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Abstract

In a recent work of ours we have solved the problem of the minimization of the spectral radius of the iteration matrix of a p-cyclic Successive Overrelaxation (SOR) method for the solution of the linear system $Ax = b$, when the matrix $A$ is block p-cyclic consistently ordered, for what is known as the "one-point" problem, for any $p \geq 3$. Particular cases of the "one-point" problem were solved by Young, Varga, Kjellberg, Kredell, Russell and others. In the present work we develop a theory using the results of our previous one and solve first the "two-point" problem special cases of which were solved by Wrigley, Eiermann, Niethammer, Ruttan, Noutsos and others. Secondly, we generalize and extend our theory to cover the "many-point" problem and develop a Young-Eidson’s type algorithm for its solution. As possible application areas we mention among others the best block p-cyclic repartitioning for the SOR method and the solution of large scale systems arising in queueing network problems in Markov analysis.

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1 Introduction and Preliminaries

In this work we consider the block SOR method for the solution of large linear systems that have matrix coefficients possessing a block \( p \)-cyclic structure. Given

\[ Ax = b, \quad A \in \mathbb{C}^{n \times n}, \quad x, b \in \mathbb{C}^n \]  

and the block splitting

\[ A = D - L - U \]  

where \( D, L \) and \( U \) are nonsingular block diagonal, strictly lower and strictly upper triangular matrices, the block SOR method for any \( \omega \neq 0 \) is defined as follows

\[ x^{(m+1)} = \mathcal{L}_\omega x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \]  

where

\[ \mathcal{L}_\omega := (D - \omega L)^{-1} [(1 - \omega)D + \omega U], \quad c := \omega(D - \omega L)^{-1}b \]

and \( x^{(0)} \in \mathbb{C}^n \) arbitrary.

For arbitrary matrix coefficient \( A \), little is known about the value of the real (optimal) relaxation factor \( \omega \in (0, 2) \) that minimizes the spectral radius of \( \mathcal{L}_\omega \), denoted by \( \rho(\mathcal{L}_\omega) \). However, when \( A \) has the block \( p \)-cyclic structure

\[ A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & A_{1p} \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{p,p-1} & A_{pp} \end{pmatrix} \]  

more is known. Let \( D \) be defined by \( D = \text{diag}(A_{11}, A_{22}, \ldots, A_{pp}) \). Then the associated block Jacobi iteration matrix \( J_p = I - D^{-1}A \) has the form

\[ J_p = \begin{pmatrix} 0 & 0 & 0 & \cdots & B_1 \\ B_2 & 0 & 0 & \cdots & 0 \\ 0 & B_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_p & 0 \end{pmatrix} \]  

Following Varga [20] we call the form (1.6) \textit{weakly cyclic of index} \( p \), and \( A \) in (1.5) \textit{block} \( p \)-\textit{cyclic consistently ordered}. For such matrices Varga [20] proved the relationship

\[ (\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{p-1} \]  

between the eigenvalues \( \mu \) of \( J_p \) and \( \lambda \) of \( \mathcal{L}_\omega \), generalizing Young's relationship for \( p = 2 \) [24].
Under the further assumption that all eigenvalues of $J^p$ satisfy

$$0 \leq \mu^p \leq \rho(J^p) < 1,$$

Young and Varga determined the unique optimal values for the parameter $\omega$, denoted from now on by $\hat{\omega}$ (see also [21], [25], [1]). Similar results have been obtained (see [11, 13, 14, 22, 4]) for the case where the eigenvalues of $J^p$ are nonpositive satisfying

$$-(\frac{p}{p-2})^p < -\rho(J^p) \leq \mu^p \leq 0.$$

The two cases above are particular cases of the one-point problem. Until very recently when our work appeared [5], the most general case for the one-point problem that was solved ([10, 18]) was for $p = 2$ when the spectrum of $J_2$, $\sigma(J_2)$, is complex and lies in a rectangle symmetric wrt the real and the imaginary axes and strictly within the infinite unit strip.

So far, particular cases of the two-point problem were considered and solved for $\sigma(J_p)$ real in [23], for $p = 2$, and in [2, 16], for $p \geq 3$. The only case where a many-point problem was solved was the one by Young and Eidson [26] (see also [25]), for $p = 2$. It is noted that for the solution of all the arising minimization problems one uses conformal mapping transformations based on (1.7). In these mappings, one deals with ellipses for $p = 2$ and with $p$-cyclic hypocycloids for $p \geq 3$ (see, e.g., [15, 22]) referred to as ”hypos” from now on.

The recent work on the best block $p$-cyclic repartitioning (see [12, 17, 2, 3]) and the work on the solution of large systems arising in queueing network problems in Markov analysis (see [9, 7, 8]) require the optimization of the convergence of the $p$-cyclic SOR for more complex spectra $\sigma(J_p)$. It is the main objective of this paper to analyze, study and solve the more general problem of the minimization of the spectral radius of the SOR matrix of the many-point problem case, for $p \geq 3$, or, equivalently, to determine the associated optimal hypo that solves it. In [5] it was proved that if an optimal (convergent) SOR exists that solves the one-point minimization problem the SOR in question will be associated with a shortened hypo (or a cusped one, in a special case). Therefore stretched hypos will not be considered here.

Our work is organized as follows. In Section 2 some background material consisting of the main results in [5] is presented. In Section 3 some new background material is developed necessary for the solution of our problems. In Sections 4 and 5 we present the solutions to the two-point and the many-point problem and give Young-Eidson's type algorithms for their numerical solution. Finally, in Section 6, a number of numerical examples are presented.

## 2 Background Material

To begin with our analysis let $\mathcal{H}$ be the class of all $p$-cyclic hypos that are symmetric wrt the real axis. The parametric equations of any member $H \in \mathcal{H}$ are given by the expressions

$$x(t) = \frac{b+a}{b-a} \cos t + \frac{b-a}{b+a} \cos(p-1)t,$$

$$y(t) = -\frac{b-a}{b+a} \sin t + \frac{b+a}{b-a} \sin(p-1)t,$$

(2.1)
where \( t \) takes all the values in \([0, 2\pi)\) (see, e.g., [19] or [22]), and \( b \) and \( a \) are the "real" and the "imaginary" semiaxes of \( H \). In view of the \( p \)-cyclic symmetry of \( H \) it suffices for the analysis to be restricted to \( t \in [0, \frac{\pi}{p}] \). Then, the associated arc of \( H \) lies in the last \((2p) – \text{ant} \) while the real and the imaginary semiaxes of \( H \) lie along the real positive semi axis and the ray with argument \(-\frac{\pi}{p} \), respectively.

Let \( P \) be a point of the complex plane strictly within its \((2p)\)th open \((2p)–\text{ant} \) with polar coordinates \((r, \psi)\), \( r > 0 \), \(-\frac{\pi}{p} < \psi < 0 \). The cartesian coordinates of \( P(\alpha, \beta) \) will satisfy

\[
\alpha = r\cos\psi, \quad \beta = r\sin\psi,
\]

\[
tan\psi = \frac{\beta}{\alpha}, \quad r = \left(\alpha^2 + \beta^2\right)^{1/2}. \tag{2.2}
\]

Let \( H_p \) denote the class of all the hypos through \( P \) and \( H_p \) denote any member of it. Suppose that \( P(r, \psi) \) (or \( P'(r, -\psi) \)) \( \in \sigma(J_p) \) and that \( \sigma(J_p) \) is contained in the intersection of the closures of all \( H_p s \).

Let \( t = \theta \) be the value of the parameter corresponding to the point \( P \) of an \( H_p \) with semiaxes \( b \) and \( a \). From (2.2) and (2.1) it can be obtained that

\[
\alpha = \frac{b+a}{2}\cos\theta + \frac{b-a}{2}\cos(p-1)\theta,
\]

\[
\beta = -\frac{b+a}{2}\sin\theta + \frac{b-a}{2}\sin(p-1)\theta,
\]

\[
r = \left\{\frac{1}{2}\left[\left(\frac{b^2 + a^2}{2} + (b^2 - a^2)\cos\theta\right)\right]\right\}^{1/2}. \tag{2.3}
\]

Solving for \( b \) and \( a \) it can be found that the semiaxes of any \( H_p \), as functions of \( \theta \), are given by the expressions

\[
b = \frac{r}{\cos\left(\frac{\theta}{2}\right)}\cos\left(\left(\frac{\theta}{2}\right) - \psi\right),
\]

\[
a = \frac{r}{\sin\left(\frac{\theta}{2}\right)}\sin\left(\left(\frac{\theta}{2}\right) - \psi\right). \tag{2.4}
\]

From (2.4) we have that either \( b > r > a \) or \( b < r < a \), in which case the hypo is said to be of type \( I \) or type \( II \), respectively. The limiting case \( b = r = a \) is a circle. From [5] it is known that there exists a unique pair of cusped hypos \( H_P \) of type \( I \) and \( II \) with values of \( \theta \) denoted by \( \theta_I \) and \( \theta_{II} \), respectively. \( \theta_I \) and \( \theta_{II} \) are the unique values of \( t \) satisfying

\[
tan\psi = \frac{\beta}{\alpha} = -\frac{(p-1)\sin\theta + \sin(p-1)t}{(p-1)\cos\theta + \cos(p-1)t} \tag{2.5}
\]

and

\[
tan\psi = \frac{\beta}{\alpha} = -\frac{(p-1)\sin\theta + \sin(p-1)t}{(p-1)\cos\theta + \cos(p-1)t}, \tag{2.6}
\]

respectively. Furthermore, for \( \theta_I \) and \( \theta_{II} \) there hold

\[
0 < \theta_{II} < -\psi < \theta_I < \frac{\pi}{p}. \tag{2.7}
\]

Let \( b_I \) and \( a_I = \frac{p-2}{p}b_I \) (resp. \( b_{II} \) and \( a_{II} = \frac{p-2}{p^2}b_{II} \)) be the real and the imaginary semiaxes of the cusped hypo \( H_P \) \( I \) (resp. \( H_P \) \( II \)). Then (2.4) and the analysis in [5] (see Lemmas 2.4
and 2.5) imply that between the three elements $\theta, b, a$ of any $H_P$ there exists a "one-to-one" correspondence. More specifically, each of them is a strictly monotone and continuous function of either of the other two elements. The monotone behavior of these functions is shown in Table 1, which is part of Table 1 of [5]. Also, in Figure 1 four hypos through a point $P$ in the last $(2p)$ – ant are depicted. The two cusped ones and also two others one of which is of type $I$ and the other of type $II$.

From the theory of $p$–cyclic SOR, to guarantee convergence, the point $(1,0)$ of the complex plane must not belong to the closure of the interior of the associated hypo $H_P$. A sufficient and necessary condition for that is $b < 1$. Consequently, if $1 \leq b_{II} < b_I$, there is no hypo $H_P$ that is associated with a convergent SOR. However, if $b_{II} < 1 \leq b_I$ or $b_{II} < b_I < 1$ all $H_P$'s with $b \in [b_{II}, \min\{1, b_I\}] \setminus \{1\}$ will be associated with convergent SORs.

As is known (see e.g., [22], [2]) the elements of a (convergent) SOR, namely its relaxation
factor \( \omega \) and its spectral radius

\[ \rho(\mathcal{L}_\omega) = \frac{1}{\eta \rho}, \quad 1 < \eta, \]  

(2.8)

are related to the real and the imaginary semiaxes of a hypo that contains \( \sigma(J_P) \) in the closure of its interior through the relationships

\[ \frac{1}{\omega \eta} = \frac{b + a}{2}, \quad \frac{(\omega - 1)\eta^2}{\omega \eta} = \frac{b - a}{2}. \]  

(2.9)

In [5] it is proved that for \( \theta \in (\theta_{II}, \theta_{I}) \) the semiaxes \( b \) and \( a \) of all the hypos \( H_P \) are twice continuously differentiable functions of \( \theta \). The same is true for \( a \) wrt \( b \) and vice versa in their respective intervals. Two useful formulas given in Lemmas 4.1 and 4.2 of [5] are presented below:

\[ D := \frac{da}{db} = \frac{(p-2)\eta \cot \frac{\theta}{2} - c^2}{(p-1)\eta \sin \frac{\theta}{2} - c}, \quad \frac{d\theta}{db} = \frac{2c}{(p-2)\eta \sin \frac{\theta}{2} - c} > 0, \]  

(2.10)

where \( c := \cot \left( \frac{\theta}{2} \right) \).

Using (2.8)-(2.9) and introducing the symbol \( x \) to denote either of the two quantities

\[ x := \frac{1}{\eta} \equiv \rho^*(\mathcal{L}_\omega), \]  

(2.11)

on elimination of \( \omega \) from the equations in (2.9) we obtain

\[ \phi := \phi(x) \equiv (b - a)x^p - 2x + b + a = 0. \]  

(2.12)

Summarizing the statements and the conclusions of Thms 4.5, 4.6, 4.7 and 4.8 of [5] we give in the theorem below the statement that will constitute the key to our analysis for the solution of the two-point and subsequently of the many-point problem.

**Theorem 2.1:** Suppose that \( b_{II} < 1 \) and let \( \tilde{\theta} \) denote \( \theta_{II} \), if \( b_I \leq 1 \), or \( \theta_{(1,0)} \), the value of \( \theta \) at \( P \) of the hypo through the points \((1,0)\) and \( P \), if \( b_I > 1 \). The unique real positive root \( x_0 \equiv x_0(b) \equiv x_0(\theta) \in (0,1) \) of equation (2.12) is a strictly decreasing function of \( b \) (resp. \( \theta \)) in \([b_{II}, b_I]\) (resp. \([\theta_{II}, \tilde{\theta}]\)) and a strictly increasing one in \([b_I, b_{I0}]\) (resp. \([\tilde{\theta}, \theta_{II}]\)), where \( b \) (resp. \( \theta \)) is the value of \( b \) (resp. \( \theta \)) at which the minimum occurs. Moreover, \( \tilde{\theta} \) is given as the unique root of the equation

\[
\left[ \frac{\sin \theta - \sin \theta \cot \frac{\theta}{2} \cos \frac{\theta}{2} - (p-1)\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\sin \frac{(p-1)\theta}{2} - (p-1)\sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right]^p
\]

(2.13)

in the interval \((\theta_{II}, \tilde{\theta})\) while the minimum value \( \tilde{x} \equiv x_0(\tilde{\theta}) \) is then given by

\[
\tilde{x} = \left[ \frac{\sin \frac{(p-1)\tilde{\theta}}{2} - (p-1)\sin \frac{\tilde{\theta}}{2} \cos \frac{\tilde{\theta}}{2}}{\sin \frac{(p-1)\tilde{\theta}}{2} - (p-1)\sin \frac{\tilde{\theta}}{2} \cos \frac{\tilde{\theta}}{2}} \right]^\frac{1}{p}.
\]

(2.14)
Note: In the limiting cases $\psi = 0$ and $\psi = -\frac{\pi}{p}$, the values $\theta = 0$ and $\theta = \frac{\pi}{p}$ can be recovered from (2.13). For this, however, one has to get rid of the denominators first and then apply (2.13) in the interval $[\theta_{11}, \theta_{1}] \equiv [0, \frac{\pi}{p}]$. Formulas (2.4) can then give the obvious limiting values for $\hat{\theta}$ and $\hat{\alpha}$, use of which into (2.12) yields $\hat{x} \equiv x_0(\hat{\theta})$.

3 Further Study of the Hypocycloids

We begin our analysis with the question of whether two distinct (shortened) hypoes $H_p$'s at $P$ are more than one common points in the last $(2p - 1)$-ant of the complex plane. For this we prove the lemma below where to simplify our notation further we will use $A \sim B$ to denote that the two quantities $A$ and $B$ are such that $\text{sign}(A) = \text{sign}(B)$.

Lemma 3.1: The slope of the tangents to the (shortened) hypoes $H_p$'s at $P$ is a strictly decreasing function of $b$.

Proof: First, we find the slope of the tangent to a hypo $H_p$ at $P$. For this, since $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$, we differentiate each of the equations in (2.1) wrt $t$, form $\frac{dy}{dx}$ and then replace $t$ by $\theta$ to obtain

$$
\frac{dy}{dx}|_{t=\theta} = \frac{(b + a)\cos \theta - (p - 1)(b - a)\cos(p - 1)\theta}{(b + a)\sin \theta + (p - 1)(b - a)\sin(p - 1)\theta},
$$

(3.1)

from which it is clear that $\frac{dy}{dx}|_{t=\theta}$ is a function of $b \in [b_{11}, b_1]$. Next, we differentiate $\frac{dy}{dx}|_{t=\theta}$ wrt $b$, omit the positive denominator, form all eight products in each of the two multiplications involved and sum up similar terms to successively obtain

$$
K := K(b) \equiv \frac{d}{db}(\frac{dy}{dx}|_{t=\theta})
$$

\sim [(b + a)\sin \theta + (p - 1)(b - a)\sin(p - 1)\theta] \times

$$
[(1 + D)\cos \theta - (b + a)\sin \theta \theta' - (p - 1)(1 - D)\cos(p - 1)\theta + (p - 1)^2(b - a)\sin(p - 1)\theta \theta']
$$

\sim [(b + a)\cos \theta - (p - 1)(b - a)\cos(p - 1)\theta] \times

$$
[(1 + D)\sin \theta + (b + a)\cos \theta \theta' + (p - 1)(1 - D)\sin(p - 1)\theta + (p - 1)^2(b - a)\cos(p - 1)\theta \theta']
$$

\sim -2(p - 1)(a - bD)\sin p\theta

\sim [(p - 1)^2 - 2[(p - 1)^2 - 1]ba + [(p - 1)^2 - 1]a^2] \theta'

\sim -(p - 2)(p - 1)(b^2 - a^2)\cos p\theta \theta'.

(3.2)

Then we replace in the rightmost expression above $\sin p\theta$ and $\cos p\theta$ by $\frac{2c}{1+c^2}$ and $\frac{c^2 - 1}{1+c^2}$, respectively, where $c = \cot(\frac{\pi}{2p})$, use the expressions for $D$ and $\theta'$ from (2.10) and multiply by
the positive expression \( \frac{(1+e^2)}{2c} [pb - (p - 2)a] \), since \( b > (\frac{p-2}{p}, \frac{p}{p-2}) \). Thus we have

\[
K \sim -2(p-1)[a(pb - (p - 2)a) + b(pa - (p - 2)b)c^2] + \{(p-1)^2 - 1\}^2 - 2[(p-1)^3 + 1]ba + [(p-1)^3 - 1]a^2\}(1 + c^2) - (p-2)(p-1)(b^2 - a^2)(c^2 - 1)
\]

\[
= p\{(p-2)pb^2 - 2[(p-1)^2 + 1]ba + (p-2)pa^2\}(1 + c^2)
\]

\[
\sim \left( \frac{b}{a} - \frac{p}{p-2}\right) \left( \frac{b}{a} - \frac{p}{p-2}\right) < 0.
\]

(3.3)

The result just obtained effectively proves our statement. □

Based on Lemma 3.1, one of our main statements regarding the maximum number of common points that any two hypos can have in the last \((2p) - \text{ant}\) of the complex plane can be proved. More specifically:

**Theorem 3.2:** Any two distinct hypos from the class \( \mathcal{H} \) can have at most one common point in the last \((2p) - \text{ant}\) of the complex plane.

**Proof:** Consider any two hypos \( H_1, H_2 \in \mathcal{H} \) with real semiaxes \( b_1, b_2 \), respectively, such that \( b_1 < b_2 \). Suppose that \( H_1 \) and \( H_2 \) have at least two common points in the \((2p) - \text{ant}\) in question, let them be \( P \) and \( P' \). Suppose that \( P' \) "precedes" \( P \) on either of the two hypos. Since either hypo belongs to both classes \( \mathcal{H}_{P} \) and \( \mathcal{H}_{P'} \), we can use for \( H_1 \) and \( H_2 \) the alternative notations \( H_{P,1} \equiv H_{P',1} \) and \( H_{P,2} \equiv H_{P',2} \), respectively. Let \( \theta_1, \theta_2 \) be their respective values of \( \theta \) at \( P \). From Table 1, the inequality \( b_1 < b_2 \) implies \( \theta_1 < \theta_2 \). On the other hand, by virtue of Lemma 3.1, for values of the parameter \( t = t_2 < \theta_2 \) in the neighborhood of \( \theta_2 \), the corresponding piece of the arc of \( H_{P,2} \) will be strictly outside the closure of the interior of the entire \( H_{P,1} \) curve. But then at \( P', H_{P,2} \) must either be tangent to \( H_{P,1} \) or intersect (in fact, "enter" the closure of the interior of the entire) \( H_{P,1} \). Then, if \( \theta'_1, \theta'_2 \) are the values of \( \theta \) at \( P' \), for \( H_{P,1} \) and \( H_{P,2} \), respectively, it will be either \( \frac{dy}{dx} = \frac{dy}{dx} = \frac{dy}{dx} \) or \( \frac{dy}{dx} < \frac{dy}{dx} \). However, by virtue of Lemma 3.1, both these relationships contradict the fact that \( b_1 < b_2 \). □

**Note:** The two cusped hypos through any point \( P \) are considered as limiting cases of the hypos in \( \mathcal{H}_P \) and can therefore be included in all the statements that have been presented so far.

As an immediate conclusion, we have the following corollary of Thm 3.2.

**Corollary 3.3:** The optimal solution to the one-point problem, as this problem was defined in [5], is the optimal solution to the problem where the spectrum of the associated Jacobi matrix \( J_p, \sigma(J_p) \), lies in the intersection of the closures of the two cusped hypos through the point \( P \).
Suppose now that we are given two points \( P_1(T_1, \psi_1) \) and \( P'_1(T'_1, \psi'_1) \) such that \( T_1, T'_1 > 0 \) and \(-\frac{\pi}{2} < \psi_2 < \psi_1 < 0\). We shall try to find the condition(s) which guarantee the existence of a hypo passing through both points. For this consider the pair of cusped hypos through \( P_1 \). Let \( b_{II,1}, a_{II,1} \) and \( b_{II,}, a_{II,} \) be the real and the imaginary semi-axes of the two cusped hypos that belong to the class \( \mathcal{H}_{P_1} \). If there exists a hypo that passes through both \( P_1 \) and \( P_2 \), this hypo will belong to both classes of hypos \( \mathcal{H}_{P_1} \) and \( \mathcal{H}_{P_2} \). The hypo in question, with real and imaginary semi-axes \( b \) and \( a \), respectively, as a member of the class \( \mathcal{H}_{P_1} \), there exists a hypo that passes through both \( P_1 \) and \( P_2 \). This hypo will belong to both classes of hypos \( \mathcal{H}_{P_1} \) and \( \mathcal{H}_{P_2} \). The hypo in question, with real and imaginary semi-axes \( b \) and \( a \), respectively, as a member of the class \( \mathcal{H}_{P_1} \), will be denoted by either \( H_1(b) \) or \( H_1(a) \). Obviously, the semi-axes of \( H_1(b) \) will satisfy the relationships, \( b_{II,1} < b < b_{II,} \) and \( a_{II,1} < a < a_{II,} \), where we have excluded the case of \( P_2 \) being on one of the two cusped hypos through \( P_1 \) as trivial. By virtue of Thm 3.2 and the previous relationships, the hypo \( H_1(b) \) will be such that its two arcs \( B_{b P_1} \) and \( P_{II}A_{b,a} \), where \((b,0)\) and \((a,-\frac{\pi}{2})\) are the polar coordinates of the points \( B_{b} \) and \( A_{a} \), respectively, must lie strictly in the interior of the two curvilinear triangles \( B_{b P_1}P_{b,1}A_{b} \) and \( A_{a}P_{II}A_{a,II} \), respectively. In view of \( \psi_2 < \psi_1 \), this implies that if \( H_1(b) \) belongs to the class \( \mathcal{H}_{P_2} \) then the point \( P'_1 \) must lie in the interior of the curvilinear triangle \( A_{a,II}P_{II}A_{a} \).

After the above analysis we can state and prove the following theorem.

**Theorem 3.4:** Let \( P_1 \) and \( P_2 \) be two points with polar coordinates \((r_1, \psi_1)\) and \((r_2, \psi_2)\), respectively, such that \( r_1, r_2 > 0 \) and \(-\frac{\pi}{2} < \psi_2 < \psi_1 < 0\). Let also \( H(b_{II,1}) = H(a_{II,1}) \in \mathcal{H}_{P_1} \) and \( H(b_{II,}) = H(a_{II,}) \in \mathcal{H}_{P_2} \) be the two cusped hypos through \( P_1 \). Let a similar notation be adopted for the two cusped hypos through \( P_2 \), where the indices 1 and 2 are interchanged. Suppose that \( P_2 \) lies strictly in the interior of the curvilinear triangle \( A_{a,II}P_{II}A_{a} \), where the points \( A_{a,II} \) and \( A_{a} \) have polar coordinates \((a_{II,1}, -\frac{\pi}{2})\) and \((a_{II,1}, -\frac{\pi}{2})\), respectively. Then, there exists a unique hypo that passes through both points \( P_1 \) and \( P_2 \).

**Proof:** Consider the two hypos \( H_1(a_{II,1}) \) and \( H_1(a_{II,}) \) and let \( b_{II}^{(1)} \) and \( b_{II}^{(1)} \) be their respective real semi-axes. Next, consider the two hypos \( H_2(b_{II}^{(1)}) \) and \( H_2(b_{II}^{(1)}) \) and let \( a_{II}^{(1)} \) and \( a_{II}^{(1)} \) be their imaginary semi-axes. Then, consider \( H_1(a_{II}^{(1)}) \) and \( H_1(a_{II}^{(1)}) \) and let \( b_{II}^{(2)} \) and \( b_{II}^{(2)} \) be their real semi-axes, and so on. By this procedure, two sequences of nested intervals are generated that satisfy the following set relationships,

\[
[b_{II}^{(-1)}, b_{II}^{(-1)}] := [b_{II,2}, b_{II,2}] \supset [b_{II}^{(0)}, b_{II}^{(0)}] := [b_{II,1}, b_{II,1}] \supset [b_{II}^{(1)}, b_{II}^{(1)}] \supset [b_{II}^{(2)}, b_{II}^{(2)}] \supset \ldots
\]

\[
[a_{II}^{(-1)}, a_{II}^{(-1)}] := [a_{II,1}, a_{II,1}] \supset [a_{II}^{(0)}, a_{II}^{(0)}] := [a_{II,2}, a_{II,2}] \supset [a_{II}^{(1)}, a_{II}^{(1)}] \supset [a_{II}^{(2)}, a_{II}^{(2)}] \supset \ldots
\]

The left and the right endpoints of the above two set sequences yield four sequences of numbers two of which are strictly increasing and bounded above while the other two sequences are strictly decreasing and bounded below. More specifically, the sequence \( \{b_{II}^{(k)}\}_{k=-1}^{\infty} \) is strictly increasing and bounded above by any term of the sequence \( \{b_{II}^{(k)}\}_{k=-1}^{\infty} \), which, in turn, is strictly decreasing and bounded below by any term of the former sequence. A completely
analogous conclusion holds true for the two sequences \( \{a_i^{(k)}\}_{k=-1}^{\infty} \) and \( \{a_{II}^{(k)}\}_{k=-1}^{\infty} \), respectively. Consequently,

\[
\lim_{k \to -\infty} b_{II}^{(k)} = b^* \leq b^{**} = \lim_{k \to -\infty} b_{I}^{(k)}, \\
\lim_{k \to -\infty} a_{I}^{(k)} = a^{**} \leq a^* = \lim_{k \to -\infty} a_{II}^{(k)}.
\]

(3.4)

Note that if \( b^* < b^{**} \) then, in view of Table 1, \( H_1(b^*) \neq H_1(b^{**}) \). Hence \( a^{**} < a^* \) and the previous procedure can continue on. In other words the strict inequality \( b^* < b^{**} \) cannot be true. Therefore \( b^* = b^{**} \) in which case \( H_1(b^*) = H_1(b^{**}) \) and \( a^{**} = a^* \). This proves the existence of a hypo, denoted by \( H_{1,2} \), that passes through both points \( P_1 \) and \( P_2 \). The uniqueness of this hypo is guaranteed by Thm 3.2. \( \square \)

4 The Solution to the "Two-Point" Problem

Suppose that two points \( P_1(r_1, \psi_1) \) and \( P_2(r_2, \psi_2) \) in the last \((2p)\) - ant of the complex plane are given and are such that \( r_1, r_2 > 0 \) and \( -\frac{\pi}{p} < \psi_2 < \psi_1 < 0 \). Suppose also that these two points and/or their symmetric ones \( \text{wrt} \) the real axis are the images of two eigenvalues of \( \sigma(J_p) \), the spectrum of the Jacobi matrix \( J_p \). Finally, suppose that all the other eigenvalues of \( J_p \) lie in the intersection of the closures of the interiors of the two pairs of the cusped hypos through \( P_1 \) and \( P_2 \). The question that arises is whether one can find an optimal convergent SOR method, if such an SOR method exists, based on the information given about \( \sigma(J_p) \).

The solution to this problem, if it exists, can be given by means of the theory in [5], presented briefly in Section 2 (mainly Table 1 and Thm 2.1), and the theory developed in the previous section. To simplify the analysis we will give the solution to the two-point problem in the most complicated case in Thm 4.1 below. Some simpler cases will be treated separately in the discussion that follows. Also, the notation adopted so far will be used throughout the rest of the present work.

Consider \( P_1(r_1, \psi_1) \) and \( P_2(r_2, \psi_2) \) as these points were defined in the beginning of this section and suppose that they also satisfy all the assumptions considered there. If \( \max\{b_{II,1}, b_{II,2}\} \geq 1 \), then at least one of the two points \( P_1(r_1, \psi_1) \) and \( P_2(r_2, \psi_2) \) will lie on or strictly outside the closure of the interior of the cusped hypo \( II \) through the point \( B(1,0) \), which separates the region of convergence from the region of divergence of the \( p \)-cyclic SOR method. Therefore, there exists no convergent SOR method associated with the pair of the given points. If, however, \( \max\{b_{II,1}, b_{II,2}\} < 1 \), then there will exist convergent SORs associated with either \( P_1 \) or \( P_2 \). Suppose that \( P_2 \) lies in the intersection of the closures of the interiors of the two cusped hypos that belong to \( \mathcal{H}_{P_2} \). In such a case and by virtue of Cor. 3.4 the optimal solution will be that of the one-point problem for \( P_1 \). The situation is reversed if \( P_1 \) lies in the intersection of the closures of the interiors of the pair of the cusped hypos that belong to \( \mathcal{H}_{P_1} \).

As is clear, the previous discussion does not cover the case where the position of the two points \( P_1(r_1, \psi_1) \) and \( P_2(r_2, \psi_2) \) \( \text{wrt} \) each other is that considered in Thm 3.4. This case is
Theorem 4.1: Let the two points \( P_1(r_1, \psi_1) \) and \( P_2(r_2, \psi_2) \) have a relative position \( \text{wrt} \) each other that described in Thm 3.4, with \( b_{I,1} < 1 \). Then, if the optimal hypo for \( P_1, \) \( \overline{H}_1 \) (resp. for \( P_2, \overline{H}_2 \)) contains \( P_2 \) (resp. \( P_1 \)) in the closure of its interior, this hypo, \( \overline{H} \), is the optimal one of the pair of points. If neither optimal hypo, \( \overline{H}_1 \) or \( \overline{H}_2 \), contains the other point in the closure of its interior then the optimal hypo of the pair of points is the hypo \( \overline{H} \equiv H_{1,2} \). (Note: As can be easily checked the case of hypo \( \overline{H}_1 \) containing \( P_2 \) in the closure of its interior and at the same time \( \overline{H}_2 \) containing \( P_1 \) in the closure of its interior does not exist.)

Proof: Let \( x_1(b) \) and \( x_2(b) \) denote the values of \( x_0 \) of Thm 2.1 associated with the hypos \( H_1(b) \) and \( H_2(b) \), respectively. (It is reminded that \( x_0 \) denotes the unique zero of (2.12) in \((0,1)\) and is in fact the \( p^\text{th} \) root of the spectral radius of the SOR iteration matrix associated with the hypo whose real semiaxis is \( b \) \((< 1)\) and passes through a point \( P \) in the last \((2p)\)-ant.) Let \( \hat{b}_1 \) and \( \hat{b}_2 \) be the values of \( b \) at which \( x_1(b) \) and \( x_2(b) \) attain their minimum values and let them be denoted by \( \hat{x}_1 \) and \( \hat{x}_2 \), respectively. Let also \( b_{1,2} \) be the real semiaxis of the unique hypo \( H_{1,2} \) that passes through \( P_1 \) and \( P_2 \). From the proof of Thm 3.4 it is known that \( b_{II,2} < b_{II,1} < b_{I,2} < b_{I,1} \). Therefore \( x_2(b_{II,1}) < x_1(b_{II,1}) \) since the hypos \( H_1(b_{II,1}) \) and \( H_2(b_{II,1}) \) have the same real semiaxis while the imaginary semiaxis of the latter is smaller than that of the former. By virtue of this observation and the analysis so far, it is readily seen that there are the possible orderings of the real semiaxes \( \hat{b}_1, \hat{b}_2, \) and \( b_{1,2} \) of the hypos \( \overline{H}_1 \equiv H_1(\hat{b}_1), \overline{H}_2 \equiv H_2(\hat{b}_2), \) and \( H_{1,2}, \) depending on their respective positions. More specifically: If \( \overline{H}_1 \) contains \( P_2 \) in the closure of its interior then we have the ordering in (4.1i) below. If \( \overline{H}_2 \) contains \( P_1 \) in the closure of its interior we have the case (4.1ii). Finally, if neither of \( \overline{H}_1 \) and \( \overline{H}_2 \) contains the other point in the closure of its interior we have the case (4.1iii). Namely,

\[
\begin{align*}
&i) \hat{b}_1 < b_{1,2}, \quad ii) b_{1,2} < \hat{b}_2, \quad \text{and} \quad \hat{b}_2 < b_{1,2} < \hat{b}_1. \\
&\text{(4.1)}
\end{align*}
\]

Having in mind the behavior of the function \( x_0(b) \) of Thm 2.1, and in view of Thm 3.2, it is easily found out that the solution to the optimization problem

\[
\min_{b \in [b_{II,1}, \min\{b_{II,1}\}]} \max\{x_1(b), x_2(b)\},
\]

for the three cases in (4.1), is achieved for

\[
\begin{align*}
&i) \hat{b} = \hat{b}_1, \quad ii) \hat{b} = \hat{b}_2, \quad \text{and} \quad iii) \hat{b} = b_{1,2}, \\
&\text{(4.3)}
\end{align*}
\]

respectively. □

Figure 2 depicts the three cases of a two-point optimal hypo. In each case both optimal hypos through the points \( P_1 \) and \( P_2 \) are given together with the unique hypo through \( P_1 \) and \( P_2 \). In Figure 2a the optimal hypo is \( \overline{H}_1 \), in 2b it is \( \overline{H}_2 \) and in 2c it is \( H_{1,2} \).
Figure 2: Optimal two-point hypos
Note: Although convergent SORs are associated with any \( b \in [b_{II,1}, 1) \) the search for the optimal one can be restricted to the interval in (4.2), in case \( b_{I,2} < 1 \).

Remark: Let \( P^* \) be the point of intersection of the cusped hypo I through \( P_1 \) and the cusped hypo II through \( P_2 \). Then, by virtue of Thm 3.2, one can observe that in all three cases examined in Thm 4.1 the curve-sided triangle \( P_1 P^* P_2 \), defined by the corresponding arcs of the cusped hypo I through \( P_1 \), the cusped hypo II through \( P_2 \), and the hypo \( H_{1,2} \), lies always in the closure of the interior of the optimal hypo \( \hat{H} \) of the two given points. This simply suggests that the region in the \((2p)\) ant considered in the beginning of this section as the one containing the eigenvalues of \( J_p \) may be extended to the curvilinear five-sided polygon \( OB_{II,1}P_1P_2A_{I,2} \) whose curved sides \( B_{II,1}P_1, P_1P_2, \) and \( P_2A_{I,2} \) are the corresponding arcs of the cusped hypo II through \( P_1 \), of the hypo \( H_{1,2} \), and of the cusped hypo I through \( P_2 \), respectively.

Note: The reader may have noticed the complete analogy between the solution to the two-point problem just given and the solution to the two-point problem given by Young and Eidson for \( p = 2 \) (see [26], [25] or [6]).

Before we give in pseudocode the algorithm for the solution to the two-point problem (as in [6]) we give the algorithm for the solution to the one-point problem taken from [5]. This is given for completeness and, furthermore, because each time an optimal hypo \( \hat{H}_j \) or \( \hat{H} \) is referred to one should bear in mind that at least a major part of the one-point problem algorithm is supposed to be applied for some of the elements of the hypo in question to be determined.

**ALGORITHM 1: One-point problem**

Given the (polar) coordinates of the point \( P \) as in Sections 2, 3 and 4;

Determine \( \theta_{II} \) from (2.6) and then \( b_{II} \) from (2.4);

if \( b_{II} \geq 1 \) then

NO CONVERGENT SOR EXISTS; stop;

endif;

Determine \( \tilde{\theta} := \theta_I \) from (2.5) and then \( b_I \) from (2.4);

if \( b_I > 1 \) then

Determine \( \theta_{(1,0)} \) from (2.4) by setting \( b := 1 \); Set \( \tilde{\theta} := \theta_{(1,0)} \);

endif;

Determine \( \hat{\theta} \in (\theta_{II}, \tilde{\theta}) \) from (2.13);

Determine \( \hat{x} \) from (2.14);

Determine \( \hat{b} \) and \( \hat{a} \) using \( \hat{\theta} \) from (2.4);

Determine \( \hat{\omega} := \frac{2\hat{x}}{\hat{b}+\hat{a}}; \rho(L_{\hat{\omega}}) := \hat{x}^{\hat{p}}; \)

end of ALGORITHM 1;
The two-point problem algorithm is as follows:

**ALGORITHM 2: Two-point problem**

Given the (polar) coordinates of the points \( P_1 \) and \( P_2 \) as in the beginning of the present section. Furthermore, assume that \( b_{II,2} < b_{II,1} < b_{I,1} < b_{I,2} \) and that \( b_{II,1} < 1 \);

Determine \( H_{1,2} \);

Determine \( H_I \);

if \( b_1 \leq b_{1,2} \) then \( \bar{H} \equiv H_I \);

else Determine \( H_2 \);

if \( b_{1,2} \leq b_2 \) then \( \bar{H} \equiv H_2 \);

else \( \bar{H} \equiv H_{1,2} \);

endif;

endif;

Determine \( \hat{\omega} := \hat{x}_{\frac{a-b}{b+a}} ; \rho(L_G) := \hat{x}_p ; \)

end of ALGORITHM 2;

**Note:** By "Determine \( H_{1,2} \)" etc we mean that all the elements of the corresponding hypo that are to be used later on are determined; among others, the real semiaxis \( b_{1,2} \), the imaginary semiaxis \( a_{1,2} \), the root \( x_{1,2} \equiv x(b_{1,2}) \) of (2.12) etc.

Before we close this section we outline very briefly the numerical solution to the following two problems which will be useful when Algorithm 2 is applied.

**Problem I:** Given a hypo \( H \in \mathcal{H} \), defined by its real and imaginary semiaxes \( b \) and \( a \) \((\alpha \in \left[ \frac{p-2}{p}, \frac{p}{p-2} \right])\), and a point \( P_1(r_1, \psi_1) \) \((r_1 > 0, -\frac{\pi}{p} < \psi_1 < 0)\). Determine whether \( P_1 \) lies strictly in the interior of \( H \), on \( H \) or outside the closure of the interior of it.

**Problem II:** Given two points \( P_1 \) and \( P_2 \) in the last \((2p) - \text{ant} \) defined as in Thm 3.4. Determine the elements of the unique hypo \( H_{1,2} \) that passes through them.

For the solution of **Problem I** it suffices to find the intersection of the polar radius of \( P_1 \) and of the hypo \( H \), let it be denoted by \( P(r, \psi = \psi_1) \), and then compare \( r_1 \) and \( r \). To find \( r \) one has to apply either the third formula of (2.3) or one of the formulas in (2.4). For this the value of \( \theta \) is needed and can be determined via the equations in (2.4), from

\[
\frac{\tan\left(\frac{\psi_1}{2}\right)}{\tan\left((\frac{\psi}{2} - 1)\theta - \psi\right)} = \frac{b}{a} \in \left[ \frac{p - 2}{p}, \frac{p}{p - 2} \right].
\]

(4.4) guarantees the uniqueness of \( \theta \). This can be proved as follows: Let

\[
Q_1 := Q_1(\theta) \equiv \frac{\tan\left(\frac{\psi}{2}\right)}{\tan\left((\frac{\psi}{2} - 1)\theta - \psi\right)}
\]
and differentiate \( \theta \) to obtain

\[
\frac{\partial Q_1}{\partial \theta} \sim p \sin((p - 2)\theta - 2\psi) - (p - 2) \sin p \theta \equiv Q_2(\theta) =: Q_2. \tag{4.6}
\]

Differentiating once again we have

\[
\frac{\partial Q_2}{\partial \theta} \sim \cos((p - 2)\theta - 2\psi) - \cos p \theta \equiv Q_3(\theta) =: Q_3. \tag{4.7}
\]

Now suppose that \( \frac{b}{a} \in \left(\frac{\pi - 2}{p}, 1\right) \) in which case \( H \) is a hypo of type II. Consider then all \( \theta \in [\theta_{II}, -\psi] \). It is readily shown that \( 0 < p \theta < (p - 2)\theta - 2\psi < \pi \), since \( 0 < \theta_{II} \leq \theta \leq -\psi < \frac{\pi}{p} \) and \( -\frac{\pi}{p} \leq \theta + \psi \leq 0 \). Therefore \( Q_3 \) is strictly negative for \( \theta \in [\theta_{II}, -\psi] \) and zero at \( \theta = -\psi \). This implies that \( Q_2 \) strictly decreases as a function of \( \theta \in [\theta_{II}, -\psi] \). It is noted that \( Q_2(-\psi) = 2 \sin(-p\psi) > 0 \). This makes \( Q_2 \) take on positive values only. Consequently, \( Q_1 \) strictly increases taking all the values in \( \left[\frac{\pi - 2}{p}, 1\right] \) and so the uniqueness of \( \theta \) is guaranteed. A similar proof holds in the case \( \frac{b}{a} \in \left[1, \frac{p}{p - 1}\right] \).

For the solution of Problem II it suffices to determine one of the elements of the unique hypo \( H_{1,2} \) for either point \( P_1 \) or \( P_2 \) since then all the others can be found very easily. If \( r_1 = r_2, H_{1,2} \) is a circle and is, therefore, already known. If \( r_1 < r_2, H_{1,2} \) is a hypo of type II while if \( r_1 > r_2 \), it is of type I. In either case let the unknown values of \( \theta \) of \( H_{1,2} \) at the points \( P_1 \) and \( P_2 \) be \( \theta_1 \) and \( \theta_2 \), respectively. If \( b \) and \( a \) are the unknown values of the semiaxes of \( H_{1,2} \) both equations in (2.4) will be satisfied by the pairs of the angles \( (\theta_1, \psi_1) \) and \( (\theta_2, \psi_2) \). Thus, on elimination of the semiaxes from (2.4), two equations in \( \theta_1 \) and \( \theta_2 \) are obtained which after some simple algebraic manipulation involving trigonometric identities yield the following nonlinear system

\[
\begin{align*}
r_1 \cos((\frac{\pi}{2}(\theta_1 - \theta_2) - \theta_1 - \psi_1) - r_2 \cos((\frac{\pi}{2}(\theta_2 - \theta_1) - \theta_2 - \psi_2) &= 0, \\
r_1 \cos((\frac{\pi}{2}(\theta_1 + \theta_2) - \theta_1 - \psi_1) - r_2 \cos((\frac{\pi}{2}(\theta_2 + \theta_1) - \theta_2 - \psi_2) &= 0.
\end{align*}
\tag{4.8}
\]

In case \( r_1 < r_2 \), we determine \( \theta_{II} \) for \( P_1 \) and \( \theta_{II} \) for \( P_2 \) from (2.6), let them be denoted by \( \theta_{II,1} \) and \( \theta_{II,2} \), respectively, while in case \( r_1 > r_2 \), we determine \( \theta_{I,1} \) and \( \theta_{I,1,2} \) for both points from (2.5). System (4.8) is solved by Newton's method for nonlinear systems with initial approximation the pair

\[
(\theta_1^{(0)}, \theta_2^{(0)}) = \left(\frac{\theta_{II,1} - \psi_1}{2}, \frac{\theta_{II,2} - \psi_2}{2}\right)
\tag{4.9}
\]

in case of a hypo II and with

\[
(\theta_1^{(0)}, \theta_2^{(0)}) = \left(\frac{\theta_{I,1} - \psi_1}{2}, \frac{\theta_{I,2} - \psi_2}{2}\right)
\tag{4.10}
\]

in case of a hypo I. It is known that if \( (\theta_1^{(0)}, \theta_2^{(0)}) \) is a good approximation to the solution, the Newton's method converges quadratically. However, if it is not, the Newton's method may diverge. In the latter case we have as an alternative an algorithm based on the idea behind
the proof of Thm 3.4 and of the Bisection method. We present this algorithm below.

**BISECTION ALGORITHM**

Given the (polar) coordinates of the points $P_1$ and $P_2$ as in the beginning of the present section. Furthermore, assume that $b_{II,2} < b_{II,1} < b_{I,1} < b_{I,2}$ and that $b_{II,1} < 1$;

\[
b_{\text{min}} := b_{II,1}; \quad b_{\text{max}} := \min\{b_{I,1}, 1\};
\]

Again: \( b = \frac{b_{\text{min}} + b_{\text{max}}}{2} \).

Determine $H_1(b) \equiv H_1(a_1); \quad H_2(b) \equiv H_2(a_2);$ if $|a_1 - a_2| > \text{tolerance}$ then

- if $a_1 < a_2$ then
  - $b_{\text{max}} := b; \quad \text{goto again};$
  - $\text{else } b_{\text{min}} := b; \quad \text{goto again};$
  - endif

else Either $H_1(b)$ or $H_2(b)$ can be considered as the desired hypo $H_{1,2}$; stop;

endif

end of BISECTION ALGORITHM;

5 The Solution to the "Many-Point" Problem

Before we develop an algorithm for the solution to the many-point problem we go through some preliminary steps in order to make sure that in the algorithm we will consider only those eigenvalues of $J_p$ that may play a role to the solution of the problem in question. For this, the eigenvalues to be eventually retained, after a preliminary procedure takes place, will be in a relative position wrt each other similar to the one considered in the solution to the two-point problem (see, e.g., Thm 3.4, Thm 4.1, or Algorithm 2). The basic steps that are to be followed for this procedure to be completed are the three steps described below:

**Step 1:** First, suppose that all the "eigenvalues" (eigenvalues of $J_p$ in the last $(2p) - \text{ant}$ and/or images of the symmetric of the eigenvalues of $J_p$ in the first $(2p) - \text{ant}$, wrt the real axis, that are not included in the previous ones) are given by their polar coordinates $P_j(\tau_j, \psi_j), \ j = 1(1)k$. Next, suppose that these eigenvalues are such that $\sigma(J_p)$ is contained in the intersection of the closures of the interiors of all cusped hyps $I$ and $II$ through the aforementioned eigenvalues (points). Finally, suppose also that these eigenvalues have been ordered in a decreasing order of their arguments, that is $0 \geq \psi_1 \geq \psi_2 \geq \cdots \geq \psi_k \geq -\pi$. If two or more eigenvalues have the same argument, we retain the one which corresponds to the largest polar radius, discard all the others and relabel the remaining eigenvalues. Let $k_1$ be number of the eigenvalues in the new set.

**Step 2:** To have convergent SORs associated with the set of the $k_1$ eigenvalues, all of them must lie in the closed interior of the curvilinear triangle $OBA$, whose vertices (in polar coordinates) are the points $O(0,0), B(1,0)$ and $A\left(\frac{\pi}{p-2}, -\frac{\pi}{p}\right)$, except on the curved side $BA$ which is nothing but the cusped hypo $II$ passing through the point $B(1,0)$. To check if this
is indeed the case, we can work as in Problem I. In other words, we check if there is an eigenvalue in the complement of the closed interior of the aforementioned cusped hypo \( II \) (\( BA \)), in which case no convergent SOR exists. Another way of checking it is to find the real semi-axes \( b_{II,j}, j = 1(1)k_1 \), of the cusped hyps \( II \) that pass through each of the points \( P_j, j = 1(1)k_1 \). If \( b_{II,j} \geq 1 \), for some \( j \), then no convergent SOR associated with the set of the \( k_1 \) eigenvalues exists.

**Step 3:** In case we have convergent SORs associated with each of the \( k_1 \) eigenvalues, we reorder and relabel the eigenvalues so that the real semi-axes of the cusped hyps \( II \) through them are in a nonincreasing order, that is \( b_{II,1} \geq b_{II,2} \geq \cdots \geq b_{II,k_1} \). In case of equality we retain the eigenvalue corresponding to the smallest \( \psi_j \), discard all the others and relabel. Let \( k_2 \) be the number of the remaining eigenvalues in the set. Then, we determine the corresponding real semi-axes of the cusped hyps \( I \) through the \( k_2 \) points. Let them be denoted by \( b_{I,j}, j = 1(1)k_2 \). If \( b_{I,j-1} \geq b_{I,j} \), for some \( j \), we discard the point \( P_j \) and relabel. Let \( k_3 \) be the number of the remaining eigenvalues in the set. To simplify the notation we drop the index 3 from \( k_3 \) and assume that we end up with a number of eigenvalues denoted by the same symbol \( k \).

After the previous three preliminary steps have been carried out the ordering of the real semi-axes of the cusped hyps \( I \) and \( II \) through the remaining eigenvalues (points) \( P_j, j = 1(1)k \), will be as follows:

\[
b_{II,k} < b_{II,k-1} < \cdots < b_{II,1} \leq b_{I,1} < b_{I,2} < \cdots < b_{I,k} \quad \text{and} \quad b_{II,1} < 1. \tag{5.1}
\]

As one can see by drawing a brief graph we are in a more general situation than that described in the two-point problem (see, e.g., Thm 3.4, Thm 4.1, or Algorithm 2). In other words, for any two points \( P_{j_1}, P_{j_2}, 1 \leq j_1 < j_2 \leq k \), \( P_j \) lies strictly within the curvilinear triangle \( P_{j_1}A_{I,j_1}A_{II,j_2} \). The two vertices \( A_{I,j_1} \) and \( A_{II,j_2} \) have polar coordinates \((\frac{\pi}{p} b_{I,j_1}, -\frac{\pi}{p})\) and \((\frac{\pi}{p} b_{II,j_2}, -\frac{\pi}{p})\), respectively, while the curved sides \( P_{j_1}A_{I,j_1} \) and \( P_{j_2}A_{II,j_2} \) are the arcs of the cusped hyps \( I \) and \( II \) through the point \( P_j \). (Note: It is understood that if at any stage of the three preliminary steps any of the numbers \( k, k_1, k_2 \), or \( k_3 \) is equal to 2 or to 1, then we are in the case of the two- or the one-point problem and Algorithm 2 or Algorithm 1, respectively, must be applied. So, for the application of the many-point problem algorithm it will be assumed that the (initial and the) final \( k \geq 3 \).)

As in the two-point problem case although convergent SORs are associated with values of \( b \in [b_{II,1}, 1) \) in some cases we may consider the interval \([b_{II,1}, \min\{b_I,k_1\}]\) instead. This is because as we will see the optimal hypo for the many-point problem has a real semi-axis \( b \) that always lies in the latter interval. So, what we will seek in the sequel will be the solution to the following optimization problem which is the complete analog to the one in (4.2)

\[
\min_{b \in [b_{II,1}, \min\{b_I,k_1\}]} \max\{x_j(b), j = 1(1)k\}. \tag{5.2}
\]

Below we state and prove two theorems referring to the many-point problem case as this is indirectly defined in (5.2). In Thm 5.1, first we prove the existence of an optimal hypo
and then show among which specific hypos the optimal one is to be sought. In Thm 5.2 the uniqueness of the optimal hypo is proved. In the proofs a different approach from that in Thm 4.1 is followed.

**Theorem 5.1:** Under the notation used and the assumptions made so far, especially those of the present section, suppose that the \( k (\geq 3) \) points \( P_j, j = 1(1)k \), given by their polar coordinates, lie in the last \((2p) - \text{ant}\) and are such that relationships (5.1) hold. Then, the solution to the optimization problem (5.2) exists and is given by the elements of a hypo that is among the \( k \) optimal hypos \( H_j \) through the points \( P_j, j = 1(1)k \), and the \( \binom{k}{2} \) hypos \( H_{j_1,j_2} \) through the pairs of points \( P_{j_1}, P_{j_2}, 1 \leq j_1 < j_2 \leq k \).

**Proof:** The existence of a hypo that solves our problem and passes through at least one of the points \( P_j \) is almost obvious. For this consider any hypo with semiaxes \( b (\leq 1) \) sufficiently close to 1 and \( a \in [\max\{\frac{\pi}{2}b, k, \frac{\pi}{2}b\}], \frac{\pi}{2}b \) such that it contains all the \( k \) points (eigenvalues) strictly in its interior. Consider next the homothetic class of hypos to the one considered previously \( \text{wrt} \) the origin (center of homothecy) and then the subclass of the homothetic class such that each member of the latter contains the set of the \( k \) eigenvalues in the closure of its interior. Obviously, all the hypos of the subclass in question are associated with convergent SORs and the smallest of them will pass through one or more than one points of the set of the \( k \) points. In the latter case the proof is concluded. In the former case consider the last hypo of the homothetic class that passes through one of the points, say \( P_{j_1} \), of the set. Let \( \bar{b} \) be its real semiaxis. If \( H_j(\bar{b}) \) is the optimal hypo for the point \( P_j \) then we are done. Otherwise, we go on starting with \( H_j(\bar{b}) \), which is a member of the family of hypos \( \mathcal{H}_{P_j} \), and consider only members of this family \( \mathcal{H}_{P_j} \) by continuously varying in a monotone fashion their real semiaxes from the value \( \bar{b} \) in the direction in which the spectral radii of the associated SOR iteration matrices decrease. Consider only those hypos that contain the set of \( k \) points in the closures of their interiors. Obviously, during this continuous transformation, one of two things will happen first. Either the real semiaxis will assume its optimal value for the point \( P_j \), in which case we are done, or as \( b \) varies in the monotone fashion explained previously there will be a first hypo of the family that will pass through a second point. This concludes the proof. Thus we have effectively proved both assertions of the present theorem. \( \Box \)

In Thm 5.1 the existence of the optimal solution to the *many-point* problem has been established and, in addition, all possible candidates for the optimal hypo are indicated. The uniqueness of the optimal hypo remains to be proved and this is done in Thm 5.2 below. After the proof of the theorem is given, an algorithm (Algorithm 3) will be developed to determine the unique optimal hypo in a systematic way that will avoid unnecessary computations.

**Theorem 5.2:** Under the notation and assumptions of Thm 5.1, there is a hypo out of the \( k + \left( \binom{k}{2} \right) \) ones mentioned in its statement that is the optimal one for the set of the
k points. Furthermore, if the optimal hypo of the set is not among the k optimal hypos \( \widehat{H}_j \), \( j = 1(1)k \), it will be the one among the \( \binom{k}{2} \) hypos \( H_{j_1,j_2} \), \( 1 \leq j_1 < j_2 \leq k \), that corresponds to the smallest \( x_{j_1,j_2} \), \( 1 \leq j_1 < j_2 \leq k \). In either case the optimal hypo is unique.

**Proof:** That an optimal hypo exists and is among the \( k + \binom{k}{2} \) hypos of Thm 5.1 is obvious from its proof. Suppose then that for some \( j \in \{1, 2, \ldots, k\} \) the optimal hypo \( \widehat{H}_j \) contains all the \( k \) points in the closure of its interior. Then this hypo is the optimal one for the entire set. To prove it, consider first any other point \( P_l \), \( l \in \{1, 2, \ldots, k\} \setminus \{j\} \) together with its associated optimal hypo \( \widehat{H}_l \) and recall Thm 4.1. If \( P_l \) lies strictly in the interior of \( \widehat{H}_j \), then \( \widehat{H}_l \) cannot contain \( P_l \) in the closure of its interior and so \( \widehat{H}_j \) is the best of the two hypos. If \( P_l \) lies on \( \widehat{H}_j \) then \( \widehat{H}_l \) can be at most as good as \( \widehat{H}_j \). In case \( \widehat{H}_l \) is as good as \( \widehat{H}_j \) then we are in a trivial case of the two-point problem where \( \widehat{H}_l \equiv \widehat{H}_i \equiv H_{j,l} \). Suppose next that there exists a hypo different from \( \widehat{H}_j \) passing through two points of the set and containing all the \( k \) points in the closure of its interior. Let it be \( H_{j_1,j_2} \), \( 1 \leq j_1 < j_2 \leq k \). If one of the two points coincides with \( P_j \) then \( \widehat{H}_j \) is already the optimal hypo through \( P_j \) and \( H_{j_1,j_2} \) cannot be better than \( \widehat{H}_j \). If neither point \( P_{j_1}, P_{j_2} \) coincides with \( P_j \) then it will be either \( j < j_1 < j_2 \) or \( j_1 < j_2 < j \). Consider the homothetic to \( H_{j_1,j_2} \) hypo through \( P_j \). This hypo will be better than \( H_{j_1,j_2} \) but not as good as \( \widehat{H}_j \). This is because the latter is the optimal hypo through \( P_j \). Consequently, we come to the same conclusion. That is, if \( \widehat{H}_j \) contains all the \( k \) points in the closure of its interior then it is the optimal hypo \( \widehat{H}_j \) of the set and is therefore unique. Suppose now that there is no hypo \( \widehat{H}_j \), \( j = 1(1)k \), that contains all the \( k \) points in the closure of its interior. Then the optimal hypo will be one of the \( H_{j_1,j_2}'s \), \( 1 \leq j_1 < j_2 \leq k \). Our claim is again that the optimal one is unique. Suppose then that there are two (or more) distinct hypos \( H_{j_1,j_2} \) and \( H_{j_3,j_4} \), where \( j_1, j_2, j_3, j_4 \in \{1, 2, \ldots, k\} \), \( j_1 < j_2, j_3 < j_4 \) and \( j_1 < j_2 \), that solve the optimization problem (5.2). By virtue of Thm 3.2, for these two hypos it cannot hold \( P_{j_1} \equiv P_{j_2} \) or \( P_{j_3} \equiv P_{j_4} \). Suppose, for example, that \( P_{j_1} \equiv P_{j_3} \). Then either \( P_{j_4} \) will be outside the closed interior of \( H_{j_1,j_2} \) or \( P_{j_2} \) will be outside the closed interior of \( H_{j_3,j_4} \). However, either of these conclusions contradicts the assumption made, that is, each of the two hypos contains all \( k \) points in the closure of its interior. A similar argument applies if \( P_{j_2} \equiv P_{j_4} \). In case \( P_{j_2} \equiv P_{j_3} \) then \( b_{1,2} < b_{3,4} \). By assumption, the hypo \( \widehat{H}_{j_2,j_4} \) is not the optimal one of the set of \( k \) points. Let \( \widehat{b}_{2=3} \) be its real semiaxis. Since \( \widehat{H}_{j_2,j_4} \) intersects both \( H_{j_1,j_2} \) and \( H_{j_3,j_4} \) at \( P_{j_2} \equiv P_{j_3} \) it will be either \( \widehat{b}_{2=3} < b_{1,2} < b_{3,4} \) or \( b_{1,2} < b_{3,4} < \widehat{b}_{2=3} \). However, by virtue of Theorem 2.1 and relationship (2.11), in the former case, \( H_{j_1,j_2} \) will be better than \( H_{j_3,j_4} \) and therefore will be the optimal hypo, while in the latter case the situation will be reversed. Consequently the uniqueness is proved. Finally, if \( j_1 < j_2 < j_3 < j_4 \) then the two hypos will intersect each other at a point \( P_{k+1} \). If \( \widehat{H}_{k+1} \) contains all the \( k \) points in the closure of its interior then there exists a point \( P_j, j_2 \leq j \leq j_3 \) such that the homothetic hypo to \( \widehat{H}_{k+1} \) through \( P_j \), contains all the \( k \) points in the closure of its interior. In such a case the corresponding optimal hypo \( \widehat{H}_j \) or a two point hypo \( H_{j,j'} \), \( j_2 \leq j' \leq j_3 \),
will be the optimal one, which contradicts our assumption, that is \( H_{j_1,j_2} \) and \( H_{j_3,j_4} \) are the optimal hypos. So, \( \tilde{H}_{ki} \) will contain only one of the two pairs of points \((P_{j_1}, P_{j_2})\) or \((P_{j_3}, P_{j_4})\) in the closure of its interior. In this case the proof is similar to that of the previous case of the three points, where now the point \( P_{ki+1} \) plays the role of the point \( P_{j_2} \equiv P_{j_3} \). Thus the examination of all possible cases has been exhausted and the uniqueness of the optimal hypo is established. \( \square \)

**Remark:** A remark similar to but more general than the one made after Thm 4.1 can also be made here. More importantly, however, it can be explored in such a way that in order to determine the optimal hypo it may not be necessary to go through all possible hypos of Thm 5.2. For example, if the hypo \( H_{j_1,j_2} \) of the two points \( P_{j_1}, P_{j_2}, 1 \leq j_1 < j_2 \leq k \), that are not consecutive ones contains all the intermediate points \( P_j, j = j_1 + 1, j_1 + 2, \ldots, j_2 - 1 \), in the closure of its interior then all intermediate points can be discarded. Consequently, for the determination of the optimal hypo of the set of the \( k \) points it will not be necessary to determine and investigate for optimality either the optimal hypos for these intermediate points or the hypos that pass through any two of them.

We conclude this section by presenting the algorithm that solves the many-point problem. As one can observe this algorithm (Algorithm 3) is a systematic extension of Algorithm 2, that solves the two-point problem \((k = 2)\), and is in complete analogy to the one developed by Young and Eidson [26] for the case \( p = 2 \) (see also [25], [6]).

**ALGORITHM 3: Many-point problem**

Given the (polar) coordinates of \( k \geq 3 \) points \( P_j \) in the last \( 2p-\text{ant} \). Suppose that these \( k \) points satisfy relationships (5.1) as well as \( b_{II,1} < 1 \).

\( y_{old} := 0; \quad x_{old} := 1; \quad l := 1; \)

again: \( y_{new} := 1; \)

for \( j := l + 1 \) step 1 to \( k \) do

Determine \( H_{ij}(b_{ij}) \);

if \( y_{new} > b_{ij} \) then

\( s := j; \quad y_{new} := b_{ls}; \)

endif;

enddo;

Determine \( \tilde{H}_{ii}(b_i); \)

if \( \tilde{b}_i < y_{old} \) or \( \tilde{b}_i > y_{new} \) then

Determine \( x_{ls} \) \((= x_0 \text{ corresponding to } H_{ls}); \)

if \( x_{ls} < x_{old} \) then

\( x_{old} := x_{ls}; \quad r := l; \quad q := s; \)

endif;

if \( s = k \) then

Determine \( \tilde{H}_k(b_k); \)

if \( \tilde{b}_k \geq y_{new} \) then

\( \tilde{H} \equiv \tilde{H}_k; \quad \tilde{x} := \tilde{x}_k; \quad \tilde{b} := \tilde{b}_k; \quad \tilde{a} := \tilde{a}_k; \)

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else $\widehat{H} \equiv H_{rq}; \hat{x} := x_{old}; \hat{b} := b_{rq}; \hat{a} := a_{rq}$
endif;
else $l := s; y_{old} := y_{new};$ goto again;
endif;
else $\widehat{H} \equiv \widehat{H}_{i}; \hat{x} := \hat{x}_{i}; \hat{b} := \hat{b}_{i}; \hat{a} := \hat{a}_{i};$
endif;
Determine $\hat{\omega} := \frac{2\hat{x}_{p}}{b_{i}+2}; \rho(L_{\hat{\omega}}) := \hat{x}_{p};$
end of ALGORITHM 3;

6 Numerical Examples

A number of nine selected numerical examples have been worked out by applying Algorithm 2 for Examples 1–3 and Algorithm 3 for Examples 4–9. For Examples 1–3, $k = 2$, for Examples 4–6, $k = 3$, for Examples 7–8, $k = 4$, and $k = 5$ for Example 9, as these are shown in Table 2. In each case $p = 3$ and the points $P_{j}$, $j = 1, 2, \ldots, k$, are given by their polar coordinates. All of the examples have been selected in such a way so that the points involved satisfy all the basic assumptions of Thm 4.1, for $k = 2$, and of Thms 5.1 and 5.2 for $k \geq 3$. In other words, if one applies the three basic steps of the preliminary procedure described in Section 5 none of the given points in any of the examples presented will be discarded.

As is illustrated in the self-explained Table 2, its fourth column gives the optimal hypo $\widehat{H}$ of the set of points considered in each case. In the other columns, some of the main elements of the optimal hypo $\widehat{H}$ and of the associated optimal SOR method are also presented.

References

### Table 2: Numerical Examples

<table>
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<tr>
<th>Example Number</th>
<th>$k$</th>
<th>$P(r, \psi)$</th>
<th>$\hat{H}$</th>
<th>$\hat{b}$</th>
<th>$\hat{a}$</th>
<th>$\hat{x}$</th>
<th>$\hat{w}$</th>
<th>$\rho(L_{\alpha})$</th>
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