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Abstract. We present a gallery of simple curvature continuous surfaces that possess the topological structure of the Platonic solids. These sphere-like surfaces consist of one cubic triangular or biquadratic quadrilateral patch per vertex of the solid and interpolate the vertices of the dual solid.

§1. Polynomial curvature continuous surfaces

Constructing low degree polynomial curvature continuous surfaces is a difficult problem. Existing parametric solutions [3, 6, 5] require both a large number of patches and high degree. A recent implicit curvature continuous construction requires only algebraic degree 4, but consists of many pieces [4]. On the other hand the existence of low degree (rational) curvature continues representations of shapes such as the sphere hints at the existence of elegant solutions for restricted geometric shapes.

This paper describes a small family of polynomial surfaces that approximate the sphere and have the symmetry structure of Platonic polyhedra. These *spheroids* embody a remarkably simple construction principle: pick a Platonic polyhedron and its dual; associate with each 3-valent vertex of the dual a triangular patch of degree 3, or with each 4-valent vertex a quadrilateral patch of degree bi-2. The Bernstein-Bézier control points of the spheroid patches are the face centroids, edge midpoints and scaled vertices of the dual. By symmetry, the spheroid interpolates the vertices of the Platonic and by scaling those of the dual Platonic. Using their simple, explicit parametrization the two highly symmetric platonic spheroids can be *proven* strictly convex. The table below collects basic spheroid properties.

Table 1 name	patches	interpolates	max Gauss curvature
Tetroid	4	Tetrahedron	3.24
Hexoid	6	Cube	1.92
Octoid	8	Octahedron	1.69
Dodecoid	20	Icosahedron	1.25

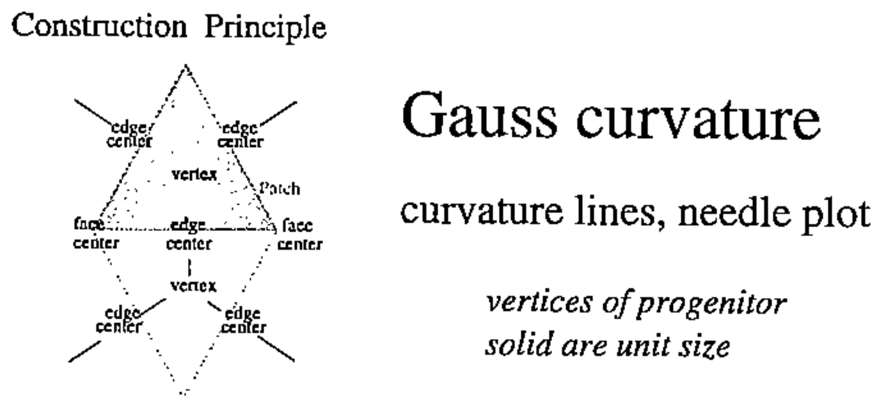

Fig. 2. Synopsis of patch transitions and curvature used in Sections 3-6.

§2. Guide to the gallery

Each of the following four sections gives the synopsis of one platonic spheroid. Pick a Platonic solid. Its dual will be interpolated and gives the particular spheroid its name. The part closer to a vertex V of the original Platonic solid is covered by a single patch (there is no dodecoid since that would require a 5-sided patch). The complete surface is obtained by applying the operations of the solid's symmetry group (see e.g. [2]) to this one patch. The patch associated with V has quadratic boundary curves and one interior control point. Each boundary curve connects the centroids of two edge-adjacent facets attached to V and uses the midpoint of their common edge as middle coefficient of the quadratic curve in Bézier form. If V is three-valent, the Bézier representation of the boundaries are degree-raised. The center control point of the biquadratic or cubic patch is always a multiple of V.

Besides the representative patch, each synopsis shows, in the *upper left* sketch, the facet centroids, edge midpoints and the two vertices that determine the tangent plane common to two patches, say p and q. Tangent continuity of the surface follows from the existence of a linear scalar polynomial λ such that the versal derivative D_1p along the common edge of p and q is a linear combination of the transversal derivatives D_2p and D_2q across the common edge, here parametrized by u. In symbols

$$p(u,0) = q(u,0) \quad \text{and} \quad \lambda(u)D_1p(u,0) = D_2p(u,0) + D_2q(u,0).$$

In the synopsis, the coefficients of D_ip are collected in a matrix. The *upper right* figure shows the spheroid textured by Gauss curvature, curvature lines and a needle plot proportional to the curvature. The normalization is chosen so that the vertices of the name-giving Platonic solid are on the unit sphere. Scale and maximal curvature are displayed on the right. Since the surfaces are symmetric across the edges and vertices, first order smoothness implies that the transitions are also curvature continuous.

Finally, the positivity of all Bézier coefficients in numerator and denominator of the expression for Gauss curvature establishes convexity of octoid and icosoid.

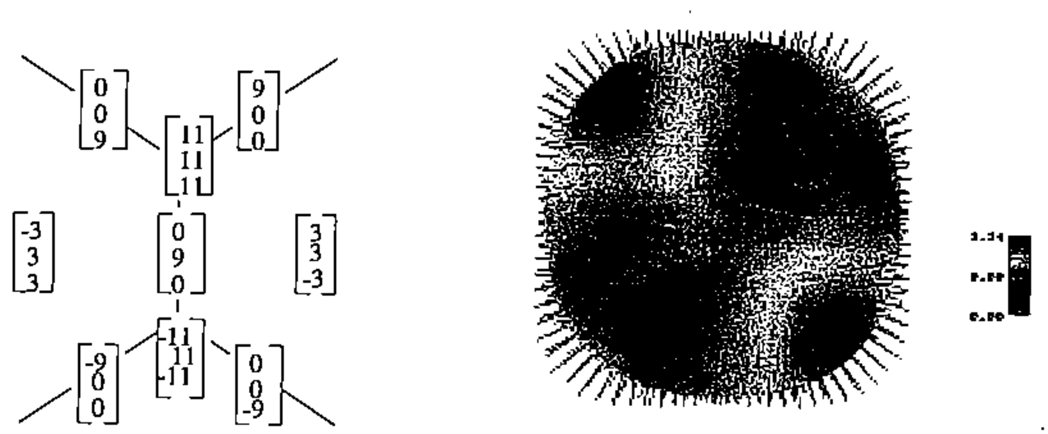

Fig. 3. Coefficients and Gauss curvature of the Tetroid.

§3. The Tetroid

A representative cubic patch has the Bézier coefficients

$$
\begin{array}{ccccccc}
\begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} & & \begin{bmatrix} 1 \\ -1 \\ -7 \end{bmatrix} & & \begin{bmatrix} -1 \\ 1 \\ -7 \end{bmatrix} & & \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} \\
& \begin{bmatrix} 1 \\ -7 \\ -1 \end{bmatrix} & & \begin{bmatrix} -11 \\ -11 \\ -11 \end{bmatrix} & & \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix} & \\
& & \begin{bmatrix} -1 \\ -7 \\ 1 \end{bmatrix} & & \begin{bmatrix} -7 \\ -1 \\ 1 \end{bmatrix} & & \\
& & & \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix} & & &
\end{array}
$$

The smoothness is characterized by

$$
D_1 p(u,0) = 2 \begin{bmatrix} 3 & 3 \\ 6 & -6 \\ -3 & -3 \end{bmatrix}, \quad D_2 p(u,0) = 3 \begin{bmatrix} 2 & 12 & 6 \\ -2 & 4 & -6 \\ 4 & 10 & 0 \end{bmatrix},
$$

$$
D_2 q(u,0) = 3 \begin{bmatrix} -4 & -10 & 0 \\ -2 & 4 & -6 \\ -2 & 12 & -6 \end{bmatrix}
$$

and hence $D_2 p(u,0) + D_2 q(u,0) = [-(1-u) + 3u] D_1 p(u,0)$.

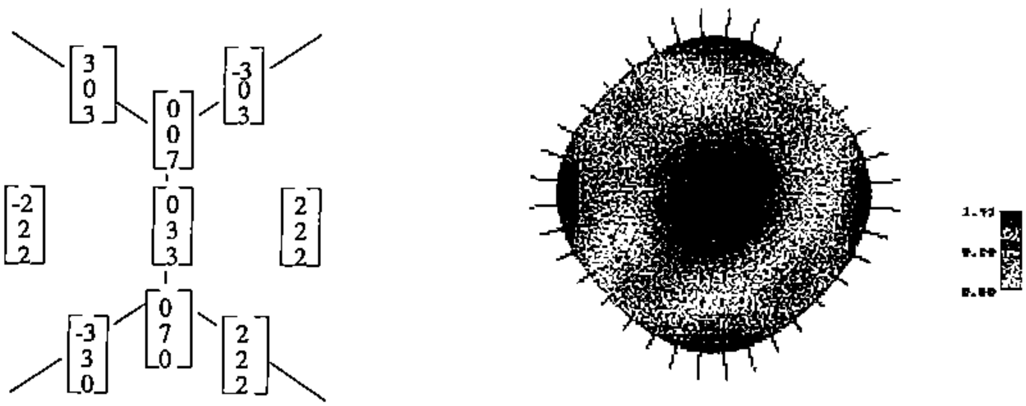

Fig. 4. Coefficients and Gauss curvature of the Hexoid.

§4. The Hexoid

A representative biquadratic patch has the Bézier coefficients

$$\begin{bmatrix}-2\\-2\\2\end{bmatrix}\quad\begin{bmatrix}-3\\0\\3\end{bmatrix}\quad\begin{bmatrix}-2\\2\\2\end{bmatrix}$$
$$\begin{bmatrix}0\\-3\\3\end{bmatrix}\quad\begin{bmatrix}0\\0\\7\end{bmatrix}\quad\begin{bmatrix}0\\3\\3\end{bmatrix}$$
$$\begin{bmatrix}2\\-2\\2\end{bmatrix}\quad\begin{bmatrix}3\\0\\3\end{bmatrix}\quad\begin{bmatrix}2\\2\\2\end{bmatrix}$$

The smoothness is characterized by

$$D_1p(u,0) = 2\begin{bmatrix}2 & 2\\1 & -1\\1 & -1\end{bmatrix},\quad D_2p(u,0) = 2\begin{bmatrix}-1 & 0 & 1\\-2 & -3 & -2\\1 & 4 & 1\end{bmatrix},$$
$$D_2q(u,0) = 2\begin{bmatrix}-1 & 0 & 1\\1 & 4 & 1\\-2 & -3 & -2\end{bmatrix}$$

and hence $D_2p(u,0) + D_2q(u,0) = [-(1-u)+u]D_1p(u,0)$.

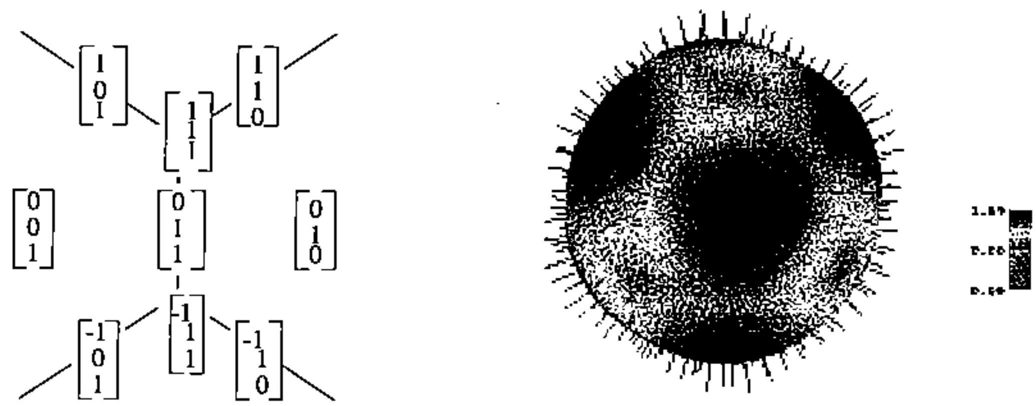

Fig. 5. Coefficients and Gauss curvature of the Octoid.

§5. The Octoid

A representative cubic patch has the Bézier coefficients

$$\begin{array}{ccccccc}
\begin{bmatrix}0\\0\\3\end{bmatrix} & & \begin{bmatrix}0\\2\\3\end{bmatrix} & & \begin{bmatrix}0\\3\\2\end{bmatrix} & & \begin{bmatrix}0\\3\\0\end{bmatrix} \\
& \begin{bmatrix}2\\0\\3\end{bmatrix} & & \begin{bmatrix}3\\3\\3\end{bmatrix} & & \begin{bmatrix}2\\3\\0\end{bmatrix} & \\
& & \begin{bmatrix}3\\0\\2\end{bmatrix} & & \begin{bmatrix}3\\2\\0\end{bmatrix} & & \\
& & & \begin{bmatrix}3\\0\\0\end{bmatrix} & & &
\end{array}$$

The smoothness is characterized by

$$D_1p(u,0) = 2\begin{bmatrix}0 & 0\\ 3 & 0\\ 0 & -3\end{bmatrix}, \quad D_2p(u,0) = 3\begin{bmatrix}2 & 3 & 2\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix},$$

$$D_2q(u,0) = 3\begin{bmatrix}-2 & -3 & -2\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

and hence $D_2p(u,0) + D_2q(u,0) = 2uD_1p(u,0)$. The surface is convex.

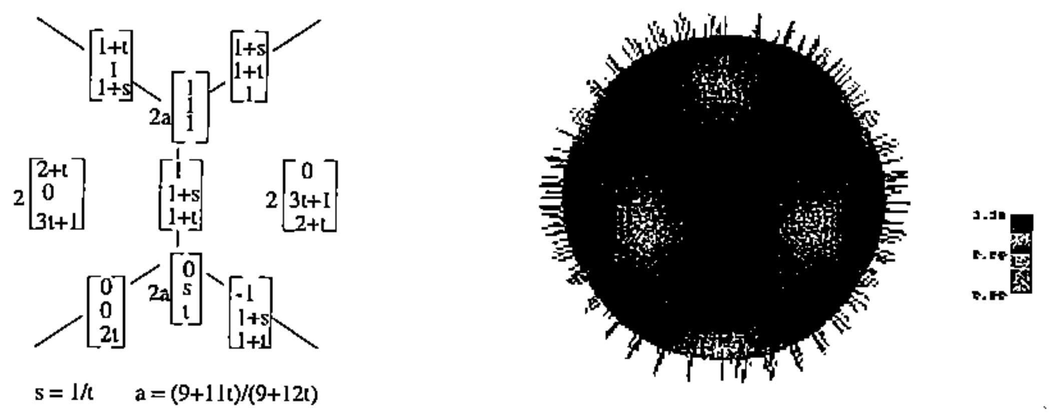

Fig. 6. Coefficients and Gauss curvature of the Icosoid.

§6. The Icosoid

With $t := (\sqrt{5}+1)/2$, $s := 1/t$, and $\gamma := 5(7+22t)/(1+4t)$, a representative cubic patch has the Bézier coefficients

$$3\begin{bmatrix}3+2t\\0\\-1+6t\end{bmatrix} \quad \begin{bmatrix}13+2t\\10t-5\\16t+4\end{bmatrix} \quad 2\begin{bmatrix}5\\8t-3\\6t+4\end{bmatrix} \quad 3\begin{bmatrix}0\\-1+6t\\3+2t\end{bmatrix}$$

$$2\begin{bmatrix}6t+4\\5\\8t-3\end{bmatrix} \quad \gamma\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}10t-5\\16t+4\\13+2t\end{bmatrix}$$

$$\begin{bmatrix}16t+4\\13+2t\\10t-5\end{bmatrix} \quad 2\begin{bmatrix}8t-3\\6t+4\\5\end{bmatrix}$$

$$3\begin{bmatrix}-1+6t\\3+2t\\0\end{bmatrix}$$

With $\delta := 5(18+22t)/(3+4t)$, the smoothness is characterized by

$$D_1p(u,0) = 2\begin{bmatrix}1/5-2/5t & -1\\ 1+s & 6/5t-3/5-s\\ 3/5-1/5t & -1/5-3/5t\end{bmatrix} \quad \text{and}$$

$$D_2p(u,0) = 3\frac{1}{15}\begin{bmatrix}2+6t & \delta-14-2t & 10s\\ 10 & \delta-10-10s & 10t-10s\\ 6+10s-12t & \delta-12-16t & -10t\end{bmatrix}$$

and symmetrically for $D_2q(u,0)$ so that $D_2p(u,0)+D_2q(u,0) = [s(1-u)+(2-s)u]D_1p(u,0)$. The surface is convex.

§7. Remarks and extensions.

It may be argued that the spheroids are a local curvature continuous construction of minimal degree since non-flat polynomial boundary curves need to be at least quadratic, and the patches require just one additional inner vertex.

With a few minor exceptions, the formulas presented here were derived, checked and typeset using symbolic programs. In particular, the scaling of the central patch coefficients was determined using the determinant condition, $\det[D_1p, D2p, D2q] = 0$, thus confirming first order continuity independently from the smoothness proof by reparametrization.

Scaled versions of the dodecoid interpolate the vertices of any Platonic solid. The spheroid properties are preserved under global affine transformations.

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