Exact mapping between classical and topological orders in two-dimensional spin systems

H. D. Chen J. P. Hu

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Han-Dong Chen[1] and Jiangping Hu[2]
[1]*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA*
[2]*Department of Physics, Purdue University, West Lafayette, Indiana 47907, USA*
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Motivated by the duality between site-centered spin and bond-centered spin in one-dimensional system, which connects two different constructions of fermions from the same set of Majorana fermions, we show that two-dimensional models with topological orders can be constructed from certain well-known models with classical orders characterized by symmetry breaking. Topology-dependent ground state degeneracy, vanishing two-point correlation functions, and unpaired Majorana fermions on boundaries emerge naturally from such construction. The approach opens a different way to construct and characterize topological orders.

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Recently, topological orders have attracted intensive interests for different reasons.[1–7] The best studied example of topological order are the fractional quantum hall (FQH) states.[8] All different FQH states have the same symmetry. Unlike classically ordered state, FQH liquids cannot be described by Landau's theory of symmetry breaking and the related order parameters.[2,9] A new theory of topological order is proposed to describe FQH liquids.[9] New nonlocal quantities, instead of local order parameters, such as ground state degeneracy,[1] the non-Abelian Berry's phase,[3] and topological entropy,[7,10] were introduced to characterize different topological orders. Topological ordered systems have also been designed and studied in the context of quantum computation as a realization of potentially fault-tolerant quantum memory and quantum computation.[6,7,11] It is the nonlocality of the topological orders that significantly reduces the effect of decoherence.[12]

Theoretically, a number of soluble or quasisoluble models which capture the topological orders have been proposed and studied.[5–7,13–16] However, unlike the conventional orders which are entirely characterized by broken symmetries, the topological orders have not been characterized in a universal way. In fact, topological orders have to be studied case by case in different models. Recently, it has also been pointed out that the spectrum is completely inconsequential to topological quantum order[17] and hidden order parameter has been suggested in Kitaev model on honeycomb lattice.[18] In this work, we show a different way to characterize topological orders, which is based on well-known conventional models. First, we show that it is possible to map a model with topological order to a model with a local order parameter in certain physical realizations through a nonlocal duality transformation. A local order parameter description of topologically ordered systems is potentially useful. For instance, thermodynamic properties and energy spectrum can be easily computed in terms of classical order parameters. Second, we show that topologically ordered systems can be constructed or designed from well-studied classically ordered states by including a topological boundary term associated with the lattice topology. In such a construction, topological properties are manifestly presented. The result provides an approach for easier and/or better design of physical implementations of topological orders for quantum computation, starting from ordered systems well understood in the framework of Landau's symmetry breaking theory. Finally, we would like to point out that the transformation used in this work only works for a limited class of models. However, we conjecture that a general connection between topological orders and classical orders might be possible and more beautiful nonlocal transformations are waiting to be discovered.

We start with examining a well-known nonlocal transformation, namely, the duality between site-centered spin and bond-centered spin in one-dimensional spin-1/2 system,[19]

$$\mu_z(n) = \sigma_x(n+1)\sigma_x(n), \tag{1}$$

$$\mu_x(n) = \prod_{m \leq n} \sigma_z(m). \tag{2}$$

The spin operators σ on the original lattice can be fermionized by a Jordan-Wigner transformation,

$$\sigma_x(n) = \left[\prod_{m<n} iA(m)B(m)\right]A(n), \tag{3}$$

$$\sigma_y(n) = -\left[\prod_{m<n} iA(m)B(m)\right]B(n), \tag{4}$$

$$\sigma_z(n) = iA(n)B(n), \tag{5}$$

where $A(n)$ and $B(n)$ are Majorana fermions on site n. Fermions can be defined as $c(n)=[A(n)+iB(n)]/2$. The duality transformation of Eqs. (1) and (2) now reads

$$\mu_z(n) = iB(n)A(n+1), \tag{6}$$

$$\mu_x(n) = \left[\prod_{m<n} iB(m)A(m+1)\right]B(n), \tag{7}$$

which is another Jordan-Wigner transformation if we introduce a new set of fermions $d(n)=[B(n)+iA(n+1)]/2$ on the dual lattice. It is thus clear that the duality transformation connects two different constructions of fermions from the same set of Majorana fermions, as illustrated in Fig. 1. The duality now appears as a very local transformation. However, in terms of spin operators, it is inherently nonlocal. In the following, we generalize this duality transformation to two-dimensional systems and show that the transformation can be used to exactly map a classically ordered system to a topo-

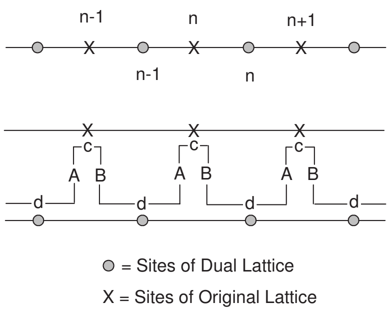

FIG. 1. Duality mapping in one dimension as a recombination of Majorana fermions. $c(n)=[A(n)+iB(n)]/2$ is the fermion defined on the original lattice site and $d(n)=[B(n)+iA(n+1)]/2$ is defined on the dual lattice site.

logically ordered one. Two specific models are discussed.

The first model is Wen's exactly soluble spin model defined on a square lattice,[5]

$$H=g\sum_{ij}F_{ij}=g\sum_{ij}\sigma^y_{i,j}\sigma^x_{i+1,j}\sigma^y_{i+1,j+1}\sigma^x_{i,j+1}, \tag{8}$$

where (i,j) is the coordinate of lattice site. It is easy to see $[F_{ij},F_{i'j'}]=0$ and the model is thus exactly soluble. This model is shown to have robust topologically degenerate ground states and gapless edge excitations.[5]

Let us first introduce the two-dimensional Jordan-Wigner transformation to fermionize the model,[20]

$$\sigma^x_{ij}+i\sigma^y_{ij}=2\left[\prod_{j'<j}\prod_{i'}\sigma^z_{i'j'}\right]\left[\prod_{i'<i}\sigma^z_{i'j}\right]c^\dagger_{ij}, \tag{9}$$

$$\sigma^z_{ij}=2c^\dagger_{ij}c_{ij}-1. \tag{10}$$

If we define the Majorana operators

$$A_{ij}=(c^\dagger_{ij}+c_{ij})\ \text{and}\ B_{ij}=i(c^\dagger_{ij}-c_{ij}), \tag{11}$$

we find F_{ij},

$$F_{ij}=A_{ij}A_{i+1,j}B_{i,j+1}B_{i+1,j+1}. \tag{12}$$

It is now interesting to generalize the duality to two dimensions and define fermions on vertical bonds $(i,j)-(i,j+1)$,

$$d_{ij}=(A_{ij}+iB_{i,j+1})/2, \tag{13}$$

$$d^\dagger_{ij}=(A_{ij}-iB_{i,j+1})/2. \tag{14}$$

It follows that

$$iA_{ij}B_{i,j+1}=2d^\dagger_{ij}d_{ij}-1\equiv\mu^z_{ij}, \tag{15}$$

where μ_{ij} is related to the fermion d through a Jordan-Wigner transformation. The Wen Hamiltonian thus reads

$$H=g\sum_{ij}\mu^z_{i,j}\mu^z_{i+1,j}. \tag{16}$$

This new Hamiltonian describes a set of decoupled Ising chains (see Fig. 2).[21] A first-order phase transition from ferromagnetism to antiferromagnetism happens at $g=0$. It is now straightforward to see that all two-point correlation

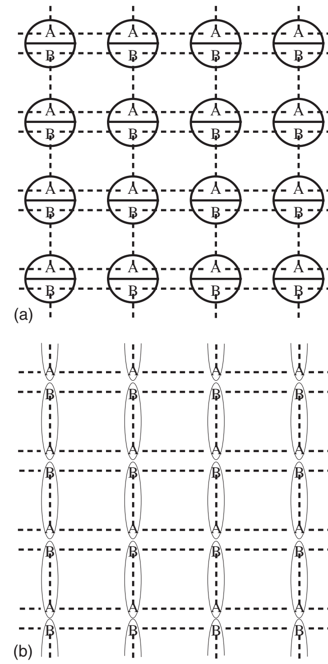

FIG. 2. (a) Graphical representation of the Wen model. Each circle denotes one original lattice site on which two Majorana fermions A and B are defined. Each dotted rectangle contributes a ring-exchange term given by Eq. (12). (b) After introducing fermions on vertical bonds, the model reduces to decoupled Ising chains.

functions are identically zero $\langle\sigma^a_1\sigma^b_2\rangle=0$ in the ground state. This is so because $\sigma^1_1\sigma^b_2$ contains A_{ij} (or $B_{i,j+1}$) that is unpaired with its partner $B_{i,j+1}$ (or A_{ij}) due to the fractionalization of σ_{ij} into A_{ij} and B_{ij} and the recombination of A_{ij} and $B_{i,j+1}$ into μ^z_{ij}.

It is also interesting to fermionize a Zeeman term,

$$b\sum_{ij}\sigma^z_{ij}=b\sum_{ij}(d_{ij}+d^\dagger_{ij})(d_{i,j-1}-d^\dagger_{i,j-1}). \tag{17}$$

We notice that Eq. (16) + Eq. (17) is the same fermionic Hamiltonian obtained by fermionizing quantum compass model using Jordan-Wigner transformation.[20] After including a Zeeman term, Wen's soluble model is thus equivalent to the quantum compass model, which is shown to have dimensional reduction[22,23] and a first-order phase transition at $b=g$.[20]

The duality mapping can also be made explicit as follows. Define $\mu^z_{i,j}$ on the bond $(i,j)-(i,j+1)$,

$$\mu^z_{i,j}=\sigma^y_{i,j}\left(\prod_{i'>i}\sigma^z_{i',j}\right)\left(\prod_{i'<i}\sigma^z_{i',j+1}\right)\sigma^y_{i,j+1}. \tag{18}$$

Let us first prove that $\mu^z_{i,j}$ commutes with $\mu^z_{k,l}$. Without losing generality, let us assume $l\geqslant j$. Let us consider the overlaps between the original lattice sites involved in μ^z_{ij} and μ^z_{kl}. (a) If $l-j\geqslant 2$, there is no overlap. Similarly, no overlap happens when $l=j+1$ and $k>i$ and two μ^z commute. (b) If $l=j+1$ and $k=i$, there is only one common site on which σ^y is involved in both μ^z. (c) If $l=j+1$ and $k<i$ or $l=j$ [see Fig. 3(b)], as far as a commutation relation is concerned, the only relevant part is

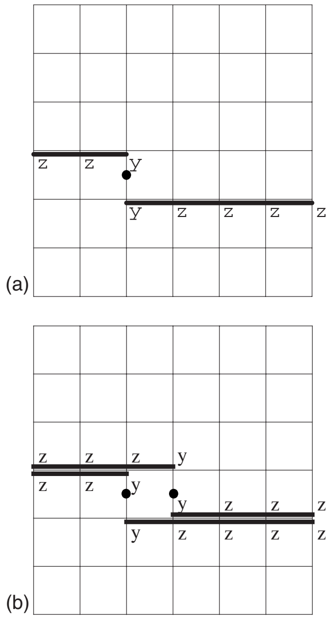

FIG. 3. (a) Graphical representation of Eq. (18). μ^z is defined on a vertical bond as the product of spin components as labeled on the original lattice sites along the thick dark line. (b) Nearest-neighbor coupling between μ_{ij}^z and $\mu_{i+1,j}^z$.

$$(\sigma_r^z\sigma_k^y)_1(\sigma_r^y\sigma_k^z)_2 = (\sigma_k^y\sigma_r^z)_1(\sigma_k^z\sigma_r^y)_2. \tag{19}$$

Here, the subscripts 1 and 2 denote two different sites of original square lattice, while r (red) and k (black) are used to keep track of which μ^z the spin operators σ come from. Based on the above consideration, we conclude that μ's are commuting with each other. It is also trivial to see $(\mu^z)^\dagger = \mu^z$ and $(\mu^z)^2 = 1$. Therefore, μ^z can be viewed as an Ising degree of freedom defined on a vertical bond. Let us now consider the interaction term between two nearest-neighboring μ^z, as illustrated in Fig. 3(b),

$$\begin{aligned}\mu_{i,j}^z\mu_{i+1,j}^z &= \sigma_{i,j}^y(\sigma^z\sigma^y)_{i+1,j}(\sigma^y\sigma^z)_{i+1,j}\sigma_{i+1,j+1}^y \\ &= \sigma_{i,j}^y\sigma_{i+1,j}^x\sigma_{i+1,j}^x\sigma_{i+1,j+1}^y.\end{aligned} \tag{20}$$

Therefore, the original Hamiltonian (8) is equivalent to Eq. (16) and the Ising order is mapped to the quantum order studied by Wen.[5] The explicit mapping (18) also allows us to examine the topological nature of the ordering. In the basis of σ^y, μ_{ij}^z creates two kinks at the two ends (i,j) and $(i,j+1)$ of the vertical bond. Therefore, the ordering is actually a condensation of kink-dipoles. The topological nature of the order in this model is thus similar to that of the Ising lattice gauge theory.[24]

The discussion of the exact mapping is so far limited to bulk terms. It is important to study the boundary conditions and demonstrate explicitly the dependence of ground state degeneracy on the topology. In general, the boundary conditions will induce a nonlocal phase factor to the coupling strength between boundary spins, due to the phase term in the Jordan-Wigner transformation. The topology-dependent phase factor has profound consequence as we shall show shortly. One immediate consequence is that it determines the ground state degeneracy.

The most interesting boundary condition is the case where we put the original spin model into a closed topology. A simple closed manifold is a torus by taking periodic boundary conditions along both directions. The boundary term along y direction is $H_b^y = g\sigma_{i,L_y}^y\sigma_{i+1,L_y}^x\sigma_{i+1,1}^y\sigma_{i,1}^x$. It is clear that the phase term cancels and the periodic condition along the y direction is mapped to a periodic boundary condition in the direction perpendicular to the Ising chain. The periodic boundary condition along x direction induces a boundary term $H_b^x = g\sigma_{L_x,j}^y\sigma_{1,j}^x\sigma_{1,j+1}^y\sigma_{L_x,j+1}^x$. This term is mapped to $H_b^x = g_x\sum_j\mu_{1,j}^z\mu_{L_x,j}^z$ with the coupling strength g_x given by

$$g_x = g\prod_i \sigma_{i,j}^z\sigma_{i,j+1}^z = g\prod_{i,j} iB_{ij}\mu_{ij}^z A_{i,j+1}. \tag{21}$$

The boundary term couples nearest-neighboring chains non-locally, which manifestly represents the hidden topological structure in the original model. A direct consequence of this coupling is the partial lift of ground state degeneracy.

Another interesting case is the ribbon structure, where periodic boundary condition in the y direction and open boundary condition in the x direction are assumed. The open boundary condition in the x direction is now mapped to the open boundary conditions in the spin chains. Consequently, the ground state now has an effect of dimension reduction and huge degeneracy 2^{L_y}, where we denote L_x and L_y as the system sizes along the x and y directions, respectively. The degeneracy can actually be related to free Majorana fermions on boundaries. This can be shown by considering an equivalent geometry where we set periodic boundary condition along x and open boundary condition along y. The ground state degeneracy is 2^{L_x} in this case. The mapping thus leads to unpaired Majorana fermions, $A_{i,L_y}(i=1,\ldots,L_x)$ on the sites of the top boundary and $B_{i,1}(i=1,\ldots,L_x)$ on the sites of the bottom boundary. A_{i,L_y} are coupled to the bulk system through the boundary term in the form of $\prod_{i=1}^{L_x}A_{i,L_y}$. Similarly, $B_{i,1}$ are coupled to bulk through $\prod_{i=1}^{L_x}B_{i,1}$. Therefore, the operators that flip even numbers of unpaired Majorana fermions on top and/or bottom boundaries are conserved quantities and consequently lead to degenerated ground states. For instance, we can combine $A_{1,L}$ and $A_{2,L}$ into a fermion whose particle number $(iA_{1,L}A_{2,L}+1)/2$ is a conserved quantity. We thus have successfully mapped the global (nonlocal) Z_2 degree of freedom of the decoupled Ising chains into a local degree of freedom of unpaired Majorana fermions at the ends of the chains.

To illustrate our approach further, we show that similar physics can also be reached for the second Kitaev model defined on a honeycomb lattice,[7]

$$H = -\sum_{\lambda=x,y,z}\sum_{\lambda\ bonds} J_\lambda S_j^\lambda S_k^\lambda. \tag{22}$$

This topologically ordered model can be mapped to a model of spinless fermions whose ground states are characterized by local order parameters.

Again, we fermionize this model using the Jordan-Wigner transformation. The idea is to deform the honeycomb lattice

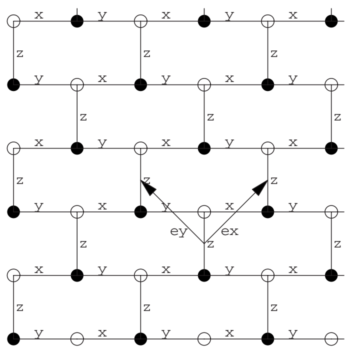

FIG. 4. Deformed honeycomb lattice and three types of bonds.

into a brick wall lattice as shown in Fig. 4. We introduce the subscripts b and w to denote the white and black sites of a bond as illustrated in Fig. 4. We also define the corresponding Majorana fermions $A_w=(c-c^\dagger)_w/i$ and $B_w=(c+c^\dagger)_w$ for white sites and $B_b=(c-c^\dagger)_b/i$ and $A_b=(c+c^\dagger)_b$ for black sites. After a Jordan-Wigner transformation given by Eqs. (9) and (10), the Hamiltonian of Kitaev model becomes

$$H=-\frac{i}{4}\left[\sum_{x\ bonds} J_x A_w A_b - \sum_{y\ bonds} J_y A_b A_w\right] - \frac{i}{4}\sum_{z\ bonds} J_z \alpha_{bw} A_b A_w, \tag{23}$$

where $\alpha=iB_bB_w$ defined on each vertical bond. It is easy to see that α is a conserved quantity[18] and can now be taken as a number that can take either $+1$ or -1. The Hamiltonian is now quadratic in A and readily to be solved exactly.

We are now ready to generalize the duality to brick wall lattice and introduce fermion on a z bond, $d=(A_w+iA_b)/2$ and $d^\dagger=(A_w-iA_b)/2$, where A_w and A_b are the Majorana fermions on the white and black sites of a given z bond. We thus have a model for fermions on a square lattice with site-dependent chemical potential,

$$4H = J_x\sum_i (d_i^\dagger + d_i)(d_{i+\hat{e}_x}^\dagger - d_{i+\hat{e}_x}) + J_y\sum_i (d_i^\dagger + d_i)(d_{i+\hat{e}_y}^\dagger - d_{i+\hat{e}_y}) + J_z\sum_i \alpha_i(2d_i^\dagger d_i - 1). \tag{24}$$

Here, $\hat{e}_y$ connects two z bonds and crosses a y bond, similar to $\hat{e}_x$, as illustrated in Fig. 4. This Hamiltonian describes a system of spinless fermions with p-wave BCS pairing and site-dependent chemical potential, where the ground states are characterized by local order parameters. Previous discussions about ground state degeneracy and vanishing two-point spin correlation functions can now be extended to this model straightforwardly. Unpaired free Majorana fermions also emerge naturally at open boundaries. For instance, a ribbon geometry can be achieved by breaking a row of z bonds. For each broken z bond, the Z_2 degree of freedom $\alpha=iB_bB_w$ is fractionalized into two unpaired free Majorana fermions, $B_b=(c-c^\dagger)_b/i$ on the top boundary and $B_w=(c+c^\dagger)_w$ on the bottom boundary. A detailed study of the fermionized Hamiltonian (24) will be presented elsewhere.[25]

In summary, we have successfully constructed exact mappings from topological orders to classical orders in two exactly soluble spin models. The topological dependence in the latter model is manifestly represented in the terms resulted from the mapping of boundary conditions. Unpaired Majorana fermions on open boundaries and vanishing two-point spin correlation functions also follow naturally from our construction. Our work suggests a different approach to construct certain topological orders from well-studied classical orders through a nonlocal transformation.

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