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SUPERCUSPIDAL REPRESENTATIONS ARISING FROM STABLE VECTORS

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SUPERCUSPIDAL REPRESENTATIONS ARISNG FROM STABLE VECTORS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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Approved by Major Professor(s): Freydoon Shahidi

Approved by: David Goldberg 7/16/15

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SUPERCUSPIDAL REPRESENTATIONS ARISING FROM STABLE VECTORS

A Dissertation

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of

Purdue University

by

Britain T. Cox

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of

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West Lafayette, Indiana

To Oakley

great things come in soft packages

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SYMBOLS

W	The Weyl group
θ	A regular, elliptic element of W
m	The order of θ
ζ_m	A fixed primitive mth root of unity
k	A finite extension of $\mathbf{Q}_p$
$\overline{k}$	A fixed algebraic extension of k
p	The residual characteristic of k
q	The residual cardinality of k
x	A point in the Bruhat Tits building
π	A uniformizer of k
Fr	A fixed Frobeneius element of k
Γ_k	The absolute Galois group of k
$G_{x,r}$	The Moy-Prasad filtration subgroup of depth r associated to x
$\mathfrak{g}_{x,r}$	The Moy-Prasad sub algebra of depth r associated to x
$\overline{G}_{x,r}$	The quotient of $G_{x,r}$ by $G_{x,r+}$
$\overline{\mathfrak{g}}_{x,r}$	The quotient of $\mathfrak{g}_{x,r}$ by $\mathfrak{g}_{x,r+}$
$\mathfrak{L}$	An abstract Lie algebra on which θ acts
$\mathfrak{L}_a$	The eigenspace of θ in $\mathfrak{L}$ with eigenvalue ζ_m^a
G_0	The reductive group $\mathrm{Aut}(\mathfrak{L}_0)$
$W(\theta, \zeta_m)$	The Litte Weyl group associated to θ, ζ_m
$\mathfrak{c}$	A fixed Cartan subspace associated to θ.

ABSTRACT

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For a reductive group G over a p-adic field k, one may grade the associated Lie algebra $\mathfrak{g}$ by an automorphism of order m. It has been shown that stable vectors $v \in \mathfrak{g}_a$ arise only when a is coprime to m. Given a stable vector $v \in \mathfrak{g}_a$, we construct packets of supercuspidal representations $\{\pi_{v,\rho}\}$ as well as discrete Langlands parameters φ_v. Both the parameter and representations are of depth a/m. We further show that for a fixed vector, $\pi_{v,\rho}$ and φ_v satisfy both sides of the formal degree conjecture.

1. Introduction

Given a group G, a primary question is to describe its representation theory. In the situation where G is a reductive group over a field k, there is a conjectural program to describe the nice representations of the group of k points $G(k)$. The specific aim of this work is to use geometric invariant theory to propose a small portion of this program. The general aim of this work is to develop tools to approach the problem in a broader context, thus informing how the program should be constituted.

1.1 History

The first case to be fully understood was for k an archimedean local field. For $k = \mathbf{C}$ the complex numbers, the full representation theory can be understood through the theory of highest weights. Specifically, for any irreducible representation, we may restrict it to a maximal torus. Since this is a representation of a semisimple abelian group, it decomposes as a direct sum of 1-dimensional representations. These *weights* determine the representation, and there is a *highest weight* which determines the full set of weights. All possible weights have a basic description with respect to the root system of the reductive group, thus effectively resolving the question of representations for complex reductive groups.

For the case where $k = \mathbf{R}$, the situation is slightly more complicated. Unlike in the complex case, the real numbers are not algebraically closed. This defect complicates matters, not least of which by forcing an explicit choice of which conjugacy class of tori to choose. From the work of Harish-Chandra, one can still parameterize representations via 1-dimensional representations of a subgroup; however, the task of actually constructing such representations is far more difficult.

One key element to understanding the representations of a real reductive group came with the advent of the Langlands Program. Understanding class field theory as the theory of 1-dimensional representations of the absolute Galois group, Robert Langlands proposed a generalization to n-dimensional representations. This eventually grew into a web of conjectures, connecting many different disciplines of mathematics with one philosophy - see the next section. Though it was conceived to better understand the representations of a Galois group, the Langlands Program could equally be taken to understand how the representations of an arbitrary reductive group are organized. Using this framework, he recast the work for real reductive groups, using it as a first demonstration of the overall program.

After archimedian fields, the next case is to consider the p-adic fields. This step increases the complexity of the problem considerably. In particular, the work of Langlands used parabolic induction to construct representations from representations of smaller rank reductive groups. In the real case, all representations arise via this method; in the p-adic case, a large number of *supercuspidal* representations arise independent of parabolic induction. Thus, in order to parrot Langlands' approach for the p-adics, one must at the very least understand the supercuspidal representations.

This task has proved formidable. The first supercuspidal representations were constructed for $\mathrm{SL}_2(\mathbf{Q}_p)$ by Mautner. After some subsequent work, Howe produced a general construction for $\mathrm{GL}_n(\mathbf{Q}_p)$, again for p large. This was later shown by Moy to be exhaustive. For general reductive group G, the first constructions came with Adler, Kim and Yu. Yu's construction, a generalization of the approach by Adler, was eventually shown by Kim to be exhaustive for large p.

Though in principle, this work should open the path for understanding the representations more generally, in practice, it is very unwieldy. The construction of [30] involves an inducing datum which, though its constituents are simple, is very complicated as a whole. In an effort to gain a better understanding of the situation, Gross and Reeder took the most basic datum, simplified the construction, and studied these *simple supercuspidal representations*. Taking advantage of the simplicity, they were able to propose a Langlands correspondence and verify some desiredata. Reeder and Yu were able to generalize this approach, bringing to bear geometric invariant theory GIT to help partially classify the minimal positive depth representations. In a precise sense, both these approaches dealt with representations of minimal depth; it is the goal of this work to apply the methods to deeper representations.

1.2 Local Langlands Correspondence

Let k be a p-adic field. In class field theory, one obtains a correspondence between the 1-dimensional representations of the absolute Galois group of k and the representations of $\mathrm{GL}_1(k) = k^\times$. The Local Langlands Correspondence (LLC) aims to generalize this. It replaces GL_1 with an arbitrary reductive group G, while representations into the L group ${}^L G(\mathbf{C})$ generalizes the 1-dimensional representations. Initially, the correspondence was designed so that the L-functions for either side agreed; as research progressed, it became clear that this requirement did not capture the full power of the approach. Where as an L-function is less sensitive for the ramified primes, important problems required consideration of ramification. In order to include these problems within the scope of the program, additional desiderata were proposed.

Following the structure found in the theory of the highest weight, one tries to understand a representation of a reductive group by restricting it to a maximal torus and studying how it decomposes into 1-dimensional sub representations. As was mentioned above, in general, there can be many different conjugacy classes of tori to choose from. Using Galois cohomology, one can parameterize these classes via representations of the absolute Galois group into the Weyl group. The approaches of [6] [24] then determine extensions of this representation to one into the full L-group; it is this extension that encodes a distinguishing weight of the representation.

The main objective of this thesis is to propose a correspondence for supercuspidal representations of greater depth than those mentioned above. In this effort, we employ the construction of [30] as well as the GIT introduced in [24]. The character for the construction as well as the twisting for the Galois representation arise from the same stable vector. Further, the torus employed in the construction can be understood via Galois cohomology as arising from the induced representation of the absolute Galois group into the Weyl group.

Initially, attempts at a correspondence were somewhat *ad hoc*; given enough of an understanding of both the representations and the Langlands parameters, one attempted to make an assignment that was sufficiently nice. The end goal would be to transition directly from a Langlands parameter to the corresponding representations. It is the hope that this thesis makes some modest progress towards this goal: with the introduction of GIT, the necessary structures on both sides of the correspondence can be related on common ground. The next step is to remove the explicit calls to GIT, avoiding the cumbersome concerns about splitting and stable vectors.

The first objective of this paper is to produce supercuspidal representations for an arbitrary reductive group of depth a/m, where m is the order of any regular elliptic element of the Weyl group and a is a positive number coprime to m. The second objective is to produce discrete Langalnds parameters of depth a/m. Given the breadth of representations and parameters considered, the scope of this paper is to focus on the more coarse predictions of the LLC, leaving the more subtle features for later. If φ is a Langlands parameter, we denote A_φ to be the centralizer. This parameter should correspond to a set (perhaps a singleton) of representations π_ρ parameterized by representations of A_φ. We let γ denote the adjoint gamma factor, ω the root number, and φ_0 the principal parameter - see Section 4.3 for definitions. The main criterion for this paper may now be stated.

Conjecture. **(Formal Degree Conjecture)** Fix a p-adic field k and reductive group G defined over k and let μ be the Euler-Poincare metric. For every discrete Langlands parameter φ there corresponds a finite set of discrete representations $\{\pi_{\varphi,\rho}\}$ such that

$$\deg_\mu \pi_{\varphi,\rho} = \frac{\deg \rho}{|A_\varphi|} \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \frac{\omega(\varphi_0)}{\omega(\varphi_0)}$$

The epitaph of Formal Degree Conjecture (FDC) reflects its leal status to the Local Langlands Conjecture. The full LLC far surpasses a basic correspondence, incorporating coarse requirements like depth [19] as well as very fine requirements like stability under endoscopic transfer [4]. The FDC can distinguish representations of the same depth, but is not precise enough to establish stability. Its main advantage is that the numerical requirement is fairly potent and, as this work will demonstrate, is reasonable to compute. Using the FDC as an organizing principle, a clearer picture of the LLC framework begins to emerge.

1.3 Supercuspidal Representations

As was mentioned earlier, the supercuspidal representations form a set of building blocks that - through parabolic induction - all admissible representations may be realized. In the case of GL_n, it was shown that this realization was compatible with the proposed correspondence. As such, it was enough to demonstrate the correspondence exists for the supercuspidal representations, and the full correspondence followed [8] [9].

For an arbitrary reductive group, no such result is known. Even with the exhaustion result of [14], there is no guarantee that parabolically inducing all the admissible representations will produce a fully compatible correspondence. To complicate matters further, the correspondence is not a bijection in general. For a general reductive group, each Langlands parameter should correspond to a finite set - an L-packet - of representations. The L-packet reflects rational conjugacy classes: since GL_n has rational canonical form, the L-packets are singletons; for a reductive group where rational canonical form fails, the L-packets will be larger.

Given these obstacles, the representations must be better understood for progress to be made. To simplify matters, we restrict ourselves to supercuspidal representations in order to shed light on how the general correspondence should manifest. Further, this thesis will focus on the relatively simple representations which arise from a stable vector. However, since this work deals with deeper representations, there exist other supercuspidal representations of equal depth as the those constructed in the sequel.

In addition to the study of these specific representations, this work will shed some light on the construction of [30]. In particular, the inducing datum required for construction requires a specific type of character. *A priori*, locating these characters is not obvious; any effort to study the representations as a group is thus limited by this deficiency. One result of this work is to classify a large portion of them via GIT. In doing so, a broader analysis of the situation is made possible.

1.4 Langlands Parameters

The Langlands parameter is a representation of the Weil-Deligne group of k into the L-group of the reductive group

$$\varphi : \mathrm{WD}_k \to {}^L G(\mathbf{C})$$

The use of the L-group is meant to encode the k-structure of the reductive group, thus some minor additional requirements arise. The use of the full Weil-Delinge group is meant to capture the Jordan decomposition of the parameter; in the sense that weights are orbits of the dual that parameterize representations of $G(\mathbf{C})$, the hope is that orbits of the dual should in some way parameterize representations for $G(k)$.

It can be shown that rational classes of tori are classified via homomorphisms from the absolute Galois group into the Weyl group of G. Under the assumption that the residual characteristic of k is larger than the rank of the group, such homomorphisms only involve the the Frobeneus and tame inertial subgroup. In this way, one sees a familiar structure, where a rational conjugacy class of tori is chosen via this Galois cohomology, while the wild inertial subgroup determines a semisimple element.

1.5 Formal Degrees

A guiding objective for the proposed correspondence is that it satisfy **Conjecture 1**, known as the Formal Degree Conjecture FDC. Proposed in [10] and recast in [6], the FDC posits that the correspondence should satisfy some numerical relation. Specifically, the formal degree of the admissible representation should be related in to the adjoint γ factor of the Langlands Parameter. If either the representation or parameter are non discrete, then both sides of the relation are zero; the discrete case is the interesting one.

In general, computing the formal degree of a discrete representation can be difficult. However, since the construction of [30] utilizes compact induction, the formal degrees are easily computed. The problem reduces to computing the dimension of the inducing representation, along with the volume of the compact subgroup. The former was computed via the theory of Weil representations, while the latter is resolved via GIT.

The computation of the adjoint γ factor is more straightforward. It was shown in [6] that the computation is intimately related to the computation of the Artin conductor of the induced representation on the Lie algebra; this in turn requires knowledge of the decomposition groups. Since the Langlands Parameter is being constructed, and the representations on the other side were chosen to be relatively uncomplicated, this knowledge is readily availed and acted upon.

It is worth noting that both sides of the numerical relation can be understood through the lens of GIT. From this view, sums appear where each term has a natural counterpart. That the two sides agree follows from the fact that the terms counter one another agree. This is a stunning fact, as one side is dealing with statistics of positive characteristic whilst the other with characteristic zero. The equality of terms is established by the fact that the GIT used is valid in both scenarios.

Since the representations and parameters have explicit constructions, we are able to compute the relevant statistics for each. Leveraging the GIT, we are able to prove the following

Main Theorem.

Let $v \in \mathfrak{L}_a$ be a stable vector. Then there exists a finite set of supercuspidal representations of depth a/m

$$\{\pi_{v,\rho}\} \qquad \rho \in \widehat{T_x T_{x,0+}}$$

and a discrete Langlands parameter of depth a/m

$$\varphi_v : \mathrm{WD}_k \to {}^L G(\mathbf{C})$$

such that the correspondence

$$\deg_\mu \pi_{v,\rho} = \frac{\dim \rho}{|A_\varphi|} \frac{\gamma(\varphi_v)}{\omega(\varphi_v)} \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}$$

Though the notation would suggest a correspondence

$$\varphi_v \leftrightarrow \pi_{v,\rho}$$

the evidence presented in this work is not sufficient to make such a claim. As will be shown, for fixed a coprime to m, all stable vectors $v \in \mathfrak{L}_a$ produce representations with the same formal degree.

This work is silent on the more subtle requirements of the LLC. In the case $a = 1$, a correspondence has been suggested by Kaletha [13] which was shown to be stable; given the similarities with this work, it is possible that the methods will extend to the representations considered here. However, the main contribution of this work is to approach breadth at the expense of subtlety. The methods employed here can be applied broadly, thus give a more complete - if coarse - understanding of how the LLC should be manifested.

We use the remainder of the introduction to summarize the work to follow.

Throughout the paper, k is a p-adic field, G a reductive group defined over k, m is the order of a regular, elliptic element θ of the Weyl group, and a is a positive integer coprime to m.

In **Section 2**, we provide the necessary background information for the work to follow, as well as the various simplifying assumptions. This section contains no new content from what is in the literature.

In **Section 3**, to each stable vector $v \in \mathfrak{L}_a$, we construct a set of supercuspidal representations $\{\pi_{v,\rho}\}$ of depth a/m using the construction of [30]. We further establish some results necessary for the later computation of formal degrees. As a consequence of this work, we give a full description of how stable vectors arise in the construction of supercuspidal representations.

In **Section 4**, to each stable vector $v \in \mathfrak{L}_a$, we construct a discrete Langlands parameter φ_v. The construction closely follows that of [24]. The major variation is that, because the depth is now a/m, we must take care to select a module with no invariants of the tame inertial generator. The parameter is constructed via a splitting, the choice of which is made using the stable vector.

In **Section 5**, we compute the formal degree of each $\pi_{v,\rho}$. This involves using the volume of parahoric subgroup, and adjusting by the inducing subgroup from **Section 3**. Here, the GIT of a graded Lie algebra plays a vital role, simplifying the complicated terms. The section concludes by demonstrating the **Main Theorem**, and making preliminary comments on where to go forward.

2. Background Information

In this section, we establish the basic results and notation to be used throughout the thesis. All the results of this section are contained within the literature, and references will be given in lieu of proofs. General references for this section are [25] [17] [26].

2.1 p-adic Fields

Let k be a p-adic field - i.e. a finite extension of $\mathbf{Q}_p$, where p is a prime number. Have $\mathfrak{o}_k$ denote the ring of integers, $\mathfrak{p}_k$ its maximal ideal, $\mathfrak{f}$ the residue field and $q = p^n$ the residual cardinality. Let

$$\mathrm{val} : k^\times \to \mathbf{Z}$$

be the usual valuation map on k, with the standard normalization. We may choose a uniformizer $\varpi \in \mathfrak{o}_k$ such that $\mathrm{val}(\varpi) = 1$ and $\mathfrak{p} = \varpi\mathfrak{o}_k$.

For any finite extension K/k of degree n, we may extend the valuation map uniquely so that

$$\mathrm{val} : K \to \frac{1}{n}\mathbf{Z}$$

Suppose that K/k is a Galois extension, with Galois group $\mathrm{Gal}(K/k) = \Gamma$. Then there is a sequence of *ramification subgroups*

$$\Gamma \supset D_0 \supset D_1 \supset D_2 \supset \dots D_a \supset D_{a+1} = \{1\}$$

Letting $\mathfrak{F}$ denote the residue field of K, then $\mathfrak{F}/\mathfrak{f}$ is a Galois extension with cyclic Galois group. Choosing a generator, we may lift it to an element $\mathrm{Fr} \in \Gamma$. This *Frobenius element* generates Γ/D_0. The group D_0/D_1 is also cyclic, where any generator θ has order m coprime to p. Finally, D_1 is a p-group.

For a p-adic field k, we may consider the unit groups

$$U_{k,a} = 1 + \mathfrak{p}^a$$

The unit groups are filtered, with

$$\frac{\mathfrak{o}_k}{U_{k,1}} \simeq \mathfrak{f}^\times \qquad\qquad \frac{U_{k,a}}{U_{k,a+1}} \simeq k^+$$

Local class field theory (CFT) describes the abelian Galois extensions K of k. Specifically, one can show an isomorphism of the Galois group

$$\mathcal{W}_k^{\mathrm{ab}} \simeq k^\times$$

Where $\mathcal{W}_k$ is the Weil group of k. This isomorphism has the added structure that if K/k is an abelian extension, then

$$\mathrm{Gal}(K/k) \simeq \frac{k^\times}{N_{K/k}(K^\times)}$$

It is the existence result from CFT that shows that every closed, normal subgroup of $k^\times$ is a norm group for some abelian class field K.

Of particular interest in this paper are the class fields associated to the groups $U_{k,a+1}$. Such fields were described explicitly in [16]. The CFT of such extensions is well known [25] [17].

In addition to the lower numbering of the Galois group, we will also be interested in the upper numbering. In order to do this, we must discuss the φ function. First, we define D_z for any positive real number z by the convention

$$D_z = D_i \qquad i \text{ is the integer} \geq z$$

We define φ as the unique piecewise linear function such that $\varphi(0) = 0$ and if z is not an integer, the slope of the linear segment is given by

$$\varphi'(u) = \frac{1}{[D_0 : D_u]}$$

We note that φ' is defined at the integers i where $D_i = D_{i+1}$. In this sense, φ measures how much the D_i change. The function can be written explicitly, where $i \leq z \leq i+1$, we have

$$\varphi(z) = \frac{1}{d_0}(d_1 + d_2 + \ldots d_i + (z - i)d_{i+1})$$

We define the upper numbering D^j of the ramification groups by

$$D^j = D_{\varphi(j)}$$

An important aspect to this work is that the Langlands parameters we construct must have depth a/m. This means that they are nontrivial on $D^{a/m}$ but trivial on D^z for $z > a/m$.

2.2 Reductive Groups

In this paper, G is a reductive group over k. Associated to any reductive group G is a root datum $(X, \Phi, X^\vee, \Phi^\vee)$. Here, $X, X^\vee$ are abelian groups of finite rank, $\Phi, \Phi^\vee$ are finite subsets of $X, X^\vee$, $X \times X^\vee$ and $\Phi \times \Phi^\vee$ are dual pairs. In practice, considering a split torus T - taking points over the algebraic closure of k, if necessary - $X, X^\vee$ are the characters and cocharacters of T, while Φ represents the roots of T, i.e. the characters of T arising from the restriction of the adjoint action of G the Lie algebra $\mathfrak{g}$.

Upon fixing a split torus T, one obtains a root datum as above. Then $\mathfrak{g}$ can be understood as the direct sum of root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the one representation of T with character α and $\mathfrak{t}$ is the vectors on which T acts trivially. If $t \in T$, since $\alpha \in \Phi$ is a character, we may consider the application $\alpha(t)$. Differentiating, we obtain an additive character for the Lie algebra $\mathfrak{t}$, which we denote H_α. Thus, if $X \in \mathfrak{t}$, we may consider the application $H_\alpha(X)$.

Given a root datum $(X, \Phi, X^\vee, \Phi^\vee)$, one has an action of the Weyl group W on X which leaves the set X invariant set wise. Following [23], we say an element $w \in W$ is *elliptic* if $X^w = 0$ and *regular* provided that it has a regular fixed vector [26].

In this work, we assume that G is absolutely simple, simply connected and k-split. The absolutely simple condition is to avoid complication in the computation of volumes, while simply connected and split are used to simplify matters for the

Langlands parameter. It should be noted that in the epipelagic case, the results have been established without these assumptions [24] [13]. It is reasonable to suppose that there should be little issue with their removal within this work.

2.3 Moy-Prasad Theory

For k a p-adic field, we would like to understand the subgroups of $G(k)$. For what follows, we fix a maximal torus $T \subset G(k)$. In order to procure a geometric description, we consider the Bruhat-Tits building $B(k, G)$. The group $G(k)$ acts on $B(k, G)$, and we consider the stabilizer subgroups of this action. If $x \in B(k, G)$ is a point, then we denote the stabilizer of x, as well as Lie algebra, by

$$G_x \qquad \mathfrak{g}_x$$

As a refinement, Moy and Prasad [19] introduced a further filtration of these stabilizers, where for any nonnegative real number r, they defined subgroups

$$G_{x,r} \subseteq G_{x,0} \qquad \mathfrak{g}_{x,r} \subseteq \mathfrak{g}_{x,0}$$

For $s > r$, then $G_{x,s} \subset G_{x,r}$. This allows the definition

$$G_{x,r+} = \bigcup_{s>r} G_{x,s} \qquad \mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$$

The subgroups are contained normally

$$G_{x,r+} \triangleleft G_{x,r}$$

so we may consider the quotient groups. It turns out that for $r = 0$, we have

$$\frac{G_{x,0}}{G_{x,0+}} = \overline{G}_x$$

is a reductive group over the residue field, while for $r > 0$

$$\frac{G_{x,r}}{G_{x,r+}} \simeq \frac{\mathfrak{g}_{x,r}}{\mathfrak{g}_{x,r+}} = \overline{\mathfrak{g}}_{x,r}$$

is isomorphic to a finite dimensional vector space over the residue field.

The major result [19] was to demonstrate that every admissible representation of $G(k)$ contains some K type - that is, a pair (K, χ) such that restriction to the compact subgroup K contains the irreducible representation χ - as well as a rational number called the *depth* - a smallest number r such that, for any point x in the building, the representation restricted to $G_{x,r+}$ contains the trivial representation. A first application of GIT to representation theory was the classification of all K-types for large residual characteristic [24].

2.4 Vinberg Theory

In this section, we describe the geometric invariant theory (GIT) of a graded Lie algebra. In application, this theory will be applied to reductive groups over the residue field of k. As such, we use the notation $\mathfrak{f}$ to ease understanding.

For $\mathfrak{f}$ an algebraically closed field, the results are due to Vinberg [28] for characteristic zero and Levy [15] for characteristic p sufficiently large. Let $\mathfrak{L}$ be a reductive Lie algebra over k and suppose $\theta \in \mathrm{Aut}(\mathfrak{L})$ is an automorphism of order m. After choosing a primitive mth root of unity ζ_m we may consider the eigen decomposition

$$\mathfrak{L} \simeq \bigoplus_{a=0}^{m-1} \mathfrak{L}_a \qquad \mathfrak{L}_a = \{v \in \mathfrak{L} : \theta v = \zeta_m v\}$$

Since θ is Lie algebra automorphism for a reducetive Lie algebra, the ramification inherits more structure than a simple vector space ramification. In particular, the group $G_0 = \mathrm{Aut}(\mathfrak{L}_0)$ is a reductive group which acts on each $\mathfrak{L}_a$. We define a *Cartan subspace* $\mathfrak{c} \subset \mathfrak{L}_1$ to be a maximal abelian subspace of semisimple elements. One can define the *little Weyl group* $W_\ell = W_\ell(\mathfrak{c}, \theta)$ to be the normalizer of $\mathfrak{c}$ modulo the centralizer in G_0. The following are the basic results in the theory

Proposition. 2.4.1

- For $v \in \mathfrak{L}_a$, the orbit $G_0 v$ is closed if and only if v is semisimple.

- Every closed orbit $G_0 v \subset \mathfrak{L}_a$ intersects a fixed $\mathfrak{c}$.

- Any two Cartan subspaces $\mathfrak{c}, \mathfrak{c}' \subset \mathfrak{L}_a$ are G_0 conjugate.

- If $t, t' \in \mathfrak{c}$ are conjugate by G_0, then they are conjugate over $W_\ell(\mathfrak{c}, \theta)$.

- The restriction of polynomial functions $k[\mathfrak{L}_1] \to k[\mathfrak{c}]$ induces an isomorphism $k[\mathfrak{L}_1]^{G_0} \to k[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$.

- $W_\ell(\mathfrak{c}, \theta)$ is a finite group in $\mathrm{GL}(\mathfrak{c})$ generated by psuedoreflections.

The GIT of graded Lie algebras was introduced to p-adic representation theory in [24]. From Moy-Prasad theory, we have a reductive group $\overline{G}_0$ acting on vector spaces $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$. This action was Galois in nature, with the tame inertial generator acting on $\overline{\mathfrak{g}}_{x,r}$. It was Lemma 4.1 of [24] that this was compatible with the algebraic action described in Vinberg's theory

Proposition. 2.4.2 Let p be sufficiently large and m coprime to p. Suppose θ is an element of the extend Weyl group of order m. There exists a point $x \in B(k, G)$ such that the breaks in the Moy-Prasad filtration are precisely at j/m for positive integer j. There exist isomorphisms

$$f_0 : (G^\theta)^0 \to \overline{G}_{x,0} \qquad f_a : \mathfrak{L}_a \to \overline{\mathfrak{g}}^*_{x,a/m} \qquad m \text{ does not divide } a$$

that commute with the representations of G_0 on $\mathfrak{L}_a$ and $\overline{G}_x$ on $\overline{\mathfrak{g}}_{x,a/m}$.

The result in [24] only mentions $a < m$, but since $\varpi \mathfrak{L}_r = \mathfrak{g}_{r+1}$, it is clear it may be extend arbitrarily.

An important aspect of the both this work and [24] is the notion of a stable vector. A vector $v \in \mathfrak{L}_a$ is said to be *stable* provided that

- The orbit $G_0.v$ is closed in $\mathfrak{L}$.

- The centralizer of v in G_0 is a finite algebraic group

If only the first condition is satisfied, it is a *semistable* vector. Both semistable and stable vectors were classified in [23]. Though a precise description of stable vectors will be recalled in **Lemma 3.1.2**, we remark here that such a vector must be contained in $\mathfrak{L}_a$, with a coprime to m, by PROPOSITION 5.1 in [23].

After choosing regular elliptic element θ, the numbering of $\mathfrak{L}_a$ is determined by which mth root of unity ζ_m one chooses. As such, there is a non canonical isomorphism $\mathfrak{L}_a \simeq \mathfrak{L}_1$, simply by choosing a different mth root of unity.

A final necessary result from the GIT is useful for computing either side of the FDC. In both the computation of the formal degree, as well as the computation of the adjoint γ factor, the difference between the dimension of the Lie algebra and the difference of a reductive sub algebra play an important role. The two appearances are of a very different nature, with the formal degree referring to a p-adic Lie algebra and the adjoint γ factor referring to a complex Lie algebra. However, the conjectural equality of the two implies that the two quantities must behave similarly. It is the following result of Panyushev [20] which establishes this fact

Lemma. 2.4.1

Suppose $x \in \mathfrak{L}_1$ is a semisimple element. Then if $\mathfrak{z}(x)$ denotes the centralizer of x in $\mathfrak{L}$,

$$\dim \mathfrak{L}_a - \dim \mathfrak{z}(x)_a = \frac{1}{m}[\dim \mathfrak{L} - \dim \mathfrak{z}(x)]$$

In particular, the expression is independent of a.

Proof The independence conclusion is **Proposition 2.1(i)** of [20]. Noting that

$$m \cdot [\dim \mathfrak{L}_1 - \dim \mathfrak{z}(x)_1] = \sum_{j=0}^{m-1} \dim \mathfrak{L}_j - \dim \mathfrak{z}(x)_j = \dim \mathfrak{L} - \dim \mathfrak{z}(x)$$

obtains the formula. We recall that, since the index of $\mathfrak{L}_1$ is dependent on choice of primitive root of unity ζ_m, changing the choice permutes the $\mathfrak{L}_a$. In particular, the result holds for all $\mathfrak{L}_a$, with a coprime to m.

The proof is for characteristic zero, though it applies to the positive characteristic case; we parrot it here for completeness. If $\langle \cdot : \cdot \rangle$ is a symmetric, invariant bilinear form, then we consider the form

$$B_x = \langle x, [y, z] \rangle$$

Since $\langle \cdot : \cdot \rangle$ is invariant to the Lie algebra action, the kernel of B must be $\mathfrak{z}(x)$. Since $[\mathfrak{L}_i, \mathfrak{L}_j] \subset \mathfrak{L}_{i+j \bmod m}$ and $x \in \mathfrak{L}_1$, the only way for $B(\mathfrak{L}_i, \mathfrak{L}_j) \neq 0$ is if $i+j = -1 \bmod m$. We may then conclude that

$$\dim \mathfrak{L}_k - \dim \mathfrak{z}(x)_k = \dim \mathfrak{L}_{-k-1} - \dim \mathfrak{z}(x)_{-k-1}$$

Thus one only needs a Lie bracket operation and a non degenerate, symmetric bilinear form invariant under the Lie algebra operation. Though this may fail for small primes, in the present context, it presents no problems. ■

The difference $\dim \mathfrak{L}_a - \dim \mathfrak{z}_a$ is very important in the computation of the formal degree, while $\frac{1}{m}[\dim \mathfrak{L} - \dim \mathfrak{z}]$ comes up in the computation of the adjoint γ factor. The former can be very erratic as a varies. In the case where m is prime, then all $\mathfrak{L}_a, \mathfrak{z}_a$ are isomorphic, so the difference $\dim \mathfrak{L}_a - \dim \mathfrak{z}_a$ is invariant with regards to a. However, if m is composite, both $\mathfrak{L}_a$ and $\mathfrak{z}_a$ may vary as a changes.

3. Supercuspidal Representations

The aim of this chapter is to relate the GIT of graded Lie algebras to supercuspidal representations. Specifically, given a stable vector $v \in \mathfrak{L}_a$, we will produce a finite set of supercuspidal representations of depth a/m. The construction will follow that of [30], where the inducing datum will be obtained from the stable vector provided. The remainder of this introduction will be spent recalling the construction we will employ.

A *supercuspidal representation* of $G(k)$ is one where the matrix coefficients are compactly supported. Given an irreducible representation σ of a compact subgroup K, we define the compact induction of σ to be

$$\operatorname{ind}_K^G \sigma = \{f : G \to V : f \text{ compactly supported and } f(kg) = \sigma(k)f(g) \quad k \in K, g \in G\}$$

By design, if $\operatorname{ind}_K^G \sigma$ is irreducible, it is supercuspidal. Accordingly, supercuspidal representations may be constructed from such pairs (σ, K) that compactly induce to irreducible representations. In this direction, the following result is useful from [2]

Lemma. 3.0.1 Suppose K is a compact (modulo the center) subgroup of G, σ a representation of K and the intertwining subgroup

$$I(G, K, \sigma) = \{g \in G : \sigma = \sigma^g \qquad \text{on } K \cap K^g\}$$

is precisely equal to K. Then the compactly induced representation

$$\operatorname{ind}_K^G \sigma$$

is irreducible and supercuspidal.

Thus, a basic strategy is to produce a couplet (K, σ) such that the full intertwining subgroup of σ is K. In [30], such a couplet is produced from the inducing datum $(\vec{G}, \pi, \phi)$. Here, we have

- $\vec{G} = (G_0, G_1, G_2 \ldots G_d)$ is a *twisted Levi sequence*. Specifically, the G_i are subgroups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_d = G$$

such that $G_i \otimes \overline{k}$ is a Levi subgroup of $G \otimes \overline{k}$ and $Z(G_0)/Z(G)$ is anisotropic. For the particular applications of this paper, G_0 will be a torus, and G will be assumed to be absolutely simple; for our purposes, this simply means G_0 is an anisotropic torus. Further, this work will primarily be concerned with sequences of length 2, thus of the form (T, G).

- π is a depth zero representation of G_0. The depth zero representations of any reductive group were fully described in [19]: they arise from cuspidal representations of the finite group over the finite field. For the particular applications of this paper, G_0 will be a torus, where all representations are vacuously cuspidal [3].

- $\vec{\phi} = (\phi_0, \phi_1, \phi_2 \ldots \phi_d)$ is a sequence of characters associated to the twisted Levi sequence. Specifically,

$$\phi_i : G_i \to \mathbf{C}$$

is a G_{i+1} *generic character of depth* r_i, with the r_i strictly increasing

$$0 < r_0 < r_1 < \cdots < r_d$$

The character ϕ_d is essentially twisting by a central character. As G is assumed to be simple, any central character must be trivial. For the particular applications of this chapter, the twisted Levi sequence will be of the form (T, G), so only one character ϕ of depth $r > 0$ must be given.

The precise definition of G-generic will be given the the next section. Though not reflected in the notation, the definition relies on the twisted Levi sequence. One goal of this work is to better understand G-generic characters with twisted Levi sequence (T, G) of non-integral depth. The case with integral depth was handled in [21].

Given the inducing datum, a sequence of compact (modulo the center) subgroups K_i are constructed, as well as a sequence of representations σ_i of K_i. The definition of the subgroups is

$$K_i = G_x^0 G_{x,r_1/2}^1 G_{x,r_2/2}^2 \cdots G_{x,r_i/2}^i$$

The representation σ_i is obtained inductively, by taking σ_{i-1} and twisting via ϕ_{i-1}. The choices incumbent in the twisting are provided by the theory of the Weil representation.

Proposition. 3.0.1 Given an inducing datum $(\vec{G}, \pi, \vec{\phi})$, with the subgroup K_i defined as above, there exists a representation σ_i of K_i such that the compactly induced representation

$$\operatorname{ind}_{K_i}^{G_i(k)} \sigma_i$$

is an irreducible, supercuspidal representation of $G_i(k)$.

This construction is a generalization of [1], which is essentially the case of a twisted Levi sequence of two elements; as this case is the primary concern of this work, the full information obtained in [1] will prove valuable. However, with an eye toward generalizing these matters, this work adopts the approach of [30].

The aim of this chapter is to use a stable vector $v \in \mathfrak{L}_a$ arising from the GIT of graded Lie algebras to produce supercuspidal representations. We will construct a twisted Levi sequence (T, G) and a G-generic character ϕ of depth a/m. By varying the depth zero representations ρ, we will produce a finite set of supercuspidal representations

$$\{\pi_{v,\rho}\}$$

Though our results only go in one direction, a converse should be available via Galois cohomology. See the remark after **Proposition 3.1.3**.

3.1 Generic Characters

The final element for the construction of [30] is the definition of a G-generic character. In order to do this, we establish notation. First, let $\overline{k}$ be the algebraic closure of k. Suppose T is a maximal torus in $G(\overline{k})$ and let $\Phi = \Phi(G, T, \overline{k})$ be the associated root system and W be the associated Weyl group.

Suppose $X \in \mathfrak{g}^*$ is a semisimple element. Let $Z_G(X)$ be the subgroup of G which fixes X under the coadjoint action and $X_W(X)$ be the subgroup of W fixing X under the induced action. Let $\Phi_X \subset \Phi$ be the set of roots $\alpha \in R$ which are trivial on X, i.e. $X(H_\alpha) = 0$. Finally, for any $w \in W$, let $n_w \in N(T)$ denote a group element representing w.

The definition of twisted Levi sequence can be understood as a series of sequences of the form (G_i, G_{i+1}). As such, in many instances, it suffices to consider the twisted Levi sequence of the form (H, G). Let $\mathfrak{h}, \mathfrak{g}$ be the associated Lie algebras, with $\mathfrak{h}^\perp \subset \mathfrak{g}$ the complimentary subspace under the usual bilinear form [1]. Since $\mathfrak{h}$ is an abelian group, all characters of $\mathfrak{h}$ arise from $\mathfrak{h}^*$. By Moy-Prasad, for $r > 0$ we have isomorphisms

$$\overline{\mathfrak{H}}_{x,r} \simeq \overline{\mathfrak{h}}_{x,r} \subset \overline{\mathfrak{h}}_{x.r} \oplus \overline{\mathfrak{h}}^\perp_{x,r} = \overline{\mathfrak{g}}_{x,r} \simeq \overline{\mathfrak{g}}_{x,r}$$

If $\phi \in \overline{\mathfrak{h}}^*_{x,r}$, then we may obtain a character of G via extending it by zero on $\overline{\mathfrak{h}}^\perp_{x,r}$.

The focus of this section is on a twisted Levi sequence of the form (T, G). If $\mathfrak{t}^*$ is the dual Lie algebra of T, then we may identify $\mathfrak{t}^* \otimes \overline{k}$ with $X^* \otimes_{\mathbf{Z}} \overline{k}$. If ϖ_r is an element of valuation r, and X is a semisimple element satisfying **GE1** below, then we may consider the element $\varpi_r X$ as an element of $X^* \otimes_{\mathbf{Z}} \overline{k}$. Projecting to $X^* \otimes_{\mathbf{Z}} \overline{\mathfrak{f}}$, we denote the residue class by $\overline{X}$. Then $\overline{X}$ is well defined, up to a constant in $\mathfrak{f}^\times$.

With this notation, we may define a G-generic character.

Definition. Let (H, G) be a twisted Levi sequence, $Z, T \subset H$ be the center and a maximal torus, respectively, of H. An element $X \in \mathrm{Lie}^*(Z)^\circ$ is called G-generic of depth r provided that it satisfies the following conditions.

GE1 $\operatorname{ord} X(H_\alpha) = -r$ for all $\alpha \in \Phi(G, T, \overline{k}) \setminus \Phi(H, T, \overline{k})$.

GE2 The subgroup $Z_W(\bar{X})$ of W fixing $\bar{X}^*$ is precisely the Weyl group of $\Phi_{\bar{X}} = \Phi(H, T, \overline{k})$.

For the purposes of this paper, H will be a maximal torus T, thus simplifying matters. In this scenario, the conditions become

GE1 $\operatorname{ord} X(H_\alpha) = -r$ for all $\alpha \in \Phi(G, T, \overline{k})$.

GE2 The subgroup $Z_W(\bar{X})$ of W fixing $\bar{X}^*$ is trivial.

The following result, due to Steinberg [27] and extended in Proposition 7.1 by Yu [30], is helpful.

Proposition. 3.1.1 The group $Z_G(X)$ is generated by T, those G_α such that $\alpha \in \Phi_X$ and the $n_w \in N_G(T)$ such that if $w \in W$ is the image under projection, $w \in Z_W(X)$. The identity component of $Z_G(X)$ is generated by T and those G_α such that $\alpha \in \Phi_X$ and is a reductive group with maximal torus T.

A linear character ϕ is said to be G *generic of depth* r if it is realized by a G-generic element. Since X comes from $\overline{G}_{x,r}$, the isomorphism of **Proposition 2.4.2** suggests using GIT to describe a G-generic vector. If $v \in \mathfrak{L}_a$, then the isomorphism associates a vector $X_v \in \overline{\mathfrak{g}}_{x,a/m}$. We would like to associate properties of v in GIT to those of X_v in the context of representation theory.

Recall, a vector $v \in \mathfrak{L}$ is *stable* provided that the orbit is closed and the stabilizer is a finite algebraic group. In the case of a graded Lie algebra, a simple description is known [23].

Proposition. 3.1.2 Suppose that $\theta \in \operatorname{Aut}(\mathfrak{L})$ is of order m with associated grading

$$\mathfrak{L} = \bigoplus_{a=0}^{m-1} \mathfrak{L}_a$$

A vector $v \in \mathfrak{L}_a$ is stable if and only if v is regular, semisimple and θ acts elliptically on $Z_G(v)$.

Proof This is Lemma 5.6 in [23] ■

In [24], stable vectors were used to construct supercuspidal representations of depth $1/m$. A first goal of this thesis was to expand this result for deeper representations. The approach employed is much different, however. [13] uses methods similar to those found in this work, though the use of stable vectors is not as apparent.

A G-generic element of depth a/m is an element of $\mathfrak{g}^*_{x,a/m}$ that is trivial on $\mathfrak{g}^*_{x,(a+1)/m}$. Given a stable vector $v \in \mathfrak{L}_a$ we use the isomorphism f_a from **Proposition 2.4.2** to obtain an element

$$\overline{X}_v = f_a(v) \in \frac{\mathfrak{g}^*_{x,a/m}}{\mathfrak{g}^*_{x,(a+1)/m}}$$

Finally, we fix a lift $X_v \in \mathfrak{g}^*_{x,a/m}$. The claim will be that this is a G-generic character of depth a/m.

Proposition. 3.1.3 Suppose $v \in \mathfrak{L}_a$ is a stable vector. Then $Z_G(X_v) = T$ is an anisotropic torus and X_v is G-generic of depth a/m with respect to the twisted Levi sequence (T, G).

Proof By **Proposition 3.1.2**, a vector $v \in \mathfrak{L}_a$ is stable if and only if it is regular, semisimple, and θ acts elliptically on the centralizer $Z_G(X)$. The anisotropic condition will follow from the elliptic action, while **GE1** and **GE2** will follow from regularity of the stable vector.

Clearly, v is semisimple if and only if $\overline{X}_v, X_v$ are semisimple.

If $Z(X_v)$ is the centralizer of a semisimple element, it is a reductive group. **Proposition 2.4.2** demonstrates that the action of θ is consistent with the action of the absolute Galois group on the the reductive group. Thus, the rational characters of $Z(X_v)$ will be fixed by θ. That θ acts elliptically demonstrates there are no rational characters of $Z(X_v)$, making it an anisotropic torus.

Finally, we must show that **GE1** and **GE2** are satisfied. Since v is stable, it is regular. This implies that $Z_G(\overline{X}_v) = T$. By **Proposition 3.1.1**, the centralizer of a semisimple element is generated by T, G_α for all $\alpha \in \Phi$ with $X_v(H_\alpha) \neq 0$ and n_w.

That the centralizer is only T implies that (1) all roots α have $\overline{X}_v(H_\alpha) \neq 0$, or simply $X_v(H_\alpha) = -r$, and (2), no $w \in W$ centralize X_v, or simply W_v is trivial. Thus, **GE1** and **GE2** are satisfied, and X_v is G-generic of depth a/m. ■

Remark. 1 All of the arguments given above are reversible, with the exception of the anisotropic tori. This requires a classification, which is available via Galois cohomology. Specifically, rational classes of tori are parameterized by homomorphisms

$$\eta : \mathcal{W}_k \to N(\hat{T})/\hat{T}$$

The latter group being the Weyl group W. In the present context, the residual characteristic is assumed to be large and will not divide $|W|$. As such, the image of η will be trivial on D_1, and thus generated by a tame inertial generator and the Frobenius element. In the case of fractional depth, the tame inertial generator must act elliptically.

3.2 Intertwining

In this section, for a stable vector $v \in \mathfrak{L}_a$, we construct a finite set of supercuspidal representations of depth a/m. In order to do this, we appeal to the construction in [30].

Proposition. 3.2.1 Given an intertwining datum $(\vec{G}, \pi_0, \vec{\phi})$, there exists a compact, modulo the center, subgroup K_i and an irreducible representation σ_i if K_i such that the compactly induced representation

$$\mathrm{ind}_{K_i}^{G^i} \sigma_i$$

is irreducible and supercuspidal of depth depth ϕ_i.

Proof This is **Theorem 0.1** of [30]. ■

Given a stable vector $v \in \mathfrak{L}_a$, **Proposition 3.1.3** produces a G-generic character ϕ_v of depth a/m with respect to the twisted Levi sequence $\vec{G} = (T, G)$. Since we are assuming G is simple, the center is trivial and has only the trivial character. We then take $\vec{\phi} = (\phi_v, \mathbf{1})$ to be the third element of the inducing datum. The final necessary element is the depth zero representation π_0 of T.

For a reductive group H, the quotient $H_{x,r}/H_{x,r+}$ has a distinct structure for $r = 0$ versus $r > 0$ - see **Section 2.3**. Given this distinction, the definition of a depth zero representation of H has a separate definition than for positive depth representations [19]. A *depth zero representation of* H is a representation which, when restricted to the compact group $H_{x,0}$, contains a representation χ, where χ is the inflation of a cuspidal representation of the finite reductive group $\overline{H}_x = H_{x,0}/H_{x,0+}$. In general, the cuspidal requirement is nontrivial; however, in the present context, $T_{x,0}/T_{x,0+}$ is a torus, where all irreducible representations are cuspidal [3]. Further, since $T_{x,0}/T_{x,0+}$ is a finite abelian group, it is non canonically isomorphic to its dual. Thus, we obtain a satisfactory description of all depth zero representations of T.

With the above, we obtain a finite set of inducing data. Upon application of **Proposition 3.2.1**, we then obtain a finite set of supercuspidal representations, as desired. However, two issues arise. First, there is no guarantee that, given two distinct inducing data, the resulting representations are distinct. Second, in order to compute the formal degree, we must understand the dimension of the representation σ. For the general representations arising in [30], these matters can become complicated - see [7] for a result on when such representations are distinct. However, in the relatively simple context of this paper, the initial work of [1] proves useful.

Proposition. 3.2.2 The irreducible, smooth representations σ_ρ of $TG_{x,a/2m}$ that contain the K-type $(\overline{\phi}, G_{x,a/2m+})$ all restrict to multiples of $\overline{\phi}$ and are naturally parameterized by the irreducible, smooth representations ρ of T which contain $(T_{x,r}, \phi)$. If ρ is such a representation of T, then the corresponding representation σ_ρ of $TG_{x,a/2m}$ has dimension

$$\dim \sigma_\rho = \dim \rho \sqrt{|\mathfrak{t}^\perp_{x,a/2m}/\mathfrak{t}^\perp_{x,a/2m+}|}$$

The irreducible, smooth representations of G which contain $(G_{x,a/m}, \phi)$ naturally correspond to the irreducible, smooth representations of T which contain $(T_{x,a/m}, \phi)$.

Proof This is a result directly from [1], with notation adjusted to fit our situation. The first portion of the statement is PROPOSITION 2.6.3 while the final result is THEOREM 2.7.1. ■

If $\tilde{\rho}$ is a depth zero supercuspidal representation of T, then we denote the tensor representation as

$$\rho = \tilde{\rho} \otimes \chi$$

Then ρ is a representation of T with contains χ. It is clear that if $\tilde{\rho}, \tilde{\rho'}$ are distinct depth zero supercuspidal representations, that the resulting representations ρ, ρ' are also distinct. By the above result, we obtain a finite set of representations σ_ρ of $TG_{x,a/2m}$, each containing ϕ_v, and may construct our representations

Proposition. 3.2.3 Suppose $v \in \mathfrak{L}_a$ is a stable vector. Then there is a finite set of smooth, irreducible, supercuspidal representations

$$\pi_{v,\rho} = \operatorname{ind}_K^G \sigma_\rho \qquad\qquad \rho \in \widehat{T_{x,0}/T_{x,0+}}$$

The formal degree of the discrete representation $\pi_{v,\rho}$ with regards to a Haar measure ν is given by

$$\begin{aligned} \deg_\nu \pi_{v,\rho} &= \frac{\dim \sigma_\rho}{\nu(K)} \\ &= \frac{\dim \rho}{\nu(K)} \sqrt{|\mathfrak{t}^\perp_{x,a/2m}/\mathfrak{t}^\perp_{x,a/2m+}|} \end{aligned}$$

Where we may compute

$$\sqrt{|\mathfrak{t}^\perp_{x,a/2m}/\mathfrak{t}^\perp_{x,a/2m+}|} = \begin{cases} q^{\frac{1}{2}\dim \bar{\mathfrak{t}}^\perp_{x,a/m}} & a \equiv 1 \bmod 2 \\ 1 & a \equiv 0 \bmod 2 \end{cases}$$

We note that by the final statement of **Proposition 3.2.2**, the representations are distinct.

A stated origin for this work was to generalize the approach of [24]. There, they use a stable vector $\lambda \in \mathfrak{L}_1$ to produce a set of supercuspidal representations $\pi_x(\lambda, \rho)$ of depth $1/m$. However, not only was the construction of their representations different from those produced in **Proposition 3.2.3**, the grouping arose by taking the irreducible terms of a representation of G. In particular, the set $\{\pi_x(\lambda, \rho)\}$, when appropriately weighted, has the structure of a representation of G.

In keeping with generalizing these results, we wished to investigate this method for the $\pi_{v,\rho}$ produced in **Proposition 3.2.3**. However, the necessary arguments would take the work too far off course. We briefly outline how one would hope to extend the result, though the rigorous results above are what will be referenced for the remainder of the work.

In order to produce the $\pi_x(\lambda, \rho)$, **Lemma 3.0.1** was generalized using Mackey theory in Proposition 2.2 of [24]. Given a subgroup J and character χ of K, we denote the intertwiner

$$I(G, J, \chi) = H_\chi$$

If H_χ is equal to the normalizer of J in G, then the representation

$$\operatorname{ind}_J^G \chi = \bigoplus_{\rho \in \operatorname{Irr}(\mathcal{H})} \dim \rho \operatorname{ind}_{H_\chi}^G \chi_\rho$$

Here, $\mathcal{H}$ is the intertwining algebra

$$\mathcal{H} = \operatorname{End}_{H_\chi}\left(\operatorname{ind}_J^{H_\chi} \chi\right)$$

and there is a bijection $\rho \mapsto \chi_\rho$ between simple $\mathcal{H}$ modules and the irreducible constituents of $\operatorname{ind}_J^{H_\chi} \chi$. Transitivity of induction then provides the desired decomposition.

There are two major obstacles to generalizing this to the depth a/m case. First, the above result is only valid for σ a character. As the formula from **Proposition 3.2.2** demonstrates, this is only true when a is even. The Mackey theory employed in [24] is general enough to resolve this, though the formula will be less straightforward.

Second, we need to obtain a subgroup J and an irreducible representation χ such that the intertwiner is the normalizer of J. A basic suggestion would be to consider K from the Adler-Yu construction, and have J be equal to the neutral component K°. The normalizer would then be K, and all we have left to do is find a suitable representation of K° whose intertwiner is K.

In order to find the suitable representation of K° we first note that, by Proposition 10.4 in [29], the neutral component of K is given by

$$K^\circ = T^\circ G_{x,a/2m}$$

The approach of Adler applies with T replaced with its neutral component, so we may choose the depth zero representation of T° arising from the neutral component, and use the natural parameterization from **Proposition 3.2.2** to procure a representation σ of $T^\circ G_{x,a/2m}$ containing $(\phi_v, G_{x,a/2m+})$. To show that

$$I(G, K^\circ, \sigma) = K$$

we note that if g intertwines σ, then it must intertwine ϕ_v.

In order to demonstrate the intertwining of such a character, the characters are parameterized by certain cosets $a + \mathfrak{s}$ of the dual Lie algebra. Then, Lemma 5.1 of [30] states that $g \in G$ intertwines the character associated to the element $a + \mathfrak{s}$ if and only if

$$\mathrm{Ad}(g).(a + \mathfrak{s}) \cap (a + \mathfrak{s}) \neq \emptyset$$

For the present work, this result reduces intertwining matter to a computation on the Lie algebra, where the concern about the neutral component is not relevant. The arguments of Section 8 and Section 9 of [30] which demonstrate that K is the full subgroup intertwining subgroup of ϕ_v go through, and we obtain that

$$I(G, K^\circ, \sigma) = K$$

4. Langlands Parameters

The aim of this chapter is to relate the the GIT of graded Lie algebras to Langlands parameters. Specifically, to a stable vector $v \in \mathfrak{L}_a$, we will construct a Langlands parameter of depth a/m. The construction will be similar to that of simple and epipelagic supercuspidal representations [6] [24]; as with these cases, the construction will be explicit. The remainder of this introduction will be spent recalling the definition of the Langlands parameter.

Let $\hat{G}$ be the complex reductive group with root datum dual to that of G. If $\hat{\mathfrak{g}}$ denotes the Lie algebra of $\hat{G}$, then we may fix a *pinning*

$$\left\{\hat{T}, \hat{B}, \{x_\alpha\}\right\}$$

where $\hat{T} \subset \hat{B}$ are a maximal torus and Borel subgroup, respectively; the x_α are the simple root vectors of $\hat{T}$. If $\hat{N} = N_{\hat{G}}(\hat{T})$, then we have the Weyl group $W = \hat{N}/\hat{T}$. Because the root data of $\hat{G}$ and G are dual, the Weyl groups are isomorphic.

In addition to $\hat{G}$, the k-form of G must be taken into account. The inner class of k-form is equivalent to choosing a group homomorphism

$$r : \Gamma_k \to \mathrm{Aut}(\Phi, \Delta)$$

whose image is generated by s, t. Here, s generates the inertial subgroup, t is the Frobenius element, and

$$tst^{-1} = s^q$$

$\mathrm{Aut}(\Phi, \Delta)$ acts on the dual root datum in the obvious way, and this action extends to an action on $\hat{G}$. This introduces the L-group

$$^{L}G = \mathcal{W}_k \ltimes \hat{G}$$

A Langlands parameter is a representation of the Weil-Deligne group WD_k into the L-group

$$\varphi : \mathrm{WD}_k \to \mathcal{W}_k \ltimes \hat{G}$$

As a further condition, if we compose φ with projection onto the first factor, the resulting map should be the identity.

As our goal is to construct Langlands parameters φ to correspond to the representations $\pi_{v,\rho}$ constructed in **Proposition 3.2.3**, we may restrict our focus on parameters. First, the $\pi_{v,\rho}$ are discrete series representations, so we will be concerned only with discrete parameters - that is, those parameters where the centralizer of $\operatorname{im}\varphi$ in $\hat{G}$ is finite. Second, $\pi_{v,\rho}$ are all supercuspidal, so the LLC predicts that the parameter φ_v be trivial on the SL_2 part of the Weil-Deligne group; as such, we need only focus on how the parameter behaves on the Weil group $\mathcal{W}_k$ of k. Finally, we invoke the simplifying assumption that G be split over k. With this assumption, the action of Γ on the pinned root datum is trivial and the L-group becomes the direct product

$${}^L G(\mathbf{C}) = \mathcal{W}_k \times \hat{G}$$

The condition that φ_v be the identity in the second factor is trivial, and thus we must find a homomorphism

$$\varphi : \mathcal{W}_k \to \hat{G}$$

It will be the aim of this section to construct a discrete Langlands parameter of depth a/m associated to a stable vector $v \in \mathfrak{L}_a$.

4.1 Artin Conductor

Let φ be a Langlands parameter. Then composing φ with the adjoint action gives an action of WD_k on $\mathrm{Lie}({}^L G)$. Noting that $\mathrm{Lie}({}^L G) = \hat{\mathfrak{g}}$, we arrive at an action of WD_k onto $\hat{\mathfrak{g}}$, which we denote by φ again

$$\varphi : \mathrm{WD}_k \to \mathrm{Aut}(\hat{\mathfrak{g}})$$

For any finite dimensional representation (σ, V) of WD_k, one can define the Artin conductor as the sum of dimensions of representations arising from the Motive - see [5] and [6] for a precise definition. In the case where the representation factors through the usual Weil group, the Artin conductor reduces to the following formula

$$\alpha(\sigma) = \sum_{j=0}^{\infty} \dim(V/V^{D_j}) \frac{d_j}{d_0}$$

where D_j is the image of the jth decomposition group under σ and $d_j = |D_j|$. For the purposes of this paper, we only consider representations trivial on the $\mathrm{SL}_2(\mathbf{C})$ factor; this formula will thus suffice.

For a fixed Langlands parameter, we can consider the Artin conductor of the associated representation on $\hat{\mathfrak{g}}$

$$\alpha(\varphi) = \sum_{j=0}^{\infty} \dim(\hat{\mathfrak{g}}/\hat{\mathfrak{g}}^{D_j}) \frac{d_j}{d_0}$$

Assuming $\hat{\mathfrak{g}}^{D_0}$ is trivial, we may isolate the contribution from $j = 0$ and rewrite

$$\begin{aligned}
\alpha(\varphi) &= \dim \hat{\mathfrak{g}} + \sum_{j=1}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_0} \\
&= \dim \hat{\mathfrak{g}} + \sum_{j=1}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_1}{d_0} \frac{d_j}{d_1} \\
&= \dim \hat{\mathfrak{g}} + \frac{d_1}{d_0} \sum_{j=1}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_1} \\
&= \dim \hat{\mathfrak{g}} + \frac{1}{m} \sum_{j=1}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_1}
\end{aligned}$$

Here we have used the fact that D_0/D_1 is a cyclic group generated by θ. Then the index

$$[D_0 : D_1] = o(\theta) = m$$

The presence of $1/m$ should be noted, as a similar factor arises when computing the formal degree. In the context of graded Lie algebras, both appearances can be understood as moving from the graded components to the whole Lie algebra. In this instance, since D_j is a group of semisimple elements, we have a reductive Lie algebra we denote $\hat{\mathfrak{g}}^{D_j} = \hat{\mathfrak{h}}$. Then by **Lemma 2.4.1** we have

$$\frac{1}{m}[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{h}}] = \dim \hat{\mathfrak{g}}_k - \dim \hat{\mathfrak{h}}_k \qquad \text{for all } k \bmod m$$

where we are considering the grading of the complex Lie algebra $\hat{\mathfrak{g}}$ by the torsion automorphism θ.

Suppose that if D_j is nontrivial, then it contains an element with centralizer $\hat{T}$. Then each $\hat{\mathfrak{g}}^{D_j} = \hat{\mathfrak{t}}$ for all $j > 0$ and the above simplifies

$$\begin{aligned} \alpha(\varphi) &= \dim \hat{\mathfrak{g}} + \frac{1}{m} \sum_{j=1}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_1} \\ &= \dim \hat{\mathfrak{g}} + \frac{1}{m} \sum_{j=1}^{N} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{t}}\right] \frac{d_j}{d_1} \end{aligned}$$

Here, N is the largest integer where $D_N \neq \{1\}$.

4.2 Langlands Parameter

In this section, to any stable vector $v \in \mathfrak{L}_a$, we construct a Langlands parameter φ_v. The objective is that the parameters should correspond with the supercuspidal representations $\pi_{v,\rho}$ constructed in **Proposition 3.2.3**. Since both are constructed with the same vectors, it would be natural to suggest the bijection

$$\varphi_v \leftrightarrow \{\pi_{v,\rho}\}$$

This work does not provide evidence necessary to make such a strong assertion. What we will demonstrate is that

- Both the parameters and the representations have the same depth a/m.

- The identification reduces to that of [24] in the $a = 1$ case.

- The identification satisfies **Conjecture 1**.

Though relevant pieces of evidence, all three are not fine enough to confidently produce a full correspondence. Indeed, if $v, v' \in \mathfrak{L}_a$ are two stable vectors, then we will show

$$\deg_\mu \pi_{v,\rho} = \deg_\mu \pi_{v',\rho'}$$

so the above facts are not enough to determine which $\varphi_v, \varphi_{v'}$ should correspond with which sets of representations. We thus limit our ambitions to the claim that the two sets should correspond, leaving the finer correspondence for later.

The parameter φ_v will be produced in two steps. First, we appeal to the LLC for tori [31] to obtain a homomorphism

$$\xi_v : \mathcal{W}_k \to {}^L T$$

with certain necessary properties. Second, we use Galois cohomology to define an L-homomorphism

$$\eta : {}^L T \to {}^L G$$

Taking the composition will yield the desired Langlands parameter.

To begin, have $T = Z_{G(k)}(X_v)$ be the anisotropic torus from SECTION 3.1. Denoting ${}^L T = \mathcal{W}_k \ltimes \hat{T}$ as the L-group of T, under the LLC of tori, any Langlands parameter

$$\xi : W_k \to {}^L T$$

corresponds to a character

$$\chi : T(k) \to \mathbf{C}^\times$$

We require that ξ have depth a/m which, by the depth preservation theorem, implies χ has depth a/m as well. In particular, this implies that when restricted to $T(k)_{x,a/m}$, χ will correspond to an element of

$$\frac{\mathfrak{t}^*_{x,-a/m}}{\mathfrak{t}^*_{x,(-a+1)/m}}$$

We require that this element be regular. Because of the choice of θ, **Proposition 3.1.2** shows that this element will be stable. Letting $v \in \mathfrak{L}_a$ denote this stable element, then have ξ_v denote the corresponding homomorphism.

Next, we construct the L-homomorphism to embed ${}^LT$ into ${}^LG$. Let T_0 be a maximal split torus of G, and have $N_0 = N(T_0)$ be the normalizer. We will be concerned with maps into the Weyl group $W = N_0/T_0$. By **Section 6** of [22], the anisotropic torus T corresponds to a homomorphism

$$\beta : \mathrm{Gal}(L/k) \to N_0$$

where L is the maximal tamely ramified extension. Recall that $\mathrm{Gal}(L/k)$ is topologically generated by a Frobenius element t and a tame generator s. The order of s is coprime to p, and they have the relation

$$tst^{-1} = s^q$$

By Galois cohomology, the image of $\beta(s)$ in N_0/T_0 is in the W-conjugacy class of the regular elliptic element θ used in the gradings considered in **Section 3**.

We may identify W with $N(\hat{T})/\hat{T}$. Denoting β_W as the composition of β with the projection map onto W, we are interested in lifting $\beta_W(s)$ up to $N(\hat{T})$. By Proposition 5.1 of [23], any lift of $\beta_W(s)$ has order m, and is thus coprime to m. We can thus lift β_W to a homomorphism

$$\hat{\beta} : \mathrm{Gal}(L/k) \to N(\hat{T})$$

If ${}^LT = \mathcal{W}_k \ltimes \hat{T}$ is the L group of T and ${}^LG = W_k \ltimes \hat{G}$ is the L-group of G, then we may define the L-homomorphism

$$\eta : {}^LT \to {}^LG \qquad\qquad w \ltimes t \mapsto w \ltimes \hat{\beta}(w)t$$

We may now construct the Langlands parameter by composing

$$\varphi_v = \eta \circ \xi_v : \mathcal{W}_k \to {}^LG$$

We denote the fixed field of $\ker(\varphi_v)$ by M, while the fixed field of β is denoted L. Then

$$k \subset L \subset M$$

are Galois extensions with L the maximally tamely ramified extension.

With the parameter constructed, we need to show that it has the relevant properties.

Corollary. 4.2.1 φ_v is a discrete Langlands parameter, with

$$\alpha(\varphi_v) = -\dim \hat{\mathfrak{g}} - \frac{aR}{m}$$

The verification of **Conjecture 1** requires the computation of the formal degree; this will be accomplished in the next chapter, with verification **Main Theorem** of **Section 5.3**. Paramount to this computation will be the Artin conductor $\alpha(\varphi_v)$, which we accomplish here.

Proof That φ_v is a Langlands parameter follows from construction. That it is discrete follows because

- Since χ restricts to a regular element, it follows that the centralizer the wild ramification is $\hat{T}$.

- Since $\beta(s)$ is regular elliptic, the centralizer $\hat{T}^{\hat{\beta}(s)}$ is finite.

The Artin conductor is

$$\begin{aligned}
\alpha(\varphi_v) &= \sum_{j=0}^{\infty} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_0} \\
&= \dim \hat{\mathfrak{g}} + \frac{1}{m} \sum_{j=1} \left[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_j}\right] \frac{d_j}{d_1} \\
&= \dim \hat{\mathfrak{g}} + \frac{1}{m} \sum_{j=1} [\dim \hat{\mathfrak{g}} - \dim \mathfrak{t}] \frac{d_j}{d_0} \\
&= \dim \hat{\mathfrak{g}} + \frac{1}{m} [\dim \hat{\mathfrak{g}} - \dim \mathfrak{t}] \sum_{j=1}^{N} \frac{d_j}{d_1}
\end{aligned}$$

Where N is the largest integer such that $D_n \neq \{1\}$. The last sum is familiar, as it is related to the φ function for the upper filtration - see **Section 2.1** - of the extension M/L. As the extension is abelian, the Hasse-Arf theorem says that the sum must be an integer. Since $V_a \subset V = \mathrm{Gal}(M/L)$ is nontrivial, this integer must be a. Thus, the computation becomes

$$\begin{aligned}\alpha(\varphi) &= \dim \hat{\mathfrak{g}} + \frac{1}{m}\left[\dim \hat{\mathfrak{g}} - \dim \mathfrak{t}\right] \sum_{j=1}^{N} \frac{d_j}{d_1} \\ &= \dim \hat{\mathfrak{g}} + \frac{1}{m} aR\end{aligned}$$

. ■

4.3 Adjoint γ Factor

The FDC relates the adjoint γ factor of a discrete Langland parameter to the formal degree of corresponding discrete representation. It can be shown that both the adjoint γ factor and the formal degree are zero in the non discrete case [12] [6], so the discreteness is not a limitation.

For any parameter φ, we may compose with the adjoint action of ${}^LG$ on $\hat{\mathfrak{g}}$ to obtain a representation

$$\mathrm{WD}_k \to {}^LG \to \hat{\mathfrak{g}}$$

By abuse of notation, we denote the above representation as φ as well. For this representation, one can define the adjoint gamma function $\gamma(\varphi, \hat{\mathfrak{g}}, s)$ [6]. We denote $\gamma(\varphi)$ as evaluation at $s = 0$ and $\omega(\varphi)$ as the adjoint root number. An important value in the succeeding chapter will be the ratio

$$\frac{\gamma(\varphi)}{\omega(\varphi)}$$

Though the general definition is quite complicated, in considering the parameters φ_v, the relatively simple nature makes the computation more tractable. Specifically, since φ_v is trivial on $\mathrm{SL}_2(\mathbf{C})$, [6] relates the above ratio to the value of the Artin conductor

$$\alpha(\varphi_v)$$

Combining this with the above computation, we can compute the adjoint γ factor of the φ_v. In our case, φ_v is totally ramified - $\hat{\mathfrak{g}}_1^D$ is trivial -and trivial on $\mathrm{SL}_2(\mathbf{C})$. It was shown in SECTION 3.2 of [6] that under these conditions

$$\frac{\gamma(\varphi)}{\omega(\varphi)} = q^{-\frac{\alpha(\varphi)}{2}}$$

Combining this with the above computation allows us to compute the adjoint γ factor for φ_v.

Corollary. 4.3.1 Let $v \in \mathfrak{L}_a$ be a stable vector. Then

$$\frac{\gamma(\varphi_v)}{\omega(\varphi_v)} = q^{-\frac{\dim \hat{\mathfrak{g}}}{2} - \frac{aR}{2m}}$$

Proof This is **Corollary 4.3.1** and the above result. ■

5. Formal Degrees

The aim of this chapter is to compute the formal degree of $\pi_{v,\rho}$ and ultimately, to show that the tacit correspondence

$$\{\pi_{v,\rho}\} \leftrightarrow \varphi_v$$

satisfies **Conjecture 1**.

Specifically, we use the isomorphism of **Proposition 2.4.2**, the dimension result of **Lemma 3.2.2** and the formula **Lemma 3.2.3**

$$\deg_\mu \operatorname{ind}_H^G \sigma = \frac{\deg \sigma}{\mu(H)}$$

to obtain the formal degree. Note that the dependence of the formal degree on the Haar measure is reflected via the volume of the inducing subgroup. H is compact in the relevant situation, so σ is finite dimensional.

5.1 Preliminary Calculations

In this section, we establish some basic results about volume. After choosing a Haar measure, the volume of a fixed subgroup and the index of Moy-Prasad quotients are computed. Both computations will be necessary in order to compute the volume of the inducing subgroup.

The approach for computing $\operatorname{vol}_\mu(K)$ will be as follows. We find a compact subgroup $G(k)_x$ which contains K. Then we may compute the volume of K by the index formula

$$\operatorname{vol}_\mu K = \frac{\operatorname{vol}_\mu G(k)_x}{[G(k)_x : K]}$$

The advantage to this approach is that K is dependent on a. Instead of making the computation for each possible a, we need only compute the volume of one subgroup, and shift the work to computing the index. This may be simplified into a product, each of whose factors can be analyzed via GIT. The end result is a uniform approach to computing the volume of a wide range of inducing subgroups.

For the compact group containing K, we use a parahoric subgroup. Given a Borel subgroup of $G(\mathfrak{f})$, one can consider the pull back under the natural projection to a subgroup I of $G(k)$. First described by Iwahori, the construction was applied to parabolic subgroups more generally. These *parahoric subgroups* were given geometric footing in the work of Bruhat-Tits, where they arise as the stabilizers of facets in the Bruhat-Tits building. The volume of such subgroups has been well studied

Lemma. 5.1.1 With regards to the Euler-Poincare metric, the volume of $G_{x,1/m}$ is

$$\mathrm{vol}_\mu G_{x,1/m} = q^{\frac{1}{2}[\dim \mathfrak{g} + \frac{1}{m}R]} \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}$$

Proof This is computed in Kaletha [13] and Reeder-Yu [24] ■

Remark. A comment is needed for the choice of the Euler-Poincare measure μ for computing volume [25]. Though there are other reasonable measures to consider for this endeavor, μ clarifies some features of the computation - for instance, by making the Steinberg representation St have degree 1. Other measures have also been used when working with formal degrees, both in the GL_n case [18] and even when developing the formal degree conjecture [10]. The FDC as stated in the **Introduction** implicitly uses the Euler-Poincare measure, so this is our choice of measure.

With the volume of the subgroup in hand, we now turn our attention to the index of the Moy-Prasad group. Suppose that $x \in B(k,T)$ is as in **Proposition 2.4.2** with the associated grading of order m. Then

$$G_{x,i/m}(k)$$

form the set of breaks in the Moy-Prasad filtration. At first glance, it is the indices in this filtration that are of interest. Unfortunately, these numbers can be very erratic as one considers different m. If m is a prime number, then all the indices for non integral i/m are the same; if m is not prime, then they can vary wildly. To this point, the approach has had little dependence on the nature of m, and it would be convenient if this feature would remain. Fortunately, the first assessment of importance for the indices is not necessarily true. The important number is the quotient

$$\frac{[G_{x,i/m}(k) : G_{x,i/m}(k)]}{[T_{x,i/m}(k) : T_{x,i/m}(k)]}$$

With the aid of Vinberg theory, this is a much more manageable computation.

Lemma. 5.1.2

$$\frac{[G_{x,i/m}(k) : G_{x,i/m}(k)]}{[T_{x,i/m}(k) : T_{x,i/m}(k)]} = q^{\frac{1}{m}R} \qquad R = |\Phi|$$

Proof By the Reeder-Yu isomorphism, for i not divisible by m, we may identify

$$\frac{G_{x,i/m}(k)}{G_{x,i/m}(k)} = \mathfrak{L}_i \qquad \frac{T_{x,i/m}(k)}{T_{x,i/m}(k)} = \mathfrak{z}(v)_i$$

as graded Lie algebas. Since $\mathfrak{z}(v)$ is a centralizer of a semisimple element, we apply the result of Panyushev **Proposition 2.4.3** to have that

$$\dim \mathfrak{L}_i - \dim \mathfrak{z}(v)_i = \frac{1}{m}[\dim \mathfrak{L} - \dim \mathfrak{z}(v)] = \frac{1}{m}R$$

is independent of i. Viewed as a vector space over $\mathfrak{f}$, this gives

$$\begin{aligned}\frac{[G_{x,i/m}(k) : G_{x,i/m}(k)]}{[T_{x,i/m}(k) : T_{x,i/m}(k)]} &= q^{\dim \mathfrak{L}_i - \dim \mathfrak{z}(v)_i} \\ &= q^{\frac{1}{m}R}\end{aligned}$$

The Reeder-Yu isomorphism is only valid for i not divisible by m. However, since

$$\sum_{i=0}^{m-1}[\dim \mathfrak{L}_i - \dim \mathfrak{z}(v)_i] = \dim \mathfrak{L} - \dim \mathfrak{z}(v)$$

the result holds for i a multiple of m as well. ∎

That the above value is constant may seem surprising. However, when viewing matters with the formal degree conjecture in mind, there are few other possibilities. As will be discussed later, the terms in question on either side of the FDC are constant by the same result, only one side is positive characteristic, and the other is for characteristic zero. The indices from the formal degree roughly correspond to terms in the Artin conductor. The sum in the Artin conductor is over a decreasing series of numbers, while the sum arising from the subgroup indices can vary wildly. In order for the FDC to hold generally, one would expect the possible variation in the latter case to not arise.

5.2 Volumes of Moy-Prasad Subgroups

In this section, we compute the formal degree of $\pi_{v,\rho}$ for any stable vector $v \in \mathfrak{L}_a$ and any ρ. By **Proposition 3.2.3**, we need to compute the dimension of ρ and the volume of the inducing subgroup K. As the former arises from the character of a torus, this only leaves the volume computation. This will consume the remainder of this section.

Proposition. 5.2.1 Let μ be the Euler-Poincare measure.

$$\operatorname{vol}_\mu Z_x(X) G_{x,a/2m} = q^{\dim \mathfrak{L}/2 - Rj/2m} \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}$$

where

$$j = \begin{cases} \frac{a}{2} & a \text{ even} \\ \frac{a-1}{2} & a \text{ odd} \end{cases}$$

Proof To ease notation, we have

$$T_x = T \qquad G_{x,0+} = G_{x,1/m} = H$$

Further, we extend this notation so that

$$T_{x,i/m}(X) = T_i \qquad G_{x,i/m} = H_i$$

Then $H = H_1 = G_{x,1/m}$ is the subgroup from **Lemma 5.1.1**, and $\operatorname{vol}_\mu H$ is known.

Then the volume can be written as

$$\begin{aligned}
\operatorname{vol}(TH_j) &= \frac{\operatorname{vol}(T)\operatorname{vol}(H_j)}{\operatorname{vol}(T\cap H_j)} \\
&= \frac{\operatorname{vol}(T)}{\operatorname{vol}(T\cap H_j)}\operatorname{vol}(H_j) \\
&= \frac{\operatorname{vol}(T)}{\operatorname{vol}(T\cap H_j)}\frac{\operatorname{vol}(H)}{[H:H_j]} \\
&= [T:T\cap H_j]\frac{\operatorname{vol}(H)}{[H:H_j]} \\
&= \frac{[T:T\cap H_j]}{[H:H_j]}\operatorname{vol}(H) \\
&= \frac{[T_1:T_2]\cdots[T_{j-1}:T_j]}{[H_1:H_2]\cdots[H_{j-1}:H_j]}\operatorname{vol}(H) \\
&= \left(\prod_{i=1}^{j}\frac{[T_{i-1}:T_i]}{[H_{i-1}:H_i]}\right)\operatorname{vol}(H)
\end{aligned}$$

From **Lemma 5.1.1**, we know $\operatorname{vol}(H)$ with regards to the Euler-Poincare measure. From **Lemma 5.1.2**, we know that each

$$\frac{[T_{i-1}:T_i]}{[H_{i-1}:H_i]} = q^{R/m}$$

is independent of i. This leaves the result

$$\begin{aligned}
\operatorname{vol}_\mu T_x G_{x,a/2m} &= \left(\prod_{i=1}^{j}\frac{[T_{i-1}:T_i]}{[H_{i-1}:H_i]}\right)\operatorname{vol}(H) \\
&= q^{-\dim\mathfrak{L}-Rj/m}\frac{\gamma(\varphi_0)}{\omega(\varphi_0)}
\end{aligned}$$

■

Now, we may combine the above result with**Proposition 2.4.2**, the dimension result of **Lemma 3.2.2** and the formula in **Lemma 3.2.1** to obtain the formal degree of $\pi_{v,\rho}$.

Corollary. 5.2.1 Let μ be the Euler-Poincare measure.

$$\deg_\mu \pi_{v,\rho} = \frac{1}{[T_x:T_{x,1/m}]}\frac{\gamma(\varphi_0)}{\omega(\varphi_0)}q^{\frac{\dim\mathfrak{L}}{2}+\frac{Rj}{2}}$$

Proof By **Proposition 3.2.3**, the formal degree of $\pi_{v,\rho}$ is

$$\frac{\dim \rho}{\nu(K)}\sqrt{|\mathfrak{t}^{\perp}_{x,a/2m}/\mathfrak{t}^{\perp}_{x,a/2m+}|}$$

Applying **Proposition 5.2.1** gives the desired result.

■

5.3 Formal Degree Conjecture

In this section, we demonstrate that the formal degree conjecture is satisfied for our correspondence. In particular, given a stable vector $v \in \mathfrak{L}_a$, we constructed a finite set of representations

$$\pi_{v,\rho}$$

in **Proposition 3.1.5**. Further, we constructed a discrete Langlands parameter

$$\varphi_v$$

in **Proposition 4.2.1**. Tacitly, we propose the correspondence

$$\{\pi_{v,\rho}\} \leftrightarrow \varphi_v$$

By construction, all of the $\pi_{v,\rho}$ as well as φ_v have depth a/m. Our other main concern is to show that this correspondence satisfies the FDC. As was mentioned earlier, this is one aspect of a potential Langlands correspondence; however, given the breadth of cases addressed, it is a sizable step forward into understanding a potential correspondence.

Main Theorem.

Let $v \in \mathfrak{L}_a$ be a stable vector. Then there exists a finite set of supercuspidal representations of depth a/m

$$\{\pi_{v,\rho}\} \qquad \rho \in \widehat{T_{x,0}/T_{x,0+}}$$

and a discrete Langlands parameter of depth a/m

$$\varphi_v : \mathrm{WD}_k \to {}^L G(\mathbf{C})$$

such that the the following equality holds

$$\deg_\mu \pi_{v,\rho} = \frac{\dim \rho}{|A_{\varphi_v}|} \frac{\gamma(\varphi_v)}{\omega(\varphi_v)} \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}$$

Proof The existence of the representations and parameters are results **Proposition 3.1.5** and **Proposition 4.2.1**. The FDC condition follows from **Corollary 5.2.1**, while

$$[T : T_{x,1/m}] = |A_{\varphi_v}|$$

follows from LEMMA 4.4.6 and LEMMA 4.4.10 of [13].

■

Remark. It should be noted that the above correspondence is a uniform approach, even as a varies. This is not *a priori* necessary under the conditions of the LLC. However, given the wide scope of the full conjectural correspondence, it is an encouraging feature.

5.4 Further Predictions

In this final section, comment on future directions for the above work. The observations here arose from the pursuit of thesis project, but did not fall into the scope of this work. The hope is that comments here will inspire further inquiry.

The first obvious direction is to generalize **Corollary 5.2.1** and compute the formal degree of the general representations constructed in [30]. That construction is a generalization of Howe's construction for GL_n [11], where the formal degrees have been computed [18]. As all the representations are compactly induced from a compact (modulo the center) subgroup, the computation follows from the basic formula

$$\deg_\mu \operatorname{ind}_K^G \sigma = \frac{\dim \sigma}{\mu(K)}$$

For the $\mu(K)$ factor, the computation follows the same method as **Proposition 5.2.1**. Each K arising in **Proposition 5.2.1** is contained in a parahoric subgroup G_x. We may take $x \in B(T,k)$ such that the grading on G_x has breaks at a/m for integral a, so that the depths of $\vec{\phi}$ are of the form

$$\phi_{j-1} = \frac{a_j}{m} \qquad a_j < a_{j-1}$$

If $\vec{G} = (G^0, G^1 \ldots G^d)$ is the twisted Levi sequence, the inducing subgroup K has the form

$$K = G^0_x G^1_{x,a_1/2m} G^2_{x,a_2/2m} \cdots G^d_{x,a_d/2m}$$

The quotient may then be decomposed into a product of indices

$$[G_x : K] = \frac{[G_{x,0} : G_{x,1/m}][G_{x,1/m} : G_{x,2/m}] \cdots [G_{x,b/m} : G_{x,a_d/2m}]}{[G^0_{x,0} : G^0_{x,1/m}][G^0_{x,1/m} : G^0_{x,2/m}] \cdots [G^{d-1}_{x,b/m} : G^{d-1}_{x,a_j/2m}]}$$

Here, the understanding is that as the depth in the denominator passes $a_j/2m$, the group in the denominator transitions from G^{j-1} to G^j. Further, b is the largest integer smaller than $a_d/2m$.

The above formula is simply a product of factors of the form

$$\frac{[G_{x,i/m} : G_{x,(i+1)/m}]}{[G^j_{x,i/m} : G^j_{x,(i+1)/m}]}$$

The q-log of which is simply

$$\log_q \frac{[G_{x,i/m} : G_{x,(i+1)/m}]}{[G^j_{x,i/m} : G^j_{x,(i+1)/m}]} = \dim \bar{\mathfrak{g}}_i - \dim \mathfrak{g}^j_i$$

In each case, $\mathfrak{g}_j$ is the centralizer of a semisimple element of $\mathfrak{g}^*$, namely the G^{j+1}-generic element with gives rise to ϕ_j. By the isomorphism of **Proposition 2.4.2** and the GIT result of **Lemma 2.4.1**, all of these differences are invariant of i, allowing the entire product to be simplified

$$\log_q [G_x : K] = \frac{1}{m} \sum_{j=0}^{d-1} [a_j - a_{j-1}] \cdot [\dim \bar{\mathfrak{g}} - \dim \bar{\mathfrak{g}}^j]$$

One should compare this to Propostion 2.2.1 of [18], where the reductive group is for GL_n.

With the index computed, **Lemma 5.1.1** allows for the computation of the full volume of $\mu(K)$, where we are now assuming μ is the Euler-Poincare measure. All that remains for the computation of the formal degree is $\dim \sigma$. In the above work, this was accomplished by **Proposition 3.2.2**, a result due to Adler [1]. No such result is available for the more general σ, and it proved outside the scope of this work to pursue. However, there is no reason to doubt that such a result is valid more generally, as it was in Lemma 2.2.1 of [18] for the GL_n case. Given the observational nature of this section, we proceed with such a generalization as a tacit hypothesis.

Limiting our scope to those representations which are expected, under the LLC, to correspond to totally ramified parameters, we may use the formula

$$\frac{\gamma(\varphi)}{\omega(\varphi)} = q^{-\alpha(\varphi)/2}$$

Turning the focus on the power of q, there are similar terms on the formal degree side

$$[\dim \overline{\mathfrak{g}} - \dim \overline{\mathfrak{g}}^j]\,(n_j - n_{j-1})$$

and the Langlands parameter side

$$[\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^{D_i}]\left(\frac{d_i}{d_1}\right)$$

As was seen in **Section 5.3**, the indices of i and j do not necessarily correspond - indeed, in that case they were referencing the upper and lower numberings. However, it is suggestive that the dimensions of reductive Lie algebras on either side should correspond. Indeed, one obtains a sequence of subalgebras on the Langlands parameter side

$$\hat{\mathfrak{g}}^{D_i}$$

which should relate to the twisted levi sequence arising in the supercuspidal construction.

The situation covered within has been relatively simple. Representations beyond the supercuspidal will eventually need to be considered, as will non-splitting Galois groups, non-abelian wild ramification, and even small residual primes. The goal of this work was to use the added structure of specificity to illuminate the structure of the space the representations, as well as the Langlands parameters. Though the methods contained within will not hold in full generality, the hope is that the wider structure has become more clear.

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VITA

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Britain Cox was born. As a reasonable hypothesis, someday he will die. Between these two events, he composed this thesis.