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The 42 Equivalence Classes of Quadratic Surfaces in Affine n-Space

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Abstract

We establish an exhaustive catalogue of 42 equivalence classes of bivariate quadratic surfaces in affine n-space. Here two surfaces belong to the same class if they differ only by zero components or by affine transformations of domain and range.

1 Introduction

Recently, elegant characterizations of quadratic surfaces in *projective* 3-space were given by Degen [Deg96], and independently by Coffman, Schwartz and Stanton [CSS96]. The subject of this report is a classification of quadratic surfaces in *affine* n-space. We establish an exhaustive catalogue of 42[1] equivalence classes, where polynomials are identified that differ only by zero components, or can be obtained from one another by a nonsingular *affine transformations* of domain and range. So, in contrast to the approach chosen in [Deg96], two surfaces may be considered different even if they admit locally the same implicit representation.

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[1]The specific finite number of equivalence classes gives a nontrivial answer (resp. question) to the famous question (resp. answer) presented in D. ADAM's book *"The Hitchhiker's Guide to the Galaxy"*.

The validity of separate equivalence classes is established by associating with each class a unique set of *invariants*. Completeness is shown constructively by explicit specification of transformations for arbitrary quadratic maps.

The case of bivariate quadratic maps, as discussed here, is special in the sense that it decomposes into a *finite* number of equivalence classes. Comparing the number of system parameters and the degrees of freedom provided by affine transformations of domain and range shows that this cannot be expected for polynomial maps in more variables or of higher degree.

2 Definitions and Notations

For A some matrix denote by $A_{p:q,r:s}$ the sub-matrix obtained by selecting all rows and columns with indices in $[p,q]$ and $[r,s]$, respectively. In sequential form A is specified line by line, separated by semi-colons. *Example:* Let $A := [1,3,0;2,0,1]$ then

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix} , \quad A_{1:2,2:3} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.1}$$

With M an $(n \times 5)$-matrix, m an n-vector, and $\underline{u} := (u^2, v^2, uv, u, v)^T$ the general form of a quadratic map $q := \mathbb{R}^2 \mapsto \mathbb{R}^n$ is

$$q(u,v) = M\underline{u} + m . \tag{2.2}$$

The set of all quadratic maps is denoted by $\mathcal{Q}$. The sub-matrices of M corresponding to the linear and quadratic monomials are

$$L := M_{1:n,4:5} , \quad Q := M_{1:n,1:3} . \tag{2.3}$$

An affine *change of coordinates* in $\mathbb{R}^n$ is given by

$$\tilde{x} = Ax + a , \tag{2.4}$$

where A is a regular $(n \times n)$-matrix and a is an n-vector. An affine *change of parameters* is expressed in terms of a (2×3)-matrix R according to

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = R \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} , \tag{2.5}$$

where R is *regular* in the sense that $\det R_{1:2,1:2} \neq 0$. With $R := [a, b, c; d, e, f]$ the induced change of $\underline{u}$ is given by

$$\tilde{\underline{u}} = \tilde{R}\begin{pmatrix}\underline{u}\\1\end{pmatrix} := \begin{pmatrix} a^2 & b^2 & 2ab & 2ac & 2bc & c^2 \\ d^2 & e^2 & 2de & 2df & 2ef & f^2 \\ ad & be & ae+bd & af+cd & bf+ce & cf \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e & f \end{pmatrix}\begin{pmatrix}\underline{u}\\1\end{pmatrix} . \tag{2.6}$$

Note that $\det \tilde{R}_{1:5,1:5} = (ae - bd)^4$, so $\operatorname{rank} \tilde{R} = 5$ by regularity of R.

The equivalence relation in $\mathbb{R}^2$ induced by (2.5) is denoted by E_2. Let $\mathcal{C}$ be the quotient set of conic sections by E_2. The elements of $\mathcal{C}$ are assigned the following icons:

$\circ$	ellipse
$\cdot$	point
$\emptyset$	empty set

$\asymp$	hyperbola
$\smile$	parabola
$\square$	plane

$\times$	crossing lines
$=$	parallel lines
$-$	single line

Further, we introduce two icons for the equivalence classes of the following point sets:

$\cdot\cdot$	two distinct points

$\cdot\cdot\cdot$	three distinct points

.

3 Equivalence classes

In Section 3.1 we define an equivalence relation E_Q in the space of quadratics. A set of invariants is specified in Section 3.2. Finally, in Section 3.3 all 42 equivalence classes of $\mathcal{Q}$ with respect to E_Q are listed and characterized in terms of invariants.

3.1 The equivalence relation

The space $\mathcal{Q}$ is endowed with an equivalence relation E_Q which identifies quadratic maps that differ only by affine transformations of domain and range or by zero coordinates. More precisely, two maps $q_i = M_i\underline{u} + m_i$ in $\mathbb{R}^{n_i}$, $i \in \{1, 2\}$ are equivalent if

i) $n_1 = n_2$ and there exists a vector $a \in \mathbb{R}^{n_1}$ such that

$$q_1 = q_2 + a . \tag{3.1}$$

ii) $n_1 = n_2$ and there exist regular matrices A and R corresponding to affine changes of coordinates and parameters such that

$$M_1 = AM_2\tilde{R} . \tag{3.2}$$

iii) $n_1 \neq n_2$ and

$$M_2 = \begin{pmatrix} M_1 \\ 0 \end{pmatrix} \quad \text{or} \quad M_1 = \begin{pmatrix} M_2 \\ 0 \end{pmatrix} . \tag{3.3}$$

It is our goal to determine the equivalence classes of Q with respect to the equivalence relation E_Q generated by *i), ii), iii)*. Each equivalence class will be associated with a particularly simple representative, also referred to as *normal form.*

Throughout, relation *i)* is applied so that $q(0,0) = 0$. Further, we may assume that the rows of M are linearly independent or that $M = [0,0,0,0,0]$. Otherwise, if $\operatorname{rank} M = n' < n$ there is a change of coordinates A such that the first n' rows of AM are linearly independent and the remaining rows are all zero. Then the zero rows can be deleted by relation *iii)*. To sum up, all quadratic maps to be considered subsequently are either trivial, i.e. $q(u,v) = 0$, or of type $q(u,v) = M\underline{u}$, where the rows of M are linearly independent. In particular, the maximal number of components of q is 5.

3.2 Invariants

In order to characterize the equivalence classes of Q we specify a set of properties which are invariant with respect to E_Q. With a slight abuse of notation, R denotes both the matrix as defined in (2.5) and the induced transformation of $\mathbb{R}^2$.

- $\operatorname{rank} M$ is invariant since A and $\tilde{R}$ are regular matrices.
- $\operatorname{rank} Q$ is invariant since A and $\tilde{R}_{1:3,1:3}$ are regular matrices.
- The singular set σ_q of a quadratic map q is defined by

$$\sigma_q := \{(u,v) \in \mathbb{R}^2 : \operatorname{rank}[q_u(u,v), q_v(u,v)] < 2\} . \tag{3.4}$$

It turns out that σ_q is either a conic section, or consists of 2 or 3 distinct points. Denote equivalence classes with respect to E_2 by $[\cdot]_{E_2}$, then the set

$$\Sigma_q := [\sigma_q]_{E_2} \in \mathcal{C} \cup \{\cdot\cdot, \cdot\cdot\cdot\} \tag{3.5}$$

is invariant, since $\sigma_{Aq\circ R+a} = R^{-1}\sigma_q$.

- Let $\lambda \in \mathbb{R}^n \backslash \{0\}$ and $\mu \in \mathbb{R}$. The preimage of the intersection of a quadratic map $q \in \mathcal{Q}$ in $\mathbb{R}^n$ with the hyperplane $h : \lambda^T x = \mu$ is a conic section determined by

$$\pi_q(\lambda, \mu) := \{(u, v) \in \mathbb{R}^2 : \lambda^T q(u, v) = \mu\} . \tag{3.6}$$

  The set of all possible types of conic sections,

$$\Pi_q := \{[\pi_q(\lambda, \mu)]_{E_2} : \lambda \in \mathbb{R}^n \backslash \{0\}, \mu \in \mathbb{R}\} \subset \mathcal{C} , \tag{3.7}$$

  is invariant, since $\pi_{Aq\circ R+a}(\lambda, \mu) = R^{-1}\pi_q(A^T\lambda, \mu - \lambda^T a)$.

- The set of different types of preimages of hyperplanes which intersect the singular set,

$$\Pi_q^* := \{[\pi_q(\lambda, \mu)]_{E_2} : \pi_q(\lambda, \mu) \cap \sigma_q \neq \emptyset, \lambda \in \mathbb{R}^n \backslash \{0\}, \mu \in \mathbb{R}\} \subset \mathcal{C} , \tag{3.8}$$

  is invariant, since $\pi_{Aq\circ R+a}(\lambda, \mu) \cap \sigma_{Aq\circ R+a} = R^{-1}(\pi_q(A^T\lambda, \mu - \lambda^T a) \cap \sigma_q)$.

3.3 Result

Tables 1 and 2 provide a complete classification of quadratic maps in $\mathbb{R}^n$ with respect to the given equivalence relation E_Q. Each class is uniquely characterized by a set of invariants. The first column contains a name for the equivalence class, coded by

$$\text{Type} = \operatorname{rank} M - \operatorname{rank} Q - \text{auxillary label} .$$

The second column shows the normal form, i.e. a particularly simple representative of the equivalence class. The next eight columns indicate whether a particular type of conic section is contained in Π_q. The type of the singular set Σ_q is specified in the last column. With two exceptions, $\operatorname{rank} M, \operatorname{rank} Q, \Pi_q, \Sigma_q$ characterize the class. Only Types 2-1-2a,b and 3-3-2a,b require a further distinction. Here and only here the elements of Π_q^* are marked by a square (▪). Plots of the normal forms in $\mathbb{R}^2$ and $\mathbb{R}^3$ are depicted in Figures 1 and 2.

Type	$q(u,v)$	$\circ$	$\cdot$	$\emptyset$	$\asymp$	$\times$	$\smile$	$=$	$-$	Σ_q
0-0	0			•						□
1-0	u								•	□
1-1-1	u^2+v^2	•	•	•						□
1-1-2	u^2-v^2				•	•				□
1-1-3a	u^2+v						•			□
1-1-3b	u^2			•				•	•	□
2-0	u,v								•	$\emptyset$
2-1-1	u^2+v^2,u	•	•	•					•	$-$
2-1-2a	$uv,u+v$				■	■			■	$-$
2-1-2b	uv,u				•	■			■	$-$
2-1-3a	u^2,v			•			•	•	•	$-$
2-1-3b	u^2,u			•				•	•	□
2-1-3c	u^2+v,u						•		•	$\emptyset$
2-2-1a	u^2,v^2	•	•	•	•	•		•	•	$\times$
2-2-1b	u^2+v,v^2+u	•	•	•	•	•	•			$\asymp$
2-2-1c	u^2+v,v^2	•	•	•	•	•	•	•	•	$\times$
2-2-2a	u^2,uv			•	•	•		•	•	$-$
2-2-2b	u^2+v,uv				•	•	•			$\smile$
2-2-2c	$u^2,uv+v$			•	•	•		•	•	$=$
2-2-3a	u^2-v^2,uv				•	•				$\cdot$
2-2-3b	$u^2-v^2,uv+u$				•	•				$\circ$

Table 1: Equivalence classes for rank $M=0,1,2$.

Example Let us briefly consider Cases 2-1-2a and 2-1-2b.

The normal form of Case 2-1-2a is given by $q(u,v)=(uv,u+v)$. Here, $M=[0,0,1,0,0;0,0,0,1,1]$ and $Q=[0,0,1;0,0,0]$, hence rank $M=2$ and rank $Q=1$. The Jacobian of q is $J(u,v)=[v,u;1,1]$, so the singular set is $\sigma_q=\{(u,v):u=v\}$ and $\Sigma_q=\{-\}$. Further, $\pi_q(\lambda,\mu)$ is defined by $\lambda_1 uv+\lambda_2(u+v)=\mu$. The quadratic form is negative definite or zero, so only hyperbola, crossing lines and single line come into account for Π_q. Indeed, all three types occur, for instance $uv=1, uv=0, u+v=0$. All three specimen intersect the singular line $u=v$, hence $\Pi_q^*=\Pi_q$.

The normal form of Case 2-1-2b is given by $q(u,v)=(uv,u)$. Here,

Type	$\mathbf{q}(u,v)$	$\circ$	$\cdot$	$\emptyset$	$\asymp$	$\times$	$\smile$	$=$	—	Σ
3-1-1	u^2+v^2, u, v	●	●	●					●	$\emptyset$
3-1-2	uv, u, v				●	●			●	$\emptyset$
3-1-3	u^2, u, v			●			●	●	●	$\emptyset$
3-2-1a	$u^2, v^2, u+v$	●	●	●	●	●	●	●	●	$\cdot$
3-2-1b	u^2, v^2, u	●	●	●	●	●	●	●	●	—
3-2-1c	u^2, v^2+u, v	●	●	●	●	●	●	●	●	$\emptyset$
3-2-2a	u^2, uv, v			●	●	●	●	●	●	$\cdot$
3-2-2b	u^2+v, uv, u				●	●	●		●	$\emptyset$
3-2-2c	u^2, uv, u			●	●	●		●	●	—
3-2-3	u^2-v^2, uv, u				●	●			●	$\cdot$
3-3-1a	u^2, v^2, uv	●	●	●	●	●		●	●	$\cdot$
3-3-1b	$u^2, v^2, uv+u$	●	●	●	●	●	●	●	●	$\cdot\cdot$
3-3-1c	$u^2, v^2, uv+u+v$	●	●	●	●	●	●	●	●	$\cdot\cdot\cdot$
3-3-2a	u^2, v^2+u, uv	■	●	●	■	■	■	●	■	$\cdot$
3-3-2b	$u^2, v^2+u, uv-v$	■	●	●	■	■	■	■	●	$\cdot$
4-2-1	u^2, v^2, u, v	●	●	●	●	●	●	●	●	$\emptyset$
4-2-2	u^2, uv, u, v			●	●	●	●	●	●	$\emptyset$
4-2-3	u^2-v^2, uv, u, v				●	●			●	$\emptyset$
4-3-1a	u^2, v^2, uv, u	●	●	●	●	●	●	●	●	$\cdot$
4-3-1b	u^2+v, v^2, uv, u	●	●	●	●	●	●	●	●	$\emptyset$
5-3	u^2, v^2, uv, u, v	●	●	●	●	●	●	●	●	$\emptyset$

Table 2: Equivalence classes for rank $M = 3, 4, 5$.

$M = [0,0,1,0,0;0,0,0,1,0]$ and $Q = [0,0,1;0,0,0]$, hence $\operatorname{rank} M = 2$ and $\operatorname{rank} Q = 1$. The Jacobian of q is $J(u,v) = [v,u;1,0]$, so the singular set is $\sigma_q = \{(u,v) : u = 0\}$ and $\Sigma_q = \{—\}$. Further, $\pi_q(\lambda,\mu)$ is defined by $\lambda_1 uv + \lambda_2 u = \mu$. The quadratic form is negative definite or zero, so only hyperbola, crossing lines and single line come into account for Π_q. Indeed, all three types occur, for instance $uv = 1, uv = 0, u = 0$. Again, the specimen for crossing lines and single line intersect the singular line $u = 0$. However, substituting $u = 0$ into $\lambda_1 uv + \lambda_2 u = \mu$ yields $\mu = 0$, and the equation $u(\lambda_1 v + \lambda_2) = 0$ cannot describe a hyperbola, hence $\Pi_q^* = \{\times, —\}$.

It remains to show that the list of equivalence classes is exhaustive. This is done in a constructive way by showing how to transform an arbitrary quadratic map to one of the given normal forms. Being somewhat technical, this part is postponed to the Appendix.

Appendix

View of the considerable number of cases to be treated, the various steps are only outlined. Verification and completion of the details is straightforward, especially when supported by a computer algebra system. For the sake of clarity, the section labels are chosen such that the first two numbers coincide with $\operatorname{rank} M$ and $\operatorname{rank} Q$, respectively.

A.1 Quadratic maps in $\mathbb{R}$

A.1.0 $\operatorname{rank} Q = 0$

$\operatorname{rank} L = 0$ yields Type 0-0. $\operatorname{rank} L = 1$ yields Type 1-0 by a straightforward change of parameters.

A.1.1 $\operatorname{rank} Q = 1$

Types 1-1-1, 1-1-2, 1-1-3a and 1-1-3b are obtained by the standard main axes transformation and quadratic completion.

A.2 Quadratic maps in $\mathbb{R}^2$

A.2.0 $\operatorname{rank} Q = 0$

$\operatorname{rank} Q = 0$ implies $\operatorname{rank} L = 2$. Thus, setting $A := L^{-1}$ yields Type 2-0.

A.2.1 $\operatorname{rank} Q = 1$

Change coordinates so that the second row of Q is zero. Then transform the first row to a one-dimensional normal form with $\operatorname{rank} Q = 1$. Note that the second row of L is non-zero since $\operatorname{rank} M = 2$.

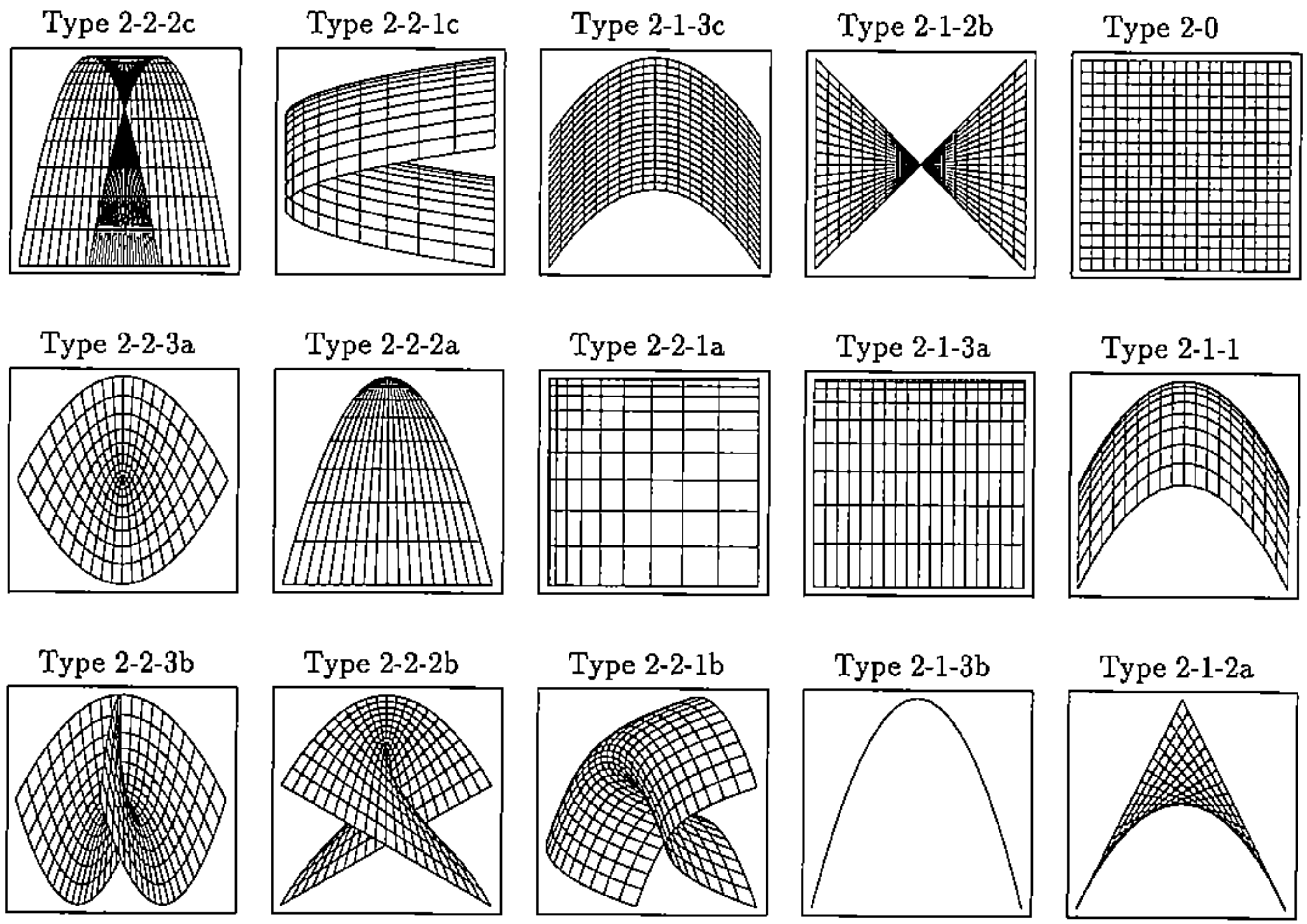

Figure 1: Quadratic maps in $\mathbb{R}^2$

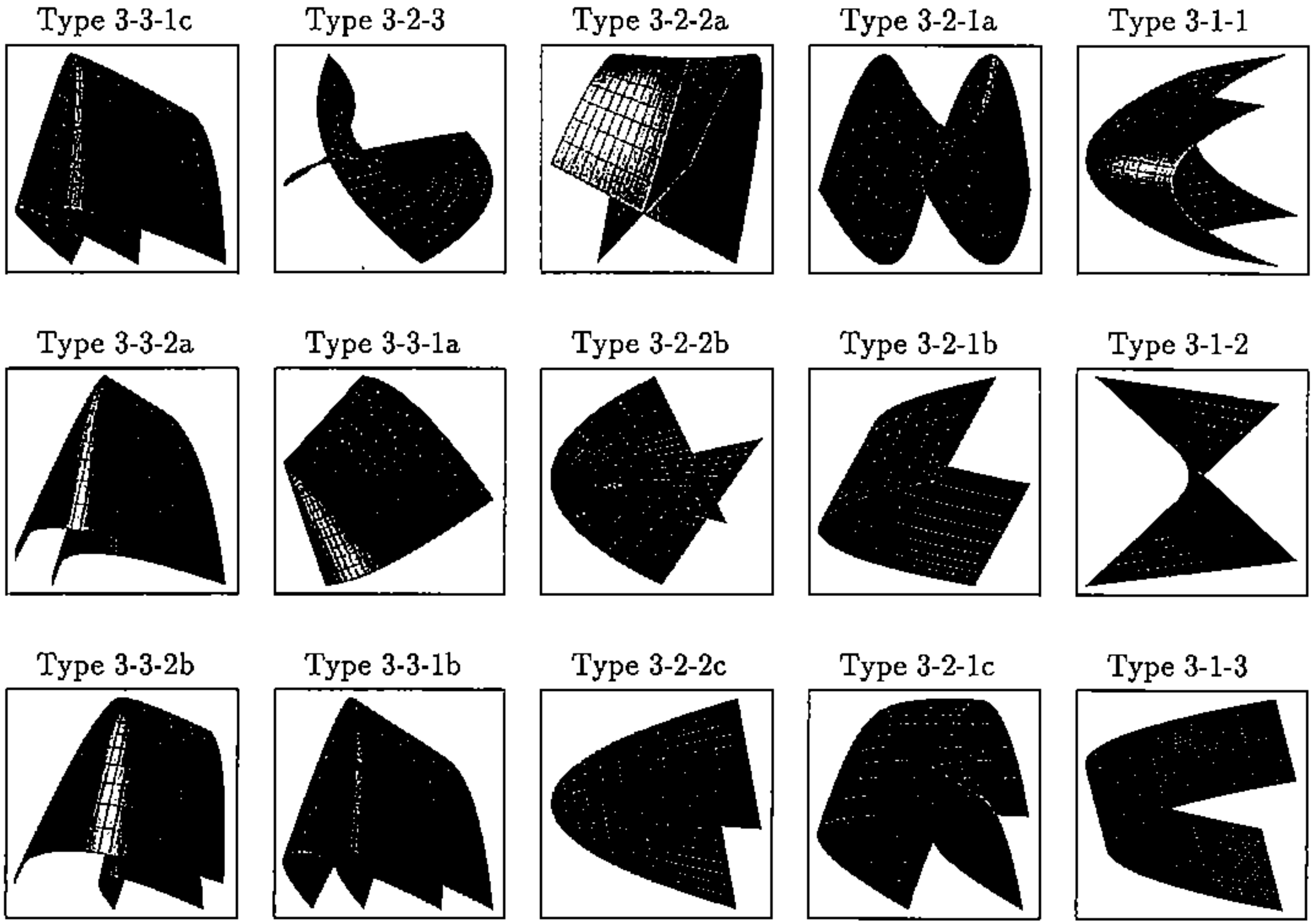

Figure 2: Quadratic maps in $\mathbb{R}^3$

A.2.1.1 First row of Type 1-1-1

The matrix

$$M := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q \end{bmatrix} \tag{A.2.1}$$

is transformed to Type 2-1-1 by means of

$$R := \frac{1}{p^2+q^2} \begin{bmatrix} p & q & 0 \\ q & -p & 0 \end{bmatrix}, \quad A := \begin{bmatrix} p^2+q^2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A.2.2}$$

A.2.1.2 First row of Type 1-1-2

The matrix

$$M := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p & q \end{bmatrix} \tag{A.2.3}$$

is transformed in dependence on p, q.

a) $pq \neq 0$ yields Type 2-1-2a by scaling.

b) $p = 0, q \neq 0$ and $p \neq 0, q = 0$ both yield Type 2-1-2b by scaling and eventually exchanging (u, v).

A.2.1.3 First row of Type 1-1-3x

The matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & p & q \end{bmatrix}, \quad r \in \{0, 1\} \tag{A.2.4}$$

is transformed in dependence on p, q, r.

a) $q \neq 0$ yields Type 2-1-3a by means of

$$R := \begin{bmatrix} 1 & 0 & rp/2q \\ -p/q & 1/q & 0 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & -r/q \\ 0 & 1 \end{bmatrix}. \tag{A.2.5}$$

b) $p \neq 0, q = 0, r = 0$ yields Type 2-1-3b by scaling.

b) $p \neq 0, q = 0, r = 1$ yields Type 2-1-3c by scaling.

A.2.2 $\operatorname{rank} Q = 2$

Denote the j-th column of Q by Q_j. We may assume that $\det[Q_1, Q_2] \neq 0$. Otherwise, if $\det[Q_1, Q_2] = 0$ then $\det[Q_1, Q_3] \neq 0$. Reparametrization with $R := [1, 1, 1;\ 0, 1, 0]$ yields $\tilde{Q} = [Q_1, Q_1 + Q_2 + Q_3, 2Q_1 + Q_3]$ and $\det[\tilde{Q}_1, \tilde{Q}_2] \neq 0$. Changing coordinates with $A := [Q_1, Q_2]^{-1}$ gives

$$Q = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}. \tag{A.2.6}$$

A.2.2.1 $pq < 1$

We may assume that $p = q = 0$. Otherwise, if $p \neq 0$ set $w := \sqrt{1 - pq}$ and

$$R := \begin{bmatrix} p & p & 0 \\ -1 - w & -1 + w & 0 \end{bmatrix}, \quad A := \frac{1}{(2pw)^2} \begin{bmatrix} (1-w)^2 & p^2 \\ (1+w)^2 & p^2 \end{bmatrix} \tag{A.2.7}$$

yielding $Q = [1, 0, 0;\ 0, 1, 0]$. If $p = 0, q \neq 0$ exchange both (u, v) and (x_1, x_2) and proceed as before. So, after quadratic completion, we have to discuss

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & s & 0 \end{bmatrix} \tag{A.2.8}$$

in dependence on r, s.

a) $r = s = 0$ yields Type 2-2-1a.

b) $r \neq 0, s \neq 0$ yields Type 2-2-1b by scaling.

c) $r = 0, s \neq 0$ and $r \neq 0, s = 0$ both yield Type 2-2-1c by scaling and eventually exchanging (u, v) and (x_1, x_2).

A.2.2.2 $pq = 1$

By scaling, we can obtain $p = q = 1$. Then setting

$$R := \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{A.2.9}$$

yields $Q = [1, 0, 0;\ 0, 0, 1]$. So, after quadratic completion, we have to discuss

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 & s \end{bmatrix} \tag{A.2.10}$$

in dependence on r, s.

a) $r = s = 0$ yields Type 2-2-2a.

b) $r \neq 0$ yields Type 2-2-2b by means of

$$R := \begin{bmatrix} 1 & 0 & -s/3 \\ 2s/3r & 1/r & -4s^2/9r \end{bmatrix} , \quad A := \begin{bmatrix} 1 & 0 \\ -2s/3 & r \end{bmatrix} . \tag{A.2.11}$$

c) $r = 0, s \neq 0$ yields Type 2-2-2c by scaling.

A.2.2.3 $pq > 1$

By scaling, we can obtain $p = q > 1$. Then setting $w := \sqrt{p^2 - 1}$ and

$$R := \begin{bmatrix} 0 & -p/w & 0 \\ 1 & 1/w & 0 \end{bmatrix} , \quad A := \begin{bmatrix} 2/p^2 - 1 & 1 \\ -w/p^2 & 0 \end{bmatrix} \tag{A.2.12}$$

yields $Q = [1, -1, 0;\ 0, 0, 1]$. So, after quadratic completion, we have to discuss

$$M := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & r & s \end{bmatrix} \tag{A.2.13}$$

in dependence on r, s.

a) $r = s = 0$ yields Type 2-2-3a.

b) $r^2 + s^2 > 0$ yields Type 2-2-3b as follows: We may assume that $r \neq 0$. Otherwise, exchange (u, v) and replace x_1 by $-x_1$. Define the polynomials

$$\begin{aligned} f(x) &:= rx^3 + 3sx^2 - 3rx - s \\ g(x) &:= sx^3 - 3rx^2 - 3sx + r\,. \end{aligned} \tag{A.2.14}$$

and choose κ so that $f(\kappa) = 0$. Note that $g(\kappa) \neq 0$ since the resultant of f and g is $\operatorname{res}(f, g) = -64(r^2 + s^2) \neq 0$. Then, the transformation matrices are

$$\begin{aligned} R &:= \frac{1}{(1+\kappa^2)^2} \begin{bmatrix} -g(\kappa) & \kappa g(\kappa) & 2\kappa(-r\kappa^2 - 2s\kappa + r) \\ \kappa g(\kappa) & g(\kappa) & 2\kappa(s\kappa^2 - 2r\kappa - s) \end{bmatrix} \\ A &:= -\frac{g(\kappa)^2}{(1+\kappa^2)^2} \begin{bmatrix} \kappa^2 - 1 & 4\kappa \\ \kappa & 1 - \kappa^2 \end{bmatrix} . \end{aligned} \tag{A.2.15}$$

A.3 Quadratic maps in $\mathbb{R}^3$

A.3.1 $\operatorname{rank} Q = 1$

$\operatorname{rank} Q = 1$ implies $\operatorname{rank} L = 2$. Change coordinates so that the second and third row of Q are zero. Transform the first row to a one-dimensional normal form with $\operatorname{rank} Q = 1$. Change coordinates so that $L = [0,0;\ 1,0;\ 0,1]$. Thus,

- First row of Type 1-1-1 yields Type 3-1-1.
- First row of Type 1-1-2 yields Type 3-1-2.
- First row of Type 1-1-3a or Type 1-1-3b yields Type 3-1-3.

A.3.2 $\operatorname{rank} Q = 2$

Change coordinates so that the third row of Q is zero. Then transform the first two rows to a two-dimensional normal form with $\operatorname{rank} Q = 2$. Note that the third row of L is non-zero.

A.3.2.1 First two rows of Type 2-2-1x

The matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & s & 0 \\ 0 & 0 & 0 & p & q \end{bmatrix}, \quad (r,s) \in \{(0,0),(1,1),(1,0)\} \tag{A.3.1}$$

is transformed in dependence on p, q, r, s.

a) $p \neq 0, q \neq 0$ yields Type 3-2-1a by means of

$$R := \begin{bmatrix} 1/p & 0 & rp/2q \\ 0 & 1/q & sq/2p \end{bmatrix}, \quad A := \begin{bmatrix} p^2 & 0 & -rp^2/q \\ 0 & q^2 & -sq^2/p \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.3.2}$$

b) $p = r = s = 0, q \neq 0$, $q = r = s = 0, p \neq 0$, and $p = s = 0, r = 1, q \neq 0$ all yield Type 3-2-1b by scaling, eventually exchanging (u, v), and eventually applying a straightforward change of coordinates.

c) $r = s = 1, p = 0, q \neq 0$, $r = s = 1, p \neq 0, q = 0$, and $q = s = 0, r = 1, p \neq 0$ all yield Type 3-2-1c by scaling, eventually exchanging (u, v), and eventually applying a straightforward change of coordinates.

A.3.2.2 First two rows of Type 2-2-2x

The matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 & s \\ 0 & 0 & 0 & p & q \end{bmatrix} \quad , \quad (r,s) \in \{(0,0),(1,0),(0,1)\} \tag{A.3.3}$$

is transformed in dependence on p, q, r, s.

a) $q \neq 0$ yields Type 3-2-2a by means of

$$R := \begin{bmatrix} 1 & 0 & rp/2q \\ -p/q & 1/q & p(rp+2sq)/2q^2 \end{bmatrix}$$
$$A := \begin{bmatrix} 1 & 0 & -r/q \\ p & q & -(3rp+2sq)/2q \\ 0 & 0 & 1 \end{bmatrix} . \tag{A.3.4}$$

b) $q = s = 0, p \neq 0, r = 1$ yields Type 3-2-2b by scaling.

c) $q = r = 0, p \neq 0$ yields Type 3-2-2c by means of

$$R := \begin{bmatrix} 1/p & 0 & -s \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad A := \begin{bmatrix} p^2 & 0 & 2sp \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{A.3.5}$$

A.3.2.3 First two rows of Type 2-2-3x

The matrix

$$M := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & r & 0 \\ 0 & 0 & 0 & p & q \end{bmatrix} \quad , \quad r \in \{0,1\} \tag{A.3.6}$$

is transformed to Type 3-2-3 by

$$R := \frac{1}{p^2+q^2} \begin{bmatrix} p & q & pqr \\ q & -p & -q^2 r \end{bmatrix}$$
$$A := \begin{bmatrix} p^2-q^2 & 4pq & 2qr(q^2-3p^2)/(p^2+q^2) \\ pq & q^2-p^2 & rp(p^2-3q^2)/(p^2+q^2) \\ 0 & 0 & 1 \end{bmatrix} . \tag{A.3.7}$$

A.3.3 $\operatorname{rank} Q = 3$

Changing coordinates by $A := Q^{-1}$ and quadratic completion yields

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & s & 0 \\ 0 & 0 & 1 & p & q \end{bmatrix} . \tag{A.3.8}$$

A.3.3.1 $(r, s) = (0, 0)$

a) $p = q = 0$ yields Type 3-3-1a by scaling.

b) $p = 0, q \neq 0$ and $p \neq 0, q = 0$ both yield Type 3-3-1b by scaling and eventually exchanging (u, v) and (x_1, x_2).

c) $pq \neq 0$ yields Type 3-3-1c by scaling.

A.3.3.2 $(r, s) \neq (0, 0)$

We may assume that $r \neq 0$. Otherwise, exchange (u, v) and (x_1, x_2). Let

$$f(x) := rx^3 - 2qx^2 - 2px + s \tag{A.3.9}$$

and choose κ, λ so that $f(\kappa) = 1$ and $f(\lambda) = 0$. Thus, $h := \lambda - \kappa \neq 0$. Applying

$$R := \begin{bmatrix} 1 & 1/h & 0 \\ \kappa & \lambda/h & 0 \end{bmatrix} , \quad A := \begin{bmatrix} \lambda^2/h^2 & 1/h^2 & -2\lambda/h^2 \\ \kappa^2 & 1 & -2\kappa \\ -\kappa\lambda/h & -1/h & (\lambda + \kappa)/h \end{bmatrix} . \tag{A.3.10}$$

and subsequent quadratic completion yields

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & p & q \end{bmatrix} , \tag{A.3.11}$$

which is transformed in dependence on the new values for p, q.

a) $p = q = 0$ yields Type 3-3-2a.

b) $p^2+2q<0$ yields Type 3-3-2b by means of $w := \sqrt{-p^2/2-q}$ and

$$R := \begin{bmatrix} -q & 0 & 0 \\ p/2 & w & -p/2 \end{bmatrix}$$
$$A := \begin{bmatrix} 1/q^2 & 0 & 0 \\ (p/2qw)^2 & 1/w^2 & p/qw^2 \\ -p/2wq^2 & 0 & -1/wq \end{bmatrix}. \tag{A.3.12}$$

c) $p^2+2q \geq 0, (p,q) \neq (0,0)$ yields no new type but either Type 3-3-1b or 3-3-1c as follows: Let

$$g(x) := x^2 - 2px - 2q \tag{A.3.13}$$

and chose a root $\kappa := p \pm \sqrt{p^2+2q}$ so that $\kappa \neq 0$. Then

$$R := \begin{bmatrix} \kappa & 0 & 0 \\ 1 & 1 & q/\kappa \end{bmatrix}, \quad A := \begin{bmatrix} 1/\kappa^2 & 0 & 0 \\ 1/\kappa & 1 & -2/\kappa \\ -1/\kappa & 0 & 1/\kappa \end{bmatrix} \tag{A.3.14}$$

yields

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2q/\kappa+p & q/\kappa \end{bmatrix}. \tag{A.3.15}$$

Obviously, $(2q/\kappa+p, q/\kappa) \neq (0,0)$ for $(p,q) \neq (0,0)$.

A.4 Quadratic maps in $\mathbb{R}^4$

A.4.2 $\operatorname{rank} Q = 2$

$\operatorname{rank} Q = 2$ implies $\operatorname{rank} L = 2$. Change coordinates so that the third and fourth row of Q are zero. Now, the third and fourth row of L are linearly independent and L can be transformed to $L = [0,0;\ 0,0;\ 1,0;\ 0,1]$ by an affine change of coordinates without altering Q.

- First two rows of Type 2-2-1x yields Type 4-2-1.
- First two rows of Type 2-2-2x yields Type 4-2-2.
- First two rows of Type 2-2-3x yields Type 4-2-3.

A.4.3 $\operatorname{rank} Q = 3$

We may assume that the first four columns of M are linearly independent. Otherwise, exchange (u, v). Thus, by an affine change of coordinates M can be transformed to

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & p \\ 0 & 1 & 0 & 0 & q \\ 0 & 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 & s \end{bmatrix} . \tag{A.4.1}$$

Applying

$$R := \begin{bmatrix} 1 & -s & -r - sq/2 \\ 0 & 1 & -q/2 \end{bmatrix} , \quad A := \begin{bmatrix} 1 & s^2 & 2s & 2(r + sq) \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & q/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.4.2}$$

yields

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & p \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{A.4.3}$$

with a new value for p.

a) $p = 0$ yields Type 4-3-1a.

b) $p \neq 0$ yields Type 4-3-1b by scaling.

A.5 Quadratic maps in $\mathbb{R}^5$

Since $\operatorname{rank} M = 5$, Type 5-3 can be obtained by an affine change of coordinates.

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