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Abstract

The distance between a Bézier segment and its control polygon is bounded in terms of the second differences of the control point sequence and a constant that depends only on the degree of the polynomial. The constant derived here is the smallest possible and is sharp for the Hausdorff distance between control polygon and curve segment.

The bound provides a straightforward proof of quadratic convergence of the sequence of control polygons to the Bézier segment under subdivision or degree-fold degree-raising and establishes the explicit convergence constants. The bound also allows analyzing the optimal choice of subdivision parameter for adaptive refinement of quadratic and cubic segments and it may be useful to establish better bounding regions.

1 Curved geometry and control polygons

A widely used, efficient, intuitive way to specify, represent and reason about curved, nonlinear geometry for design and modeling is the control point or control polyline paradigm: for popular representations like the B-spline and the Bernstein-Bézier representation the curve-shape is outlined by the broken line connecting the control points. For many applications, e.g. rendering, intersection testing, design, this raises the question just *how well* the control line approximates the exact curved geometry.

This paper gives a simple, optimal *quantitative* answer to the question in terms of the second differences of the control point sequence of the Bézier representation and a constant that depends only on the degree of the polynomial. In these terms the bound is generically sharp, i.e. there exist commonly used curves such that the bound is taken on and any reduction of the constant would not yield a bound. Remarkably, the bound remains sharp under degree-raising and subdivision, i.e. refinement of the piecewise linear control structure to better approximate the curved geometry. This yields for example a sharp *a priori*

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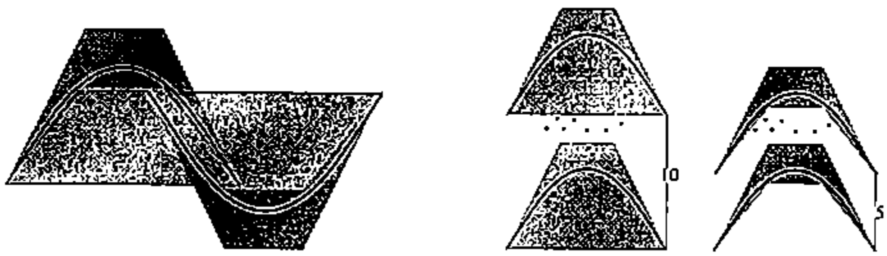

Figure 1: Improved bounds for intersection testing (*left*) and creating tolerance envelopes (*right*). Shaded region corresponds to convex hull, darker portion to new bound derived from the results in this paper.

bound on the number of subdivision steps needed to bring curve and control polyline within a prescribed Hausdorff distance of one another.

While the focus of this paper is on capturing the essence of the bound and its implications for the general toolkit of computer aided geometric design, the two example scenarios sketched in Figure 1 illustrate the potential impact of the new result on a wide range of applications. Complemented by the convex hull bound and an improved bound at the ends of a curve segment, the result confines the curve segment to a region bounded by at most $2d+2$ line segments where d is the degree of the component functions of the curve. Standard localization of the curve to the convex hull, here depicted as the union of shaded regions gives more conservative estimates than localization to the darker shaded region implied by the new bound of this paper (c.f. Section 6). In Figure 1 (*left*) non-intersection follows immediately from the new bounds, while the convex hull estimate requires several refinements to separate the bounding regions. On the right, the curve and its translate can be chosen closer together while still guaranteeing the inclusion of the given point set.

For the general computer aided design toolkit, the tight bound reveals the constants that scale the quadratic rate of convergence of the sequence of control polylines to the curve under subdivision, respectively under degree-fold degree-raising. This allows, for example, determining the optimal subdivision parameter for segments of low degree.

After reviewing prior work, Section 3 represents the technical heart of the paper, a bound for functions in Bernstein-Bézier form. Section 4 extends the result to the Hausdorff distance between control polyline and curve segment. Section 5 discusses subdivision and Section 6 the relevance of the bound to estimating the effect of degree-raising.

2 Prior Bounds

Two properties lie at the heart of control point representations of curves: variation diminuition and the subdivision property. The variation diminishing property, that any line crosses the control polygon at least as often as it does the curve, makes precise the notion that the features of the curve are exaggerated by the control polygon. Variation diminuition also implies the convex hull property, which states that all points on the curve segment are convex combinations of the control points. Thus the convex hull yields a bound on the distance between curve segment and control polyline.

The subdivision property gives a stable way of approximating the curve through a sequence of refinements of the control polygon using fixed-weight, finite averaging. Approximation rates for this process have been established in [1] and by the careful analysis in [11]. Either result yields *qualitative* assurance that the approximation will improve under subdivision, but the corresponding *quantitative* estimates are too coarse for practical use. For example, the estimate in [11], exceeds the bound implied by the convex hull property. In [7], Filip, Magedson and Markot derive bounds for the distance between a curve and its piecewise linear *interpolant* to the end points. For a Bézier curve of degree d, this bound is $d-1$ times the bound derived in this paper. In [9] upper and lower bounds for the modulus of continuity of polynomial and rational curves in Bézier form are derived. In [6] Farin points out that for rational curves, the convex hull can be tightened to include only rational weight points and end points. A similar projection argument applies to the joint intersection of convex hull and the new tight bound. In [12], an arc is subtracted from the curve segment prior to generating a min-max bounding box. The authors of [12] call the arc offset by the bounding box a fat arc.

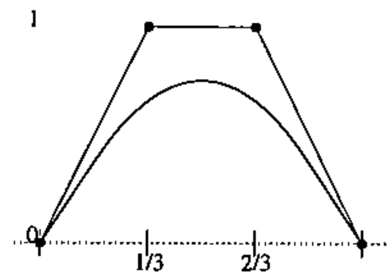

Figure 2: Cubic and its control polygon with coefficient sequence $(b_0, b_1, b_2, b_3) = (0, 1, 1, 0)$ and corresponding $\|\Delta_2 b\|_\infty = |\Delta_2 b_1| = |\Delta_2 b_2| = 1$.

3 Bounding functions

This section contains the central estimate for localizing the graph of a function in Bernstein-Bézier form with respect to the control polyline. The estimate is easily computed in terms of a constant $\mathtt{N}(d)$ that depends only on the degree d and the maximum second difference of the coefficient sequence. Some definitions are in order (c.f. [5], [2]).

A univariate, scalar-valued, polynomial of degree d is in *Bernstein-Bézier form* if

$$p(t) := \sum_{i=0}^{d} b_i B_i^d(t)$$

where $B_i^d(t) := \binom{d}{i}(1-t)^{d-i}t^i$.
The *control polyline* ℓ of p is a broken line connecting the points (t_k, b_k) where the first components $t_k := \frac{k}{d}$ are the Greville abscissae. Its kth segment $\ell_{[t_k,t_{k+1}]}$ on the interval $[t_k, t_{k+1}]$, is defined by

$$\ell_{[t_k,t_{k+1}]}(t) := b_k \frac{t_{k+1} - t}{t_{k+1} - t_k} + b_{k+1} \frac{t - t_k}{t_{k+1} - t_k}.$$

The ith *centered second difference* of the coefficient sequence $b_i, i = 0..d$ is abbreviated

$$\Delta_2 b_i := b_{i-1} - 2b_i + b_{i+1} \quad \text{and} \quad \|\Delta_2 b\|_\infty := \max_{0<i<d} |\Delta_2 b_i|.$$

Finally, $\|p(t) - \ell(t)\|_{\infty,[0,1]}$ denotes the maximal absolute difference between p and ℓ on the interval $[0, 1]$.

With these definitions the main result reads as follows.

Theorem 3.1 *The distance from the univariate, scalar-valued, degree d polynomial p to its control polyline ℓ is bounded as*

$$\|p(t) - \ell(t)\|_{\infty,[0,1]} \le \mathtt{N}(d)\|\Delta_2 b\|_\infty$$

where

$$\mathtt{N}(d) := \frac{\lfloor d/2 \rfloor \lceil d/2 \rceil}{2d}$$

For example, $[\mathtt{N}(0), \ldots, \mathtt{N}(8)] = [0, 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{3}{4}, \frac{6}{7}, 1]$.

Proof On the interval $[t_k, t_{k+1}]$, $p(t) - \ell(t) = \sum_i \alpha_{ki}(t) b_i$, where

$$\alpha_{ki}(t) := B_i^d(t) - \begin{cases} k+1-dt & \text{if } i = k \\ dt - k & \text{if } i = k+1 \\ 0 & \text{else} \end{cases}$$

since $t_{k+1} - t_k = t_k - t_{k-1} = 1/d$. The formulae for conversion to power form, $\sum_{i=k}^d \binom{i}{k} B_i^d(t) = \binom{d}{k} t^k$, implies that the that the Bernstein operator reproduces linear polynomials, i.e. that

$$\sum_{i=0}^{d} \alpha_{ki}(t) = 0 \quad \text{and} \quad \sum_{i=1}^{d} i\alpha_{ki}(t) = 0.$$

It follows that $\sum_{j=0}^{i} (i-j)\alpha_{kj}(t) = \sum_{j=i}^{d} (j-i)\alpha_{kj}(t)$, and hence for $0 \le i \le d$ and all k

$$\beta_{ki}(t) := \sum_{j=0}^{i} (i-j)\alpha_{kj}(t) = \begin{cases} \sum_{j=0}^{i} (i-j) B_j^d(t) & \text{for } i \le k \\ \sum_{j=i}^{d} (j-i) B_j^d(t) & \text{for } i \ge k+1. \end{cases}$$

The sequence β_{ki} (cf. Figure 3), extended to $\beta_{ki} = 0$ for $i < 0$ or $i > d$ is a nonnegative second antidifference to α_{ki} on $[t_k, t_{k+1}]$. That is

$$\Delta_2 \beta_{ki} = \beta_{k,i+1} - 2\beta_{k,i} + \beta_{k,i-1} = \alpha_{k,i} \quad \text{for } 0 \le i \le d,$$

since $\beta_{ki}(t) > 0$ for $0 < i < d$, but $\beta_{ki}(t) = 0$ for all other i. Furthermore

$$\begin{aligned} \sum_{i=0}^{d} \beta_{ki}(t) &= \sum_{i=0}^{d} \sum_{j=0}^{i} (i-j)\alpha_{kj}(t) = \sum_{j=0}^{d} \sum_{i=j}^{d} (i-j)\alpha_{kj}(t) = \sum_{j=0}^{d} \sum_{i=0}^{d-j} i\alpha_{kj}(t) \\ &= \sum_{j=0}^{d} \binom{d-j}{2} \alpha_{kj}(t) = \sum_{j=2}^{d} \binom{j}{2} \alpha_{kj}(t) = \sum_{i=2}^{d} \binom{i}{2} B_i^d(t) + \frac{k}{2}(k+1-2dt) \\ &= \binom{d}{2} t^2 + \frac{k}{2}(k+1-2dt). \end{aligned}$$

On its interval $[t_k, t_{k+1}]$, $\sum_{i=0}^{d} \beta_{ki}(t)$ is a positive quadratic polynomial with positive leading coefficient and therefore takes on its maximum either at t_k or

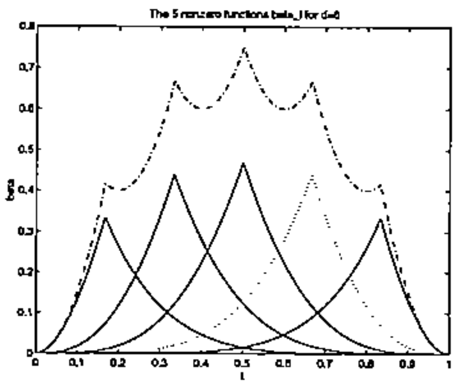

Figure 3: The antidifferences β_{ki} for $d = 6$, $k = 0, \dots, 5$, $i = 1, \dots, 5$ and their piecewise quadratic sum (*dash-dotted*). Defining β_i as the function β_{ki} on the interval $[t_k, t_{k+1}] = [k, k+1]/d$, e.g. β_4 the *dotted* curve, the ith (*solid* or *dotted*) peak separates the monotonically increasing part of β_i on $[0, t_i]$ from the decreasing part on $[t_k, 1]$.

t_{k+1} implying

$$\begin{aligned}
\max_{0 \le k < d} \max_{\max_{t_k \le t \le t_{k+1}}} \sum_i \beta_{ki}(t) &= \max_{0 \le k < d} \max\{\sum_{i=2}^{d} \binom{i}{2} \alpha_{ki}(t_k), \sum_{i=2}^{d} \binom{i}{2} \alpha_{ki}(t_{k+1})\} \\
&= \max_{0 \le k \le d} \binom{d}{2} \frac{k^2}{d^2} - \binom{k}{2} = \max_{0 \le k \le d} \frac{k}{2d}(d-k) \\
&= \frac{\lfloor d/2 \rfloor \lceil d/2 \rceil}{2d}
\end{aligned}$$

Abbreviating $\|\cdot\|_k := \|\cdot\|_{\infty,[t_k,t_{k+1}]}$, the bound follows from the substitutions

$$\begin{aligned}
\|p(t) - \ell(t)\|_{\infty,[0,1]} &= \max_k \|p(t) - \ell(t)\|_k \\
&= \max_k \|\sum_{i=0}^{d} \alpha_{ki}(t) b_i\|_k \\
&= \max_k \|\sum_{i=0}^{d} \Delta_2 \beta_{ki}(t) b_i\|_k \\
&= \max_k \|\sum_{i=1}^{d-1} \beta_{ki}(t) \Delta_2 b_i\|_k \\
&\le \max_k \|\sum_i \beta_{ki}(t)\|_k \|\Delta_2 b\|_\infty \\
&= \frac{\lfloor d/2 \rfloor \lceil d/2 \rceil}{2d} \|\Delta_2 b\|_\infty .
\end{aligned}$$

⋈

The constant $\mathtt{N}(d)$ is not just a better, i.e. $d-1$ times smaller than the previous best estimate in [7], but it is optimal.

Corollary 3.1 *The bound in Theorem 3.1 is sharp.*

Proof If p is the degree-raised representation of a quadratic polynomial then all second differences of the degree-raised representation of p are equal, i.e. for each i, $|\Delta_2 b_i| = \|\Delta_2 b\|_\infty$ since differencing and degree-raising commute. Since the β_{ki} are nonnegative, we have equality throughout the proof, in particular

$$\max_k \|\sum_{i=0}^{d-2} \beta_{ki}(t)\Delta_2 b_i\|_k = \max_k \|\sum_i \beta_{ki}(t)\|_k \|\Delta_2 b\|_\infty.$$

Small perturbations of the coefficients of the degree-raised quadratics yield, for any given degree d, polynomials that asymptotically match the bound. In other words, for any ϵ there are polynomials of degree d that match the bound up to ϵ. ⋈

4 Bounding the Hausdorff distance

The Hausdorff metric μ, introduced by Felix Hausdorff in 1914, and used e.g. in fractal approximation [10] and non-smooth optimization [3], is the natural metric for measuring the distance of two sets $\mathcal{L}$ and $\mathcal{P}$. It is defined as ([8], [4])

$$\mu(\mathcal{P},\mathcal{L}) := \max\{\sup_{L\in\mathcal{L}} \inf_{P\in\mathcal{P}} \|L-P\|, \sup_{P\in\mathcal{P}} \inf_{L\in\mathcal{L}} \|L-P\|\}.$$

The two point sets of interest here are the curve segment $\mathcal{P}$ parametrized by p and its control polyline $\mathcal{L}$ parametrized by ℓ. The two numbers whose maximum is the Hausdorff distance, measure respectively the maximal distance of a point on the control polygon to the curve segment (over-drawing) and the maximal distance of a point on the curve segment to the control polygon (under-drawing). Since the Hausdorff distance is independent of the parametrization it is bounded above by all parametric distance measurements:

$$\mu(\mathcal{P},\mathcal{L}) \leq \|p-\ell\|_{\infty,[0,1]}.$$

The bound for functions is also a sharp bound on the Hausdorff distance between the two point sets.

Lemma 4.1 *The bound*

$$\mu(\mathcal{P},\mathcal{L}) \leq \mathtt{N}(d)\|\Delta_2 b\|_\infty$$

is sharp for the Hausdorff distance of a curve segment $\mathcal{P}$ to its Bézier control polygon $\mathcal{L}$.

Proof Set both components $x(t) = y(t) = q(t)$ for some quadratic q, e.g. $q(t) = 4(1-t)t$. Then the sharpness proof and perturbation argument of the function case apply directly. ⋈

For a more 2-dimensional example consider the distance of the control point $(0.5, 2)$ to the curve segment $(t, q(t)) := (t, 4(1-t)t)$. Here the Hausdorff distance is attained as the furthest distance of a point on the control polygon to the curve. The distance of any point on the curve to the control polygon is less, giving hope that a smaller bound on this distance might exist. However this hope is, at least asymptotically, not justified. Choose $x(u) = u$, $y(u) = \alpha q(u)$. As $\alpha \downarrow 0$, the Hausdorff distance is taken on at $u = 0.5$ as the distance from $(0.5, q(0.5))$ to the control polyline.

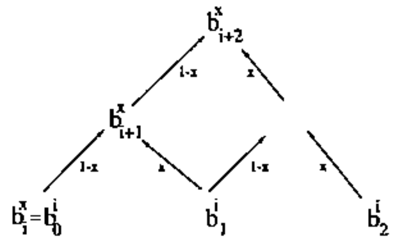

Figure 4: The coefficients b_i^x of the restriction of the polynomial p to $[0,x]$ are obtained as convex combinations of the coefficients b_j^i of the ith step of de Casteljau's algorithm.

5 Bounding subdivision

Refinement, in particular adaptive refinement of the control point sequence to the function or curve can be achieved by creating control polylines for subintervals of the domain. Specifically, we consider

$$p(xt) = p_{[0,x]}(t) := \sum_{i=0}^{d} b_i^x B_i^d(t)$$

the restriction of $p(t) := \sum_{i=0}^{d} b_i B_i^d(t)$ to the interval $[0,x]$, $0 < x < 1$. The coefficients of the restriction can be computed by DeCasteljau's algorithm

$$\begin{aligned} &b_i^0 := b_i, i = 0..d \\ &\text{for } j = 1..d \\ &\quad b_i^j := (1-x)b_i^{j-1} + xb_{i+1}^{j-1},\ i = 0..d-j. \end{aligned}$$

The recurrence expands to

$$b_i^x = b_0^i = \sum_{k=0}^{i} B_k^i(x) b_k^0 = \sum_{k=0}^{i} B_k^i(x) b_k.$$

For $i = 0..d-2$, with Figure 4 illustrating the second equality,

$$\begin{aligned} \Delta_2 b_{i+1}^x = b_i^x - 2b_{i+1}^x + b_{i+2}^x &= x^2(b_0^i - 2b_1^i + b_2^i) \\ &= x^2(\Delta_2 b_1^i) = x^2\ [\sum_{k=0}^{i} B_k^i(x)\Delta_2 b_{k+1}]. \end{aligned}$$

The bound on the restriction is therefore just a scaled version of the original bound.

Lemma 5.1 *The distance between $p_{[0,x]}(t)$, the restriction of p to the interval $[0,x]$, and $l_{[0,x]}(t)$, the corresponding polyline, is bounded by*

$$||p_{[0,x]}(t) - \ell_{[0,x]}(t)||_{\infty,[0,x]} \le x^2 \mathrm{N}(d)||\Delta_2 b||_\infty$$

where $\|\Delta_2 b\|_\infty$ *is the maximal absolute second difference of the coefficient sequence of* $p_{[0,1]}$.

Since the bound is sharp for any quadratic we have the following corollary.

Corollary 5.1 *The constants* $\mathtt{N}(d)$ *are sharp under subdivision.*

For example, subdividing q at $0 < x < 1$ into $q_{[0,x]}$ and $q_{[x,1]}$, we get $q_{[0,x]} = 2x \cdot 2(1-t)t + 4(1-x)xt^2$, and

$$\|q_{[0,x]}(1/2) - \ell_{[0,x]}(1/2)\| = |(x + (1-x)x) - 2x| = x^2$$

equals the bound

$$\frac{2}{8}[2(2x) - 4(1-x)x].$$

The next lemma establishes the quadratic rate of convergence of the control polygon to the curve segment under subdivision.

Lemma 5.2 *The distance between the polynomial and its control polylines after* m*-fold subdivision at the local parameter* x *is bounded by*

$$\mathrm{x}^{2m}\mathtt{N}(d)\|\Delta_2 b\|_\infty \text{ where } \mathrm{x} := \max\{x, 1-x\}.$$

Proof By symmetry, the bound for the polynomial restricted to the interval $[x, 1]$ is

$$\|p_{[x,1]}(t) - \ell_{[x,1]}(t)\|_{\infty,[x,1]} \le (1-x)^2 \mathtt{N}(d)\|\Delta_2 b\|_\infty$$

and hence the distance of the curve segment to the union of the control polylines of $p_{[0,x]}$ and $p_{[x,1]}$ is bounded by $\mathrm{x}^2\mathtt{N}(d)\|\Delta_2 b\|_\infty$. ⋈

With the identity $\Delta_2 b^x_{i+1} = x^2 \left[\sum_{k=0}^{i} B^i_k(x)\Delta_2 b_{k+1}\right]$ derived earlier, the problem of *finding the optimal subdivision parameter* x becomes

$$\min_{x\in(0,1)} \max_{i=1,\ldots,d-1} \{|\Delta_2 b^x_i|, |\Delta_2 b^{1-x}_i|\}$$
$$= \min_x \max_i \{x^2 |\sum_{k=0}^{i} B^i_k(x)\Delta_2 b_{k+1}|, (1-x)^2 |\sum_{k=0}^{i} B^i_k(1-x)\Delta_2 b_{d-1-k}|\}$$

- For $d = 2$, after scaling by $\Delta_2 b_0$, the problem becomes

  $$\min_x \max_i \{x^2, (1-x)^2\}$$

  and $x = 1/2$ is optimal.

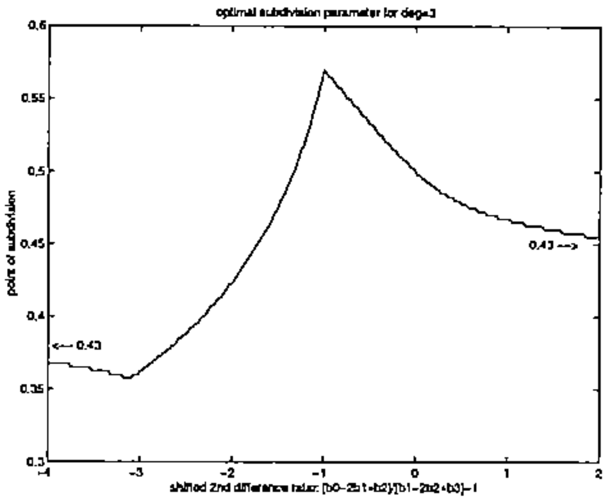

Figure 5: The optimal subdivision parameter x of a cubic as a function of δ, where $\Delta_2 b_0 = 1 + \delta$ and $\Delta_2 b_1 = 1$.

- For $d = 3$, assuming the curve is not a straight line, the second difference can be normalized by dividing by a nonzero $\Delta_2 b_i$, without loss of generality $\Delta_2 b_2$. We may therefore assume that $\Delta_2 b_1 = 1 + \delta$ and $\Delta_2 b_2 = 1$. The problem becomes

  $$\min_x \max_i \{x^2|1+\delta|, (1-x)^2, x^2|(1+\delta) - \delta x|, (1-x)^2|1+\delta(1-x)|\}$$

  The numeric solution to the problem is displayed in Figure 5. The limiting optimal value $x_{\pm\infty} \approx 0.43$, is the solution of $(1-x)^3 = x^2$.

A popular criterion for determining the subdivision parameter for adaptive subdivision is the curvature of the Bézier segment. We note that in neither case is the point of maximal curvature necessarily the optimal parameter.

6 Bounding degree-raising

Expressing a polynomial of degree d in Bernstein-Bézier form in the basis B_j^{d+1} by multiplying the polynomial by $B_0^1 + B_1^1 = (1-t) + t$ is called degree-raising. Clearly the number of coefficients increases by one and since the new coefficients are obtained as convex combinations of the original coefficients, it is possible to show convergence of the sequence of control polygons corresponding to repeated degree-raising to the graph of the polynomial on $[0,1]$ (see e.g. the fine analysis in [11]). The next lemma reveals the exact rate and constant of convergence.

Lemma 6.1 *Let ℓ^{2d} be the control polyline of the d-fold degree-raised representation of the polynomial p of degree d. Then*

$$\|p - \ell^{2d}\|_{\infty,[0,x]} \leq \frac{1}{2}\mathtt{N}(d)\|\Delta_2 b\|_\infty$$

where $\|\Delta_2 b\|_\infty$ is the maximal absolute second difference of the original coefficient sequence.

Proof Define the coefficients b_i^{d+1} by

$$(1-t+t)\sum_{i=0}^{d} b_i^d B_i^d(t) = \sum_{i=0}^{d+1} b_i^{d+1} B_i^{d+1}(t).$$

Differentiating twice

$$d(d-1)\sum_{i=0}^{d-2} \Delta_2 b_{i+1}^d B_i^{d-2} = (d+1)d\sum_{i=0}^{d-1} \Delta_2 b_{i+1}^{d+1} B_i^{d-1}.$$

Since degree-raising averages and commutes with differentiation, $\|\Delta_2 b^{d+1}\|_\infty$ is maximal when all second differences $\Delta_2 b_i^d$ are equal, implying

$$\|\Delta_2 b^{d+1}\|_\infty \leq \frac{d-1}{d+1}\|\Delta_2 b^d\|_\infty.$$

The distance between the control polyline ℓ^{d+1} of the degree-raised Bernstein representation and p is therefore

$$\|p - \ell^{d+1}\|_{\infty,[0,1]} \leq K(d, d+1)\mathtt{N}(d)\|\Delta_2 b^d\|_\infty$$

where $K(d, d+1) := \frac{d-1}{d+1}\frac{\mathtt{N}(d+1)}{\mathtt{N}(d)}$ and d-fold degree-raising increases the bound to

$$K(d, 2d) = \frac{d(d-1)}{2d(2d+1)}\frac{\mathtt{N}(2d)}{\mathtt{N}(d)} = \frac{1}{2}\begin{cases} \frac{d-1}{d+1/2} & \text{if } d \text{ is even.} \\ \frac{d^2}{(d+1/2)(d+1)} & \text{if } d \text{ is odd.} \end{cases} < \frac{1}{2}.$$

⋈

Analysis of the parabola $q(t) := 4(1-t)t$ shows sharpness of the bound.

As a means of creating control polygon sequences that converge to the graph of the function we can compare d-fold degree-raising with subdivision at the midpoint. Single degree-raising requires work comparable to computing one level in the subdivision scheme, but slightly more since the number of coefficients computed increases rather than decreases by one. Also, the asymptotic constant, 1/2, implies slower guaranteed convergence than the constant 1/4 obtained from subdivision at the midpoint.

7 Conclusion

The explicit bound on the distance of the control polyline to its Bézier segment presented in this paper facilitates a constructive, quantitative derivation of fundamental properties of the Bernstein-Bézier representation. Ongoing research shows the feasibility of an extension of the approach to several variables, not just by tensoring.

In conjunction with the convex hull the new bound has potential to yield tighter, more effective localization and bounding boxes. Whether such bounding boxes are effective in practice is currently being studied and depends on the domain specific trade-off of computational effort *vs* accuracy.

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