1998

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Report Number:
98-017

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CSD-TR #98-017
June 1998

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Abstract

Two simple interface relaxation techniques for solving elliptic differential equations are considered. Their theoretical analysis is carried out at the differential level is carried out and "optimal" relaxation parameters are obtained for 1–dimensional model problems. A comprehensive experimental numerical study is also presented.

1 Introduction

Domain decomposition has proven an effective means of partitioning the task of solving Differential Equation (DE) problems numerically. It is mainly an algebraic approach and works by splitting the discrete DE domain into subdomains which can be coupled in many ways. The well established additive and multiplicative Schwartz methods are examples of typical domain decomposition approaches that have been analyzed extensively. Interface Relaxation (IR) is a step beyond domain decomposition. IR methods are characterized by the fact that they can be easily formulated as numerical procedures for solving differential equation problems while all their actions involve continuous data. They assume a splitting of the domain into a set of non-overlapping subdomains and consider the associated DE problem defined on them. These subproblems are coupled through relaxation mechanisms on the interfaces. IR methods naturally apply to multi-physics problems when the DE may change

[1] Work supported in part by PENED grants 95-602 and 95-107, NSF grants CCR-9202536 and CDA-9123502, and AFOSR FM 49620-92-J-0069. An early version of this paper was presented at the Ninth International Domain Decomposition Conference.

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from one subdomain to another; we do not consider this application here. For a general introduction to the IR methodology the reader is refereed to [9,10].

A review study of a large collection of IR methods can be found in [11]. The convergence of these schemae depends, as expected, on the differential operator, the geometry of the original domain, and in addition on the geometry of the subdomains chosen. This makes the selection of "optimum" values for the relaxation parameters a hard and challenging problem. On the other hand, the local subdomain discretization schemae do not affect the convergence properties of the IR schemae which gives these methods great versatility; one can select the most appropriate discretization parameters or numerical method for the differential problem defined on each subdomain.

The development of an automated and adaptive procedure that dynamically estimates "good" relaxation parameters, using automatic differentiation techniques, for general differential operators and arbitrarily shaped subdomains is underway [12]. Nevertheless, in order for this parameter selection procedure to be effective, theoretical results for simple model problems are needed that provide a better understanding of the convergence mechanisms involved.

The main objective of our study is to estimate values analytically for the parameters involved in two recently proposed and analyzed IR methods. Namely we consider an averaging scheme [13,16,17] (denoted by **AVE** in the sequel) and a Robin–type IR scheme [8] (denoted by **ROB**). We restrict ourselves to Helmholtz boundary value problems. In both schemes the error involved in each interface can be analytically given in terms of the error in the previous iteration. This leads us to a system of linear algebraic equations that represents the relation between the errors on all interfaces in two consequent iterations. Then we minimize the spectral radius of the iteration matrix involved using a different approach for each method. For the **AVE** scheme we minimize the area of the associated Gershgorin discs, which is equivalent of bounding the max norm of the iteration matrix. The iteration matrix associated with the **ROB** scheme is quite sparse, and so we were able to make its spectral radius zero, by selecting appropriate values for the relaxation parameters involved.

The rest of this paper is organized as following. In the next section we formulate the two IR schemae whose theoretical convergence analysis is given in Section 3. Section 4 presents results from an experimental study numerically which confirm our theoretical results; they also show that these hold for more general problems and presents various interesting characteristics of the methods. Section 5 contains our conclusions.

2 Two interface relaxation methods

We consider the Helmholtz boundary value problem

$$Lu \equiv -u''(x) + \gamma(x)^2 u(x) = f, \qquad x \in \Omega \equiv [a,b] \tag{1}$$

with $a, b, \gamma \in \mathbb{R}$, subject to boundary conditions on a and b which, for simplicity, are taken to be homogeneous Dirichlet. Assume that Ω is decomposed into the p non-overlapping subdomains $\Omega_i \equiv [x_{i-1}, x_i]$, $i = 1, \ldots, p$ with $x_0 = a$, $x_p = b$ and $x_{i-1} < x_i \in \Omega$ for $i = 1, \ldots, p-1$. We denote the size of a subdomain Ω_i by $\ell_i = x_i - x_{i-1}$ and the restrictions of L, f and γ in Ω_i by L_i, f_i, γ_i, respectively. We further assume that $\gamma(x) = \gamma_i$ for $x \in \Omega_i$, $i = 1, \ldots, p$, where the γ_i's are real constants.

2.1 *The* ROB *method.*

The **ROB** scheme is defined, for the model problem under consideration, by the following algorithm:

1. Define:

$$\left.\begin{aligned} g_i^i &= \left.\frac{du_{i+1}^{(k)}}{dx}\right|_{x=x_i} + \lambda_i \left. u_{i+1}^{(k)}\right|_{x=x_i} \\ g_i^{i+1} &= -\left.\frac{du_i^{(k)}}{dx}\right|_{x=x_i} + \lambda_i \left. du_i^{(k)}\right|_{x=x_i} \end{aligned}\right\} i = 1, \ldots, p-1.$$

2. Choose initial guesses $u_i^{(0)}(x)$ for the solutions on each subdomain Ω_i, $i = 1, 2, \ldots, p$.

3. Define the sequence of subdomain solutions $u_i^{(k)}(x)$, $k = 1, 2, \ldots$ as follows:

$$\left.\begin{aligned} & L_1 u_1^{(k+1)} = f_1 \quad \text{in} \quad \Omega_1 \\ & \left. u_1^{(k+1)}\right|_{x=x_0} = 0 \\ & \left.\frac{du_1^{(k+1)}}{dx}\right|_{x=x_1} + \lambda_1 \left. u_1^{(k+1)}\right|_{x=x_1} = g_1^1 \end{aligned}\right| \left.\begin{aligned} & L_p u_p^{(k+1)} = f_p \quad \text{in} \quad \Omega_p \\ & -\left.\frac{du_p^{(k+1)}}{dx}\right|_{x=x_{p-1}} + \lambda_{p-1} \left. u_p^{(k+1)}\right|_{x=x_{p-1}} = g_{p-1}^p \\ & \left. u_p^{(k+1)}\right|_{x=x_p} = 0 \end{aligned}\right.$$

$$\left.\begin{aligned} & L_i u_i^{(k+1)} = f_i \ \ in \ \ \Omega_i \\ & -\left.\frac{du_i^{(k+1)}}{dx}\right|_{x=x_{i-1}} + \lambda_{i-1} \left. u_i^{(k+1)}\right|_{x=x_{i-1}} = g_{i-1}^i \\ & \left.\frac{du_i^{(k+1)}}{dx}\right|_{x=x_i} + \lambda_i \left. u_i^{(k+1)}\right|_{x=x_i} = g_i^i \end{aligned}\right\} i = 2, \ldots, p-1.$$

This scheme, first proposed in [8], is based on a simple relaxation technique that involves the Robin interface conditions shown above. The DE problem is solved in each subdomain where the boundary conditions are provided from the previously computed solution and its normal derivative from the adjacent subdomains. The *relaxation parameter* λ_i controls the influence of the value of the function and/or its normal derivative on the smoothing Robin interface conditions.

This method was first analyzed in [8] where, through energy estimates, the convergence of the method at differential level has been established for arbitrary decompositions and elliptic operators. Later in [2,6] this method was been further analyzed at discrete level in a finite element framework. Several variations of this method have been also appeared. In [5] an ADI based modification is considered and analyzed at discrete level for model problems and decompositions. A second variation of ROB method that extends its applicability and frees it from the cross–point trouble is formulated and analyzed in [1]. In [14] the addition of tangential derivatives in the smoothing procedure is proposed and analyzed and, recently, in [18] a finite difference variation is presented and analyzed. In some of these studies optimal values for the relaxation parameters have been obtained but only for model problems and only assuming a discrete formulation of the method (i.e., first discretize and then decompose the linear algebra problem). Therefore the determination of effective choices for λ_i's in the IR framework and for general domains and decompositions is, *in general, an open problem.*

2.2 The two step average AVE *method.*

The AVE [11,13,16,17] IR method is a two-step iterative scheme described by the following algorithm:

1. Choose initial guesses $u_i^{(0)}(x)$ for the solution on each subdomain Ω_i, $i = 1, 2, \ldots, p$.

2. Define the odd terms of the sequence of subdomain solutions $u_i^{(2k+1)}(x)$ as follows:

$$g_i^i = \beta_i \left.\frac{du_i^{(2k)}}{dx}\right|_{x=x_i} + (1-\beta_i) \left.\frac{du_{i+1}^{(2k)}}{dx}\right|_{x=x_i}, \; i = 1, \ldots, p-1.$$

$$\begin{array}{l|l|l}
L_1 u_1^{(2k+1)} = f_1 \text{ in } \Omega_1 & \text{for } i = 2, \ldots, p-1 & L_p u_p^{(2k+1)} = f_p \text{ in } \Omega_p \\
u_1^{(2k+1)}\Big|_{x=x_0} = 0 & L_i u_i^{(2k+1)} = f_i \text{ in } \Omega_i & \left.\frac{du_p^{(2k+1)}}{dx}\right|_{x=x_{p-1}} = g_{p-1}^{p-1} \\
\left.\frac{du_1^{(2k+1)}}{dx}\right|_{x=x_1} = g_1^1 & \left.\frac{du_i^{(2k+1)}}{dx}\right|_{x=x_{i-1}} = g_{i-1}^{i-1} & u_p^{(2k+1)}\Big|_{x=x_p} = 0 \\
 & \left.\frac{du_i^{(2k+1)}}{dx}\right|_{x=x_i} = g_i^i &
\end{array}$$

3. Define the even terms of the sequence of subdomain solution $u_i^{(2k)}(x)$ as follows:

$$h_i^i = \alpha_i \, u_i^{(2k+1)}\Big|_{x=x_i} + (1-\alpha_i) \, u_{i+1}^{(2k+1)}\Big|_{x=x_i} \;, \; i = 1, \ldots, p-1.$$

$$\begin{array}{l|l|l}
L_1 u_1^{(2k+2)} = f_1 \text{ in } \Omega_1 & \text{for } i = 2, \ldots, p-1 & L_p u_p^{(2k+2)} = f_p \text{ in } \Omega_p \\
u_1^{(2k+2)}\Big|_{x=x_0} = 0 & L_i u_i^{(2k+2)} = f_i \text{ in } \Omega_i & u_p^{(2k+2)}\Big|_{x=x_{p-1}} = h_{p-1}^{p-1} \\
u_1^{(2k+2)}\Big|_{x=x_1} = h_1^1 & u_i^{(2k+2)}\Big|_{x=x_{i-1}} = h_{i-1}^{i-1} & u_p^{(2k+2)}\Big|_{x=x_p} = 0 \\
 & u_i^{(2k+2)}\Big|_{x=x_i} = h_i^i &
\end{array}$$

The relaxation parameters α_i and β_i are to smooth the function and its normal derivative respectively and they both take values in $(0,1)$. In the first step (odd terms), the Dirichlet problem is solved for each subdomain. The boundary values are computed as a convex combination of the previously computed solutions on adjacent domains. Then a convex combination of the normal derivatives of the previously computed solutions in each subdomain are used to smooth the derivatives on each interface. Using these estimates of the normal derivatives, the Neumann problem is solved in the second step (even terms) for all subdomains.

There are already several theoretical results concerning the above method which we briefly list below. In [17], two finite element approaches (a Galerkin and a hybrid mixed) have been employed to analyze the convergence of the method at a discrete level setting both relaxation parameters equal to $1/2$. A convergence analysis of the method at the differential level using Hilbert space techniques is given in [16]. A simple model problem with a two subdomain decomposition is considered in [13] where Fourier analysis at the differential level is used to obtain "good" values for the interface relaxation parameter λ_1 while set equal to $1/2$.

It is worth pointing out the inherent parallelism in both the λ_2 is algorithms. In each one the PDE solver or interface task steps can be executed on different processing elements. The only synchronization needed is a barrier at the end of each step and then only data on the interfaces need to be communicated to the neighboring processors.

3 Selection of relaxation parameters

We start our analysis by stating the following simple lemma that can be easily verified.

Lemma 1 *The solution of the boundary values problem*

$$Lu = 0 \text{ in } (a,b), \quad c_1u'(a) + c_2u(a) = v_1 \text{ and } c_3u'(b) + c_4u(b) = v_2$$

with $c_i \in \mathbb{R}$, $i = 1, \ldots, 4$ *is given by*

$$\begin{aligned} u(x) = \Big[&(-(c_3\gamma + c_4)e^{\gamma(b-x)} + (-c_3\gamma + c_4)e^{-\gamma(b-x)})v_1 + \\ &(-(-c_1\gamma + c_2)e^{\gamma(x-a)} + (c_1\gamma + c_2)e^{-\gamma(x-a)})v_2\Big] \\ &\Big[(c_1\gamma + c_2)(-c_3\gamma + c_4)e^{-\gamma(b-a)} - (c_3\gamma + c_4)(-c_1\gamma + c_2)e^{\gamma(b-a)}\Big]^{-1}. \end{aligned} \tag{2}$$

Let us now introduce notation for the sequence of values of the solutions, their derivatives and their errors at the interface points: $u_{i,j}^{(k)} \equiv u_i^{(k)}(x_j)$, $du_{i,j}^{(k)} \equiv \left.\frac{du_i^{(k)}}{dx}\right|_{x=x_j}$, $\epsilon_i^{(k)}(x) \equiv u_i^{(k)}(x) - u(x)$, $\epsilon_{i,j}^{(k)} \equiv u_{i,j}^{(k)} - u(x_j)$ and $d\epsilon_{i,j}^{(k)} \equiv du_{i,j}^{(k)} - u'(x_j)$.

3.1 Optimum relaxation parameters for the ROB *method*

Consider the following differential problems associated with the error functions in each subdomain which can be easily obtained from the ROB algorithm given in the previous section.

$$\begin{aligned} &L_1\epsilon_1^{(k+1)}(x) = 0, \quad x \in \Omega_1, \\ &\epsilon_{1,0}^{(k+1)} = 0, \quad d\epsilon_{1,1}^{(k+1)} + \lambda_1\epsilon_{1,1}^{(k+1)} = d\epsilon_{2,1}^{(k)} + \lambda_1\epsilon_{2,1}^{(k)}, \end{aligned} \tag{3}$$

$$\begin{aligned} &\text{for} \quad i = 2, \ldots, p-1, \\ &\quad L_i\epsilon_i^{(k+1)}(x) = 0, \quad x \in \Omega_i, \\ &\quad -d\epsilon_{i,i-1}^{(k+1)} + \lambda_{i-1}\epsilon_{i,i-1}^{(k+1)} = -d\epsilon_{i-1,i-1}^{(k)} + \lambda_{i-1}\epsilon_{i-1,i-1}^{(k)}, \\ &\quad d\epsilon_{i,i}^{(k+1)} + \lambda_i\epsilon_{i,i}^{(k+1)} = d\epsilon_{i+1,i}^{(k)} + \lambda_i\epsilon_{i+1,i}^{(k)}, \end{aligned} \tag{4}$$

$$\begin{aligned} &L_p\epsilon_p^{(k+1)}(x) = 0, \quad x \in \Omega_p, \\ &\epsilon_{p,p}^{(k+1)} = 0, \quad -d\epsilon_{p,p-1}^{(k+1)} + \lambda_{p-1}\epsilon_{p,p-1}^{(k+1)} = -d\epsilon_{p-1,p-1}^{(k)} + \lambda_{p-1}\epsilon_{p-1,p-1}^{(k)}. \end{aligned} \tag{5}$$

Using (2) we observe that these error functions are given by

$$\epsilon_1^{(k+1)}(x) = \frac{-e^{\gamma_1 x} + e^{-\gamma_1 x}}{(-\gamma_1+\lambda_1)e^{-\gamma_1 x_1} - (\gamma_1+\lambda_1)e^{\gamma_1 x_1}} \left(d\epsilon_{21}^{(k)} + \lambda_1 \epsilon_{21}^{(k)}\right) \tag{6}$$

for $i = 2, \ldots, p-1$,

$$\begin{aligned}\epsilon_i^{(k+1)}(x) = &\left[(-\gamma_i+\lambda_{i-1})(-\gamma_i+\lambda_i)e^{-\gamma_i \ell_i} - (\gamma_i+\lambda_i)(\gamma_i+\lambda_{i-1})e^{\gamma_i \ell_i}\right]^{-1} \\ &\left[\left(-(\gamma_i+\lambda_i)e^{\gamma_i(x_i-x)} + (-\gamma_i+\lambda_i)e^{-\gamma_i(x_i-x)}\right)\left(-d\epsilon_{i-1,i-1}^{(k)} + \lambda_{i-1}\epsilon_{i-1,i-1}^{(k)}\right) + \right. \\ &\left.\left(-(\gamma_i+\lambda_{i-1})\,e^{\gamma_i(x-x_{i-1})} + \left(-\gamma_i+\lambda_{i-1}\right)e^{-\gamma_i(x-x_{i-1})}\right)\left(d\epsilon_{i+1,i}^{(k)} + \lambda_i \epsilon_{i+1,i}^{(k)}\right)\right]\end{aligned} \tag{7}$$

and

$$\epsilon_p^{(k+1)}(x) = \frac{-e^{\gamma_p(x_p-x)} + e^{-\gamma_p(x_p-x)}}{(-\gamma_p+\lambda_{p-1})e^{-\gamma_p \ell_p} - (\gamma_p+\lambda_{p-1})e^{\gamma_p \ell_p}} \left(-d\epsilon_{p-1,p-1}^{(k)} + \lambda_{p-1}\epsilon_{p-1,p-1}^{(k)}\right). \tag{8}$$

From these we obtain

$$\begin{aligned}
\varepsilon_{1,1}^{(k+1)} &= \tfrac{m_1}{\gamma_1 n_1 + \lambda_1 m_1}\left(d\varepsilon_{2,1}^{(k)} + \lambda_1 \varepsilon_{2,1}^{(k)}\right), \\
\varepsilon_{i,i-1}^{(k+1)} &= \tfrac{1}{d_i}\left[(\gamma_i n_i + \lambda_i m_i)\left(-d\varepsilon_{i-1,i-1}^{(k)} + \lambda_{i-1}\varepsilon_{i-1,i-1}^{(k)}\right) + 2\gamma_i\left(d\varepsilon_{i+1,i}^{(k)} + \lambda_i \varepsilon_{i+1,i}^{(k)}\right)\right] \\
\varepsilon_{i,i}^{(k+1)} &= \tfrac{1}{d_i}\left[(\gamma_i n_i + \lambda_{i-1} m_i)\left(d\varepsilon_{i+1,i}^{(k)} + \lambda_i \varepsilon_{i+1,i}^{(k)}\right) + 2\gamma_i\left(-d\varepsilon_{i-1,i-1}^{(k)} + \lambda_{i-1}\varepsilon_{i-1,i-1}^{(k)}\right)\right], \\
\varepsilon_{p,p-1}^{(k+1)} &= \tfrac{m_p}{\gamma_p n_p + \lambda_{p-1} m_p}\left(-d\varepsilon_{p-1,p-1}^{(k)} + \lambda_{p-1}\varepsilon_{p-1,p-1}^{(k)}\right)
\end{aligned}$$

where

$$d_i = (\gamma_i^2 + \lambda_i\lambda_{i-1})m_i + \gamma_i(\lambda_i + \lambda_{i-1})n_i, \quad i = 2, \ldots, p-1,$$

$$n_i = e^{\gamma_i \ell_i} - e^{-\gamma_i \ell_i} \quad \text{and} \quad m_i = e^{\gamma_i \ell_i} + e^{-\gamma_i \ell_i}, \quad i = 1, \ldots, p.$$

By differentiating equations (6)–(8) we obtain expressions similar to the above that relate $d\varepsilon_{1,1}^{(k+1)}$, $d\varepsilon_{i,i-1}^{(k+1)}$, $d\varepsilon_{i,i}^{(k+1)}$, $i = 2, \ldots, p-1$, and $d\varepsilon_{p,p-1}^{(k+1)}$ with associated values from the iteration k; these are

$$\begin{aligned}
d\varepsilon_{1,1}^{(k+1)} &= \tfrac{\gamma_1 n_1}{\gamma_1 n_1 + \lambda_1 m_1}\left(d\varepsilon_{2,1}^{(k)} + \lambda_1 \varepsilon_{2,1}^{(k)}\right), \\
d\varepsilon_{i,i-1}^{(k+1)} &= \tfrac{\gamma_i}{d_i}\left[(\gamma_i m_i + \lambda_i n_i)\left(d\varepsilon_{i-1,i-1}^{(k)} - \lambda_{i-1}\varepsilon_{i-1,i-1}^{(k)}\right) + 2\lambda_{i-1}\left(d\varepsilon_{i+1,i}^{(k)} + \lambda_i \varepsilon_{i+1,i}^{(k)}\right)\right], \\
d\varepsilon_{i,i}^{(k+1)} &= \tfrac{\gamma_i}{d_i}\left[(\gamma_i m_i + \lambda_{i-1} n_i)\left(d\varepsilon_{i+1,i}^{(k)} + \lambda_i \varepsilon_{i+1,i}^{(k)}\right) + 2\lambda_i\left(d\varepsilon_{i-1,i-1}^{(k)} - \lambda_{i-1}\varepsilon_{i-1,i-1}^{(k)}\right)\right], \\
d\varepsilon_{p,p-1}^{(k+1)} &= \tfrac{\gamma_p n_p}{\gamma_p n_p + \lambda_{p-1} m_p}\left(d\varepsilon_{p-1,p-1}^{(k)} - \lambda_{p-1}\varepsilon_{p-1,p-1}^{(k)}\right).
\end{aligned}$$

Now we order the errors on the interface points to create a sequence of error vectors as follows, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}\underline{\varepsilon}^{(k)} \equiv \Big[&d\varepsilon_{1,1}^{(k)}, \varepsilon_{1,1}^{(k)}, \varepsilon_{2,1}^{(k)}, d\varepsilon_{2,1}^{(k)}, d\varepsilon_{2,2}^{(k)}, \varepsilon_{2,2}^{(k)}, \varepsilon_{3,2}^{(k)}, d\varepsilon_{3,2}^{(k)}, \ldots, \\ &d\varepsilon_{i,i}^{(k)}, \varepsilon_{i,i}^{(k)}, \varepsilon_{i+1,i}^{(k)}, d\varepsilon_{i+1,i}^{(k)}, \ldots, d\varepsilon_{p-1,p-1}^{(k)}, \varepsilon_{p-1,p-1}^{(k)}, \varepsilon_{p,p-1}^{(k)}, d\varepsilon_{p,p-1}^{(k)}\Big]^T\end{aligned}$$

We obtain the following relation between the vectors of interface errors in the two consecutive iteration steps k and $k+1$.

$$\underline{\epsilon}^{(k+1)} = M\underline{\epsilon}^{(k)}, \quad k = 0, 1, \ldots, \tag{9}$$

where the iteration matrix $M \in \mathbb{R}^{4(p-1)\times 4(p-1)}$ has the form

$$M = \begin{bmatrix}
0 & M_{1,2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
M_{2,1} & 0 & 0 & M_{2,4} & 0 & 0 & \cdots & 0 \\
M_{3,1} & 0 & 0 & M_{3,4} & 0 & 0 & \cdots & 0 \\
0 & 0 & M_{4,3} & 0 & 0 & M_{4,6} & \cdots & 0 \\
0 & 0 & M_{5,3} & 0 & 0 & M_{5,6} & \cdots & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & \cdots & 0 & M_{2(p-1)-2,2(p-1)-3} & 0 & 0 & M_{2(p-1)-2,2(p-1)} \\
0 & 0 & \cdots & 0 & M_{2(p-1)-1,2(p-1)-3} & 0 & 0 & M_{2(p-1)-1,2(p-1)} \\
0 & 0 & \cdots & 0 & 0 & 0 & M_{2(p-1),2(p-1)-1} & 0
\end{bmatrix} \tag{10}$$

and where

$$M_{1,2} = \frac{1}{\gamma_1 n_1 + \lambda_1 m_1} \begin{bmatrix} \gamma_1 n_1 \lambda_1 & \gamma n_1 \\ m_1 \lambda_1 & m_1 \end{bmatrix},$$

and, for $i = 2, \ldots, p-1$,

$$M_{2(i-1),2(i-1)-1} = \frac{1}{d_i} \begin{bmatrix} -(\gamma_i n_i + \lambda_i m_i) & \lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) \\ \gamma_i(\gamma_i m_i + \lambda_i n_i) & -\gamma_i \lambda_{i-1}(\gamma_i m_i + \lambda_i n_i) \end{bmatrix},$$

$$M_{2(i-1)+1,2(i-1)+2} = \frac{1}{d_i} \begin{bmatrix} \gamma_i \lambda_i(\gamma_i m_i + \lambda_{i-1} n_i) & \gamma_i(\gamma_i m_i + \lambda_{i-1} n_i) \\ \lambda_i(\gamma_i n_i + \lambda_{i-1} m_i) & (\gamma_i n_i + \lambda_{i-1} m_i) \end{bmatrix},$$

$$M_{2(i-1),2(i-1)+2} = \frac{2\gamma_i}{d_i} \begin{bmatrix} \lambda_i & 1 \\ \lambda_i \lambda_{i-1} & \lambda_{i-1} \end{bmatrix}, \quad M_{2(i-1)+1,2(i-1)-1} = \frac{2\gamma_i}{d_i} \begin{bmatrix} \lambda_i & -\lambda_i \lambda_{i-1} \\ -1 & \lambda_{i-1} \end{bmatrix},$$

and

$$M_{2(p-1),2(p-1)-1} = \frac{1}{\gamma_p n_p + \lambda_{p-1} m_p} \begin{bmatrix} -m_p & \lambda_{p-1} m_p \\ \gamma_p n_p & -\gamma_p n_p \lambda_{p-1} \end{bmatrix}.$$

For the rest of the analysis in this section we use a methodology similar to the one found in [7]. In the following lemma we construct a matrix $\tilde{M} \in \mathbb{R}^{2(p-1)\times 2(p-1)}$ of reduced size which is spectrally equivalent to the iteration matrix M and whose special non-zero structure lets us select optimum values for the relaxation parameters λ_i.

Lemma 2 *The two matrices M and $\bar{M}$ have the same non-zero eigenvalues, i.e.,*

$$\sigma(M) = \sigma(\bar{M}) \cup 0, \tag{11}$$

where

$$\tilde{M} = \begin{bmatrix} 0 & \tilde{M}_{1,2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{M}_{2,1} & 0 & 0 & \tilde{M}_{2,4} & 0 & 0 & \cdots & 0 \\ \tilde{M}_{3,1} & 0 & 0 & \tilde{M}_{3,4} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{M}_{4,3} & 0 & 0 & \tilde{M}_{4,6} & \cdots & 0 \\ 0 & 0 & \tilde{M}_{5,3} & 0 & 0 & \tilde{M}_{5,6} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \tilde{M}_{2(p-1)-2,2(p-1)-3} & 0 & 0 & \tilde{M}_{2(p-1)-2,2(p-1)} \\ 0 & 0 & \cdots & 0 & \tilde{M}_{2(p-1)-1,2(p-1)-3} & 0 & 0 & \tilde{M}_{2(p-1)-1,2(p-1)} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{M}_{2(p-1),2(p-1)-1} & 0 \end{bmatrix} \tag{12}$$

and

$$\tilde{M}_{1,2} = \frac{-\gamma_1 n_1 + \lambda_1 m_1}{\gamma_1 n_1 + \lambda_1 m_1},$$

and where, for $i = 2, \ldots, p-1$,

$$\tilde{M}_{2(i-1),2(i-1)-1} = \frac{\lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) - \gamma_i(\gamma_i m_i + \lambda_i n_i)}{d_i},$$

$$\tilde{M}_{2(i-1)+1,2(i-1)+2} = \frac{\lambda_i(\gamma_i n_i + \lambda_{i-1} m_i) - \gamma_i(\gamma_i m_i + \lambda_{i-1} n_i)}{d_i},$$

$$\bar{M}_{2(i-1),2(i-1)+2} = \frac{4\gamma_i \lambda_{i-1}}{d_i}, \quad \bar{M}_{2(i-1)+1,2(i-1)-1} = \frac{4\gamma_i \lambda_i}{d_i},$$

and

$$\tilde{M}_{2(p-1),2(p-1)-1} = \frac{-\gamma_p n_p + \lambda_{p-1} m_p}{\gamma_p n_p + \lambda_{p-1} m_p}.$$

Proof: We define the non-singular matrix

$$Q = diag(\, Q_1, Q_1^T, Q_2, Q_2^T, \cdots, Q_{p-1}, Q_{p-1}^T \,),$$

where

$$Q_i = Q_i^{-1} = \begin{bmatrix} 1 & -\lambda_i \\ 0 & -1 \end{bmatrix}, \quad Q_i^T = Q_i^{-T} = \begin{bmatrix} 1 & 0 \\ -\lambda_i & -1 \end{bmatrix}$$

and consider the similarity transformation matrix $Q^{-1}MQ$ whose elements are specified by the following relations.

$$Q_1^{-1} M_{1,2} Q_1^T = \begin{bmatrix} 0 & \frac{-\gamma_1 n_1 + \lambda_1 m_1}{\gamma_1 n_1 + \lambda_1 m_1} \\ 0 & \frac{m_1}{\gamma_1 n_1 + \lambda_1 m_1} \end{bmatrix},$$

$$Q_{i-1}^{-T} M_{2(i-1),2(i-1)-1} Q_{i-1} = \frac{1}{d_i} \begin{bmatrix} -(\gamma_i n_i + \lambda_i m_i) & 0 \\ \lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) - \gamma_i(\gamma_i m_i + \lambda_i n_i) & 0 \end{bmatrix},$$

$$Q_{i-1}^{-T} M_{2(i-1),2(i-1)+2} Q_i^T = \frac{2\gamma_i}{d_i} \begin{bmatrix} 0 & -1 \\ 0 & 2\lambda_{i-1} \end{bmatrix},$$

$$Q_i^{-1} M_{2(i-1)+1,2(i-1)-1} Q_{i-1} = \frac{2\gamma_i}{d_i} \begin{bmatrix} 2\lambda_i & 0 \\ 1 & 0 \end{bmatrix},$$

$$Q_i^{-1} M_{2(i-1)+1,2(i-1)+2} Q_i^T = \frac{1}{d_i} \begin{bmatrix} 0 & -\gamma_i(\gamma_i m_i + \lambda_{i-1} n_i) + \lambda_i(\gamma_i n_i + \lambda_{i-1} m_i) \\ 0 & \gamma_i p_i + \lambda_{i-1} m_i \end{bmatrix},$$

and

$$Q_{p-1}^{-T} M_{2(p-1),2(p-1)-1} Q_{p-1} = \frac{1}{\gamma_p n_p + \lambda_{p-1} m_p} \begin{bmatrix} -m_p & 0 \\ m_p \lambda_{p-1} - \gamma_p n_p & 0 \end{bmatrix}.$$

A simple comparison of the above relations with the elements of the matrix $\bar{M}$ and the fact that there exists (Lemma 3.2 in [7]) a permutation matrix P such that

$$P^T M P = \begin{bmatrix} 0 & * \\ 0 & \bar{M} \end{bmatrix},$$

complete the proof of the lemma. □

We conclude this section with the main theorem that presents analytic expressions for the relaxation parameters.

Theorem 3 *Consider the model problem (1) and a non-overlapping decomposition of Ω into p subdomains Ω_i of length ℓ_i, $i = 1, \ldots, p$. If the parameters λ_i involved in the* ROB *interface relaxation method are selected as follows:*

$$\lambda_{i-1} = \frac{\gamma_i(\gamma_i m_i + \lambda_i n_i)}{\gamma_i n_i + \lambda_i m_i}, \quad i = p-1, \ldots, 2, \qquad \lambda_{p-1} = \frac{\gamma_p \ell_p}{m_p}, \tag{13}$$

then the spectral radius of the iteration matrix M is zero.

Proof: It can been seen (Lemma 3.2 in [7]) that if we set $\bar{M}_{2(i-1),2(i-1)-1} = 0, i = 2,\ldots,p$, then we obtain that $\sigma(\tilde{M}) = 0$. This leads to the following equations.

$$\lambda_{p-1} m_p - \gamma_p n_p = 0$$

and

$$\lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) - \gamma_i(\gamma_i m_i + \lambda_i n_i) = 0, \quad i = 2,\ldots,p. \tag{14}$$

To conclude the proof, we back solve for $\lambda_i, i = p-1,\ldots,1$ and use the previous Lemma. □

3.2 Good relaxation parameters for the AVE *method*

Using the notation adopted in the previous section and the **AVE** algorithm given in Section 2 we easily see that the error functions involved satisfy the following differential equations:

For the odd steps the equations are:

$$\begin{aligned} L_1 \epsilon_1^{(2k+1)}(x) &= 0 \; x \in \Omega_1, \\ \epsilon_{1,0}^{(2k+1)} &= 0, \quad d\epsilon_{1,1}^{(2k+1)} = \beta_1 d\epsilon_{1,1}^{(2k)} + (1-\beta_1) d\epsilon_{2,1}^{(2k)}, \end{aligned} \tag{15}$$

and, for $i = 2,\ldots,p-1$,

$$\begin{aligned} L_i \epsilon_i^{(2k+1)}(x) &= 0 \; x \in \Omega_i, \\ d\epsilon_{i,i-1}^{(2k+1)} &= \beta_{i-1} d\epsilon_{i-1,i-1}^{(2k)} + (1-\beta_{i-1}) d\epsilon_{i,i-1}^{(2k)}, \\ d\epsilon_{i,i}^{(2k+1)} &= \beta_i d\epsilon_{i,i}^{(2k)} + (1-\beta_i) d\epsilon_{i+1,i}^{(2k)}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} L_p \epsilon_p^{(2k+1)}(x) &= 0 \; x \in \Omega_p, \\ d\epsilon_{p,p-1}^{(2k+1)} &= \beta_{p-1} d\epsilon_{p-1,p-1}^{(2k)} + (1-\beta_{p-1}) d\epsilon_{p,p-1}^{(2k)}, \\ \epsilon_{p,p}^{(2k+1)} &= 0. \end{aligned} \tag{17}$$

For the even steps the equations are

$$
\begin{aligned}
&L_1\epsilon_1^{(2k+2)}(x) = 0 \; x \in \Omega_1, \\
&\epsilon_{1,0}^{(2k+2)} \qquad = 0, \quad \epsilon_{1,1}^{(2k+2)} = \alpha_1\epsilon_{1,1}^{(2k+1)} + (1-\alpha_1)\epsilon_{2,1}^{(2k+1)},
\end{aligned} \tag{18}
$$

and, for $i = 2, \ldots, p-1$,

$$
\begin{aligned}
&L_i\epsilon_i^{(2k+2)}(x) = 0 \; x \in \Omega_i, \\
&\epsilon_{i,i-1}^{(2k+2)} \qquad = \alpha_{i-1}\epsilon_{i-1,i-1}^{(2k+1)} + (1-\alpha_{i-1})\epsilon_{i,i-1}^{(2k+1)}, \\
&\epsilon_{i,i}^{(2k+2)} \qquad = \alpha_i\epsilon_{i,i}^{(2k+1)} + (1-\alpha_i)\epsilon_{i+1,i}^{(2k+1)},
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
&L_p\epsilon_p^{(2k+2)}(x) = 0 \; x \in \Omega_p, \\
&\epsilon_{p,p-1}^{(2k+2)} \qquad = \alpha_{p-1}\epsilon_{p-1,p-1}^{(2k+1)} + (1-\alpha_{p-1})\epsilon_{p,p-1}^{(2k+1)}, \quad \epsilon_{p,p}^{(2k+2)} = 0.
\end{aligned} \tag{20}
$$

The solutions to the Neumann problems (15)–(17) are given by (see Lemma 1)

$$
\begin{aligned}
&\epsilon_1^{(2k+1)}(x) = \tfrac{1}{\gamma_1 m_1}\left(e^{\gamma_1(x-x_0)} - e^{-\gamma_1(x-x_0)}\right)\left(\beta_1 d\epsilon_{1,1}^{(2k)} + (1-\beta_1)d\epsilon_{2,1}^{(2k)}\right), \\
&\epsilon_i^{(2k+1)}(x) = \tfrac{1}{\gamma_i n_i}\left\{\left(-e^{\gamma_i(x_i-x)} - e^{-\gamma_i(x_i-x)}\right)\left(\beta_{i-1}d\epsilon_{i-1,i-1}^{(2k)} + (1-\beta_{i-1})d\epsilon_{i,i-1}^{(2k)}\right)\right. \\
&\left. + \left(e^{\gamma_i(x-x_{i-1})} + e^{-\gamma_i(x-x_{i-1})}\right)\left(\beta_i d\epsilon_{i,i}^{(2k)} + (1-\beta_i)d\epsilon_{i+1,i}^{(2k)}\right)\right\}, \quad i = 2, \ldots, p-1 \\
&\epsilon_p^{(2k+1)}(x) = \tfrac{1}{\gamma_p m_p}\left(-e^{\gamma_p(x_p-x)} + e^{-\gamma_p(x_p-x)}\right)\left(\beta_{p-1}d\epsilon_{p-1,p-1}^{(2k)} + (1-\beta_{p-1})d\epsilon_{p,p-1}^{(2k)}\right).
\end{aligned} \tag{21}
$$

This solution to the Dirichlet problems (18)–(20) are given by

$$
\begin{aligned}
&\epsilon_1^{(2k+2)}(x) = \tfrac{1}{n_i}\left(e^{\gamma_1(x-x_0)} - e^{-\gamma_1(x-x_0)}\right)\left(\alpha_1\epsilon_{1,1}^{(2k+1)} + (1-\alpha_1)\epsilon_{2,1}^{(2k+1)}\right), \\
&\epsilon_i^{(2k+2)}(x) = \tfrac{1}{n_i}\left\{\left(e^{\gamma_i(x_i-x)} - e^{-\gamma_i(x_i-x)}\right)\left(\alpha_{i-1}\epsilon_{i-1,i-1}^{(2k+1)} + (1-\alpha_{i-1})\epsilon_{i,i-1}^{(2k+1)}\right) + \right. \\
&\left. \left(e^{\gamma_i(x-x_{i-1})} - e^{-\gamma_i(x-x_{i-1})}\right)\left(\alpha_i\epsilon_{i,i}^{(2k+1)} + (1-\alpha_i)\epsilon_{i+1,i}^{(2k+1)}\right)\right\}, \quad i = 2, \ldots, p-1 \\
&\epsilon_p^{(2k+2)}(x) = \tfrac{1}{n_p}\left(e^{\gamma_p(x_p-x)} - e^{-\gamma_p(x_p-x)}\right)\left(\alpha_{p-1}\epsilon_{p-1,p-1}^{(2k+1)} + (1-\alpha_{p-1})\epsilon_{p,p-1}^{(2k+1)}\right).
\end{aligned} \tag{22}
$$

If, for $k = 0, 1, \ldots$, we define the vectors

$$
\underline{\epsilon}^{(k)} \equiv \left[\epsilon_{1,1}^{(k)}, \epsilon_{2,2}^{(k)}, \ldots, \epsilon_{p-1,p-1}^{(k)}\right]^T \quad \text{and} \quad \underline{d\epsilon}^{(k)} \equiv \left[d\epsilon_{1,1}^{(k)}, d\epsilon_{2,2}^{(k)}, \ldots, d\epsilon_{p-1,p-1}^{(k)}\right]^T \tag{23}
$$

then we get from the above that

$$
\underline{\epsilon}^{(2k+2)} = M^D\underline{d\epsilon}^{(2k+1)} \text{ and } \underline{d\epsilon}^{(2k+1)} = M^N\underline{\epsilon}^{(2k)} \tag{24}
$$

where the Dirichlet and Neumann iteration matrices $M^D, M^N \in \mathbb{R}^{(p-1)\times(p-1)}$ are tridiagonal with elements

$$\begin{aligned}
&M^D_{1,1} = \tfrac{\alpha_1 m_1}{n_1\gamma_1} - \tfrac{(1-\alpha_i)n_2}{m_2\gamma_2}, \quad M^D_{p-1,p-1} = \tfrac{\alpha_{p-1}n_{p-1}}{m_{p-1}\gamma_{p-1}} - \tfrac{(1-\alpha_{p-1})m_p}{n_p\gamma_p},\\
&M^D_{i,i} = \tfrac{\alpha_i n_i}{m_i\gamma_i} - \tfrac{(1-\alpha_i)n_{i+1}}{m_{i+1}\gamma_{i+1}}, \quad i = 2,\ldots,p-2,\\
&M^D_{i,i+1} = \tfrac{2(1-\alpha_i)}{m_{i+1}\gamma_{i+1}}, \quad i = 2,\ldots,p-1,\\
&M^D_{i+1,i} = -\tfrac{2\alpha_i}{m_i\gamma_i}, \quad i = 1,\ldots,p-2,
\end{aligned} \tag{25}$$

$$\begin{aligned}
&M^N_{1,1} = \tfrac{\beta_1 n_1\gamma_1}{m_1} - \tfrac{(1-\beta_1)n_2\gamma_2}{m_2}, \quad M^N_{p-1,p-1} = \tfrac{\beta_{p-1}n_{p-1}\gamma_{p-1}}{m_{p-1}} - \tfrac{(1-\beta_{p-1})n_p\gamma_p}{m_p},\\
&M^N_{i,i} = \tfrac{\beta_i n_i\gamma_i}{m_i} - \tfrac{(1-\beta_i)n_{i+1}\gamma_{i+1}}{m_{i+1}}, \quad i = 2,\ldots,p-2,\\
&M^N_{i,i+1} = \tfrac{2(1-\beta_i)\gamma_{i+1}}{m_{i+1}}, \quad i = 2,\ldots,p-1,\\
&M^N_{i+1,i} = -\tfrac{2\beta_i\gamma_i}{m_i}, \quad i = 1,\ldots,p-2.
\end{aligned} \tag{26}$$

For $p = 2$ it is easy to see (make the roots of the characteristic polynomial of M^D or M^N be zero) that $\alpha_1 = \frac{m_2 n_1\gamma_1}{m_2 n_1\gamma_1 + m_1 n_2\gamma_2}$ or $\beta_1 = \frac{m_1 n_2\gamma_2}{m_1 n_2\gamma_2 + m_2 n_1\gamma_1}$ are optimum values and imply immediate convergence. For $p > 2$ we were unable to solve for the optimum parameters. Instead we obtain values for the relaxation parameters that are optimum in the max-norm.

Theorem 4 *Consider the iteration matrix $M \equiv M^N M^D$ of the* **AVE** *method associated with the model problem (1) and a non-overlapping decomposition of Ω into p subdomains Ω_i of length $\ell_i, i = 1,\ldots,p$ with $\gamma_i = \gamma$ in $i = 1,\ldots,p$. The max-norm of M is minimized by the relaxation parameters*

$$\alpha_1 = \tfrac{n_1 n_2}{n_1 n_2 + m_1 m_2}, \qquad \alpha_{p-1} = \tfrac{m_p m_{p-1}}{m_p m_{p-1} + n_p n_{p-1}}, \tag{27}$$

$$\alpha_i = \tfrac{m_i n_{i+1}}{m_i n_{i+1} + m_{i+1} n_i}, \quad i = 2,\ldots,p-2 \tag{28}$$

and

$$\beta_1 = \tfrac{m_1 n_2}{m_1 n_2 + m_2 n_1}, \qquad \beta_{p-1} = \tfrac{m_{p-1} n_p}{m_{p-1} n_p + m_p n_{p-1}} \tag{29}$$

$$\beta_i = \tfrac{m_i n_{i+1}}{m_i n_{i+1} + m_{i+1} n_i}, i = 2,\ldots,p-2, \tag{30}$$

provided that $\ell_i > \frac{\ln(1+\sqrt{2})}{\gamma}, \quad i = 1, \ldots, p.$

Proof: To minimize the max-norm of the iteration matrix, it is sufficient to minimize the quantity

$$
\begin{aligned}
&f(\alpha_i, \beta_i, \beta_{i-1}, \beta_{i+1}) = 4\frac{\alpha_i \beta_{i-1}}{m_i m_{i-1}} + \\
&\frac{2}{m_i}\left|\alpha_i \frac{\beta_{i-1}(m_i n_{i-1} + m_{i-1} n_i) - m_{i-1} n_i}{m_{i-1} m_i} + \beta_i \frac{\alpha_i(m_{i+1} n_i + m_i n_{i+1}) - m_i n_{i+1}}{m_i m_{i+1}}\right| + \\
&\left|-4\frac{\alpha_i(1-\beta_{i-1})}{m_i^2} + \frac{\alpha_i(m_{i+1} n_i + m_i n_{i+1}) - m_i n_{i+1}}{m_i m_{i+1}} \frac{\beta_i(m_{i+1} n_i + m_i n_{i+1}) - m_i n_{i+1}}{m_i m_{i+1}} - 4\frac{(1-\alpha_i)\beta_{i+1}}{m_{i+1}^2}\right| + (31) \\
&\frac{2}{m_{i+1}}\left|\frac{\alpha_i(m_{i+1} n_i + m_i n_{i+1}) - m_i n_{i+1}}{m_i m_{i+1}}(1-\beta_i) + \frac{\beta_{i+1}(m_{i+2} n_{i+1} + m_{i+1} n_{i+2}) - m_{i+1} n_{i+2}}{m_{i+1} m_{i+2}}(1-\alpha_i)\right| + \\
&4\frac{(1-\alpha_i)(1-\beta_{i+1})}{m_i m_{i+1}}.
\end{aligned}
$$

One can determine values for $\alpha_i, \beta_i, \beta_{i-1}$ and β_{i+1} that minimize f by an elementary but very lengthly and tedious analysis which involves splitting the absolute values and considering several different cases. We do not present this analysis here. Instead, we give an indication why this theorem is true.

Set $\alpha_i = \alpha_i^* \equiv \frac{m_i n_{i+1}}{m_{i+1} n_i + m_i n_{i+1}}, \beta_i = \beta_i^* \equiv \frac{m_i n_{i+1}}{m_{i+1} n_i + m_i n_{i+1}}, \beta_{i-1} = \beta_{i-1}^* \equiv \frac{m_{i-1} n_i}{m_{i-1} n_i + m_i n_{i-1}}$ and $\beta_{i+1} = \beta_{i+1}^* \equiv \frac{m_{i+1} n_{i+2}}{m_{i+1} n_{i+2} + m_{i+2} n_{i+1}}$. Then the expressions in the absolute values of (31) become zero and so we have

$$
f(\alpha_i^*, \beta_i^*, \beta_{i-1}^*, \beta_{i+1}^*) = \frac{4}{m_{i+1} n_i + m_i n_{i+1}} \left(\frac{n_{i+1}(n_i + n_{i-1})}{m_{i-1} n_i + m_i n_{i-1}} + \frac{n_i(n_{i+1} + n_{i+2})}{m_{i+1} n_{i+2} + m_{i+2} n_{i+1}} \right).
$$

Under the constraint that $\ell_j > \frac{\ln(1+\sqrt{2})}{\gamma}$ we have that $m_j > 2, \quad j = i-1, i, i+1$, and therefore we have

$$
f(\alpha_i^*, \beta_i^*, \beta_{i-1}^*, \beta_{i+1}^*) < \frac{4}{2(n_i + n_{i+1})} \left(\frac{n_{i+1}(n_i + n_{i-1})}{2(n_i + n_{i-1})} + \frac{n_i(n_{i+1} + n_{i+2})}{2(n_{i+2} + n_{i+1})} \right) = 1.
$$

Continuing in the same way for the $1^{st}, 2^{nd}, (p-2)^{th}$ and $(p-1)^{th}$ rows of the iteration matrix, we get values for the relaxation parameters for all the interface points. □

4 Numerical experiments

The purpose of the numerical experiments performed in this study is twofold. First to verify and elucidate our theoretically determined relaxation parameter values on a class of 1-dimensional problems and then to examine how effective these parameters values are for 2-dimensional PDEs with skyline domains. All experiments reported here were performed on a SUN workstation using MATLAB. All MATLAB files

we use to produce the data in this section are available through our web page[3]. Implementations of several other relaxation schemes also can be found there. We use zero as initial guess and consider the following model problem:

$$u''(x) - \gamma^2 u(x) = f(x),\ x \in (0,1),\ \ u(0) = 0,\ u(1) = 0, \tag{32}$$

where the right hand side function f is selected such that the true solution $u(x)$ is either

DP1 $u(x) = \cosh(2\,x - 1) - \cosh(1.0)$, or

DP2 $u(x) = e^{x+4}x(x-1)(x-.7)$.

In Table 1 we present the max norm of the error $||u^{(k)} - u||_\infty$ and the computed convergence factor

$$r_k = \sqrt[k]{||Lu^{(k)} - f||_\infty / ||Lu^{(0)} - f||_\infty}, k = 1, 2, \ldots$$

of the **ROB** method applied to the model problem (32). We assume the solution DP1 with $\gamma = 2$ and that the domain is decomposed into $p = 2, 4, 10, 20$ domains of equal size. We use the 5-point-star difference approximation with two different global discretization steps $h = .01$ and $h = .005$ to solve the DE. Similarly in Table 2 we consider the **AVE** method and set $\gamma = 10$. The rapid rate of convergence is easily observed as one moves down along any column. Note that this convergence is not immediate (1 iteration) as our theory might indicate. It can be shown [15] that this is mainly due to the particular block structure of the Jordan form of the iteration matrices but a formal analysis of this is beyond the scope for this paper. It can be also observed that, as the computed convergence factors indicate, the rate of convergence of both methods does not seem to depend on the finess of the domain discretization. Nevertheless, the order h^2 finite difference discretization convergence rate is preserved. The rate of convergence does depend, as expected, on both the number of subdomains and the PDE coefficient γ. Extensive numerical experiments (some of them presented in Figure 3 below, and some others that are not included in this paper) show that the rate of convergence increases as γ increases for both methods but much more rapidly in the **AVE** case. The **AVE** method diverges for $\gamma = 10$ and $p = 20$. This is in good agreement with the restriction $\ell_i > \frac{ln(1+\sqrt{2})}{\gamma}$, $i = 1, \ldots, p$ imposed by Theorem 4. This restriction seems to be necessary as well as sufficient (see also our following discussion of the figures).

In Figures 1, 2 and 3 we consider the model problem (32)-DP1, with a splitting of the domain Ω into 3 subdomains. We present the contour plots of the experimentally determined number of iterations required to reduce the max norm of the difference of two successive iterands smaller than 10^{-5} as a function of the various relaxation parameters involved. The stars in these plots indicate

[3] `http:www.cs.purdue.edu/homes/mav/projects/m_code.html`

	h=.01				h=.005			
iter	p=2	p=4	p=10	p=20	p=2	p=4	p=10	p=20
2	3.08E-5	1.48E-1	2.74E-1	3.87E-1	7.83E-6	1.48E-1	2.74E-1	3.87E-1
	(.2966)	(.2997)	(.2424)	(.1811)	(.2966)	(.2992)	(.2421)	(.1809)
3	1.19E-5	7.18E-2	1.72E-1	3.00E-1	3.07E-6	7.18E-2	1.72E-1	3.00E-1
	(.4447)	(.4635)	(.4409)	(.3723)	(.4447)	(.4635)	(.4408)	(.3721
4	1.19E-5	3.18E-2	1.42E-1	2.27E-1	3.07E-6	3.18E-2	1.42E-1	2.27E-1
	(.5446)	(.5524)	(.5051)	(.5106)	(.5446)	(.5524)	(.5651)	(.5105)
5	1.19E-5	1.41E-2	1.53E-1	1.75E-1	3.07E-6	1.41E-2	1.53E-1	1.75E-1
	(.6150)	(.6167)	(.6425)	(.6063)	(.6150)	(.6166)	(.0425)	(.6003)
8	1.19E-5	6.11E-5	7.25E-2	1.04E-1	3.07E-6	1.52E-5	7.24E-2	1.64E-1
	(.7379)	(.7379)	(.7494)	(.7576)	(.7379)	(.7379)	(.7494)	(.7576)
16	1.19E-5	6.14E-5	2.04E-3	7.43E-2	3.07E-6	1.52E-5	2.05E-3	7.41E-2
	(.8590)	(.8590)	(.8590)	(.8658)	(.8590)	(.8590)	(.8590)	(.8658)
20	1.19E-5	6.14E-5	2.13E-4	3.09E-2	3.07E-6	1.52E-5	5.28E-5	3.98E-2
	(.8855)	(.8855)	(.8855)	(.8883)	(.8855)	(.8855)	(.8855)	(.8883)
32	1.19E-5	6.14E-5	2.14E-4	2.23E-3	3.07E-6	1.52E-5	5.32E-5	2.18E-3
	(.9268)	(.9268)	(.9268)	(.9268)	(.0269)	(.9269)	(.9268)	(.9268)
36	1.19E-5	6.14E-5	2.14E-4	4.72E-4	3.07E-6	1.52E-5	5.32E-5	2.54E-4
	(.9347)	(.9347)	(.9347)	(.9347)	(.9347)	(.9347)	(.9347)	(.9347)

Table 1
The max norm of the error and the computed values of the convergence factor of the **ROB** method applied to model problem (32)-DP1 ($\gamma = 2$). In the first column we have the iteration number, in the first row the discretization step-size and in the second row the number of equal subdomains.

	h=.01				h=.005			
iter	p=2	p=4	p=10	p=20	p=2	p=4	p=10	p=20
2	1.39E-6	2.32E-4	1.43E-2	1.03E-1	3.48E-7	2.34E-4	1.43E-2	1.04E-1
	(.0965)	(.0965)	(.0966)	(.0966)	(.0965)	(.0965)	(.0966)	(.0966)
3	1.39E-6	5.23E-6	7.18E-3	2.23E-1	3.48E-7	3.71E-6	7.23E-3	2.25E-1
	(.2103)	(.2104)	(.2097)	(.2104)	(.2103)	(.2104)	(.2097)	(.2104)
4	1.39E-6	1.99E-6	4.58E-3	5.39E-1	3.48E-7	4.65E-7	4.62E-3	5.44E-1
	(.3106)	(.3106)	(.3113)	(.3419)	(.3106)	(.3106)	(.3113)	(.3423)
5	1.39E-6	2.04E-6	3.15E-3	1.41E+0	3.48E-7	5.10E-7	3.10E-3	1.43E+0
	(.3924)	(.3924)	(.3920)	(.4934)	(.3924)	(.3924)	(.3920)	(.4945)
8	1.39E-6	2.04E-6	8.61E-4	4.22E+1	3.48E-7	5.09E-7	8.76E-4	4.32E+1
	(.5573)	(.5573)	(.5574)	(.9617)	(.5573)	(.5573)	(.5574)	(.9644)
16	1.39E-6	2.04E-6	2.93E-5	5.18E+5	3.48E-7	5.09E-7	2.65E-5	5.43E+5
	(.7465)	(.7465)	(.7465)	(1.830)	(.7465)	(.7465)	(.7465)	(1.770)
20	1.39E-6	2.04E-6	9.18E-6	1.68E+6	3.48E-7	5.09E-7	5.58E-6	6.19E+7
	(.7915)	(.7915)	(.7915)	(1.993)	(.7915)	(.7915)	(.7915)	(2.001)
32	1.39E-6	2.04E-6	7.44E-6	2.68E+14	3.48E-7	5.09E-7	1.86E-6	2.95E+14
	(.8640)	(.8640)	(.8640)	(2.487)	(.8640)	(.8640)	(.8640)	(2.491)
36	1.39E-6	2.04E-6	7.44E-6	4.41E+16	3.48E-7	5.09E-7	1.86E-6	6.29E+16
	(.8781)	(.8781)	(.8812)	(2.586)	(.8781)	(.8781)	(.8749)	(2.597)

Table 2
The max norm of the error and the computed values of the convergence factor of the **AVE** method applied to model problem (32)-DP1 ($\gamma = 10$). In the first column we have the iteration number, in the first row the discretization step-size and in the second row the number of equal subdomains.

the theoretically optimum relaxation parameters computed by using the formulas (13) and (27) of the **ROB** and **AVE** methods respectively. In all plots associated with the **AVE** method, we set $\gamma = 2$. The Neumann relaxation parameters the β_1 and β_2 computed by formula (29) while we systematically

vary the Dirichlet parameters α_1 and α_2 in $(0,1)$. For the **ROB** method, we set $\gamma = 2$ while the relaxation parameters vary in a larger interval since there no bounds for them. For this method we see that there is curve on the $\lambda_1\lambda_2$ plane with optimum values for the relaxation parameters. The stars in **ROB** plots represent the optimum values computed using formula (13), which is located at the intersection of the above curve and the solution of equation (14) for $p = 3$, i.e.,

$$\lambda_1(\gamma_2 n_2 + \lambda_2 m_2) = \gamma_2(\gamma_2 m_2 + \lambda_2 n_2).$$

We note that at the points indicated by stars in all the following graphs, the experimentally observed number of iterations were always in the range of $5 - 8$ confirming the theoretical optimality of the parameter values. It is also interesting to observe that this optimality seems to be independent of the uniformity of the decomposition and of changes in the value of γ in the subdomains.

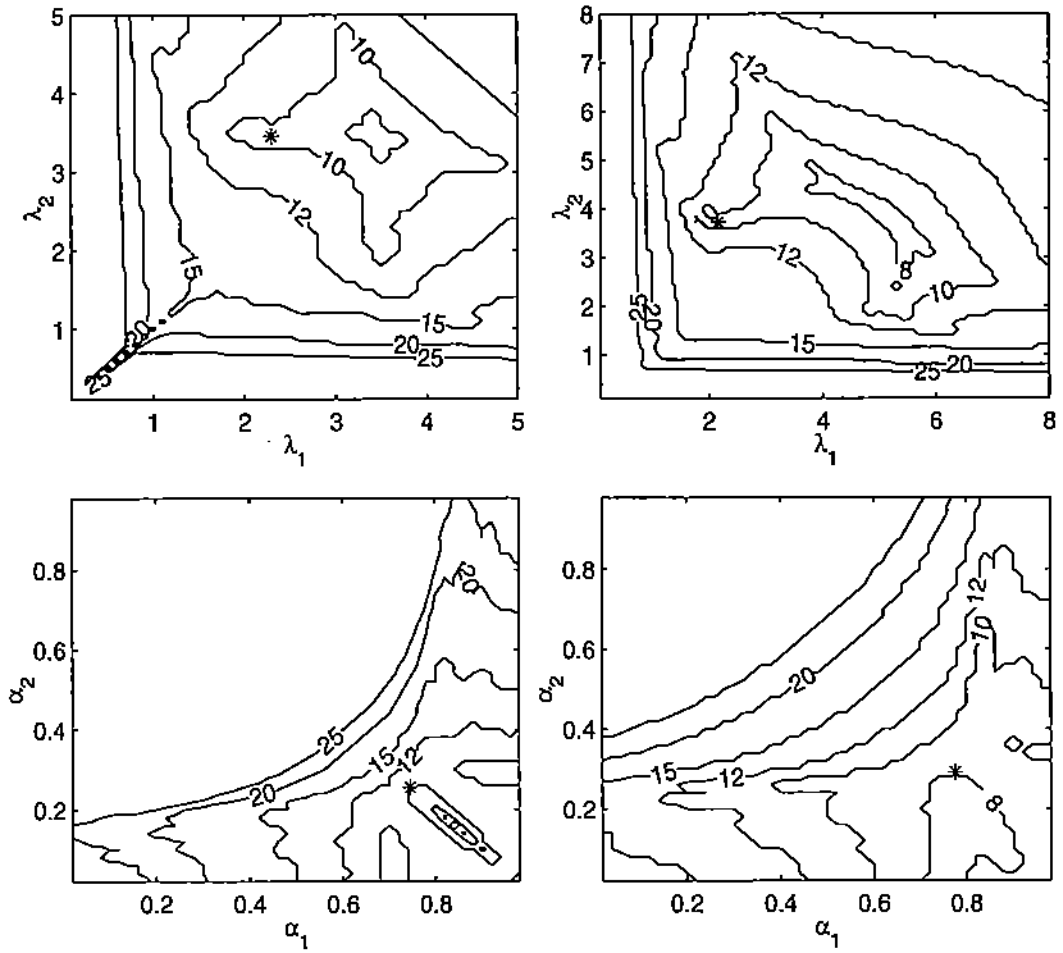
In particular in Figure 2, we have the same non-uniform decomposition as in the bottom two plots in Figure 1, but here the coefficient of u in the DE is discontinuous at the interface points. Specifically in the first subdomain $\gamma^2 = 2$, in the second $\gamma^2 = 10$ and in the third $\gamma^2 = 4$. The right plot for **AVE** is made using, as before, Neumann relaxation parameters (β_1, β_2) computed by formula 29 and letting α_1 and α_2 vary in $(0,1)$.

In general, the **AVE** method seems to converge faster than **ROB** but Theorem 4 imposes a restriction on its convergence region. In Figure 3 we experimentally verify the results in Theorem 4 and we clearly see that the restriction on the size of the subdomains imposed is not only sufficient but required too. The restriction $\gamma\ell_i > ln(1+\sqrt{2})$ of Theorem 4 is, for the 6 cases in Figure 3, top row (γ/p = 2.24, 1.19, .89 > .881) and bottom (γ/p = .913, 1.05, 1.17 > .881). The convergence region (the area where the spectral radius of the iteration matrix is less than 1) shrinks as one either increases the number of subdomains keeping γ constant or decreases γ assuming a constant number of subdomains. The imposed bound on the size of subdomains seems to be a sharp one since in all our experiments we observe divergence every time we make $\ell_i\gamma_i$ slightly less than $\ln(1+\sqrt{2})$ while we always obtained convergence otherwise.

To obtain additional information on the convergence behavior of the two methods we now switch to the model problem (32)-DP2. In Figure 4 we set $\gamma^2 = 20$ and plot the true solution and the first three iterands. We observe that both methods converge in a non-monotone way, but **AVE** follows a much smoother path.

Finally in Figure 5 we consider the model problem (32)-DP2, with a two subdomain partition. We set all relaxation parameters equal to .5 and experimentally compare the effect that the size of γ and the location of the interface

Fig. 1. Contour plots for case DP1 of the number of iterations required by the **ROB** (top two plots) and **AVE** (bottom two plots) methods to make the max norm of the difference of two successive iterands smaller than 10^{-5} as a function of associated relaxation parameters. We assume a uniform 3 subdomain partition in the graphs on the left and non-uniform partition with $x_1 = .2$ and $x_2 = .7$ on the right ($\gamma = 2$). The stars point the theoretically determined optimum values of the parameters.



point have on the convergence rates for the two methods. We plot the logarithm of the max norm of the error (on the y-axis) versus the number of iterations (on the x-axis). The interface point is fixed at .5 for the two plots on the left of the figure while $\gamma^2 = 20$ for the two on the right. We observe that the **AVE** method is significantly affected by both parameters while the **ROB** method converges in a smoother but slower way.

5 Concluding remarks

We have presented a theoretical and experimental study of two interface relaxation methods. Although both methods were considered in several previous

Fig. 2. Contour plots for case DP1 of the number of iterations required by the **ROB** (left) and the **AVE** (right) methods to make the max norm of the difference of two successive iterands smaller than 10^{-5} as a function of the associated relaxation parameters. We assume a uniform 3 subdomain partition in the graph on the left and non-uniform partition with $x_1 = .2$ and $x_2 = .7$ on the right. Here the coefficient of u is $\gamma^2 = 2$ for the first subdomain, $\gamma^2 = 10$ for the second and $\gamma^2 = 4$ for the third subdomain. The stars represent the theoretical optimum values.

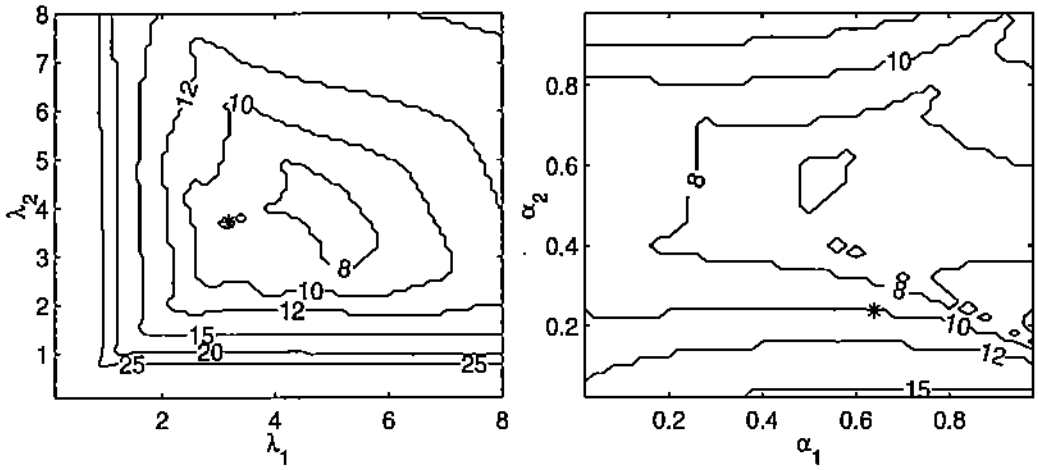
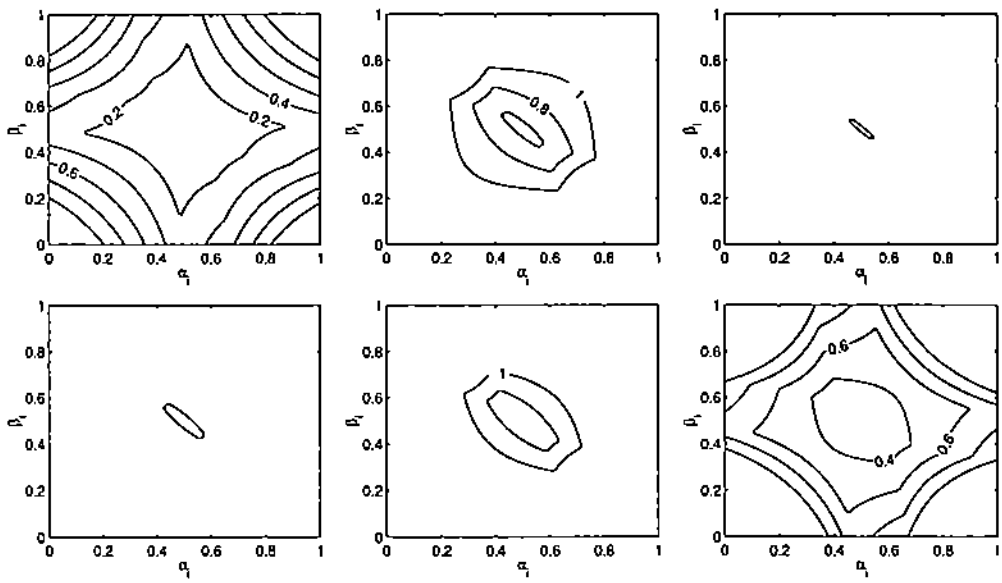


Fig. 3. Contour plots for case DP1 of the upper bounds of the spectral radius for the uniform case for the **AVE** method. In the top three plots $\gamma^2 = 20$ while the number of subdomains is equal to 2 (left), 4 (middle) and 5 (right). In the bottom three figures we fix the number of subdomains at $p = 6$ and γ^2 is equal to 30 (left), 40 (middle) and 80 (right).



studies more understanding is needed. The main objective of our study is to provide to set up the background for

– The comprehensive analysis of these methods for arbitrary decompositions

Fig. 4. Convergence history for case DP2 with $\gamma^2 = 20$ and a 4 subdomain uniform decomposition. The graph shows the true solution and the first three iterants for the **ROB** (on the left plot) and the **AVE** (on the right) methods.

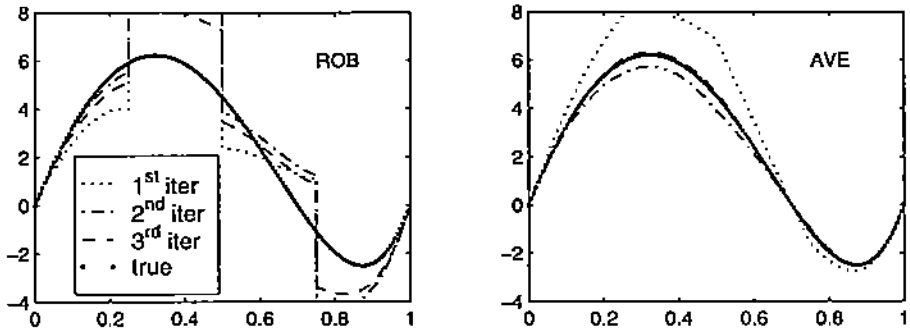
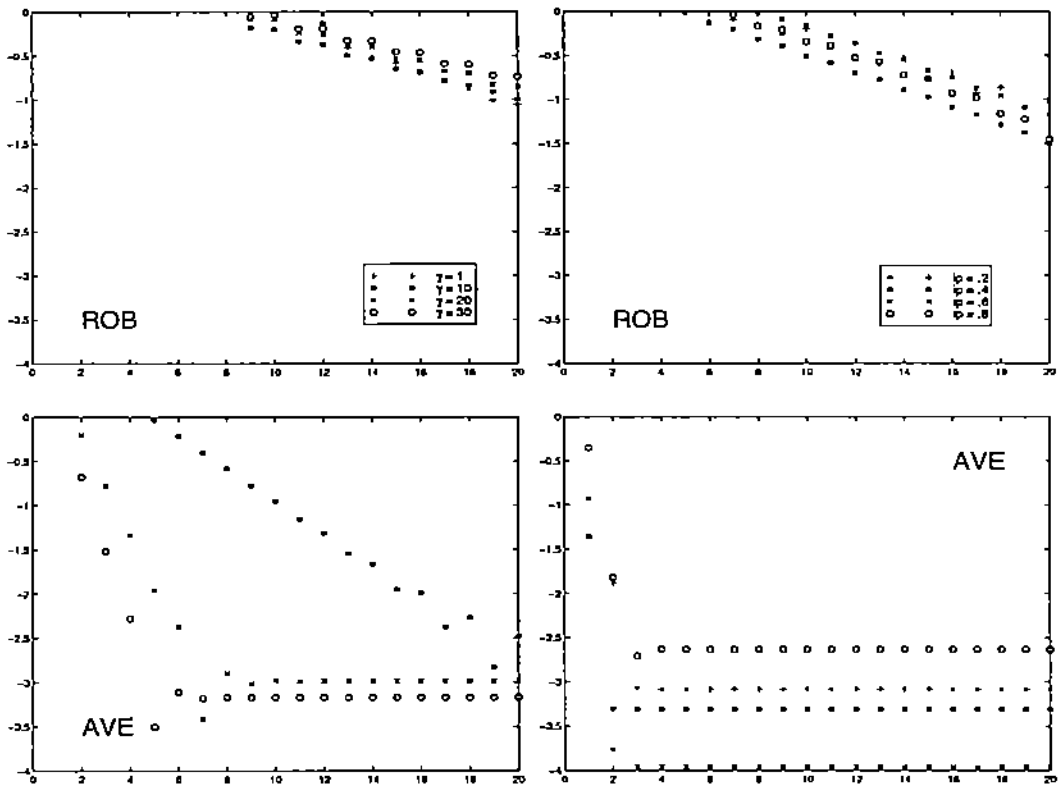


Fig. 5. The effect of the coefficient γ (left graph $\gamma = 1, 10, 20, 30$) and of the location of the interface point (right, $x = .2, .4, .6, .8$) on the convergence rates for the **ROB** (top) and **AVE** (bottom) applied to case DP2. The y-axis is the max norm of the difference of successive solutions and the x-axis is the number of iteratives.



and complicated 2-D and 3-D PDE problems.
- The implementation and use of these methods in an agent [3] framework that is already operational [4].

Our analysis is at the continuous level and our theoretical results have been fully confirmed with extensive experimental data. We used simple mathematical tools to obtain interesting theoretical results. In particular, Theorem 3

gives optimum values of the relaxation parameters involved in **ROB** methods, which are independent both of the particular discretization of the differential operator, its domain and of the subdomain splitting. Furthermore, similar results were obtained for the **AVE** method in Theorem 4 where an important relation between the size of the subdomains and the coefficient of the differential equation is established that determines the domain of convergence of this method.

Acknowledgments

We are grateful to Prof. A. Hadjidimos for the constructive discussion which help us to derive the "optimum" values for the **ROB** method.

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