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Abstract

Several efficient constructions of smooth surfaces following the outlines of a polyhedral mesh are based on three-sided patches. To demonstrate that these constructions are compatible with existing software based on tensor-product patches, the particular scheme in [3] is expressed in terms of linearly-trimmed bicubic patches. Explicit formulas relating the coefficients of the patches to the vertices of an arbitrary input polyhedron are given. Four of these patches can be grouped together into a NURBS surface.

1 Introduction

A main criticism leveled at many new and old surfacing schemes, for example rational blending schemes [2], generalized subdivision [1] and three-sided patches [3], is that they can not be represented exactly or efficiently in the dominant patch representation, tensor-product B-splines, which serves as the standard for storage, transmission and high-level rendering. To show that three-sided patches fit rather nicely into the B-spline standard, the author has worked out the details of representing the three-sided surface splines defined in [3] as linearly-trimmed bicubic patches in Bernstein-Bézier form. Figure 2 displays such a patch with its control net as part of a smooth surface. Figure 1 shows how four of the resulting patches can be grouped together into a single NURBS surface, a trick the author uses to interface with OpenGL routines. This paper gives the formulas defining the Bernstein-Bézier control net in terms of the input polyhedron.

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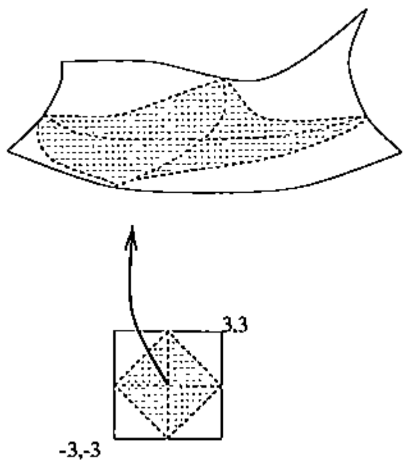

Figure 1: NURBS patch with trim region in the range and the domain. The formulas in Section 2.2 describe one quadrant of the patch in Bernstein-Bézier representation.

2 High-level description of the algorithm

Let A_i, C_i, and V_i be points in R^3 as shown in Figure 2. We first refine the polyhedron to a planar-cut polyhedron with coefficients C_i by applying a sparse linear map M_A. Then we map the planar-cut polyhedron to the bicubic patches with coefficient vector B using a second local linear map M_B. In short, the Bernstein-Bézier coefficients of the bicubic surface patches, B, are obtained from the input polyhedron, A, by

$$B = M_B C = M_B M_A A.$$

Since the matrices M_A and M_B have more rows than columns more points are output then input, and just storing A and the local components of M_A and M_B (see the sections below) is more space efficient than the expansion into Bernstein-Bézier or NURBS form.

2.1 Computing the planar-cut polyhedron: $C = M_A A$

The map M_A locally averages and projects the input polyhedral mesh to obtain a *planar-cut polyhedron*, i.e. a polyhedron such that every interior vertex is surrounded by four facets, the first and third of which are four-sided and the second and fourth are planar if they have more than four edges. For example, M_A can represent the following geometric construction

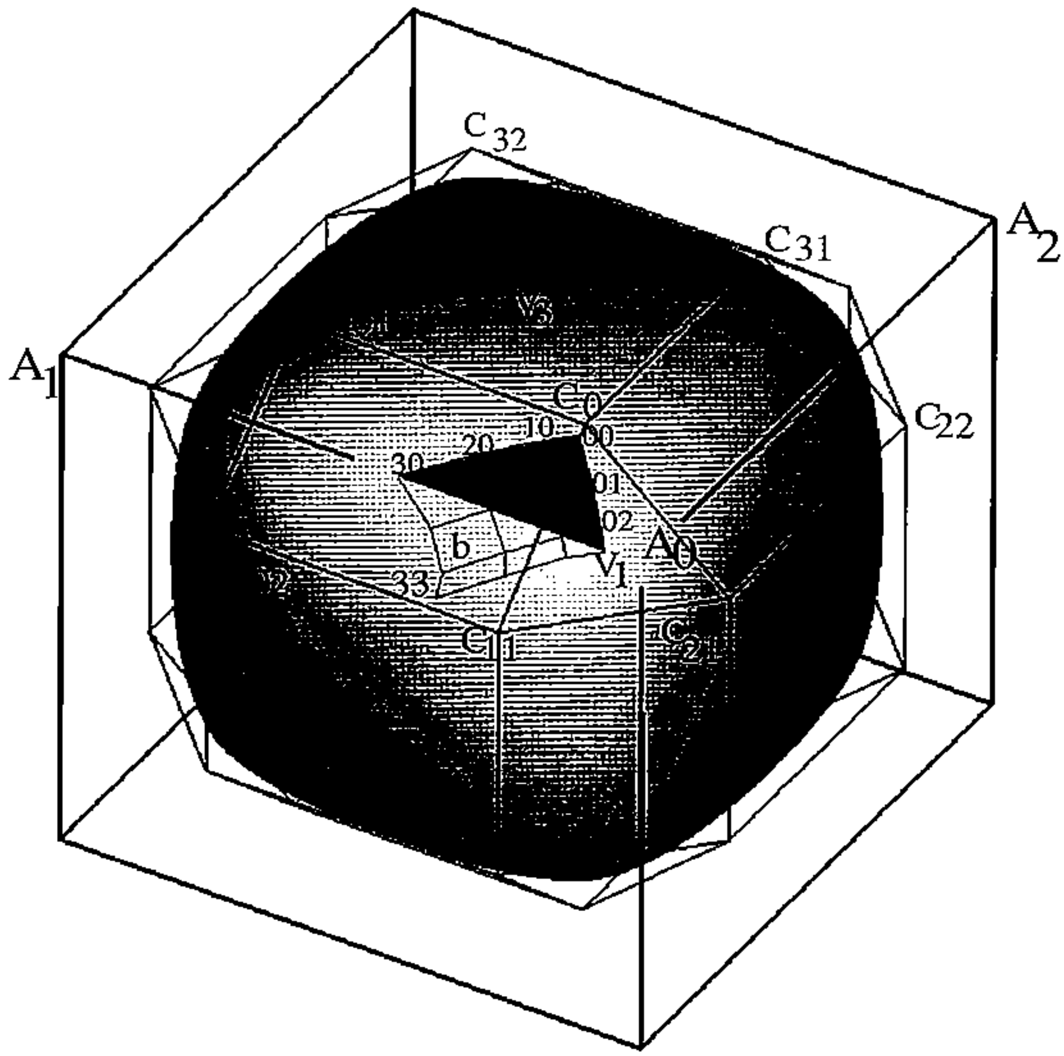

Figure 2: Naming conventions: A_i – vertices of the input polyhedron, C_i – vertices of the planar-cut polyhedron, and ij – indices of the bicubic patch. The trimmed patch is shaded purple (dark).

illustrated in Figure 2. First, averaging generates new mesh points by the rule

$$C_0' = A_0 + \frac{\alpha_1}{2}(A_1 - A_0) + \frac{\alpha_2}{2}(A_2 - A_0)$$

where the *blend ratios* $\alpha_i \in [0,1]$ are local to each face-vertex pair. As displayed in Figure 2, the points of type C are connected as a *dual mesh* such that C_0, C_{01}, C_{02}, C_{11} and C_0, C_{21}, C_{22}, C_{31} form quadrilaterals. The other two patches have n_1 and n_3 (here 3 and 4) edges and centroids V_1 and V_3. The points of type C are obtained from those of type C' for example by the projection

$$V := \frac{1}{n}\sum C_i',$$

$$C_j := \begin{cases} C_j' & \text{if } n < 5, \\ V + \frac{2}{n}\sum_i \cos(2\pi(i+j)/n)C_i', & \text{else} \end{cases} \qquad (*)$$

Note that the blend ratios are an intuitive handle for distributing curvature across the surface. Since they determine the placement of the points C_i on the original facet, small blend ratios correspond to small cuts of the input polyhedron and hence fast changes of the normal on the final surface.

2.2 Computing Bernstein-Bézier coefficients: $B = M_B C$

Where the mesh is *regular*, i.e. where all four subfacets abutting at a point C_0 are quadrilateral, the points

$$\begin{matrix} C_{32} & C_{31} & C_{22} \\ C_{01} & C_0 & C_{21} \\ C_{02} & C_{11} & C_{12} \end{matrix}$$

may be interpreted as the control points of a biquadratic (B-)spline. Here C_{32} and C_{12} are the points completing the quadrilateral with points C_{01}, C_0, C_{31} and C_{21}, C_0, C_{11} respectively. At non-regular vertices, a complex of four trimmed bicubics surrounding C_0 is generated that seemlessly, i.e. tangent continuously, integrates with any of the above biquadratic B-spline surfaces. The Bernstein-Bézier control net of one of the four patches is shown in Figure 2. A three-sided patch is obtained from the tensor-product patch by restricting the evaluation to $(u,v) \in [0,1]^2, u + v <= 1$. The vector of its Bernstein-Bézier coefficients $B_{3i+j} := b_{ij}$ is computed from the planar-cut polyhedron

$$C := [C_0, C_{01}, C_{02}, C_{11}, V_1, C_{21}, C_{22}, C_{31}, V_3]^T$$

as $B = M_B C$ where

$$M_B := \frac{1}{288}[M_1, M_2],$$

n_i is the number of $C's$ averaging to form V_i, $c_i := \cos(2\pi/n_i)$ and

$$M_1 := \begin{bmatrix}
156 - 6c_3 - 6c_1 & 18 - 3c_3 + 3c_1 & 6 + 3c_3 + 3c_1 & 18 - 3c_1 + 3c_3 \\
156 - 12c_1 & 6 + 6c_1 & 6 + 6c_1 & 30 - 6c_1 \\
96 & 0 & 0 & 48 \\
0 & 0 & 0 & 0 \\
156 - 6c_3 - 6c_1 & 36 - 6c_3 + 6c_1 & 12 + 6c_3 + 6c_1 & 36 - 6c_1 + 6c_3 \\
156 - 12c_1 & 11c_1 - c_3 + 16 & c_3 + 11c_1 + 12 & c_3 - 11c_1 + 56 \\
96 & 4c_1 + 4 & 4c_1 + 4 & 84 - 4c_1 \\
0 & 0 & 0 & 48 \\
120 & 72 & 24 & 72 \\
4c_3 - 4c_1 + 120 & 4c_3 + 4c_1 + 40 & 24 + 4c_1 - 4c_3 & -4c_1 - 4c_3 + 104 \\
2c_3 + 10c_1 + 60 & -7c_1 + 3c_3 + 18 & -3c_3 + 14 - 7c_1 & 7c_1 + 146 - 3c_3 \\
12c_1 - 36 & -18c_1 + 6 & -18c_1 + 6 & 126 + 18c_1 \\
72 & 72 & 72 & 72 \\
72 & 24 & 72 & 120 \\
-6c_3 + 18c_1 + 12 & -6c_3 - 18c_1 - 12 & 60 - 18c_1 + 6c_3 & 18c_1 + 6c_3 + 180 \\
-12c_3 + 24c_1 - 84 & -39c_1 - 15c_3 - 36 & 15c_3 - 39c_1 + 48 & 15c_3 + 39c_1 + 180
\end{bmatrix}$$

$$M_2 := \begin{bmatrix}
24 & 18 - 3c_1 + 3c_3 & 6 + 3c_3 + 3c_1 & 18 - 3c_3 + 3c_1 & 24 \\
48 & 30 - 6c_1 & 6 + 6c_1 & 6 + 6c_1 & 0 \\
96 & 48 & 0 & 0 & 0 \\
288 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 24 \\
48 & 4 - c_1 - c_3 & -c_3 + c_1 & c_3 - 4 + c_1 & 0 \\
96 & 12 + 4c_1 & -4 - 4c_1 & -4 - 4c_1 & 0 \\
288 & -48 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
16 & 0 & 0 & 0 & -16 \\
56 & 3c_1 + 2 + c_3 & -2 + c_3 - 3c_1 & 2 - c_3 - 3c_1 & -8 \\
240 & -66 - 6c_1 & 6 + 6c_1 & 6 + 6c_1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 24 \\
192 & -72 - 15c_1 - 3c_3 & -3c_3 + 12 + 15c_1 & 15c_1 + 3c_3 & 48
\end{bmatrix}$$

3 Properties of the surface

The surface consisting of the trimmed patches inherits the properties of surface splines made up from 3- and 4-sided patches (c.f. [3]). These include tangent plane continuity, the convex hull property, locality, and affine invariance. An example illustrating the two stages of the algorithm is given in Figure 3.

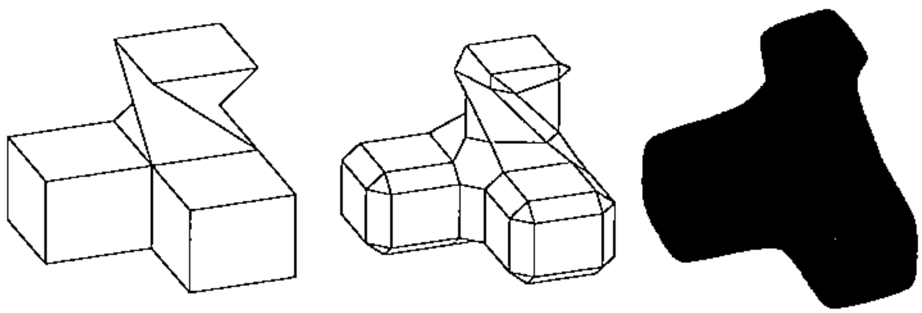

Figure 3: Input polyhedron, planar-cut polyhedron and smooth NURBS surface.

References

[1] CATMULL, E., AND CLARK, J. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design 10* (1978), 350–355.

[2] LOOP, C., AND DEROSE, T. Generalized b-spline surfaces of arbitrary topology. *Computer Graphics 24*, 4 (1990), 347–356.

[3] PETERS, J. C^1-surface splines. *SIAM J. of Numer. Anal. 32*, 2 (1995), 645–666.