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Abstract

We discuss the numerical solution of a free boundary parabolic partial differential problem based on the Black-Scholes model for the pricing of derivative securities. We present the formulation and numerical behavior of the linear complementarity, and front-tracking solutions to the option pricing problem for American style options. We show experimentally that the class of front-tracking methods produce efficient and accurate approximations to the pricing of derivative securities compared to the binomial pricing method.

1 Introduction

An option is a financial derivative product which in its simplest form is a contract giving the right to the owner to buy or sell a predefined quantity of an *underlying asset*, on a predefined date, the *expiration date*, in a predefined price, the *strike price*. A primary classification of option products is in *puts* and *calls*[1]. The underlying asset can be almost any kind of asset whose price fluctuates over time. Most commonly the underlying asset is a stock, index, currency or interest rate. All these entities are traded in the world financial markets and their prices are publicly quoted. Depending on whether the owner of an option contract has the right to exercise the option on an earlier date than the the expiration date of the contract, options can be furthered classified in *European* and *American*.[2] The objective of option pricing is to come up with a 'fair' value for the option

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[1]A call (put) option is a financial contract giving the right, but not the obligation, to the owner to buy (sell) the underlying asset on the specified date at the specified price.

[2]An American option contract can be exercised on any date prior to the expiration date. A European option contract can be exercised only on the expiration date specified by the contract.

along with a number of other parameters such as the *greeks* of the option that are used for the marketing and management of the financial derivative products.

The origins of the mathematical modeling for the option pricing problem are found in the seminal work of Black and Scholes [2], widely known as the Black-Scholes model for option pricing. In this study we are interesting in the numerical approximation of the pricing of American options which is reduced to solving a free boundary parabolic PDE problem derived by the Black-Scholes model. In Section 2 we review the mathematical formulation of the option pricing problem. Section 3 presents a number of techniques that are used to solve numerically the Black-Scholes model and discuss a number of variations of the option pricing problem that can be solved using these techniques. We focus on the solution of the free boundary problem for American options and briefly discuss the linear complementarity formulation of the problem and the family of solutions that can be obtained using this particular technique. We also describe a different approach to the solution of the free boundary problem using front-tracking techniques. In Section 4 we discuss the various methods and in Section 5 we summarize our findings and discuss further directions and research objectives.

2 Mathematical Modeling of Option Pricing

One of the challenging parts in option pricing is to come up with an appropriate model describing the physical system as accurately as possible. In order to arrive to models tractable by mathematical and computational techniques one has to make a number of assumptions and approximations for the physical system. In the case of option pricing the physical system is the financial market place and the particular object of observation is the price of the option's underlying asset. A simplistic approach would suggest that the value of a stock is a deterministic function of time. Unfortunately this is not true, because had it been it would imply that we are in a position to predict accurately and with absolute certainty the prices of the underlying asset and profit accordingly. Reality suggests that the price is not any known deterministic function of any set, however big, of variables observed in the physical system. In as early as 1900 Louis Bachellier [1] described the asset price as a *Brownian motion*. In recent years substantial progress has been made in financial modeling by approximating the price process as a discrete time binomial process [5] and in continuous time as a log-normal Ito process [2].

In this study we focus to the continuous time modeling of option pricing. Throughout we employ the Black-Scholes model to formulate the American option pricing problem. Assuming that the price of the option is a function of the underlying asset and the time to expiration, and under the condition that there exists a risk free replicating portfolio which duplicates the returns of the option, the Black-Scholes PDE model is as follows

$$\frac{\partial V(S,\tau)}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,\tau)}{\partial S^2} + (r-\delta)S\frac{\partial V(S,\tau)}{\partial S} - rV(S,\tau) = 0$$
$$S \in [0,\infty), \quad \text{and} \quad \tau \in [0,T] \tag{2.1}$$

where $V(S,\tau)$ is the price of the option at time τ, S is the price of the underlying asset, τ is the time from the initiation of the option, T is the duration of the option, σ is the volatility of the underlying asset, δ is the continuous dividend yield, and r is the risk free interest rate. One other parameter that does not appear directly in the equation is the strike price denoted by E. Depending on the boundary conditions and the terminal value of the option (i.e. the option *payoff*) both call and put options as well as a variety of other more complicated option products can be priced. A rigorous and detailed presentation as well as a thorough analysis of the assumptions inherent in this mathematical model and its derivation can be found in [9].

Equation (2.1) along with the appropriate initial and boundary conditions that are implied by the particular option being priced is an adequate model for European style options, and has been used extensively by practitioners. It accepts a closed form solution and apart from that it is easily solvable by numerical techniques. When the Black-Scholes model is to be used for pricing American options a number of changes are necessary. Because of the early exercise feature, (2.1) must give the correct price for all times before expiration, avoiding arbitrage[3] opportunities. At each time not only the option price must be computed but also whether it is optimal to exercise the option or hold it. The price of the underlying asset for which it is advantageous to exercise the option prescribes an optimal exercise boundary which is a function of time and not known in advance. Equation (2.1) holds for a region of asset prices, while out of the region the option price equals its payoff. An early discussion of the free boundary problem for the American option pricing can be found in [10] where some of the numerical analysis issues involved are reviewed. Recent progress in the free boundary formulation of the problem is reported in [8] and [11].

3 Numerical Methods for Option Pricing

The PDE model for the option pricing problem does not usually have closed form solutions except in a few cases such as for plain (vanilla) options. Most interesting option products require the application of approximation techniques. A number of computational methods have been used for option pricing with varying degree of applicability and efficiency. The most widely used methods include the binomial discrete time approximations [5], the Monte-Carlo simulation [3] and PDE methods based on the Black-Scholes model. When it comes to

[3]Lack of arbitrage implies that there can never be opportunities to make an instantaneous risk-free profit.

American options the binomial method can be easily converted to take into account the early exercise feature, although it results in an intensive and memory consuming computation. An overview of discrete time pricing methods can be found in [13]. Monte-Carlo approaches are slightly more difficult to use in the context of American options although they retain the advantage in terms of the generality of price process they can incorporate. When applicable, PDE methods are very efficient in pricing American style options, giving the possibility to compute both the option price and optimal exercise boundary.

Below we present two different techniques that are used for solving the free boundary PDE model describing the pricing problem for an American option. Notice that we have converted Equation (2.1) into a forward parabolic PDE by introducing the transformation $t = T - \tau$. Consequently, the payoff of the option is taken as the initial value of PDE problem. We assume that the space domain is discretized in intervals of length $h = \frac{1}{N}$, where N is the resolution of the space discretization, and the time step is of length $\Delta t = \frac{T}{M}$, where M is the resolution of the time discretization. A superscript n indicates that the value of the superscripted function is taken at time step $n\Delta t$, for $n = 1(1)\{\frac{T}{M}\}$ and a subscript j indicates that the value of the subscripted function is taken at the point jh, for $j = 1(1)\{\frac{1}{N}\}$. Throughout, we approximate the partial derivatives with respect to the space variable involved in (2.1) as follows

$$\frac{\partial V^{(n)}}{\partial S} = \frac{V_{j+1}^{(n)} - V_{j-1}^{(n)}}{2h}, \quad \frac{\partial^2 V^{(n)}}{\partial S^2} = \frac{V_{j+1}^{(n)} - 2V_j^{(n)} + V_{j-1}^{(n)}}{h^2}. \tag{3.1}$$

3.1 Linear Complementarity

In [14] and [15] (2.1) is solved using a linear complementarity formulation of the free boundary problem. Equation (2.1) is transformed to the heat equation and the resulting set of linear inequalities is solved using the projected SOR method [7]. Here we present the linear complementarity formulation based on the Black-Scholes operator without any transformations. The advantage is that a number of different option products can be more easily incorporated for which there is no obvious transformation of the corresponding equation to the heat equation. One disadvantage however is that the coefficient matrix of the linear system is not symmetric as shown below and the application of the Projected SOR method becomes difficult and in many cases impossible. To solve the particular system one has to resort to variations of finite difference methods such as those suggested in [4].

The linear complementarity formulation of the option pricing problem can be summarized by

$$A \equiv \frac{\partial V(S,t)}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - (r-\delta)S\frac{\partial V(S,t)}{\partial S} + rV(S,t) \geq 0 \tag{3.2}$$

$$B \equiv \big(V(S,t) - g(S,t)\big) \geq 0 \tag{3.3}$$

$$A \cdot B = 0 \tag{3.4}$$

where $g(S,t)$ is the payoff of the option product.The optimal exercise boundary does not appear in (3.2). Option products that differ only in their payoff can be priced with the above formulation. A number of different numerical schemes can be used for the solution of (3.2) with the given constraints. Using a Crank-Nicholson second order implicit scheme (3.2) is described as

$$a_j V_{j-1}^{(n+1)} + b_j V_j^{(n+1)} + c_j V_{j+1}^{(n+1)} \geq d_j V_{j-1}^{(n)} + e_j V_j^{(n)} + f_j V_{j+1}^{(n)} \tag{3.5}$$

with

$$\begin{aligned} a_j &= \frac{1}{4}\Delta t\Big(j(r-\delta) - \sigma^2 j^2\Big), \quad b_j = 1 + \frac{1}{2}\Delta t(\sigma^2 j^2 + r), \\ c_j &= -\frac{1}{4}\Delta t\Big(j(r-\delta) + \sigma^2 j^2\Big), \quad d_j = -\frac{1}{4}\Delta t\Big(j(r-\delta) - \sigma^2 j^2\Big), \\ e_j &= 1 - \frac{1}{2}\Delta t(\sigma^2 j^2 + r), \quad f_j = \frac{1}{4}\Delta t\Big(j(r-\delta) + \sigma^2 j^2\Big). \end{aligned} \tag{3.6}$$

The constraints are described as

$$\begin{aligned} &V_j^{(n+1)} \geq g_j^{(n+1)}, \quad \text{and,} \\ &(V_j^{(n+1)} - g_j^{(n+1)}) \cdot \\ &(a_j V_{j-1}^{(n+1)} + b_j V_j^{(n+1)} + c_j V_{j+1}^{(n+1)} - d_j V_{j-1}^{(n)} - e_j V_j^{(n)} - f_j V_{j+1}^{(n)}) = 0 \end{aligned}$$

We see that in the case of the original Black-Scholes operator the coefficient matrix is not symmetric, although it is straightforward to see that it is positive definite since it is strictly diagonally dominant. Consequently the behavior of the Projected SOR method and its convergence is not guaranteed by the theory of SOR. It is preferable in this case to solve the model using an explicit finite difference scheme described as

$$V_j^{(n+1)} \geq a_j V_{j-1}^{(n)} + b_j V_j^{(n)} + c_j V_{j+1}^{(n)} \tag{3.7}$$

with

$$\begin{aligned} a_j &= \frac{1}{2}\Delta t\Big(j(r-\delta) + \sigma^2 j^2\Big), \quad b_j = 1 - \Delta t(\sigma^2 j^2 + r), \\ c_j &= -\frac{1}{2}\Delta t\Big(j(r-\delta) - \sigma^2 j^2\Big). \end{aligned} \tag{3.8}$$

The constraints are described as

$$\begin{aligned} &V_j^{(n+1)} \geq g_j^{(n+1)}, \quad \text{and,} \\ &(V_j^{(n+1)} - g_j^{(n+1)}) \cdot (V_j^{(n+1)} - a_j V_{j-1}^{(n)} - b_j V_j^{(n)} - c_j V_{j+1}^{(n)}) = 0 \end{aligned}$$

The model can be solved by

$$\begin{aligned} Y_j^{(n+1)} &= a_j V_{j-1}^{(n)} + b_j V_j^{(n)} + c_j V_{j+1}^{(n)} \\ V_j^{(n+1)} &= \max(g_j^{(n+1)}, Y_j^{(n+1)}). \end{aligned} \tag{3.9}$$

3.2 Front-Tracking

Another technique for solving the free boundary problem is based on an explicit tracking of the optimal exercise boundary. Front-tracking methods for Stefan problems are described in [6]. An application of front-tracking methods in the American option pricing problem is described in [12]. The pricing of an American option on a dividend paying asset with explicit reference to the free boundary can be described by the parabolic initial/boundary value problem

$$\frac{\partial V(S,t)}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r-\delta) S \frac{\partial V(S,t)}{\partial S} - rV, \qquad (3.10)$$
$$S \in [0, f_S(t)), \quad t \in [0,T]$$

with initial and boundary conditions

$$V(S,0) = \max(S-E,0), \quad S \in [0, f_S(0)) \qquad (3.11)$$
$$V(0,t) = 0, \qquad (3.12)$$
$$V(f_S(t),t) = f_S(t) - E, \qquad \frac{\partial V(f_S(t),t)}{\partial S} = 1 \qquad (3.13)$$

in the case of a call option and

$$V(S,0) = \max(E-S,0), \quad S \in (f_S(0), \infty) \qquad (3.14)$$
$$V(0,t) = E, \qquad (3.15)$$
$$V(f_S(t),t) = E - f_S(t), \qquad \frac{\partial V(f_S(t),t)}{\partial S} = -1 \qquad (3.16)$$

in the case of a put option. V denotes the value of the American option and f_S the optimal exercise boundary. The complete option value is given as

$$V_{complete}(S,t) = \begin{cases} V(S,t) & \text{if} \quad S \in [0, f_S(t)), \\ \max(S-E,0) & \text{if} \quad S \in [f_S(t), \infty) \end{cases} \qquad (3.17)$$

in the case of the American call and

$$V_{complete}(S,t) = \begin{cases} \max(E-S,0) & \text{if} \quad S \in [0, f_S(t)], \\ V(S,t) & \text{if} \quad S \in (f_S(t), \infty) \end{cases} \qquad (3.18)$$

in the case of the American put option. Equations (3.17) and (3.18) make explicit that the American option has an optimal exercise boundary, $f_S(t)$, which indicates whether the option should be held or exercised at time t.

Observe that the explicit formulation is now dependent on the type of option being priced. As an example we develop the front-tracking model for the American put. We first formulate the American put model (3.10) onto a rectangular domain $[0,1] \times [0,T]$ by introducing the new space variable

$$x = \frac{S}{f_S(t)}. \qquad (3.19)$$

Equation (3.10) and the corresponding initial/boundary conditions become

$$\frac{\partial V}{\partial t} = \left((r-\delta) + \frac{1}{f_S(t)}\frac{df_S(t)}{dt}\right)x\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - rV \tag{3.20}$$

subject to initial condition

$$V(x,0) = \max(E - xf_S(0), 0), \qquad f_S(0) = \frac{rE}{\delta} \tag{3.21}$$

and boundary conditions

$$V(\infty, t) = 0, \tag{3.22}$$

$$V(1,t)|_{S=f_S(t)} = E - f_S(t), \qquad \frac{\partial V(1,t)}{\partial x}|_{S=f_S(t)} = -f_S(t).$$

To obtain the unknown free boundary function $f_S(t)$ at each time step a secondary equation, the *front tracking* equation is used. For the case of the American put the front-tracking equation is given as

$$\frac{df_S(t)}{dt} = rE - \delta f_S(t) - \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}|_{x=1}. \tag{3.23}$$

Several finite difference schemes can be developed to implement the front-tracking technique. A number of such methods are described in detail in [12].

4 Discussion

The advantage of the linear complementarity approach is that it does not make explicit reference of the free boundary in the problem formulation. The free boundary is found in a postmortem fashion from the obtained solution and the prevailing inequalities. Front-tracking methods on the other hand, lack the uniformity of the linear complementarity and require more elaborate work in the formulation of the problem. However, they provide the optimal exercise boundary as part of the solution without need of any further processing. This in general results in a faster computational process. The accuracy of both techniques is comparable. In Table 1 we show the solution of an American option pricing problem with the two different techniques. For comparison we include the solution provided by the discrete time binomial algorithm. Table 2 gives the execution time for the three algorithms.

5 Conclusions and Future work

In this paper we have described the mathematical modeling of the option pricing problem, and discussed two main techniques for solving the resulting PDE model. Linear complementarity and variational inequalities have been used for

Asset Price	Binomial	Complementarity	Front-Tracking
2.0	0.004	0.004	0.004
4.0	0.176	0.176	0.176
6.0	0.815	0.813	0.813
8.0	1.925	1.925	1.925
10.0	3.373	3.371	3.371
12.0	5.022	5.025	5.021

Table 1: The binomial, linear complementarity and front-tracking solutions to the pricing of an American call problem for $E = 8.0$, $\sigma = 0.75$, $\delta = 0.06$, $r = 0.25$, $T = 0.5$. The price has been calculated using the binomial method for 256 time steps, the Crank-Nicholson implementation of the linear complementarity method with $\Delta t = 0.8 \times 10^{-4}$, and the Crank–Nicholson implementation of the front-tracking method for $\Delta t = 1.0 \times 10^{-4}$.

Binomial	Complementarity	Front-tracking
1.31 sec	0.80 sec	0.33 sec

Table 2: Time taken by the three indicated methods to compute the price of an American call with $E = 8$, $\sigma = 0.75$, $\delta = 0.06$, $r = 0.25$ and $T = 0.5$, and a similar level of accuracy.

some time and present several advantages. Front-tracking techniques are relatively new in the solution of the particular problem, although they have been used successfully for many years in the similar Stefan problem. We plan to extend our research in generalizing the front-tracking method for more complicated American options, such as exotic options, and for several extensions to the original Black-Scholes model such as discrete dividends, transaction costs and time-dependent parameters. Extensions to the model such as stochastic volatility or interest rates demand the solution of a multidimensional PDE problem. Also, some effort is needed for establishing the convergence of the methods in the context of the option pricing problem.

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