t^*_F(j')][(1-c)p_D-p_F(j')]+p_F(j')t_F(j')\}$$

Then we have,

$$t^*_A(i)[1-cp_B(i)]$$

$$= p_B(i)\{t_B(i)(1-c)+[f+t^*_F(j')][1-c-\frac{p_F(j')}{p_D}]+\frac{p_F(j')t_F(j')}{p_D}\}$$

$$+p_A(i)t_A(i)-(1-c)p_B(i)t_D+cep_B(i)-e$$

$$= p_A(i)t_A(i)+p_B(i)\{[t_B(i)-t_D](1-c)+[f+t^*_F(j')]\times$$

$$[1-c-\frac{p_F(j')}{p_D}]+\frac{p_F(j')t_F(j')}{p_D}+ce\}-e$$

So we get,

$$t^*_A(i) = \frac{p_A(i)t_A(i) + p_B(i)\{[t_B(i) - t_D](1-c) + [f + t^*_F(j')][1 - c - \frac{p_F(j')}{p_D}] + \frac{p_F(j')t_F(j')}{p_D} + ce\} - e}{1 - cp_B(i)}$$

b) Conversely, suppose for all $1 \le i \le I$, and $1 \le j \le J$, there exists

$p^*_A, p^*_B, p^*_D, p^*_F(j), t^*_A, t^*_B, t^*_D(i), t^*_F(j)$ such that

$$0 \le p^*_A, p^*_B, p^*_D, p^*_F(j) \le 1$$

$$t^*_A, t^*_B, t^*_D(i), t^*_F(j) \ge 0$$

With all the following equations,

$p^*_B(i) = cp_B(i)$,

$p^*_F(j) = p_F(j)/c + (c-1)p_D/c$,

$p^*_D = p_D$

$t^*_B(i) = t_B(i) + f$,

$t^*_F(j) =$

$$\frac{p_F(j)t_F(j) + f \cdot [p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j) + (c-1)p_D},$$

$t^*_D = t_D + e$,

and

$$t^*_A(i) = \frac{p_A(i)t_A(i) + p_B(i)\{[t_B(i) - t_D](1-c) + [f + t^*_F(j')][1 - c - \frac{p_F(j')}{p_D}] + \frac{p_F(j')t_F(j')}{p_D} + ce\} - e}{1 - cp_B(i)}$$

as well as the bounds for *c, e* and *f*,

$$0 < c \le 1/\max\{p_B(i)\}$$

$$p_D - \min\{p_F(j)\} \le p_D \cdot c$$

$$\max\{p_F(j)\} - p_D \le (1 - p_D)c$$

(Schweickert & Chen, 2008)

$$\max\{-t_D\} \le e \le \min\{t^*_D\}$$

$$\max\{-t_B(i)\} \le f \le \min\{t^*_B(i)\}$$

Then

$$p^*_A\, p^*_D\,[t^*_A(i) + t^*_D]$$

$$= [1 - p^*_B(i)]p^*_D\,[t^*_A(i) + t^*_D]$$

$$= p_A(i)t_A(i)p_D + p_B(i)\{[t_B(i) - t_D](1-c)p_D + [f + t^*_F(j')][(1-c)p_D - p_F(j')]$$

$$+ p_F(j')t_F(j') + cep_D\} - ep_D + (t_D + e)[1 - cp_B(i)]p_D$$

$$= p_A(i)p_D[t_D + t_A(i)] + p_B(i)\{t_B(i)(1-c)p_D + [f + t^*_F(j')][(1-c)p_D - p_F(j')] + p_F(j')t_F(j')\}$$

Meanwhile,

$$p^*_B(i)\,p^*_F(j)[t^*_B(i) + t^*_F(j)]$$

$$= cp_B(i)[p_F(j)/c + (c-1)p_D/c] \times [t_B(i) + f$$

$$+ \frac{p_F(j)t_F(j) + f\cdot[p_F(j') - p_F(j)] + t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')}{p_F(j) + (c-1)p_D}]$$

$$= p_B(i)\{[t_B(i) + f][p_F(j) + (c-1)p_D] + p_F(j)t_F(j) + f\cdot[p_F(j') - p_F(j)]$$

$$+ t^*_F(j')[p_F(j') + (c-1)p_D] - p_F(j')t_F(j')\}$$

$$=$$

$$p_B(i)\{t_B(i)[p_F(j) + (c-1)p_D] + p_F(j)t_F(j) + [t^*_F(j') + f][p_F(j') + (c-1)p_D]$$

$$- p_F(j')t_F(j')\}$$

Put these two parts together, we get

$$p*_A\ p*_D\,[t*_A\,(i)+t*_D]+p*_B\,(i)\,p*_F\,(j)[t*_B\,(i)+t*_F\,(j)]$$

$$= p_A(i)p_D[t_D+t_A(i)]+p_B(i)\{t_B(i)(1-c)p_D+[f+t*_F\,(j')][(1-c)p_D-p_F(j')]$$

$$+p_F(j')t_F(j')\}$$

$$+p_B(i)\{t_B(i)[p_F(j)+(c-1)p_D]+p_F(j)t_F(j)+[t*_F\,(j')+f][p_F(j')+(c-1)p_D]$$

$$-p_F(j')t_F(j')\}$$

$$= p_A(i)p_D[t_A(i)+t_D]+p_B(i)p_F(j)[t_B(i)+t_F(j)]$$

$$= p(i,j)t(i,j)$$

Additionally, the degree of freedom is calculated as below

$$3IJ-(I+J+1)-(2+2I+2J)+3+3=3IJ-3I-3J+3$$

for the total observations are $3IJ$, the number of arc probabilities is $(I+J+1)$, the number of arc times is $(2+2I+2J)$, and there are three constants c, e, f in the calculation for situation of correct response as well as 3 constants for situation of incorrect response. Here we exclude the situation where $p_A(i)$ is 0 or 1, for that leads to a totally different processing tree.

APPENDIX C

Proof for Theorem 6

(a) Suppose $p*_B(i), p*_D(i), p*_F(j), t*_B(i), t*_D(i),$ and $t*_F(j)$ exist with

$$p(i,j)t(i,j) = p*_D(i)t*_D(i) + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

Also, for any *i* and *j*, there exists

$$p(i,j)t(i,j) = p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

We have

$$\frac{p(i,j)t(i,j)}{p_B(i)} = \frac{p_D(i)t_D(i)}{p_B(i)} + p_F(j)[t_B(i) + t_F(j)] \qquad (2.3.1)$$

$$\frac{p(i,j)t(i,j)}{p*_B(i)} = \frac{p*_D(i)t*_D(i)}{p*_B(i)} + p*_F(j)[t*_B(i) + t*_F(j)] \qquad (2.3.2)$$

Because $p*_B(i) = cp_B(i)$, Equation (2.3.2) can be transformed into

$$\frac{p(i,j)t(i,j)}{p_B(i)} = \frac{p*_D(i)t*_D(i)}{p_B(i)} + cp*_F(j)[t*_B(i) + t*_F(j)] \qquad (2.3.3)$$

Then according to Equation 1, for any *j'* which is a different value from *j*, we have

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i,j')t(i,j')}{p_B(i)}$$

$$= \frac{p_D(i)t_D(i)}{p_B(i)} + p_F(j)[t_B(i) + t_F(j)] - \frac{p_D(i)t_D(i)}{p_B(i)} - p_F(j')[t_B(i) + t_F(j')]$$

$$= p_F(j)[t_B(i) + t_F(j)] - p_F(j')[t_B(i) + t_F(j')] \qquad (2.3.4)$$

Meanwhile, according to Equation (2.3.3), there is

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i,j')t(i,j')}{p_B(i)}$$

$$= \frac{p*_D(i)t*_D(i)}{p_B(i)} + cp*_F(j)[t*_B(i) + t*_F(j)]$$

$$- \frac{p*_D(i)t*_D(i)}{p_B(i)} - cp*_F(j')[t*_B(i) + t*_F(j')]$$

$$= cp*_F(j)[t*_B(i) + t*_F(j)] - cp*_F(j')[t*_B(i) + t*_F(j')] \quad (2.3.5)$$

Then replace $p*_F(j)$ with $p_F(j)$ with $p*_F(j) = p_F(j)/c + d$, we get

$$\frac{p(i,j)t(i,j)}{p_B(i)} - \frac{p(i,j')t(i,j')}{p_B(i)}$$

$$= c[p_F(j)/c + d][t*_B(i) + t*_F(j)] - c[p_F(j')/c + d][t*_B(i) + t*_F(j')]$$

$$= p_F(j)[t*_B(i) + t*_F(j)] + cd[t*_F(j) - t*_F(j')] - p_F(j')[t*_B(i) + t*_F(j')] \quad (2.3.6)$$

Because Equation (2.3.4) shares the same left side as Equation (2.3.6), there should be

$$p_F(j)[t_B(i) + t_F(j)] - p_F(j')[t_B(i) + t_F(j')]$$

$$= p_F(j)[t*_B(i) + t*_F(j)] + cd[t*_F(j) - t*_F(j')] - p_F(j')[t*_B(i) + t*_F(j')]$$

which can be simplified into

$$cd[t*_F(j) - t*_F(j')]$$

$$= p_F(j)[t_B(i) - t*_B(i) + t_F(j) - t*_F(j)]$$

$$- p_F(j')[t_B(i) - t*_B(i) + t_F(j') - t*_F(j')]$$

Equivalently,

$$cd[t*_F(j) - t*_F(j')]$$

$$= [p_F(j) - p_F(j')][t_B(i) - t*_B(i)] + p_F(j)[t_F(j) - t*_F(j)] - p_F(j')[t_F(j') - t*_F(j')]$$

Hence, for any *j* such that $p_F(j) \neq p_F(j')$, we get

$$t_B(i) - t*_B(i) = \frac{cd[t*_F(j) - t*_F(j')] - p_F(j)[t_F(j) - t*_F(j)] + p_F(j')[t_F(j') - t*_F(j')]}{p_F(j) - p_F(j')}$$

The left hand side of the above equation cannot change when j changes. So the left hand side must be a constant, denote it as *e*.

So we have

$$\frac{cd[t*_F(j) - t*_F(j')] - p_F(j)[t_F(j) - t*_F(j)] + p_F(j')[t_F(j') - t*_F(j')]}{p_F(j) - p_F(j')} = e$$

and

$$t*_B(i) = t_B(i) - e$$

Because $t_B(i) \geq 0$ and $t_B*(i) \geq 0$, we get the bound of constant *e* as

$$\max\{-t*_B(i)\} \leq e \leq \min\{t_B(i)\}$$

Further,

$$t*_F(j) = \frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] + cdt*_F(j') + p_F(j)t_F(j)}{cd + p_F(j)}$$

Then, because

$$\boldsymbol{p(i,j)t(i,j)} = p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$= p*_D(i)t*_D(i) + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

There is

$$p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$= p*_D(i)t*_D(i) + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

Put all these values in to the equation:

$$p*_B(i) = cp_B(i),$$

$$p*_D(i) = p_D(i) - cdp_B(i),$$

$$p*_F\,(j) = p_F(j)/c + d \ ,$$

(Schweickert & Chen, 2008)

$$t*_B\,(i) = t_B(i) - e\ ,$$

and

$$t*_F\,(j) = \frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F\,(j')] + cdt*_F\,(j') + p_F(j)t_F(j)}{cd + p_F(j)}\ ,$$

we get

$$p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$= [p_D(i) - cdp_B(i)]t*_D\,(i) + cp_B(i)[p_F(j)/c + d][t_B(i) - e +$$

$$\frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F\,(j')] + cdt*_F\,(j') + p_F(j)t_F(j)}{cd + p_F(j)}$$

$$= [p_D(i) - cdp_B(i)]t*_D\,(i) + p_B(i)[p_F(j) + cd]\times$$

$$[t_B(i) + \frac{-ep_F(j') - p_F(j')[t_F(j') - t*_F\,(j')] + cd[t*_F\,(j') - e] + p_F(j)t_F(j)}{cd + p_F(j)}]$$

Then

$$[cdp_B(i) - p_D(i)]t*_D\,(i)$$

$$= p_B(i)[p_F(j) + cd]\frac{-ep_F(j') - p_F(j')[t_F(j') - t*_F\,(j')] + cd[t*_F\,(j') - e] + p_F(j)t_F(j)}{cd + p_F(j)}$$

$$+ p_B(i)p_F(j)t_B(i) + cdp_B(i)t_B(i) - p_D(i)t_D(i) - p_B(i)p_F(j)t_B(i) - p_B(i)p_F(j)t_F(j)$$

$$= p_B(i)\{-ep_F(j') - p_F(j')[t_F(j') - t*_F\,(j')] + cd[t*_F\,(j') - e] + p_F(j)t_F(j)\}$$

$$+ cdp_B(i)t_B(i) - p_D(i)t_D(i) - p_B(i)p_F(j)t_F(j)$$

$$= -p_D(i)t_D(i) + p_B(i)[p_F(j') + cd][t*_F\,(j') - e] - p_B(i)p_F(j')t_F(j') + cdp_B(i)t_B(i)$$

Hence,

$$t*_D(i) = \frac{-p_D(i)t_D(i) + p_B(i)[p_F(j') + cd][t*_F(j') - e] - p_B(i)p_F(j')t_F(j') + cdp_B(i)t_B(i)}{cdp_B(i) - p_D(i)}$$

b) Conversely, suppose there exists $p*_B(i), p*_D(i), p*_F(j), t*_B(i), t*_D(i), t*_F(j)$

such that

$$0 \le p*_B(i), p*_D(i), p*_F(j) \le 1$$

$$t*_B(i), t*_D(i), t*_F(j) \ge 0$$

With all the following equations,

$p*_B(i) = cp_B(i)$,

$p*_D(i) = p_D(i) - cdp_B(i)$,

$p*_F(j) = p_F(j)/c + d$,

(Schweickert & Chen, 2008)

$t*_B(i) = t_B(i) - e$,

$$t*_D(i) = \frac{-p_D(i)t_D(i) + p_B(i)[p_F(j') + cd][t*_F(j') - e] - p_B(i)p_F(j')t_F(j') + cdp_B(i)t_B(i)}{cdp_B(i) - p_D(i)}$$

and

$$t*_F(j) = \frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] + cdt*_F(j') + p_F(j)t_F(j)}{cd + p_F(j)},$$

as well as the bounds for c, d and e,

$$0 < c \le 1/\max\{p_B(i)\}$$

$$-\min\{p_F(j)\} \le cd$$

$$\max\{p_F(j)\} \le c(1-d)$$

$\max\{[p_D(i) - 1]/p_B(i)\} \le cd \le \min\{p_D(i)/p_B(i)\}$ (Schweickert & Chen, 2008)

$$\max\{-t*_B(i)\} \le e \le \min\{t_B(i)\}$$

$$p*_D(i)t*_D(i) + p*_B(i)p*_F(j)[t*_B(i) + t*_F(j)]$$

$$= [p_D(i) - cdp_B(i)]\times$$

$$\frac{-p_D(i)t_D(i) + p_B(i)[p_F(j') + cd][t*_F(j') - e] - p_B(i)p_F(j')t_F(j') + cdp_B(i)t_B(i)}{cdp_B(i) - p_D(i)}$$

$$+ cp_B(i)[p_F(j)/c + d]\times$$

$$[t_B(i) - e + \frac{e[p_F(j) - p_F(j')] - p_F(j')[t_F(j') - t*_F(j')] + cdt*_F(j') + p_F(j)t_F(j)}{cd + p_F(j)}]$$

$$= p_D(i)t_D(i) - p_B(i)[p_F(j') + cd][t*_F(j') - e] + p_B(i)p_F(j')t_F(j') - cdp_B(i)t_B(i)$$

$$+ p_B(i)\{[p_F(j) + cd][t_B(i) - e] + e[p_F(j) - p_F(j')]$$

$$- p_F(j')[t_F(j') - t*_F(j')] + cdt*_F(j') + p_F(j)t_F(j)\}$$

$$= p_D(i)t_D(i) - p_B(i)p_F(j')[t*_F(j') - e] - cdp_B(i)[t*_F(j') - e]$$

$$+ p_B(i)p_F(j')t_F(j') - cdp_B(i)t_B(i)$$

$$+ p_B(i)p_F(j)[t_B(i) - e] + cdp_B(i)[t_B(i) - e] + ep_B(i)[p_F(j) - p_F(j')]$$

$$- p_B(i)p_F(j')[t_F(j') - t*_F(j')] + cdp_B(i)t*_F(j') + p_B(i)p_F(j)t_F(j)$$

$$= p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$- p_B(i)p_F(j')[t*_F(j') - e - t_F(j') + e + t_F(j') - t*_F(j')]$$

$$- cdp_B(i)t*_F(j') + cdp_B(i)t*_F(j') - cdp_B(i)[t_B(i) - e] + cdp_B(i)[t_B(i) - e]$$

$$+ ep_B(i)p_F(j) - ep_B(i)p_F(j)$$

$$= p_D(i)t_D(i) + p_B(i)p_F(j)[t_B(i) + t_F(j)]$$

$$= p(i,j)t(i,j)$$

Additionally, the degrees of freedom are calculated as below

$$3IJ-(2I+J)-(3I+2J)+3+3=3IJ-5I-3J+6$$

for the total observations are $3IJ$, the number of arc probabilities is $(2I+J)$, the number of arc times is $(3I+2J)$, and there are three constants *c*, *d*, *e* in the calculation for situation of correct response as well as 3 constants for situation of incorrect response. Here we suppose that $p_B(i)$ or $p_D(i)$ cannot be neither 0 nor 1, for that leads to a totally different processing tree.