1995

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Report Number:

95-074

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CSD TR-95-074
November 1995

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Abstract

In this work we consider the Successive Overrelaxation (SOR) method for the solution of a linear system $Ax = b$, when the matrix A has a block $p \times p$ partitioned p-cyclic form and its associated block Jacobi matrix J_p is weakly cyclic of index p. Following the pioneering work by Young and Varga in the 50s many researchers have considered various cases for the spectrum $\sigma(J_p)$ and have determined (optimal) values for the relaxation factor $\omega \in (0,2)$ so that the SOR method converges as fast as possible. After the most recent work on the best block p-cyclic repartitioning and that on the solution of large scale systems arising in queueing network problems in Markov analysis, the optimization of the convergence of the p-cyclic SOR for more complex spectra $\sigma(J_p)$ has become more demanding. Here we state the *"one-point"* problem for the general p-cyclic complex SOR case. The existence and the uniqueness of its solution are established by analyzing and developing further the theory of the associated hypocycloidal curves. For the determination of the optimal parameter(s) an algorithm is presented and a number of illustrative numerical examples are given.

Subject Classifications: AMS(MOS): 65F10. CR Category: 5.14

Key Words: iterative methods, p-cyclic matrices, successive overrelaxation, hypocycloidal curves

Running Title: Optimal p-cyclic SOR

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[2]The work of this author was supported in part by NSF grant CCR 86-19817, AFOSR 91-F49620 and ARPA grant DAAH04-94-G-0010.

1 Introduction

Block iterative methods are suitable for the solution of large sparse linear systems having matrices that possess a special structure. In the present work we consider the block p–cyclic SOR. Given

$$Ax = b, \quad A \in \mathbb{C}^{n \times n}, \ x, b \in \mathbb{C}^n \tag{1.1}$$

and the block decomposition

$$A = D - L - U \tag{1.2}$$

where D, L and U are nonsingular block diagonal, strictly lower and strictly upper triangular matrices, respectively, the block SOR method for any $\omega \neq 0$ is defined as follows

$$x^{(m+1)} = \mathcal{L}_\omega x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \tag{1.3}$$

where

$$\mathcal{L}_\omega := (D - \omega L)^{-1}\left[(1-\omega)D + \omega U\right], \quad c := \omega(D - \omega L)^{-1} b \tag{1.4}$$

and $x^{(0)} \in \mathbb{C}^n$ arbitrary. It is well known that, for nonsingular linear systems (1.1), SOR converges iff $\rho(\mathcal{L}_\omega) < 1$; also that, for $\omega \in \mathbb{R}$, $\omega \in (0, 2)$ constitutes a necessary condition for SOR to converge.

For arbitrary matrix coefficient A in (1.1), little is known about the value of the optimal relaxation factor ω that minimizes $\rho(\mathcal{L}_\omega)$. However, for the case where A has a special block cyclic structure more is known. In this case we assume, without loss of generality, that A has the block form

$$A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & A_{1p} \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{p,p-1} & A_{pp} \end{pmatrix}. \tag{1.5}$$

With D in (1.2) defined by $D \equiv \operatorname{diag}(A_{11}, A_{22}, \ldots, A_{pp})$, the associated block Jacobi iteration matrix $J_p \equiv I - D^{-1}A$ has the form

$$J_p = \begin{pmatrix} 0 & 0 & 0 & \cdots & B_1 \\ B_2 & 0 & 0 & \cdots & 0 \\ 0 & B_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_p & 0 \end{pmatrix}. \tag{1.6}$$

Matrices of the form (1.6) were defined by Varga [20] to be *weakly cyclic of index p*, and in this case A in (1.5) is termed *block p–cyclic and consistently ordered.* For such matrices Varga [20] proved the relationship

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{p-1} \tag{1.7}$$

between the eigenvalues μ of J_p and λ of $\mathcal{L}_\omega$, generalizing in this way Young's relationship for $p = 2$ [23]. Under the further assumption that all eigenvalues of J_p^p satisfy

$$0 \leq \mu^p \leq \rho(J_p^p) < 1,$$

Young and Varga determined the unique optimal values for the parameter ω, denoted from now on by $\widehat{\omega}$ (see also [21], [24], [1]). Similar results have been obtained (see [10, 12, 13, 22, 5]) for the case where the eigenvalues of J_p^p are nonpositive, that is,

$$-(\frac{p}{p-2})^p < -\rho(J_p^p) \leq \mu^p \leq 0.$$

For the case where the eigenvalues of J_p^p are real the corresponding problem has been solved very recently [2, 15]. Meanwhile and *only* for $p = 2$ the known as the *"one-point"* problem that is when $\sigma(J_2)$ is complex and lies in a rectangle, which is symmetric *wrt* the real and the imaginary axes and is strictly within the infinite unit strip, was solved in [9, 18]. (*Note:* The term *"one-point"* comes from the fact that the only information needed to find the optimal ω ($\widehat{\omega}$) is the coordinates of the vertex of the rectangle in the first quadrant.) Later, Young and Eidson [25] obtained the solution to the more general *"many-point"* problem (see also [24]). It is also important to note here that for the solution of all the arising minimization problems one uses, directly or indirectly, conformal mapping transformations. Because of the transformations involved one deals with ellipses for $p = 2$ and with hypocycloidal curves of cusped, shortened and stretched type for $p \geq 3$ (see, e.g., [14, 22]) which are depicted in Figure 1.

After the most recent work on the best block $p-$cyclic repartitioning by Markham, Neumann and Plemmons [11], Pierce, Hadjidimos and Plemmons [16], Eiermann, Niethammer and Ruttan [2] and Galanis and Hadjidimos [4] and the work on the solution of large scale systems arising in queueing network problems in Markov analysis, with direct applications to computer, communication and transportation systems, by Kontovasilis, Plemmons and Stewart [8] and Hadjidimos and Plemmons [6, 7] the optimization of the convergence of the $p-$cyclic SOR for more complex spectra $\sigma(J_p)$ has become more demanding.

It is the main objective of this paper to study and solve the more general problem of the minimization of the spectral radius of the SOR iteration matrix in the case of the *"one-point"* problem for $p \geq 3$. The present work is organized as follows. In Section 2 the main problem is described and various properties of a class of hypocycloidal curves, referred to as *"hypos"* from now on, associated with the problem of interest are analyzed and studied. Based on the properties of the hypos through a given point and their associated SOR methods it is proved in Section 3 that the optimal solution to the *"one-point"* problem, if it exists, is given by means of a shortened hypo. In Section 4, the existence and the uniqueness of the solution to our problem are established. Finally, in Section 5, a numerical algorithm is presented for the determination of the solution and a number of numerical examples are given.

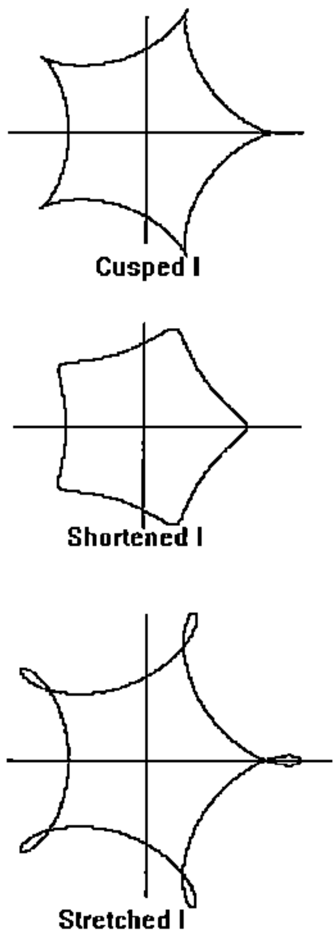

Figure 1: Hypocycloidal Curves of all kinds and types for $p = 5$

2 Analysis and Study of a Class of Hypocycloids

Let $\sigma(J_p)$ be the spectrum of the Jacobi matrix J_p in (1.6). This has a $p-$cyclic symmetry about the origin (see [21]). Let P be a point of the complex plane strictly within the $(2p)^{th}$ open $(2p)-ant$ of it with polar coordinates (r,ψ), $r>0$, $-\frac{\pi}{p}<\psi<0$. (*Note:* The extreme cases $r=0$, $\psi=0$ and $\psi=-\frac{\pi}{p}$ can be treated as limiting cases of the ones to be studied.) Let (α,β) be the cartesian coordinates of P. There will be

$$\begin{array}{ll} \alpha = rcos\psi, & \beta = rsin\psi, \\ tan\psi = \frac{\beta}{\alpha}, & r=(\alpha^2+\beta^2)^{1/2}. \end{array} \tag{2.1}$$

Suppose that either $P(r,\psi)$ or its symmetric *wrt* the real axis, $P'(r,-\psi)$, is an element of $\sigma(J_p)$. Suppose also that all possible hypos with p *"vertices"* that are symmetric *wrt* the real axis and pass through P contain $\sigma(J_p)$ in the closure of their interior. Our main problem is then that of determining, among all the associated convergent block SOR methods in case such SORs exist, the one that is asymptotically faster. For this, an analysis and a study of some further properties of the class of all the aforementioned hypos through P must be made.

The parametric equations of the cartesian coordinates of the points of a hypo, when the parameter t takes all values in $[0,2\pi)$, are given by the expressions

$$\begin{array}{ll} x(t) = & \frac{b+a}{2}cost + \frac{b-a}{2}cos(p-1)t, \\ y(t) = & -\frac{b+a}{2}sint + \frac{b-a}{2}sin(p-1)t \end{array} \tag{2.2}$$

(see, e.g., [19] or [22]). In (2.2), b and a are to be called the *"real"* and the *"imaginary"* semiaxes of the hypo because in the trivial case $p=2$ they are nothing but the corresponding semiaxes of an ellipse. In view of the $p-$cyclic symmetry of the hypo we will consider that $t\in[0,\frac{\pi}{p}]$. So the associated arc of the hypo will lie in the last $(2p)-ant$ since it is described in a clockwise fashion when t increases. The real and the imaginary semiaxes of the hypo will lie then along the real positive semiaxis and the ray with argument $-\frac{\pi}{p}$, respectively.

Let $t=\theta$ be the value of the parameter corresponding to the point P of a hypo passing through it with semiaxes b and a (see Fig. 2). From (2.1) and (2.2) it can be obtained that

$$\begin{array}{ll} \alpha = & \frac{b+a}{2}cos\theta + \frac{b-a}{2}cos(p-1)\theta, \\ \beta = & -\frac{b+a}{2}sin\theta + \frac{b-a}{2}sin(p-1)\theta, \\ r = & \left\{\frac{1}{2}[b^2+a^2+(b^2-a^2)cosp\theta]\right\}^{1/2}. \end{array} \tag{2.3}$$

We begin our analysis with a number of propositions that will be stated and proved. In order to simplify the notation we will be using "$A\sim B$" to denote that the two quantities or expressions A and B are of the same sign, that is $sign(A)=sign(B)$.

Lemma 2.1 The semiaxes of any hypo passing through P are given by the expressions

$$\begin{array}{lll} b = & \frac{1}{cos(\frac{p\theta}{2})}[\alpha cos(\frac{p}{2}-1)\theta + \beta sin(\frac{p}{2}-1)\theta] = & \frac{r}{cos(\frac{p\theta}{2})}cos((\frac{p}{2}-1)\theta-\psi), \\ a = & \frac{1}{sin(\frac{p\theta}{2})}[\alpha sin(\frac{p}{2}-1)\theta - \beta cos(\frac{p}{2}-1)\theta] = & \frac{r}{sin(\frac{p\theta}{2})}sin((\frac{p}{2}-1)\theta-\psi). \end{array} \tag{2.4}$$

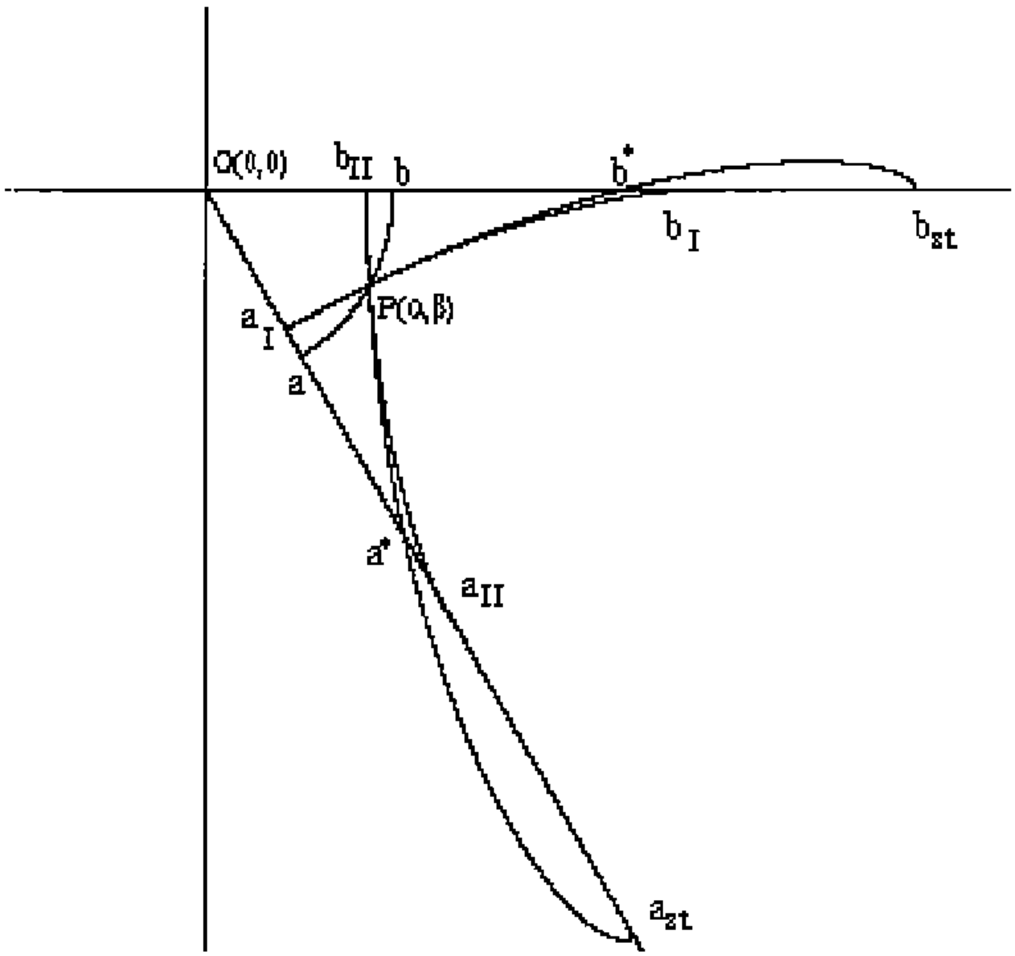

Figure 2: Hypocycloidal Curves passing through the point P

Proof: From the first two equations of (2.3), solving first for $\frac{b+a}{2}$ and $\frac{b-a}{2}$ and then for b and a, using simple trigonometric identities, the middle expressions in (2.4) are readily obtained. Then, from the latter expressions the ones on the right are easily obtained by plugging in the expressions for α and β from (2.1). □

Lemma 2.2 For any hypo passing through P there hold

$$b \quad \begin{cases} > r > a & if \quad \theta \in (-\psi, \frac{\pi}{p}), \\ = r = a & if \quad \theta = -\psi, \\ < r < a & if \quad \theta \in (0, -\psi). \end{cases} \tag{2.5}$$

Proof: In view of the rightmost expressions in (2.4) we easily obtain after some simple manipulation, where positive common factors are omitted, that $b - a \sim sin(\theta + \psi)$. However, since $0 < \theta, -\psi < \frac{\pi}{p}$ and $-\frac{\pi}{p} < \theta + \psi < \frac{\pi}{p}$, the relationships between b and a depending on the values of θ in (2.5) are obtained. Based on these relationships and the third one in (2.3) the assertions regarding r in (2.5) readily follow. □

From now on a hypo through P will be termed *hypo of type I*, or simply *hypo I*, *iff* $b > a$ or, equivalently, *iff* $\theta \in (-\psi, \frac{\pi}{p})$. It will be termed *hypo II iff* $b < a$ or *iff* $\theta \in (0, -\psi)$ (see Fig. 1). If $b = a$ the hypo is a circle and constitutes a limiting case of both types of hypos I and II.

Lemma 2.3 There exist uniquely determined cusped hypos I and II through P corresponding to $t = \theta_I \in (0, \frac{\pi}{p})$ and $t = \theta_{II} \in (0, \frac{\pi}{p})$, respectively. For these hypos there hold

$$\begin{aligned} (p-1)sin(\theta_I + \psi) - sin((p-1)\theta_I - \psi) &= 0, \\ (p-1)sin(\theta_{II} + \psi) + sin((p-1)\theta_{II} - \psi) &= 0. \end{aligned} \tag{2.6}$$

Furthermore, θ_I and θ_{II} satisfy the inequalities

$$0 < \theta_{II} < -\psi < \theta_I < \frac{\pi}{p}. \tag{2.7}$$

Proof: The proof will be given for the cusped hypo I since the corresponding proof for the cusped hypo II is similar. As is known for the cusped hypo I it is $a = \frac{p-2}{p}b$. Hence, using the expressions from (2.1) and (2.2) one obtains

$$tan\psi = \frac{\beta}{\alpha} = \frac{-(p-1)sint + sin(p-1)t}{(p-1)cost + cos(p-1)t} =: K_I(t) \tag{2.8}$$

(see also (3.1) of [3]). Differentiating the function $K_I(t)$ *wrt* t, omitting positive common factors and using simple trigonometric identities, we obtain, after some manipulation, that

$$\frac{\partial K_I(t)}{\partial t} \sim -(p-2)(1 - cospt). \tag{2.9}$$

Since $t \in [0, \frac{\pi}{p}]$, $K_I(t)$ is a strictly decreasing function of t taking values from 0 to $-tan\frac{\pi}{p}$. Thus there will exist a unique value of t, denoted by θ_I such that $K_I(\theta_I) = tan\psi$. From the expression for $K_I(\theta_I)$ in (2.8) one can very easily obtain the first equation in (2.6). Also, in view of Lemma 2.2 and the concept of a hypo I the validity of the two rightmost inequalities in (2.7) is readily established. For the cusped hypo II we simply note that the corresponding to (2.8) relationships are

$$tan\psi = \frac{\beta}{\alpha} = \frac{(p-1)sint + sin(p-1)t}{-(p-1)cost + cos(p-1)t} =: K_{II}(t) \tag{2.10}$$

(see also (3.1') of [3]). □

Having done the analysis so far we are now able to prove that the semiaxes b and a of the hypos through P are continuous functions of $\theta \in (0, \frac{\pi}{p})$. Moreover, they have derivatives *wrt* θ that are of constant sign in specified subintervals of θ. The latter will establish the strictly monotonic behavior of each of the three elements (variables) a, b and θ in terms of either of the others.

For this, we define the functions F and G below by using the middle expressions in (2.4)

$$\begin{aligned} F := F(b,\theta) \equiv \quad & \alpha cos(\tfrac{p}{2}-1)\theta + \beta sin(\tfrac{p}{2}-1)\theta - bcos(\tfrac{p\theta}{2}) = 0, \\ G := G(a,\theta) \equiv \quad & \alpha sin(\tfrac{p}{2}-1)\theta - \beta cos(\tfrac{p}{2}-1)\theta - asin(\tfrac{p\theta}{2}) = 0. \end{aligned} \tag{2.11}$$

First, we work with the function F, differentiate it *wrt* b and then *wrt* θ. So, we get

$$F_b' = -cos(\frac{p\theta}{2}) < 0, \quad \forall\, \theta \in (0, \frac{\pi}{p}), \tag{2.12}$$

and by simple manipulation, using the middle expression for b from (2.4) and the expressions for α and β from (2.1), we obtain that

$$\begin{aligned} F_\theta' \ &= \ \beta(\tfrac{p}{2}-1)cos(\tfrac{p}{2}-1)\theta - \alpha(\tfrac{p}{2}-1)sin(\tfrac{p}{2}-1)\theta + \tfrac{p}{2}bsin(\tfrac{p\theta}{2}) \\ &\sim \ (p-1)sin(\theta+\psi) + sin((p-1)\theta - \psi) =: k(\theta,\psi). \end{aligned} \tag{2.13}$$

However, from (2.5) for hypos I we have that both *sines* in $k(\theta, \psi)$ in (2.13) are positive. Thus

$$F_\theta' > 0, \quad \forall\, \theta \in (-\psi, \frac{\pi}{p}). \tag{2.14}$$

On the other hand, for hypos II, we readily obtain from (2.13) that

$$\frac{\partial k}{\partial \theta} \sim cos(\theta+\psi) + cos((p-1)\theta - \psi) > 0, \quad \forall\, \theta \in (0, -\psi). \tag{2.15}$$

Therefore, k strictly increases with $\theta \in (0, -\psi)$. Since, from (2.6), $k(\theta_{II}, \psi) = 0$, k takes on only negative values in the previous interval of θ. Consequently, we have proved that the function $k(\theta,\psi)$ or, equivalently, F_θ' satisfies the relationships

$$F_\theta' \quad \begin{cases} < 0, & \forall\, \theta \in (0, \theta_{II}), \\ = 0, & \theta = \theta_{II}, \\ > 0, & \forall\, \theta \in (\theta_{II}, -\psi). \end{cases} \tag{2.16}$$

Relationships (2.14) and (2.16) can be written together as follows:

$$F'_\theta \quad \begin{cases} < 0, & \forall\, \theta \in (0, \theta_{II}), \\ = 0, & \theta = \theta_{II}, \\ > 0, & \forall\, \theta \in (\theta_{II}, \frac{\pi}{p}), \end{cases} \tag{2.17}$$

where the case $\theta = -\psi$ has been incorporated since, in view of (2.13), the corresponding value of F'_θ is strictly positive.

Let now b_I, a_I and b_{II}, a_{II} denote the real and the imaginary semiaxes of the cusped hypos I and II, respectively (see Fig. 2). Then, according to the Implicit Function Theorem (see, e.g., Thm 14.1 of [17]) we will have the following statement.

Lemma 2.4 For θ increasing in $(0, \theta_{II}]$, b strictly decreases from $rcos\psi$ to b_{II}. For θ increasing in $[\theta_{II}, \frac{\pi}{p})$, b strictly increases from b_{II} to ∞.
Proof: By virtue of the *Implicit Function Theorem*, using (2.12) and (2.17), it can be obtained that

$$\frac{\partial b}{\partial \theta} = -\frac{F'_\theta}{F'_b} < 0, \quad \forall\, \theta \in (0, \theta_{II}) \tag{2.18}$$

and also that

$$\frac{\partial b}{\partial \theta} = -\frac{F'_\theta}{F'_b} > 0, \quad \forall\, \theta \in (\theta_{II}, \frac{\pi}{p}). \tag{2.19}$$

The limiting cases for $\theta = \theta_{II}$ and $\theta = 0, \frac{\pi}{p}$ can be readily obtained from (2.17) and by using continuity arguments, respectively. □

Working now with the function G in (2.11) and following a similar analysis one can end up with similar results regarding the behavior of the imaginary semiaxis a as a function of θ. Some of the intermediate results and final conclusions and a lemma (Lemma 2.5) analogous to Lemma 2.4 are given below without any further explanations. Thus we have:

$$G'_a = -sin(\frac{p\theta}{2}) < 0, \quad \forall\, \theta \in (0, \frac{\pi}{p}), \tag{2.20}$$

$$G'_\theta \sim (p-1)sin(\theta + \psi) - sin((p-1)\theta - \psi) =: l(\theta, \psi), \tag{2.21}$$

and

$$G'_\theta \quad \begin{cases} < 0, & \forall\, \theta \in (0, \theta_I), \\ = 0, & \theta = \theta_I, \\ > 0, & \forall\, \theta \in (\theta_I, \frac{\pi}{p}). \end{cases} \tag{2.22}$$

Lemma 2.5 For θ increasing in $(0, \theta_I]$, a strictly decreases from ∞ to a_I. For θ increasing in $[\theta_I, \frac{\pi}{p})$, a strictly increases from a_I to $rcos(\frac{\pi}{p} + \psi)$.

Table 1: Behavior of the semiaxes b and a as functions of θ

θ	0		θ_{II}		$-\psi$		θ_I		$\frac{\pi}{p}$
b	$rcos\psi$	$\searrow$	b_{II}	$\nearrow$	r	$\nearrow$	b_I	$\nearrow$	∞
a	∞	$\searrow$	a_{II}	$\searrow$	r	$\searrow$	a_I	$\nearrow$	$rcos(\frac{\pi}{p}+\psi)$
Type of Hypo		Stretch II	Cusp II	Short II	Circle	Short I	Cusp I	Stretch I	

The behavior of the semiaxes b and a as functions of θ together with the kind and type of the corresponding hypo are given in the self-explained Table 1. Some names of the hypos are given in an abbreviated form.

Remarks: From Lemmas 2.4, 2.5 and Table 1 one can readily draw the following conclusions: i) For each $b \in (b_{II}, rcos\psi)$ there are two hypos passing through the given point P. One is a stretched of type II while the other is a shortened II. Their corresponding $\theta's$ belong to the intervals $(0, \theta_{II})$ and $(\theta_{II}, -\psi)$, respectively, and can be uniquely determined from the first equations in (2.4). The corresponding values of their $a's$ are in the intervals (a_{II}, ∞) and (r, a_{II}), respectively, and, having found the $\theta's$, can be determined from the second equations in (2.4). ii) For each $a \in (a_I, rcos(\frac{\pi}{p}+\psi))$ there are also two hypos of type I, a shortened and a stretched one, passing through P. Their corresponding $\theta's$ lie in the intervals $(-\psi, \theta_I)$ and $(\theta_I, \frac{\pi}{p})$, and can be uniquely determined from the second equations in (2.4) while their $b's$ are in the intervals $[r, b_I)$ and (b_I, ∞) and can be determined from the first equations in (2.4). iii) For any other possible value of b the hypo through P is a unique shortened one. It is of type II for $b \in (b_{II}, r)$ and of type I for $b \in (r, b_I)$.

Based on the Remarks made previously and having always in mind the conclusions of Lemmas 2.4 and 2.5 as well as of Table 1, one notes that if one restricts oneself to considering only one kind of hypos through P, either shortened or stretched, then there is a *one-to-one* correspondence between any two of the three elements (variables) b, a and θ of them so that any one of them can be given as a function of either of the others.

As is known from the analysis of convergent $p-$cyclic SOR methods, associated with shortened and stretched hypos, the elements that play the most important role are the two semiaxes of the hypos (see [2] and [15], respectively). In the case of shortened hypos the two semiaxes are directly involved in the formulas that give the corresponding parameters of the SOR. However, in the case of the stretched hypos the parameters that are directly involved are for hypos I the imaginary semiaxis, the intercept on the real semiaxis, denoted from now on by b^*, and the value of the parameter t that corresponds to b^*, denoted by θ^*, and for hypos II the real semiaxis, the intercept on the imaginary semiaxis a^* and the value of $t = \theta^*$ corresponding to a^* (see Fig. 2).

For stretched hypos I, not necessarily passing through P, with semiaxes b and a we have from (2.2) that

$$\begin{aligned} b^* = & \frac{b+a}{2}cos\theta^* + \frac{b-a}{2}cos(p-1)\theta^*, \\ 0 = & \frac{b+a}{2}sin\theta^* - \frac{b-a}{2}sin(p-1)\theta^*. \end{aligned} \tag{2.23}$$

In view of (2.23), assuming that the hypo in question has intercept b^* on the real semiaxis, in other words it passes through $P^*(b^*, 0)$, it is implied, after some simple manipulation, that

$$\begin{aligned} \frac{b}{a} = & \frac{tan(\frac{p\theta^*}{2})}{tan(\frac{p}{2}-1)\theta^*} =: s_1(\theta^*) \equiv s_1, \\ \frac{b^*}{a} = & \frac{sin(\frac{p\theta^*}{2})}{sin(\frac{p}{2}-1)\theta^*} =: s_2(\theta^*) \equiv s_2. \end{aligned} \tag{2.24}$$

A statement very useful in the subsequent analysis based on the functions s_1 and s_2, defined in (2.24), is given in the sequel. Part of it can also be found in [15].

Lemma 2.6: Let $P^*(b^*, 0)$ be the closest to the origin point of intersection of a stretched hypo I, of semiaxes b and a, with the real semiaxis. The function s_1 defined in (2.24) is a strictly increasing function of $\theta^* \in (0, \frac{\pi}{p})$ taking values in the interval $(\frac{p}{p-2}, \infty)$. The function s_2 is a strictly decreasing function of $\theta^* \in (0, \frac{\pi}{p})$ taking values in the interval $(\frac{1}{cos\frac{\pi}{p}}, \frac{p}{p-2})$. On the other hand, for a fixed a, b strictly increases from $a\frac{p}{p-2}$ to ∞ while b^* strictly decreases from $a\frac{p}{p-2}$ to $\frac{a}{cos\frac{\pi}{p}}$. Their common limiting value is assumed for $\theta^* = 0$ and corresponds to the cusped hypo I.
Proof: Differentiate s_1 *wrt* to $\theta^* \in (0, \frac{\pi}{p})$. After some little algebra we take

$$\frac{\partial s_1}{\partial \theta^*} \sim psin(p-2)\theta^* - (p-2)sinp\theta^* =: s_1'. \tag{2.25}$$

Differentiating now s_1' *wrt* θ^* we have $\frac{\partial s_1'}{\partial \theta^*} \sim cos(p-2)\theta^* - cosp\theta^* > 0$. Since $s_1'(0) = 0$ and s_1' strictly increases in $[0, \frac{\pi}{p}]$, it is concluded that s_1' is strictly positive implying, in turn, that s_1 is strictly increasing. The end points of the interval for the values of s_1 are readily obtained by considering the limits of s_1 as θ^* tends to 0 and $\frac{\pi}{p}$, respectively. For the function s_2, the proof is similar and is not given here. (See, e.g., Lemma 1 of [15]). For a fixed a, the monotonic behavior of b and b^* and their common limiting value as θ^* tends to 0 are trivially verified. □

Remarks: i) In Lemma 2.6, for a given ratio $\frac{b}{a} \in (\frac{p}{p-2}, \infty)$ a family of stretched hypos I are defined for which θ^* exists in $(0, \frac{\pi}{p})$ and is unique. This unique value of θ^* is shared by all the members of the family. ii) If the stretched hypo I of Lemma 2.6 passes through P then $0 < \theta^* < \theta$ since the point P^* precedes P on the hypo.

An analogous analysis for stretched hypos II reveals that the corresponding point of intersection, $P^*(a^*, -\frac{\pi}{p})$, with intercept a^* on the imaginary semiaxis, will satisfy the relationships

$$\begin{array}{ll} \frac{a}{b} = & \frac{tan(\frac{\pi}{p}+(\frac{p}{2}-1)\theta^*)}{tan(\frac{p\theta^*}{2})} =: s_3(\theta^*) \equiv s_3, \\ \frac{a^*}{b} = & \frac{cos(\frac{p\theta^*}{2})}{cos(\frac{\pi}{p}+(\frac{p}{2}-1)\theta^*)} =: s_4(\theta^*) \equiv s_4. \end{array} \quad (2.26)$$

The following statement analogous to Lemma 2.6 is given without proof.

Lemma 2.7: Let $P^*(a^*, -\frac{\pi}{p})$ be the closest to the origin point of intersection of a stretched hypo II, of semiaxes b and a, with the imaginary semiaxis. The function s_3 defined in (2.26) is a strictly decreasing function of $\theta^* \in (0, \frac{\pi}{p})$ taking values in the interval $(\frac{p}{p-2}, \infty)$. The function s_4 is a strictly increasing function of $\theta^* \in (0, \frac{\pi}{p})$ taking values in the interval $(\frac{1}{cos\frac{\pi}{p}}, \frac{p}{p-2})$. On the other hand, for a fixed b, a strictly increases from $b\frac{p}{p-2}$ to ∞ while a^* strictly decreases from $b\frac{p}{p-2}$ to $\frac{b}{cos\frac{\pi}{p}}$. Their common limiting value is assumed for $\theta^* = \frac{\pi}{p}$ and corresponds to the cusped hypo II.

Remarks analogous to the ones after Lemma 2.6 can be made. This time, however, if the stretched hypo II passes through P it will be $\theta < \theta^* < \frac{\pi}{p}$.

To close this section we give below one more statement that will be used in Section 3. It is closely related to the two previous Lemmas.

Lemma 2.8: The function

$$s := s(\theta^*) \equiv \frac{sin(p-1)\theta^*}{sin\theta^*}, \quad \theta^* \in [0, \frac{\pi}{p}] \quad (2.27)$$

is a strictly decreasing function of θ^* with its extreme values being $p-1$ and 1, respectively.
Proof: For the strictly decreasing character of s, see Lemma 3 of [15]. Its extreme values are readily found. Note that the value at the left end of the interval is a limiting one. □

3 Convergent SORs and Associated Hypocycloids

From the convergence SOR theory and the analysis so far it is known that for $1 \leq b_{II} < b_I$ there is **no** hypo through P of any kind (cusped [20, 22, 5], shortened [2], or stretched [15]) that is associated with a convergent SOR. On the other hand, for $b_{II} < 1 \leq b_I$ there are at least shortened hypos II that lead to convergent SORs (see [2]) while for $b_{II} < b_I < 1$ all kinds and types of hypos may lead to convergent SORs .

The elements of a convergent SOR, namely its relaxation factor ω and its spectral radius

$$\rho(\mathcal{L}_\omega) = \frac{1}{\eta^p}, \ 1 < \eta, \quad (3.1)$$

are related to the real and the imaginary semiaxes of the associated hypo through the relationships below

$$\frac{1}{\omega\eta} = \frac{b+a}{2}, \quad \frac{(\omega-1)\eta^p}{\omega\eta} = \frac{b-a}{2}. \tag{3.2}$$

For a given point P, defined in the beginning of Section 2, let $a \in (a_I, rcos(\frac{\pi}{p}+\psi))$ and b_{st}, θ_{st}, b^* and θ^* be the real semiaxis, the value of the parameter t at P, the intercept on the real semiaxis and the value of t at $P^*(b^*, 0)$, respectively, of the stretched hypo I with imaginary semiaxis a. Let also b_{sh} and θ_{sh} be the real semiaxis and the value of t at P of the shortened hypo I through P with imaginary semiaxis a. For the SOR that is associated with the previously defined stretched hypo to be convergent there must hold

$$b^* \frac{p-2}{p} < a < b^* cos\frac{\pi}{p} \ \text{ and } \ b^* < \frac{cos(\frac{p\theta^*}{2})}{cos(\frac{p}{2}-1)\theta^*} \tag{3.3}$$

(see Thm 4a of [15]). The first set of inequalities in (3.3) are satisfied due to the fact that $a \in (a_I, rcos(\frac{\pi}{p}+\psi))$ and the theory developed so far. In view of (2.24), however, the second inequality holds *iff*

$$b_{st} < 1 \tag{3.4}$$

or, equivalently, *iff*

$$a < \frac{tan(\frac{p}{2}-1)\theta^*}{tan(\frac{p\theta^*}{2})}. \tag{3.5}$$

Under the assumption (3.4), or its equivalent (3.5), it is obvious that

$$r < r\frac{cos(\frac{\pi}{p}+\psi)}{cos\frac{\pi}{p}} < b_{sh} < b_I < b_{st} < 1$$

and therefore the SOR method associated with the shortened hypo with semiaxes b_{sh} and a does also converge.

Together with the two hypos above (stretched and shortened) that lead to convergent SORs we also consider the cusped hypo I which shares with the two previous hypos their imaginary semiaxis a. This cusped hypo will have a real semiaxis $b = a\frac{p}{p-2}$. It is clear that both b^* and b_{sh} will be strictly less than $b = a\frac{p}{p-2}$. However, in view of (3.5), (2.24) also gives

$$a < \frac{tan(\frac{p}{2}-1)\theta^*}{tan(\frac{p\theta^*}{2})} \le sup_{\theta^* \in [0,\frac{\pi}{p}]} \frac{tan(\frac{p}{2}-1)\theta^*}{tan(\frac{p\theta^*}{2})} = \frac{p-2}{p}. \tag{3.6}$$

Consequently, $b = a\frac{p}{p-2} < 1$ and the cusped hypo I with semiaxes $b = a\frac{p}{p-2}$ and a is also associated with a convergent SOR. Note that for a fixed and θ^* varying to produce a convergent SOR associated with a stretched hypo I, the latter cannot take all the values in $[0, \frac{\pi}{p}]$. but only those in $[0, \overline{\theta^*})$, where $\overline{\theta^*}$ is the unique value of θ^* that makes the strict inequality in (3.6) be an equality.

Let $\widehat{\omega}_{st}$, $\widehat{\omega}_{sh}$ and $\widehat{\omega}_c$ denote the (optimal) relaxation factors of the stretched, shortened and cusped hypos that share their imaginary semiaxis $a \in (a_I, rcos(\frac{\pi}{p}+\psi))$ and the first two pass through P. The respective (optimal) spectral radii of the three associated SORs will be given by the following expressions (see [2, 20, 22, 5, 15]):

$$\rho(\mathcal{L}_{\widehat{\omega}_{sh}}) = \frac{(b_{sh}+a)}{(b_{sh}-a)}(\widehat{\omega}_{sh}-1) = \Big[\frac{(b_{sh}+a)}{2}\widehat{\omega}_{sh}\Big]^p, \tag{3.7}$$

$$\rho(\mathcal{L}_{\widehat{\omega}}) = \frac{p^p}{(p-1)^{p-1}}(\widehat{\omega}_c-1) = \Big[\frac{p}{(p-2)}a\widehat{\omega}_c\Big]^p, \tag{3.8}$$

and

$$\rho(\mathcal{L}_{\widehat{\omega}_{st}}) = \frac{sin(p-1)\theta^*}{sin\theta^*}(\widehat{\omega}_{st}-1) = \Big[\frac{sin(p-1)\theta^*}{sinp\theta^*}b^*\widehat{\omega}_{st}\Big]^p. \tag{3.9}$$

To prove one of the main results of this section we need the following statement whose proof will only be outlined.

Lemma 3.1 Consider the point P and the shortened and stretched hypos I with a common imaginary semiaxis $a \in (a_I, rcos(\frac{\pi}{p}+\psi))$ that pass through it. Let b_{sh}, b_{st} be their real semiaxes and $b = a\frac{p}{p-2}$ be the corresponding semiaxis of the cusped hypo I. Suppose that the condition (3.4) is satisfied. Then the optimal relaxation factors of the associated (convergent) SORs will be in the following order of magnitude

$$1 < \widehat{\omega}_{sh} < \widehat{\omega}_c < \widehat{\omega}_{st} < 2. \tag{3.10}$$

Proof: The inequalities in (3.10) and the proofs of our main assertions almost duplicate the corresponding ones in Thms 4a and 7a of [15] but refer to quite different hypos. The inequalities $1 < \widehat{\omega}_{sh}, \widehat{\omega}_{st} < 2$ trivially hold because of the middle expressions in (3.7) and (3.9) and in view of the convergence of the associated SOR method. The second inequality from the left can be easily verified to be true in a way similar to that in [15] with the only difference being that in our case a is fixed and b_{sh} varies instead of the other way around. Very briefly, differentiate the two rightmost members of the equation in (3.7) *wrt* to b_{sh}, next replace the expression $\left[\frac{(b_{sh}+a)}{2}\widehat{\omega}_{sh}\right]^p$ appearing in the resulting equation by using the second expression in (3.7) and finally solve for $\frac{\partial\widehat{\omega}_{sh}}{\partial b_{sh}}$ to obtain

$$\frac{\partial\widehat{\omega}_{sh}}{\partial b_{sh}} = \frac{\widehat{\omega}_{sh}(\widehat{\omega}_{sh}-1)[pb_{sh}-(p-2)a]}{[p-(p-1)\widehat{\omega}_{sh}](b_{sh}^2-a^2)} \;>\; 0. \tag{3.11}$$

To prove the second inequality from the right we consider the function

$$f := f(\widehat{\omega}_{st}, \theta^*) \equiv \Big[\frac{sin(p-1)\theta^*}{sinp\theta^*}b^*\widehat{\omega}_{st}\Big]^p - \frac{sin(p-1)\theta^*}{sin\theta^*}(\widehat{\omega}_{st}-1) \tag{3.12}$$

defined for any $\theta^* \in [0, \overline{\theta^*})$. If we replace b^* from (2.24) and use the function s in (2.27), (3.12) can be rewritten as

$$f := f(\widehat{\omega}_{st}, \theta^*) \equiv \Big[\frac{s}{s-1} a\widehat{\omega}_{st}\Big]^p - s(\widehat{\omega}_{st} - 1). \tag{3.13}$$

Note that from Lemma 2.8, s strictly decreases while $\frac{s}{s-1}$ strictly increases in $[0, \overline{\theta^*})$. Take any $\theta^* \in (0, \overline{\theta^*})$. Then it will be

$$\begin{aligned} &\Big[\tfrac{s(0)}{s(0)-1} a\widehat{\omega}_{st}(\theta^*)\Big]^p < \Big[\tfrac{s(\theta^*)}{s(\theta^*)-1} a\widehat{\omega}_{st}(\theta^*)\Big]^p \\ &= s(\theta^*)(\widehat{\omega}_{st}(\theta^*) - 1) < s(0)(\widehat{\omega}_{st}(\theta^*) - 1) \end{aligned} \tag{3.14}$$

and since $s(0) = lim_{\theta^* \to 0^+} s(\theta^*) = p - 1$, it is proved that $f(\widehat{\omega}_{st}(\theta^*), 0) < 0$. On the other hand, $f(1, 0) > 0$, $f(\widehat{\omega}_c, 0) = 0$ and $f(1 + \frac{1}{p-1}, 0) < 0$. It is therefore concluded that $\widehat{\omega}_{st}(\theta^*)(> 1)$ lies strictly outside the interval $[1, \widehat{\omega}_c]$ and therefore the second inequality from the right holds true. This concludes the proof of the present lemma. □

Having obtained the result (3.10) of Lemma 3.1 one can prove the following statement.

Theorem 3.2: Under the assumptions of Lemma 3.1 there holds

$$\rho(\mathcal{L}_{\widehat{\omega}_{sh}}) < \rho(\mathcal{L}_{\widehat{\omega}_{st}}) \ (< 1). \tag{3.15}$$

Proof: In view of (3.10), to prove (3.15) it suffices to prove that the bases of the powers in (3.7) and (3.9) without the ω factors satisfy a similar inequality. Namely, that

$$\frac{(b_{sh} + a)}{2} < \frac{sin(p-1)\theta^*}{sinp\theta^*} b^*. \tag{3.16}$$

For this we substitute b_{sh} and a of the shortened hypo in (3.16) by using the rightmost expressions in (2.4). Next, for b^* and a for the stretched hypo in (3.16) first we use the second equation of (2.24) and then the rightmost one in (2.4), recalling that a is the same regardless of the kind of hypo we are considering. Therefore we can use θ_{sh} in the place of θ_{st} in the expression for a. So, after some little algebra, (3.16) gives equivalently that

$$\frac{sin((p-1)\theta_{sh} - \psi)}{2cos(\frac{p\theta_{sh}}{2}) sin((\frac{p}{2} - 1)\theta_{sh} - \psi)} < \frac{sin(p-1)\theta^*}{2cos(\frac{p\theta^*}{2}) sin(\frac{p}{2} - 1)\theta^*}, \tag{3.17}$$

or

$$\frac{sin(p-1)\theta^*}{sin\theta^*} < \frac{sin((p-1)\theta_{sh} - \psi)}{sin(\theta_{sh} + \psi)}. \tag{3.18}$$

However, in view of Lemma 2.8 the left hand side of (3.18) is strictly less than $p - 1$. On the other hand, by virtue of the first inequality of (2.22) the function

$$l(\theta_{sh}, \psi) = -sin((p-1)\theta_{sh} - \psi) + (p-1) sin(\theta_{sh} + \psi), \ \ \theta_{sh} \in (0, \theta_I),$$

defined in (2.21), takes on strictly negative values. This implies that the right hand side of (3.18) is strictly greater than $p-1$. The two previous results, regarding the values of the two members of the inequality (3.18) *wrt* the number $p-1$, effectively show that the inequality (3.18) is a valid one and so is (3.15). □

Similarly, statements analogous to Lemma 3.1 and Thm 3.2 can be stated and proved for stretched hypos II that pass through the point P and are associated with convergent SOR methods. We simply note that one uses the theory developed so far and also the corresponding results of [15] in a similar way. Here we only present the main results pertaining to stretched hypos II. For this we must bear in mind that we consider the point P again and a stretched hypo II trough P that is associated with a convergent SOR. Let $b \in (b_{II}, r\cos\psi)$, its real semiaxis, be fixed and let a_{st} be its imaginary semiaxis with intercept a^* on it. Let θ^* be the value of the parameter t at $P^*(a^*, -\frac{\pi}{p})$. Consider also the shortened hypo II through P, with the same real semiaxis b, and imaginary semiaxis a_{sh} as well as the cusped hypo II with real semiaxis b. Using the same notation to denote the optimal relaxation factors and the spectral radii of the SORs associated with the three hypos it can be proved that:

$$0 < \hat{\omega}_{st} < \hat{\omega}_c < \hat{\omega}_{sh} < 1 \tag{3.19}$$

and

$$\rho(\mathcal{L}_{\hat{\omega}_{sh}}) < \rho(\mathcal{L}_{\hat{\omega}_{st}}) \ (< 1). \tag{3.20}$$

The previous discussion and the identical results (3.15) and (3.20) lead us to the following general conclusion which we give in the form of a theorem.

Theorem 3.3: For any stretched hypo that passes through the point P and is associated with a convergent SOR method there is a unique shortened hypo of the same type (I or II) that passes through P, shares with the stretched hypo one of the semiaxes (a or b, respectively) and is associated with a faster convergent SOR method.

In view of Thm 3.3, it is obvious that in order to solve the *one-point* problem described in Sections 2 and 3 it suffices to find among all the **shortened** hypos that pass through P and are associated with convergent SORs, if any, the one that corresponds to the (asymptotically) fastest SOR. The analysis, the study and the determination of the *optimal one-point* shortened hypo, in the sense just explained, will be done in the next section.

4 Optimal One-Point Shortened Hypocycloid

We begin our analysis by considering the point P as in the two previous sections and all the shortened hypos through it as well as the two cusped ones. From Lemmas 2.4, 2.5 and Table 1 it is clear that for $\theta \in (\theta_{II}, \theta_I)$ the semiaxes b and a of the class of all the hypos we consider are differentiable functions of θ. Also, a is differentiable *wrt* b and vice versa in their respective intervals. In the sequel the analysis is facilitated if one considers all the elements (parameters) of the hypos of interest as differentiable functions of b.

For the various derivatives involved analytic expressions can be found which are of constant sign. These expressions are presented in a series of lemmas that follow.

Lemma 4.1: The derivative of a *wrt* $b \in (b_{II}, b_I)$ is a continuous function of b that takes on strictly negative values. It is given by the expression

$$D := \frac{da}{db} = \frac{[pa-(p-2)b]}{[(p-2)a-pb]} cot^2(\frac{p\theta}{2}) \ < 0. \tag{4.1}$$

Proof: Differentiating each of the first two equations in (2.3) *wrt* b, solving each one for $\frac{d\theta}{db}$ and then equating the two equivalent expressions one obtains (4.1). From (4.1) it is directly concluded that D is negative since for shortened hypos there holds $\frac{a}{b} \in (\frac{p-2}{p}, \frac{p}{p-2})$. □
Note: It is noted that from (4.1) it can be readily obtained that

$$\lim_{b \to r} D = -cot^2(\frac{p\psi}{2}). \tag{4.2}$$

Lemma 4.2: The derivative of θ *wrt* $b \in (b_{II}, b_I)$ is a continuous function of b that takes on strictly positive values. It is given by the following expression

$$\frac{d\theta}{db} = \frac{2cot(\frac{p\theta}{2})}{[pb-(p-2)a]} \ > 0. \tag{4.3}$$

Proof: Considering either of the two expressions for $\frac{d\theta}{db}$ found during the proof of Lemma 4.1 and plugging in it the expression for D from (4.1), (4.3) is obtained. Obviously, $\frac{d\theta}{db}$ is strictly positive. □

Lemma 4.3: The second derivative of a (or the derivative of D) *wrt* $b \in (b_{II}, b_I)$ is a continuous function of b, is given by the expression below and takes on strictly positive values.

$$\frac{dD}{db} = \frac{d^2a}{db^2} = \frac{2cot^2(\frac{p\theta}{2})}{[pb-(p-2)a]^2}\left\{2(p-1)(a-bD)+p[pa-(p-2)b][1+cot^2(\frac{p\theta}{2})]\right\} \ > 0. \tag{4.4}$$

Proof: Differentiating D in (4.1) and replacing $\frac{d\theta}{db}$ from (4.3) one obtains the expression in (4.4). Recalling from (4.1), that $D < 0$ and also that $\frac{b}{a} \in (\frac{p-2}{p}, \frac{p}{p-2})$, both terms in the braces in the expression just found are positive. This proves the second part of our statement. □

As is known for a shortened hypo, which is a member of the class we have been studying so far, to be associated with a convergent SOR method the point $(1,0)$ of the complex plane must not belong to the closure of the interior of the hypo in question. This suggests that the real semiaxis b of the hypo considered must be strictly less than 1. Consequently, if $b_{II} < 1 \leq b_I$ or if $b_{II} < b_I < 1$ all hypos with $b \in (b_{II}, 1)$ in the former case and with

$b \in (b_{II}, b_I]$ in the latter one will be associated with convergent SORs.

Recalling the relationships (3.1) and (3.2) of a convergent SOR, we introduce the symbol x to denote either of the two equivalent quantities

$$x := \frac{1}{\eta} \equiv \rho^{\frac{1}{p}}(\mathcal{L}_\omega). \tag{4.5}$$

On elimination of ω from the equations in (3.2) the polynomial equation in x below

$$\phi := \phi(x) \equiv (b-a)x^p - 2x + b + a = 0 \tag{4.6}$$

is obtained. It is readily checked that $\phi(0) = b + a > 0$ and $\phi(1) = 2(b-1) < 0$, since the only case of convergence is when $b < 1$. By Descartes' rule of signs there is a unique real positive root x of (4.6) in $(0,1)$, let it be denoted by x_0, whose p^{th} power gives the spectral radius of the convergent SOR iteration matrix. The aforementioned root x_0 is a continuous function of the coefficients of the polynomial equation (4.6). However, recalling that for the class of the hypos through P, a is a continuous function of b it is implied that x_0 is also a continuous function of b (or of a or even of θ) only. To find intervals, different from $(b_{II}, 1)$ and $(b_{II}, b_I]$ considered previously, in which x_0 also lies we note that $\phi(b) = (b-a)(b^p - 1)$, $\phi(a) = (b-a)(a^p + 1)$ and $\phi(\frac{b+a}{2}) = (b-a)(\frac{b+a}{2})^p$. Based on the values just obtained we can state the following lemma.

Lemma 4.4: Under the assumption $b < 1$, the unique real positive root $x_0 \in (0,1)$ of equation (4.6) lies in the following intervals, respectively,

$$x_0 \quad \begin{cases} \in (b, \frac{b+a}{2}), & if \;\; b < a \; (< \frac{p}{p-2}), \\ = \frac{b+a}{2}, & if \;\; b = a \; (< 1), \\ \in (\frac{b+a}{2}, b), & if \;\; a < b \; (< 1). \end{cases} \tag{4.7}$$

(*Note:* From (4.7) It becomes clear that in the first case x_0 is given by means of a shortened hypo II, in the second by a circle and in the third one by a shortened hypo I.)

In the case of a convergent SOR consider the continuous function $x_0 \equiv x_0(b)$ defined on the closed interval $[b_{II}, min\{1, b_I\}]$. Obviously, for any b in this interval, except for its right endpoint when it is 1, the corresponding shortened hypo through P will be associated with a convergent SOR. In what follows we examine the behavior of the function x_0 in the above interval in the neighborhoods of b_{II} and $min\{1, b_I\}$. For this we differentiate (4.6) *wrt* b and solve for $\frac{dx_0}{db}$ to get

$$\frac{dx_0}{db} = \frac{(1-D)x_0^p + (1+D)}{2 - p(b-a)x_0^{p-1}}. \tag{4.8}$$

Since $x \in (0,1)$, $\frac{a}{b} \in (\frac{p-2}{p}, \frac{p}{p-2})$ and $b < 1$, is readily checked that $2 - p(b-a)x_0^{p-1} > 0$. Therefore

$$\frac{dx_0}{db} \sim (1 + x_0^p) + (1 - x_0^p)D. \tag{4.9}$$

However, from (4.1) we have that for $b \to b_{II}^+$, $\theta \to \theta_{II}^+$ and $D \to -\infty$. On the other hand, if $b_I < 1$ then for $b \to b_I^-$, $\theta \to \theta_I^-$ and $D \to 0^-$. In the two cases just examined we will have

$$\lim_{b \to b_{II}^+} \frac{dx_0}{db} \sim -\infty, \quad \lim_{b \to b_I^-} \frac{dx_0}{db} \sim 1 + x_0^p(b_I) \; (> 0). \tag{4.10}$$

Note that in the case $min\{1, b_I\} = 1$ since for $b \to 1^-$ the corresponding SOR converges it is implied that $x_0(b)$ strictly increases in the neighborhood of 1. In concluding, (4.10) and the note just made give as a consequence the following statement which establishes the existence of a minimum point in the interval considered. Thus we have:

Theorem 4.5: Suppose that $b_{II} < 1$. The function $x_0 \equiv x_0(b)$, the root of (4.6) in $(0,1)$ defined on $[b_{II}, min\{1, b_I\}]$, strictly decreases in a right neighborhood of b_{II} and strictly increases in a left neighborhood of $min\{1, b_I\}$. It therefore possesses at least one minimum point in the corresponding open interval.

To find the minimum point(s) of Thm 4.5 we have first to find the points in the above interval at which $\frac{dx_0}{db} = 0$. However, from (4.8), setting $\frac{dx_0}{db} = 0$, it is obtained

$$x_0^p = \frac{D+1}{D-1}. \tag{4.11}$$

In the right hand side of the above equation we replace D using (4.1) and then in the expression obtained we replace b and a using the rightmost expressions in (2.4). So after some algebraic manipulation involving simple trigonometric transformations we take

$$x_0^p = \frac{sin((p-1)\theta - \psi)cosp\theta - (p-1)sin(\theta + \psi)}{sin((p-1)\theta - \psi) - (p-1)sin(\theta + \psi)cosp\theta}. \tag{4.12}$$

Since $x_0 \in (0,1)$ must be a root of equation (4.6) the value of it just obtained must verify this equation. Thus, if we use (4.12) and the rightmost expressions in (2.4) for b and a in (4.6), we have that

$$\begin{aligned} &\left[r \frac{sinp\theta - psin(\theta+\psi)cos((p-1)\theta - \psi)}{sin((p-1)\theta-\psi) - (p-1)sin(\theta+\psi)cosp\theta}\right]^p \\ &= \frac{sin((p-1)\theta-\psi)cosp\theta - (p-1)sin(\theta+\psi)}{sin((p-1)\theta-\psi) - (p-1)sin(\theta+\psi)cosp\theta} \end{aligned} \tag{4.13}$$

from which a value of $\theta \in (\theta_{II}, \theta_I)$ can be obtained. Therefore, we have effectively proved the following statement.

Theorem 4.6: Suppose that $b_{II} < 1$ and let $\tilde{\theta}$ denote θ_I, if $b_I \leq 1$, otherwise denote $\theta_{(1,0)}$, the value of θ at P of the hypo through the points $(1,0)$ and P. If for the real positive root $x_0 \in (0,1)$ of (4.6), $\frac{dx_0}{db}$ vanishes for some $b = \hat{b} \in (b_{II}, min\{1, b_I\})$ (resp. $\theta = \hat{\theta} \in (\theta_{II}, \tilde{\theta})$), then $\hat{x} \equiv x_0(\hat{b}) \equiv x_0(\hat{\theta})$ is given by (4.12), where $\hat{\theta} \in (\theta_{II}, \tilde{\theta})$ is a root of (4.13).
Notes: i) For the limiting cases $\psi = 0$ and $\psi = -\frac{\pi}{p}$, the corresponding values $\theta = 0$ and $\theta = \frac{\pi}{p}$ are recovered from (4.13) but one has first to get rid of the denominators and then apply (4.13) in the interval $[\theta_{II}, \theta_I] \equiv [0, \frac{\pi}{p}]$. After this, one can readily obtain the very

simple expessions for $\widehat{b}$ and $\widehat{a}$ from the limiting cases of the formulas in (2.4) and then use (4.6) to obtain $\widehat{x} \equiv x_0(\widehat{b})$. ii) One has to have in mind and apply the previous note even in cases where ψ is very close to 0 or to $-\frac{\pi}{p}$ to avoid possible instabilities due to round-off errors.

In the following theorem we establish the uniqueness of the minimum point.

Theorem 4.7: Under the assumptions of Thm 4.6, the real positive root $x_0 \in (0,1)$ of equation (4.6), as a function of $b \in [b_{II}, min\{1, b_I\}]$ (resp. $\theta \in (\theta_{II}, \widetilde{\theta})$), attains a minimum at some unique value $b = \widehat{b}$ (resp. $\theta = \widehat{\theta}$) strictly in the interior of the corresponding interval. This value of x_0, $\widehat{x} \equiv x_0(\widehat{b}) \equiv x_0(\widehat{\theta})$, can be expressed explicitly in terms of $\widehat{\theta}$ by (4.12) and implicitly in terms of either $\widehat{b}$ or $\widehat{a}$.

Proof: To prove our assertion we take the second derivative of x_0 *wrt* b ($\frac{d^2x_0}{db^2}$) and find its value at the point x_0 at which $\frac{dx_0}{db} = 0$. For this, first we differentiate $\frac{dx_0}{db}$ in (4.8), next omit the positive denominator, then set $\frac{dx_0}{db} = 0$ and, finally, omit again obvious positive factors. Thus, we obtain that

$$\frac{d^2x_0}{db^2}\Big|_{\frac{dx_0}{db}=0} \sim \frac{dD}{db} > 0, \tag{4.14}$$

which holds in view of (4.4). If there were more than one points b at which $\frac{dx_0}{db} = 0$, let b_1 and b_2 be any two consecutive ones, then by Rolle's Theorem there would be an intermediate point $b \in (b_1, b_2)$ at which $\frac{d^2x_0}{db^2} = 0$. But then at least at one of the two points b_1 or b_2 we would have either $\frac{d^2x_0}{db^2} = 0$ or $\frac{d^2x_0}{db^2} < 0$. This, however, contradicts the inequality in (4.14) which completes the proof. □

As an immediate consequence of the last three theorems we can have the following one whose proof is obvious.

Theorem 4.8: Under the assumptions of Thms 4.5, 4.6, and 4.7, the function $x_0 \equiv x_0(b)$ ($\equiv x_0(\theta)$) is a strictly decreasing function of b (resp. θ) in $[b_{II}, \widehat{b}]$ (resp. $[\theta_{II}, \widehat{\theta}]$) and a strictly increasing one in $[\widehat{b}, min\{1, b_I\}]$ (resp. $[\widehat{\theta}, \widetilde{\theta}]$), where $\widehat{b}$ (resp. $\widehat{\theta}$) is the value of b (resp. θ) at which the minimum occurs. Moreover, the value of $\widehat{\theta}$ is given as the unique root of (4.13) in the interval $(\theta_{II}, \widetilde{\theta})$ while the value $\widehat{x}$ of the minimum attained is then given by (4.12).

Based on the theory so far we would like to comment on the common region in the last $(2p) - ant$ enclosed by all the shortened (and cusped) hypos through P. Since, as a by-product of the analysis done, one can show that a shortened hypo through P cannot have more than one common point with either cusped hypo I or II through P in the last $(2p) - ant$, the aforementioned common region is the curvilinear quadrilateral with vertices $O(0,0)$, $P_{b_{II}}(b_{II}, 0)$, $P(r, \psi)$ and $P_{a_I}(a_I, -\frac{\pi}{p})$. Its two sides $OP_{b_{II}}$ and $P_{a_I}O$ are straight line segments while the other two are the arcs $P_{b_{II}}P$ and PP_{a_I} of the two cusped hypos of types II and I, respectively. Clearly, the optimal solution to the *one-point* problem of this section is also optimal if all the elements of $\sigma(J_p)$ in the last and the first $(2p) - ants$ lie in the union of the curvilinear quadrilateral $OP_{b_{II}}PP_{a_I}$ and its symmetric one *wrt* the real axis.

We close this section with the following remark regarding the case $p = 2$.

Remark: The theory of this paper holds also in the case $p = 2$. We have confirmed it both theoretically and computationally. Very briefly, for $p = 2$ the parametric equations (2.2) give the corresponding ones of an ellipse. Obviously, to have convergent SORs ($b < 1$) the polar coordinates of the point P must satisfy $r\cos\psi < 1$. Then, following step by step the theory developed we can verify that the optimal value of $\rho(\mathcal{L}_\omega)$ is given by the expression in (4.12), where θ is the unique solution of (4.13) in $(0, \frac{\pi}{p})$. It is noted that the cusped hypo I corresponds to $\theta_I = \frac{\pi}{p}$ with $b_I = \infty$, $a_I = -r\sin\psi$ and the ellipse in question becomes a pair of straight lines parallel to the real axis. The cusped hypo II corresponds to $\theta_{II} = 0$ with $b_{II} = r\cos\psi$, $a_{II} = \infty$ and is a similar pair parallel to the imaginary axis. So, we can consider the ellipses through P as shortened hypos and the elements of their associated optimal SORs can be obtained by our theory or by the theory described in [24]. There are also two limiting cases corresponding to $\psi = 0$ and $\psi = -\frac{\pi}{p}$ that give degenerate ellipses (double straight line segments) along the real and the imaginary axes, with $b = r$, $a = 0$ and $b = 0$, $a = r$, respectively. Using (4.6), (4.5), and either of (3.2), the well-known optimal SOR formulas and results associated with the degenerate ellipses are readily recovered.

5 Algorithm and Numerical Examples

Having completed the analysis in the previous sections we can now give in pseudocode an algorithm which will allow us to solve computationally the *one-point* problem in the general case.

THE ALGORITHM

Given the (polar) coordinates of the point P as in Sections 2, 3 and 4;
Determine θ_{II} from (2.10) and then b_{II} from (2.4);
if $b_{II} \geq 1$ **then**
  NO CONVERGENT SOR EXISTS; **stop**;
endif;
Determine $\tilde{\theta} := \theta_I$ from (2.8) and then b_I from (2.4);
if $b_1 > 1$ **then**
  Determine $\theta_{(1,0)}$ from (2.4) by setting $b = 1$; Set $\tilde{\theta} := \theta_{(1,0)}$;
endif;
Determine $\hat{\theta} \in (\theta_{II}, \tilde{\theta})$ from (4.13);
Determine $\hat{x}$ from (4.12);
Determine $\hat{b}$ and $\hat{a}$ using $\hat{\theta}$ from (2.4);
Determine $\hat{\omega} := \frac{2\hat{x}}{\hat{b}+\hat{a}}$, $\rho(\mathcal{L}_{\hat{\omega}}) := \hat{x}^p$;
end of ALGORITHM;

The algorithm just presented is applied to a number of numerical examples for the values of $p = 3, 4, 5$. For each p three different cases have been worked out. All the input and the output information is presented in Table 2. In the very first case, $p = 3$, $P(0.5, -\frac{\pi}{12})$, it is found that $b_{II} < b_I < 1$ so that for all $b \in [b_{II}, b_I]$ all shortened and cusped hypos through P lead to convergent SORs. For the optimal hypo it is $\widehat{b} > \widehat{a}$ (hypo I) that gives $\widehat{\omega} > 1$ (see (3.2)). In the second case, $p = 3$, $P(1.0, -\frac{\pi}{12})$, it is $b_{II} < 1 < b_I$ hence for all $b \in [b_{II}, 1)$ all shortened (and cusped II) hypos through P are associated with convergent SORs. Since $\widehat{b} < \widehat{a}$ (hypo II), $\widehat{\omega} < 1$. Finally, in the third case $p = 3$, $P(1.5, -\frac{\pi}{12})$, it is $b_{II} > 1$ so that no convergent SOR, yielded by a shortened (cusped) hypo through P, can exist. All the other examples illustrated in Table 2 can be explained in an analogous way.

Table 2: Numerical examples

p	$P(r,\psi)$	b_{II}	b_I	a_{II}	a_I	$\widehat{b}$	$\widehat{a}$	$\widehat{\omega}$	$\rho(\mathcal{L}_{\widehat{\omega}})$
3	$P(0.5, -\frac{\pi}{12})$	.4807834	.9881485	1.442350	.3293827	.5380996	.3995527	1.015969	.10806300
	$P(1.0, -\frac{\pi}{12})$	.9615669	1.976297	2.884700	.6587655	.9699653	1.409681	.8251402	.94630350
	$P(1.5, -\frac{\pi}{12})$	1.442350	NO CONVERGENT SOR EXISTS						
	$P(0.5, -\frac{\pi}{6})$	.4237433	1.271230	1.271230	.4237433	.4882581	.5133052	.9968889	.12441870
	$P(1.0, -\frac{\pi}{6})$	.8474866	2.542460	2.542459	.8474866	.8764933	1.345895	.8326846	.79215710
	$P(1.5, -\frac{\pi}{6})$	1.271230	NO CONVERGENT SOR EXISTS						
	$P(0.5, -\frac{\pi}{4})$	.3293827	1.442351	.9881483	.4807834	.3822809	.5551888	.9820156	.09752972
	$P(1.0, -\frac{\pi}{4})$	.6587655	2.884701	1.976297	.9615668	.7048243	1.232513	.8486724	.55557940
	$P(1.5, -\frac{\pi}{4})$	.9881483	4.327052	2.964445	1.442350	.9884788	2.811749	.5246571	.99075340
4	$P(0.5, -\frac{3\pi}{16})$	.3806604	.9742557	.7613209	.4871286	.4178384	.5367081	.9936990	.05059268
	$P(1.0, -\frac{3\pi}{16})$	.7613208	1.948511	1.522642	.9742572	.7876535	1.168418	.8888338	.57108530
	$P(1.5, -\frac{3\pi}{16})$	1.141981	NO CONVERGENT SOR EXISTS						
5	$P(0.5, -\frac{3\pi}{20})$	.4080800	.8172027	.6801333	.4903212	.4369032	.5274884	.9975731	.02575377
	$P(1.0, -\frac{3\pi}{20})$	.8161599	1.634405	1.360267	.9806423	.8342583	1.131831	.9122255	.57993250
	$P(1.5, -\frac{3\pi}{20})$	1.224240	NO CONVERGENT SOR EXISTS						

We conclude the present work by noting the various difficult issues one had to address and resolve for the solution of the *"one-point"* problem to be accomplished. However, the solution of this problem may undoubtedly constitute the basis for one to attack the more challenging and the much more complicated *"two-point* and *"many-point"* problems, in the general case, which still remain open.

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