SCHOOL OF
CIVIL ENGINEERING

INDIANA

DEPARTMENT OF TRANSPORTATION

JOINT HIGHWAY RESEARCH PROJECT

JHRP-75-8

Final Report

COMPUTER ANALYSIS OF GENERAL
SLOPE STABILITY PROBLEMS

Ronald A. Siegel

PURDUE UNIVERSITY

Digitized by the Internet Archive
in 2011 with funding from
LYRASIS members and Sloan Foundation; Indiana Department of Transportation

JOINT HIGHWAY RESEARCH PROJECT

JHRP-75-8

Final Report

**COMPUTER ANALYSIS OF GENERAL
SLOPE STABILITY PROBLEMS**

Ronald A. Siegel

Final Report

COMPUTER ANALYSIS OF GENERAL SLOPE STABILITY PROBLEMS

TO: J. F. McLaughlin, Director
Joint Highway Research Project

June 4, 1975

FROM: H. L. Michael, Associate Director
Joint Highway Research Project

Project: C-36-36K

File: 6-14-11

The Final Report "Computer Analysis General Slope Stability Problems" is submitted for acceptance in fulfillment of the objectives of the approved JHRP Research Study titled "A Computer Analysis of an Irregular Failure Surface by the Method of Slices". The research and report were performed by Mr. Ronald A. Siegel, Graduate Instructor in Research on our staff, under the direction of Professors C. W. Lovell and W. D. Kovacs.

The report includes a review of methods available for slope stability analysis, and a description of the computer program developed. The program has great versatility and should be of considerable value to the ISHC in economically analyzing cut and embankment slopes. Another report, JHRP-75-9, is the "STABL USER MANUAL" and will be useful to anyone who uses the computer programs of this research.

The Report is submitted for approval and acceptance.

Respectfully submitted,

Harold L. Michael

Harold L. Michael
Associate Director

HLM:mf

cc: W. L. Dolch
R. L. Eskew
G. D. Gibson
W. H. Goetz
M. J. Gutzwiller
G. K. Hallock

M. L. Hayes
C. W. Lovell
G. W. Marks
R. F. Marsh
R. D. Miles
G. T. Satterly

C. F. Scholer
M. B. Scott
K. C. Sinha
H. R. J. Walsh
L. E. Wood
E. J. Yoder
S. R. Yoder

Final Report
COMPUTER ANALYSIS OF GENERAL SLOPE STABILITY PROBLEMS

by

Ronald A. Siegel
Graduate Instructor in Research

Joint Highway Research Project

Project No.: C-36-36K

File No.: 6-14-11

Prepared as Part of an Investigation
Conducted by

Joint Highway Research Project
Engineering Experiment Station
Purdue University

in cooperation with the
Indiana State Highway Commission

Purdue University
West Lafayette, Indiana
June 4, 1975

ACKNOWLEDGMENTS

The author wishes to express his gratitutde to Dr. W. D. Kovacs, Assistant Professor of Civil Engineering, and Dr. C. W. Lovell, Professor of Civil Engineering at Purdue University, for their valued assistance and guidance during the course of the project.

The project was funded by the Joint Highway Research Project, Engineering Experiment Station, Purdue University, Dr. J. F. McLaughlin, Director, in cooperation with the Indiana State Highway Commission.

Special thanks are due to Mr. W. J. Sisiliano, Soils Engineer, Indiana State Highway Commission, for his assistance in defining the capabilities required of the computer program developed, and to Mr. John Bellinger of the Indiana State Highway Commission Computer Center for his assistance in establishing the program on line at that center.

Finally, the author wishes to thank Mr. Yogesh D. Shah, who drafted the figures, Miss Janice Wait for her efforts in typing the draft, and Mrs. Paul W. Velten who has typed the final manuscript.

TABLE OF CONTENTS

	Page
LIST OF TABLES.	v
LIST OF FIGURES	vi
LIST OF SYMBOLS	viii
ABSTRACT.	xiii
INTRODUCTION.	1
METHODS AVAILABLE FOR SLOPE STABILITY ANALYSIS.	4
Limiting Equilibrium Methods	5
Limiting Equilibrium and Fundamental Assumptions	6
Logarithmic Spiral Shear Surface	14
$\phi = 0$ Case	17
Friction - Circle Method	18
Method of Slices	19
Ordinary Method of Slices	22
Simplified Bishop Method.	30
Bishop's Method, Rigorous	31
Janbu's Generalized Procedure of Slices	39
Morgenstern and Price Method.	44
Spencer's Method.	49
Carter's Method	53
METHOD CHOSEN FOR COMPUTER PROGRAM.	57
PROGRAM STABL	64
Division of Mass into Slices above the Assumed Shear Surface	64
Total Weight Calculation of Each Slice	66
Water Forces	70
SEARCHING TECHNIQUES.	75
Random Numbers	76
Circular Surface Generation.	77
General Irregular Shear Surface Generation	89
Searching for the Critical Circular or General Shear Surfaces.	92
Sliding Block Surface Generation	96

COMPARISON OF CARTER'S METHOD WITH SPENCER'S METHOD ASSUMING PARALLEL SIDE FORCES	Page 102
SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK	108
LIST OF REFERENCES.	110
APPENDIX A: LISTING OF PROGRAM STABL	113
STABL Main Program	113
Subroutine READER.	117
Subroutine QUIT.	121
Subroutine PROFIL.	122
Subroutine ANISO	131
Subroutine SURFAC.	135
Subroutine WATER	138
Subroutine LOADS	141
Subroutine EQuAKE.	145
Subroutine LIMITS.	146
Subroutine EXECUT.	149
Subroutine SLICES.	151
Subroutine INTSCT.	156
Subroutine WEIGHT.	158
Subroutine SOILWT.	163
Subroutine FACTR	165
Subroutine RANDOM.	170
Subroutine RANSUF.	184
Subroutine SORT.	197
Subroutine SCALER.	198
Subroutine PLOTIN.	200
Subroutine PLTN.	205
Subroutine PSTN.	210

LIST OF TABLES

Table	Page
1. Equations and Unknowns Associated with Procedure of Slices	24
2. Factors of Safety Calculated Using Carter's Method	104

LIST OF FIGURES

Figure	Page
1. Basic Forces Associated With Limiting Equilibrium	7
2. Planar Shear Surface.	13
3. Logarithmic Spiral Shear Surface and Force Orientation	16
4. Mass Above Assumed Shear Surface Divided into Slices and Linearization of the Slices' Bases .	21
5. Basic Forces Acting upon Individual Slice . . .	23
6. Forces Considered for Ordinary Method of Slices	25
7. Boundary Forces Acting on the End Slices. . . .	33
8. Lever Arms of Nonveiller Extension of Bishop's Approach.	38
9. Infinitesimal Slice and Force System.	40
10. Slice and Resultant of the Side Forces.	50
11. Slice Forces Referenced to an Arbitrary Point .	54
12. Forces Considered for Computer Model.	61
13. Locations of Slice Divisions.	65
14. Geometry of Slices.	67
15. Heterogeneous Slices Partially Submerged. . . .	68
16. Water Forces Acting Upon Submerged Slices . . .	71
17. Approximation of Head for Pore Pressure Calculation	73
18. Selecting the Initial Line Segment.	78
19. Probability of the Location of the Initial Line Segment	80

Figure	Page
20. Family of Circles Having Initial Line Segment as a Chord.	82
21. Back-Calculating a Deflection Limit for Circular Surface	83
22. Limitations Imposed on Deflection Angle	85
23. Limitations Imposed on Inclination of Final Line Segment.	87
24. Deflection Limits for Successive Line Segments of Irregular Surface.	90
25. 100 Generated Circular-Shaped Surfaces.	93
26. 10 Most Critical Surfaces of Those in Figure 25	95
27. Simple Sliding Block Problem.	97
28. Sliding Block Generator Using More than Two Boxes	99
29. Generation of Active and Passive Portions of Sliding Surface	100
30. Comparison of Carter's Method to Spencer's Method.	105
31. Curvature of Slope Composed of $c' - \phi'$ Soil . .	107

LIST OF SYMBOLS

a	= point of application of N' measured from the left side of slice.
$\bar{a}$	= $\bar{y} - h$
A	= slope of slice base.
b_1	= height of left side of slice.
b_2	= height of right side of slice.
B	= intercept of slice base with respect to coordinate system.
c'_a	= effective stress Mohr-Coulomb strength intercept available.
c'_f	= peak effective stress Mohr-Coulomb strength intercept.
c'_m	= mobilized "cohesion."
c'_a	= component of the available shear resistance along an assumed shear surface which is independent of N' .
$\Delta C'_a$	= available shear resistance at base of slice which is independent of N' .
E'	= internal effective interslice normal force.
$\Delta E'$ (dE')	= change of E' across width of finite (infinitesimal) slice.
E'_o	= effective horizontal force acting on the left of the first slice.
E'_n	= effective horizontal force acting on the right of the last slice.
$f(x)$	= distributional relationship assumed for the side force inclinations.
F	= factor of safety (strength reduction factor).

- h = height of slice measured at centerline.
- h_{eq} = location of horizontal component of earthquake load with respect to midpoint of slice base.
- k = slope of a linear relationship for $f(x)$ within a slice.
- k_h = horizontal earthquake coefficient.
- k_v = vertical earthquake coefficient.
- K = $-\lambda k (\tan \phi'_a / F - A)$
- L = $-\lambda m (\tan \phi'_a / F - A) + 1 + A \tan \phi'_a / F$
- m = intercept of a linear relationship for $f(x)$ within a slice.
- M = internal moment within mass above assumed shear surface
- M_n = moment at right end of mass above assumed shear surface
- n = number of slices into which mass above assumed shear surface is divided.
- N = $p (\tan \phi'_a / F - A)$
- N' = resultant of effective normal stress distribution acting on the mass above the assumed shear surface
- $\Delta N' (dN')$ = effective normal force acting on the base of a finite (infinitesimal) slice
- O = point about which moments are summed.
- P = slope of linear relationship of the first derivative of weight with respect to position within a slice.
- P = $(1 + A^2) c'_a / F + q (\tan \phi'_a / F - A)$
- q = intercept of linear relationship of the first derivative of weight with respect to position within a slice.
- Q = resultant of side forces of a slice.

- ΔQ = resultant of uniform surcharge boundary load acting upon the top of a slice.
- r = length of radial line of a logarithmic spiral; or radius of a circle.
- r_d = lever arm of R_d about an arbitrary point (O).
- r_n = lever arm of N' about an arbitrary point (O).
- r_o = reference radius of logarithmic spiral.
- r_s = lever arm of S_r about an arbitrary point (O).
- r_u = pore pressure coefficient $(\frac{u}{\gamma h})$.
- R = lever arm to Q of a slice about a moment center; or random number of range 0 to 1; or random force acting upon logarithmic spiral shear surface.
- R_d = resultant of driving forces acting on mass above the assumed shear surface.
- S_a = resultant of the available shear strength which could be mobilized along an assumed shear surface.
- ΔS_a = available shear resistance at the base of a slice.
- S_r = resultant of the shear resistance mobilized along the assumed shear surface.
- ΔS_r (dS_r) = resisting shear force required for limiting equilibrium at the base of a finite (infinitesimal) slice.
- T = length of shear surface line segments.
- ΔU_α = water force acting on base of slice.
- ΔU_β = water force acting upon top of slice.
- ΔW (dW) = total weight of a slice.
- $\bar{x}$ = distance in the x direction from the center of slice base to a moment center.
- Δx (dx) = width of finite (infinitesimal) slice.

- x_n = lever arm of $\Delta N'$ about an arbitrary point (O).
- x_s = lever arm of ΔS_r about an arbitrary point (O).
- x_w = lever arm of ΔW about an arbitrary point (O).
- X = internal interslice shear force.
- ΔX (dx) = change of X across width of a finite (infinitesimal) slice.
- x_o = vertical force acting on the left of the first slice.
- x_n = vertical force acting on the right of the last slice.
- $\bar{y}$ = distance in y direction from the center of a slice base to the moment center.
- y = elevation of the shear surface at left side of slice.
- Δy (dy) = change of elevation of the shear surface across finite (infinitesimal) slice.
- y'_t = elevation of the line of effective thrust at left side of slice.
- $\Delta y'_t$ (dy'_t) = change of elevation of the line of effective thrust across finite (infinitesimal) slice.
- α = inclination of slice base.
- α_{dc} = angle between R_d and C'_a .
- α_{dn} = angle between R_d and N' .
- α_{ds} = angle between R_d and S_r .
- α_1 = inclination of counterclockwise direction limit for initial line segment of shear surface.
- α_2 = inclination of clockwise direction limit for initial line segment of shear surface.
- β = inclination of slice top; or inclination of ground surface to right of initiation point.

- θ = angle between reference radius and radial line to point on logarithmic spiral; inclination of resultant of slice side forces; inclination of line segment defining trial shear surface; or scaling factor for resultant side force inclination, used in Spencer's procedure for slope stability analysis.
- θ_1 = inclination of counterclockwise direction limit for line segment of irregular-shaped shear surface.
- θ_2 = inclination of clockwise direction limit for line segment of irregular-shaped shear surface.
- $\Delta\theta$ = angle subtended about center of circle by initial line segment.
- $\Delta\theta'$ = randomly selected deflection angle for circular-shaped shear surface.
- $\Delta\theta_1$ = counterclockwise direction limit of line segment of irregular-shaped shear surface.
- $\Delta\theta_2$ = clockwise deflection limit of line segment of irregular-shaped shear surface.
- $\Delta\theta_{\min}$ = minimum deflection limit for line segment of circular-shaped shear surface.
- $\Delta\theta_{\max}$ = maximum deflection limit for line segment of circular-shaped shear surface.
- λ = scaling factor for side force inclinations, used in the Morgenstern and Price procedure for slope stability analysis.
- ϕ'_a = effective stress Mohr-Coulomb shear strength angle available.
- ϕ'_f = peak effective stress Mohr-Coulomb strength angle.
- ϕ'_m = mobilized strength angle.
- σ'_n = effective normal stress.
- τ_a = available shear strength which can be mobilized at a point
- τ_f = peak shear strength at point on failure surface at verge of failure.

ABSTRACT

Siegel, Ronald A., MSCE, Purdue University, May 1975.
Computer Analysis of General Slope Stability Problems.
Major Professor: C. W. Lovell.

A computer program, named STABL, for performing limiting equilibrium analyses by the method of slices for general slope stability problems has been developed. A discussion of previously available theories and programs precedes the presentation of the model of stability analysis used in this research and its computer simulation.

The program can handle slope profiles having multiple slope ground surfaces. Any arrangement of subsurface soil types having different soil properties can be specified. Pore pressures may be related to a steady state flow domain, related to the overburden, or specified within zones. Surcharge boundary loads and pseudo-static earthquake loadings are also considered.

Three trial shear surface generators have been developed to generate surfaces either of circular shape, of general irregular shape, or of specified character such as sliding blocks. Restrictions may be imposed to generate only surfaces of interest.

Graphical output provides means for checking input and for quick evaluation of results. Print character plots are

processed on line printers or teletype terminals. Plotting devices provides plots of higher resolution.

The program has been written in Fortran IV source language, and is routinely used on Purdue University's Computing Center's Control Data Corporation 6500 computer. With full plotting capabilities 66K octal of core storage is required for execution. Program length is 6000+ statements including components. Another version is being modified for use at the Indiana State Highway Commission's Computer Center's IBM 360 facility.

INTRODUCTION

Wherever there is a difference in the elevation of the earth's surface, whether it be the result of man's actions or natural processes, there are forces which act to restore the earth to a level surface. The process in general is referred to as mass movement. A particular event of special interest to the geotechnical engineer is the landslide. The geotechnical engineer is often given the task of ensuring the safety of human lives and property from the destruction which landslides can cause.

In order to accomplish this task he must have a thorough understanding of the mechanics of the failure mechanisms involved. He must be able to reasonably ascertain the insitu shearing stress and resistance of the materials involved, and if water is present, he must be able to define the flow regime and the water's relationship to strength. Knowing these, he can formulate a model which can be used to make a prediction as to whether a slope will perform safely or not.

The primary cause of motion is a component of gravity acting on the sliding mass in the direction of movement along a failure surface. Other forces may also act. Seepage forces, which result from energy lost by ground water flowing through permeable earth, can be and often are

significant. In regions of seismic activity, earthquake displacements can impart significant forces which can initiate mass movement in an earth slope.

These forces produce shear stresses within the slope. Unless the strength of the earth material is sufficient along any possible shear surface, a movement of mass down slope will occur. Except for a few special cases, the position and shape of the surface most likely to fail or most likely to approach failure is not obvious by examination of slope profiles. Therefore an analysis using limiting equilibrium usually requires that a number of trial surfaces be evaluated in order to determine the position and shape of the most likely path of failure.

The method of slices, particularly applicable to heterogeneous soil systems, requires a great amount of computation per trial surface, regardless of the assumptions made for analyses. If performed by hand, time in general will allow only a few trial surfaces to be analyzed economically. However, with the aid of a programmed computer, hundreds of trial surfaces can be analyzed quickly and economically.

In highway design, it is common to be concerned with a large number of cuts and embankments, each often having characteristics sufficiently unique to require individual analysis. A civil engineering project such as a highway, having great length, invariably encounters geologic

conditions which vary considerably from location to location along its route, e.g., groundwater conditions, surface drainage, material properties, and economical construction techniques. The need to analyse an adequate number of typical slope profiles, in a routine and efficient manner, requires implementation of a computer programmed to handle general problems.

The primary purpose of this thesis was to develop a computer program which will enable a geotechnical engineer with an understanding of slope stability to define a problem and then be provided some measure of its stability, a factor of safety. The program named STABL is a model with flexibility able to handle most stability problems, without requiring gross simplification to render them tractable for solution.

A number of computer programs were available at the time of this undertaking. Their capabilities and limitations vary widely, as well as their requirements for input data. It is difficult to assess their capabilities because many are still being developed, modified and improved by their authors or successors of their authors. Therefore, this author does not wish to generalize about their capabilities. However, it was felt that a new program, expanding beyond the capabilities of existing programs, with few limitations, and with an emphasis on ease of use was needed.

METHODS AVAILABLE FOR SLOPE STABILITY ANALYSIS

There are basically two deterministic approaches to slope stability analysis. The first concerns limiting equilibrium analyses where strain considerations are of no consequence; and the second, elastic methods where strain and its relationship to stress is of importance.

The latter can be more realistic for modeling the mechanics of slope behavior. However, our inability to adequately ascertain the insitu nonlinear stress-strain behavior of materials composing slopes, severely limits the application of these methods. More often than not, an adequate assessment of a slope's stability can be made using a cruder limiting equilibrium model where the uncertainty of material properties is confined to the shear strength parameters selected.

Methods using theories of elasticity are either exact or approximate. Exact solutions have been determined for simple cases only. For example the Boussinesq solution (1855) determines the distribution of stresses within a linear elastic half-space induced by a concentrated force placed upon its surface. A problem somewhat more complicated has been solved for the stress distribution within the body of a slope having a cylindrical face (Romani, Lovell and Harr, 1972). Exact solutions are not available for

application to general slope stability problems encountered in engineering practice.

Numerical approximations such as finite difference methods and finite element methods offer greater flexibility for handling complicated problems, including treatment of nonlinear stress strain relationships, but require a great amount of attention to detail, and often the models become too complicated for practical use in engineering problems. Practical engineering applications require simplifications, and when they are introduced, inaccurate results may be obtained (Hansen, 1953).

Limiting Equilibrium Methods

Limiting equilibrium methods follow one of two approaches (Harr, 1966). The first, originating with Rankine (1857), assumes full mobilization of the available shear strength at all points within a slope. The methods utilizing this approach are mathematical treatments, in closed form, solving statically determinate systems of equations satisfying given boundary conditions. The problems that have been solved by this approach are limited, due to analytical difficulties encountered while arriving at solutions.

The second approach, originating earlier with Coulomb (1776), requires assessment of the stability of the material above an assumed shear surface. A number of assumed shear surfaces are analysed to determine the critical shear

surface. Equilibrium of the material above each shear surface is examined by assuming enough shear strength is mobilized to maintain the slope at the verge of failure. The factor of safety with respect to failure along a shear surface being investigated is usually defined as the ratio of the available shear strength to the shear strength mobilized.

Limiting Equilibrium and Fundamental Assumptions

In general the limiting equilibrium approach is statically indeterminate. To illustrate this, a discussion, of what is known and not known relevant to a general slope stability problem, is presented. Figure 1 may be referred to while following this discussion.

The resultant of the driving forces, R_d , and its line of action can be readily calculated. The driving forces include the total weight of all the mass located above the assumed shear surface, earthquake forces which may be imparted to this mass, buoyant forces if the mass is partially or totally submerged, and boundary surcharge loads which may be located upon the ground surface within the extent of the assumed shear surface.

The distribution of effective normal stress, which is unknown, can be represented by a resultant effective normal force, N' . However, its magnitude and line of action are unknown. The resisting shear strength mobilized along the

FIGURE 1 - BASIC FORCES ASSOCIATED WITH LIMITING EQUILIBRIUM.

assumed shear surface can also be replaced by a single unknown resisting shear force, S_r , whose line of action is also unknown.

The shear strength mobilized is that required to maintain the mass, located above the assumed shear surface, in a state of limiting equilibrium. If the slope is stable, the shear strength mobilized is only a portion of what is available.

A factor of safety may be defined as the ratio of available shear strength to the shear strength mobilized to maintain limiting equilibrium. In terms of forces,

$$F = S_a / S_r \quad (1)$$

where S_a is the resultant shear force of all the shear strength available along the assumed shear surface. The resultant shear force required for limiting equilibrium then becomes,

$$S_r = S_a / F \quad (2)$$

where the factor of safety is a strength reduction factor.

The value of the factor of safety, as expressed, is an average value. It is implied that the shear strength mobilized along the shear surface is proportional to the available shear strength. The limiting equilibrium model does not account for stress concentrations.

Having defined the factor of safety, the problem becomes a matter of determining a unique value for it. Relationships between the forces involved must be established. The first of these relationships may be determined by satisfying static equilibrium.

If moment equilibrium of the free body isolated by the assumed shear surface is to be satisfied, the three resultant forces shown in Figure 1(a) must not produce a net moment about any point. Below is an expression for moment equilibrium about an arbitrary point 0,

$$R_d r_d - N' r_n - S_r r_s = 0 \quad (3)$$

where r_d , r_n , and r_s are lever arms about point 0 to the resultant forces R_d , N' , and S_r , respectively. Since the lines of action of N' and S_r are unknown, r_n and r_s are also unknown.

If force equilibrium is to be satisfied for the same free body, the summation of forces in two mutually exclusive directions must not result in a force imbalance. Expressions for force equilibrium in directions parallel and perpendicular to the resultant driving force, R_d , are as follow,

$$R_d - S_r \cos \alpha_{ds} - N' \cos \alpha_{dn} = 0 \quad (4a)$$

$$S_r \sin \alpha_{ds} - N' \sin \alpha_{dn} = 0 \quad (4b)$$

where α_{ds} and α_{dn} are the angular differences of inclination (Figure 1(b)) between the resultant driving force R_d and the

resultant forces S_r and N' , respectively. The angles α_{ds} and α_{dn} are unknown, because of no information with regard to the lines of action of S_r and N' .

An audit of the unknowns and the independent equations available to relate them, when the free body is at limiting equilibrium, shows a total of seven unknowns; F , S_a , N' , r_n , r_s , α_{dn} , α_{ds} , but only three equations from which a solution for a unique value of the factor of safety is desired. The problem is clearly statically indeterminate.

In order to establish static determinacy, assumptions are required regarding the forces involved, or stress-strain considerations must be introduced. Wishing to avoid the complications and uncertainties involved with the latter, assumptions are made regarding the unknowns.

The basic assumption, made with regard to limiting equilibrium methods to be discussed, is that the Mohr-Coulomb failure criteria is valid. A linear relationship between shear strength and effective normal stress is commonly used.

$$\tau_f = c'_f + \sigma'_n \tan \phi'_f \quad (5)$$

where τ_f is the shear strength on the failure surface at the verge of failure, c'_f and ϕ'_f are effective stress parameters describing the intercept and angle of inclination of the Mohr-Coulomb failure envelope, and σ'_n is the effective normal stress on the failure surface at failure.

Other criteria may be used instead of peak stress failure, such as residual shear strength, for example. Therefore expression (5) is rewritten as,

$$\tau_a = c'_a + \sigma'_n \tan \phi'_a \quad (5a)$$

where τ_a is the available shear strength, c'_a and ϕ'_a are effective stress parameters describing the intercept and angle of inclination of the Mohr-Coulomb envelope for some criteria other than peak stress failure.

A nonlinear assumption could be made for the relationship between shear strength and effective normal stress. However, such an assumption will not be presented here.

The relationship between available shear strength and effective normal stress can be expressed in terms of forces,

$$S_a = C'_a + N' \tan \phi'_a \quad (6)$$

where C'_a is the resultant of the component of available shear strength which is independent of the normal stress distribution. The resisting shear force required to maintain limiting equilibrium is then,

$$S_r = \frac{C'_a + N' \tan \phi'_a}{F} \quad (7)$$

By examining the two components of the resisting shear force, C'_a / F and $(N' \tan \phi'_a) / F$, two observations can be made (Figure 1(c)). First, the magnitude of the component

C'_a / F is dependent only upon the value of the factor of safety, and its line of action is dependent only upon the geometry of the assumed shear surface; α_{dc} (Figure 1(c)) is known. Although its magnitude will vary depending upon the value of factor of safety, its direction is fixed.

The second observation which can be made, is that the component of shear resistance assumed dependent upon the normal stress distribution must act perpendicular to the resultant effective normal force, N' . These two observations do not add any significant information which will assist in arriving at a solution for the factor of safety, F , for a shear surface of general shape. However they do, if the assumed shear surface is a plane.

For a planar shear surface, the frictional and cohesive components of the resisting shear force, S_r , act in the same direction, along the plane of the shear surface (Figure 2(a) and 2(b)). The angles α_{dc} and α_{ds} (Figure 1(b) and 1(c)) are equal. The resultant effective normal force, N' , acts perpendicular to the planar shear surface. Realizing this, α_{dn} (Figure 2(b)) becomes a known which can be calculated.

By summing forces parallel and normal to the planar shear surface, the following two equations for force equilibrium are obtained.

$$S_r - R_d \sin \alpha_{dn} = 0 \quad (8a)$$

$$N' - R_d \cos \alpha_{dn} = 0 \quad (8b)$$

FIGURE 2 - PLANAR SHEAR SURFACE.

The solution becomes statically determinate for a planar shear surface. The resisting shear force, S_r , and the effective normal force, N' , can be solved for by use of the two independent equations of force equilibrium (8). Having calculated the effective normal force, N' , the available shear resistance, S_a , can be calculated with expression (6), and finally the factor of safety, F , can be calculated using expression (1).

For shear surfaces of general shape, the problem remains statically indeterminate. If a solution is to be arrived at, a distribution of effective normal stress along the assumed shear surface must be determined or assumed, so that the magnitude and line of action may be established for the resultant effective normal force, N' .

If this is done, N' , α_{dn} , and r_n will become knowns. Thus the unknowns are reduced to four. With the three equilibrium equations, plus the assumed shear strength relationship with normal stress, a solution can be arrived at. The value determined for the factor of safety is dependent upon what assumption is used with regard to the effective normal stress distribution.

Logarithmic Spiral Shear Surface

The limiting equilibrium approach originating with Coulomb is a statically indeterminate one, except when the shear surface assumed is a plane, as previously mentioned,

or a logarithmic spiral (Frohlich, 1953) of the form

$$r = r_0 e^{\theta \tan \phi'_m} \quad (9)$$

where r is the radial distance from the center of the spiral to a point on the spiral, r_0 is a reference radius, θ is the angle between the reference radius and a radial line to the point on the spiral, and ϕ'_m is the effective mobilized friction angle of the material composing the slope. A plane can be considered a special case of expression (9), where r_0 is of infinite magnitude.

The shear surface is dependent upon the mobilized friction angle which is unknown. Therefore the procedure requires that the spiral's center and reference radius be fixed while determining the effective mobilized friction angle. Given a spiral center point and reference radius, an effective mobilized friction angle is assumed.

When moments are summed about the center of the spiral, the moment contributed by the resultant of the effective normal stresses, N' , and the resultant of the frictional components of the mobilized shear strength, $N' \tan \phi'_m$, cancel each other. The resultant of these two forces, R , acts through the center of the spiral (Figure 3). Therefore by summation of moments about the spiral's center, equilibrium can be examined without requiring knowledge of the normal stress distribution along the shear surface. The average component of shearing resistance due to "cohesion" required to

FIGURE 3 - LOGARITHMIC SPIRAL SHEAR SURFACE AND FORCE ORIENTATION.

maintain equilibrium of the slope along the assumed shear surface can then be assessed by satisfying moment equilibrium about the center of the spiral.

The factor of safety, F , is defined as the ratio of the "cohesive" shear strength available, c'_a , to the "cohesive" shear strength mobilized, c'_m . This calculated factor of safety must be equivalent to that defined by the ratio of the available frictional shear strength, $\tan \phi'_a$, to the frictional shear strength mobilized, $\tan \phi'_m$.

$$F = \frac{c'_a}{c'_m} = \frac{\tan \phi'_a}{\tan \phi'_m} \quad (10)$$

If the above relationship is not satisfied, other values of ϕ'_m must be assumed, repeating the procedure, trial and error, until it is satisfied. To determine the critical shear surface, a number of spiral centers and reference radii should be evaluated.

$\phi = 0$ Case

The shearing resistance may be written as a value independent of the state of stress, rather than in terms of c'_a and ϕ'_a . This is equivalent to stating that the value of ϕ'_a in equation (5) is zero. For this case the radius of the spiral becomes a constant, i.e., the spiral degenerates into a circle. When the shear strength of a slope is so assumed to be independent of the normal stresses acting upon the shear surface, and these stresses all act toward the center of the circle, their contribution to the moment about the

center of the circle is eliminated. Statical determinacy with respect to moment equilibrium is achieved, and the average shear strength required for equilibrium can be calculated from the moment equilibrium equation. The factor of safety for the $\phi = 0$ condition, assuming a circular surface, is defined as the ratio of the available shear strength to the shear strength mobilized.

Surfaces, other than logarithmic spirals of the form discussed, planar surfaces, and circular surfaces when the frictional component of shear strength is not considered require assumptions to determine the normal stress distribution along the shear surface. A number of methods have been developed.

Friction-Circle Method

The friction-circle method provides a value of the factor of safety for any reasonable assumption for the distribution of the normal stresses. By making an assumption equivalent to assuming that all the normal stress is concentrated at a single point on the failure surface, a lower bound value can be established for the factor of safety (Frochlich, 1955; Whitman and Moore, 1963). By assuming all the normal stress to be concentrated at the two end points of the trial shear surface (Frochlich, 1955; Whitman and Moore, 1963) an upper bound value for the factor of safety can also be established.

These two cases are extremes which are physically unreasonable. However, the value of the factor of safety resulting from the assumption of any reasonable normal stress distribution will lie between these two bounds. Values of factors of safety calculated by other methods may be judged as being reasonable, by comparison with those obtained by the lower and upper bound solutions.

A sinusoidal normal stress distribution is often used in practice for the friction-circle method, because it is intuitively more reasonable (Taylor, 1937). The upper bound solution has greater influence on the value of the factor of safety than the sinusoidal distribution when comparing both to the lower bound solution (Taylor, 1937; Wright, 1969). This may suggest using the value determined by the lower bound solution as a conservative value. However, the difference, in the values of the factor of safety for the lower bound solutions and that for solutions using the more reasonable assumption of a sinusoidal normal stress distribution, varies. The difference increases with an increase of the subtended angle of the circular arc defining the shear surface. The lower bound solution may therefore be too conservative for economical design purposes.

Method of Slices

The most commonly used procedure of slope stability analysis are methods which divide the mass above an assumed

shear surface into slices (Figure 4). These methods can be conveniently applied to problems dealing with heterogeneous slope profiles. The mass is divided so that the base of each slice is characterized by a single set of shear strength parameters. The shear strength at the base of each slice can therefore be described independently of the other slices.

Additional divisions of the mass above the assumed shear surface are made to simplify calculations. It is convenient to assume that the base of each slice is the chord of the curved shear surface, rather than the curved shear surface itself (Figure 4). The weight calculations are thus greatly simplified. Also the inclination of the chord may be used as the average inclination of the actual curved base of each slice. Both these simplifications introduce an error which can be minimized if an adequate number of slices is used. The net error summed for all slices decreases as the number of slices increases, providing an adequate number of significant figures are used for the calculations.

The mathematical models presented initially will ignore consideration of forces frequently encountered in practical slope stability problems such as those produced by buoyancy and seepage, earthquake displacements, and ground surface boundary loads. Introduction of these considerations do not add to the statical indeterminacy of the procedures to be discussed, but simply complicate the expressions derived.

FIGURE 4 - MASS ABOVE ASSUMED SHEAR SURFACE DIVIDED INTO SLICES AND LINEARIZATION OF THE SLICES' BASES.

This being the case, only the fundamental forces involved will be considered at this time.

Each slice is acted upon by a system of forces as shown in Figure 5. Table 1 lists the equations available for analysing the stability of a slope by the method of slices procedure, as well as the unknowns involved. It is seen, that unless only one slice is used, the problem is statically indeterminate requiring some simplifying assumptions to be made.

Ordinary Method of Slices

When considering overall moment equilibrium of the mass above a circular shear surface, the need to know the position of the effective normal force acting on the base of each slice, $\Delta N'$, is not required when moments are summed about the center of the assumed circular shear surface. All the effective normal stress will act through the circle's center, producing no moment about that particular point.

The overall moment equilibrium equation, taking moments about the center of the circle, (Figure 6), becomes

$$\sum_{i=1}^n \Delta W r \sin \alpha - \sum_{i=1}^n \Delta S_r r = 0 \quad (11)$$

in which ΔW is the weight of an individual slice, r is the radius of the circle, α is the inclination of the base of the slice, ΔS_r is the resisting shear force acting along the base of the slice, and n is the number of slices. The side

FIGURE 5 - BASIC FORCES ACTING UPON INDIVIDUAL SLICE.

Table 1
Equations and Unknowns Associated with Procedure of Slices

Equations	
n^*	Moment equilibrium equations for each slice.
$2n$	Force equilibrium equations in each of two mutually exclusive directions for each slice.
n	Mohr-Coulomb shear strength relationship with normal effective stress of the material at the base of each slice.
$4n$	Total number of equations
Unknowns	
1	Factor of safety (F).
n	Effective force acting on the base of each slice ($\Delta N'$).
n	Position of the effective normal force on the base of each slice (a).
n	Available shear resistance at base of each slice (ΔS_a).
$n-1$	Interslice normal forces (E').
$n-1$	Interslice shear forces (X).
$n-1$	Location of interslice forces (y'_t).
$5n-2$	Total number of unknowns.
* (n is the number of slices).

FIGURE 6 - FORCES CONSIDERED FOR ORDINARY METHOD OF SLICES.

forces are internal forces that do not contribute to the overall external moment.

An error has been introduced into the above expression. The moment arm to the weight force, ΔW , of each slice, $r \sin \alpha$, implies that the weight of each slice acts at the center of each slice. This is generally not the case. The error can be eliminated by using the appropriate moment arm to the center of gravity of each slice. If the simplification $r \sin \alpha$ is used for the moment arm, the magnitude of the error introduced will decrease as the number of slices used increases.

The resisting shear force, ΔS_r , acting along the base of each slice is defined as the sum of the cohesive and frictional strength available at the base of a slice, divided by a factor of safety

$$\Delta S_r = \frac{\Delta C'_a + \Delta N' \tan \phi'_a}{F} \quad (12)$$

where $\Delta N'$ is the resultant of the effective normal stress acting on the base of the slice, and $\Delta C'_a$ is that component of the resultant of the shear strength available at the base of the slice, which is independent of the state of effective stress.

By substitution of the expression for shearing resistance (12) into the expression for overall moment equilibrium (11) and solving for the factor of safety, the following expression is obtained,

$$F = \frac{\sum_{1}^n (\Delta C'_a + \Delta N' \tan \phi'_a)}{\sum_{1}^n \Delta W \sin \alpha} \quad (13)$$

Not needing to know the location of the effective normal force acting on the base of each slice, the number of unknowns reduces to $4n-2$. The $3n$ equations of equilibrium are still not adequate for statical determinacy, and additional assumptions are required. These final assumptions are generally made with respect to the side forces.

If forces are summed normal to the base of the slice shown in Figure 5 it will be observed that the only unknowns which would contribute to a resulting expression would be the interslice side forces and the effective normal force at the base of the slice. Therefore any assumptions made with regard to the side forces would enable determination of the effective normal force acting on the base of each slice.

An assumption that simplifies the calculations the most was first proposed by Fellenius (1927). He assumed that the interslice forces acting on each slice have no net effect. That is, the effective normal and shear interslice forces acting on the left side of each slice are equal in magnitude but opposite in action to the respective interslice forces acting on the right side (Taylor, 1937).

When no external side forces act upon the outside edge of the two end slices, the usual case for slope stability problems, the assumption requires that the interslice forces

acting upon these two slices be zero. Conformity to Newton's Third Law, stating that every action is always opposed by an equal reaction, requires that the remaining interslices also be zero, which is physically unreasonable.

A water filled tension crack at the top of a slope is a situation where an external force would act upon the side of an end slice. For this case it is generally impossible to satisfy Newton's Third Law with the Fellenius assumption.

Elimination of the side forces and the line of thrust from consideration, reduces the unknowns by a total of $3n-3$. The number of unknowns are thus reduced to $n+1$, making the problem highly overdeterminate.

The n force equilibrium equations obtained by summing forces normal to the base of each slice, while ignoring the side forces, are used to obtain the effective normal force acting on the base of each slice,

$$\Delta N' = \Delta W \cos \alpha \quad (14)$$

Including the overall moment equilibrium equation, $n+1$ equilibrium equations are used.

The expression for the factor of safety becomes,

$$F = \frac{\sum_{i=1}^n (\Delta C_a + \Delta W \cos \alpha \tan \phi'_a)}{\sum_{i=1}^n \Delta W \sin \alpha} \quad (15)$$

This expression does not satisfy force equilibrium parallel to the base of each individual slice. Also, since it is not

necessary to know or to assume a position for the effective normal force at the base of each slice to obtain a solution, satisfaction of moment equilibrium of each individual slice can not be shown.

U The procedure as described above is known as the Ordinary Method of Slices. It is also called the Fellenius Method or Swedish Circle Method. The values of the factor of safety calculated by the above expression have been shown to be smaller than those calculated by the lower bound solution (Whitman and Moore, 1963). The error range of calculated values of the factor of safety may vary from five to forty percent from more accurate methods to be described later. The variation depends upon the value of the factor of safety, the subtended angle of the circular shear surface, and the magnitude and distribution of pore pressure when considered (Whitman and Bailey, 1967). It may be concluded then, that the assumption of no side forces used by the Ordinary Method of Slices is inaccurate. The unreasonably low values of the factor of safety could lead to conservative and uneconomical designs.

Alternative expressions for the factor of safety can be obtained by use of overall horizontal or vertical force equilibrium. But it has been shown (Spencer, 1967; Wright, 1969) that the value of the factor of safety by overall moment equilibrium is not as sensitive to the side force assumption.

Simplified Bishop Method

Bishop (1955) proposed an alternative method for obtaining a value of the factor of safety without including the effect of interslice forces. Overall moment equilibrium is satisfied as with the Ordinary Method of Slices. However the expression for the effective normal force at the base of each slice is arrived at in an alternate manner.

Forces are summed vertically to obtain an equilibrium expression containing the effective normal force for each slice (Figure 5), ignoring the side forces,

$$\Delta W - \Delta S_r \sin \alpha - \Delta N' \cos \alpha = 0 \quad (16)$$

rather than summed normal to the base of each slice. By substituting the expression for required shear resistance (12) and solving, the resulting expression is obtained for the effective normal force at the base of each slice.

$$\Delta N' = \frac{\Delta W - \Delta C'_a \sin \alpha / F}{\cos \alpha (1 + \tan \phi'_a \tan \alpha / F)} \quad (17)$$

Substitution of this expression into the overall moment equilibrium equation (11) yields an implicit expression for the factor of safety.

$$F = \frac{\sum_{i=1}^n [(\Delta C'_a \cos \alpha + \Delta W \tan \phi'_a) / \cos \alpha (1 + \tan \phi'_a \tan \alpha / F)]}{\sum_{i=1}^n \Delta W \sin \alpha} \quad (18)$$

The solution for a value of the factor of safety requires an iterative approach, assuming a value for the

factor of safety and testing for equality. Although the Simplified Bishop Method, also called the Modified Bishop Method, does not provide a direct method for obtaining the factor of safety as does the Ordinary Method of Slices, the values obtained agree much closer to those obtained by more accurate methods considering interslice forces (Whitman and Bailey, 1967).

Bishop's Method, Rigorous

A rigorous method capable of analysing circular shear surfaces for slopes composed of soil having both frictional and cohesive components of shear strength was first proposed by Bishop (1955). The method satisfies complete equilibrium and provides a means for judging the reasonableness of any solution arrived at.

When arriving at an expression for the effective normal force acting on the base of each slice, the interslice forces are not ignored for this rigorous method. Forces are summed vertically to obtain the following equation of equilibrium for an individual slice (Figure 5),

$$\Delta W + \Delta X - \Delta S_r \sin \alpha - \Delta N' \cos \alpha = 0 \quad (19)$$

where ΔX is the difference of magnitude between the two shear forces acting on the vertical sides of the slice.

Substituting the expression for shear resistance (12) into this vertical equilibrium equation and solving for the effective normal force, the following expression is obtained,

$$\Delta N' = \frac{W + \Delta X - \Delta C'_a \sin \alpha / F}{\cos \alpha (1 + \tan \phi'_a \tan \alpha / F)} \quad (20)$$

Substitution of this expression into that of the overall moment equilibrium, equation (11), yields an expression for the factor of safety,

$$F = \frac{\sum_1^n \Delta C'_a \cos \alpha + (\Delta W + \Delta X) \tan \phi'_a}{\sum_1^n \Delta W \sin \alpha} \quad (21)$$

If internal equilibrium is to be satisfied, the internal forces must satisfy the following boundary conditions,

$$\sum_1^n \Delta X = X_o - X_n \quad (22a)$$

$$\sum_1^n \Delta E' = E'_o - E'_n \quad (22b)$$

where $\Delta E'$ is the difference in magnitude between the two effective normal forces acting on the vertical sides of each slice, and X_o , E'_o , X_n and E'_n are shear and effective normal forces, which may be present on the extreme left and right boundaries of the sliding mass, respectively (Figure 7).

By summing forces parallel to the base of each slice, an equation for force equilibrium in directions exclusive of the vertical direction, previously used, is obtained,

$$(\Delta W + \Delta X) \sin \alpha + \Delta E' \cos \alpha - \Delta S_r = 0$$

FIGURE 7 - BOUNDARY FORCES ACTING ON THE END SLICES.

Solving for $\Delta E'$,

$$\Delta E' = (\Delta W + \Delta X) \tan \alpha - \Delta S_r \sec \alpha \quad (23)$$

and summing for all slices,

$$\sum_1^n \Delta E' = \sum_1^n [(\Delta W + \Delta X) \tan \alpha - \Delta S_r \sec \alpha]$$

an expression satisfying equation (14b) is then arrived at,

$$\sum_1^n [(\Delta W + \Delta X) \tan \alpha - \Delta S_r \sec \alpha] = E'_n - E'_o \quad (24)$$

Vertical force equilibrium, force equilibrium in a direction other than vertical for each slice, and overall moment equilibrium will be satisfied if appropriate values for ΔX can be selection to satisfy expressions (21), (22a) and (24). If not, complete equilibrium cannot be satisfied.

Moment equilibrium of the individual slices was not considered when arriving at a solution for the factor of safety. Use of $n-1$ equations for individual slice moment equilibrium will provide the location of the line of thrust. Since overall moment equilibrium was considered, one equation for slice moment equilibrium is redundant and therefore not used.

The position of the effective normal force, $\Delta N'$, at the base of each slice must be assumed in order to establish an expression for moment equilibrium for each slice. Bishop (1955) assumed it to be at the center of the base of each slice. The weight of each slice, ΔW , was also taken at the

center of each slice for moment equilibrium, as is implied by the overall moment equilibrium expression (3).

If moments are summed about the midpoint of the base of each slice, the contribution of the weight, ΔW , and the effective normal force, $\Delta N'$, to the resulting moment equilibrium equation is eliminated. The equation for moment equilibrium

$$\begin{aligned}
 & E' [(y'_t - y) - \frac{\Delta Y}{2}] \\
 & - (E' + \Delta E') [(y'_t + \Delta y'_t) - (y + \Delta y) + \frac{\Delta Y}{2}] \\
 & + X \frac{\Delta x}{2} + (X + \Delta X) \frac{\Delta x}{2} = 0
 \end{aligned} \tag{25}$$

where Δx is the width of the slice, and Δy and $\Delta y'_t$ are the incremental changes, across the width of the slice, of the functions defining the shear surface, $y(x)$, and the line of effective thrust, $y'_t(x)$, respectively.

Solving for the location of the line of effective thrust at the right face of a slice, the following expression is obtained,

$$(y'_t + \Delta y'_t) = \frac{E' y'_t + \Delta E' (y + \frac{\Delta Y}{2}) + X \Delta x + \Delta X \frac{\Delta x}{2}}{E' + \Delta E'} \tag{26}$$

The calculation of the location of the line of effective thrust is performed slice by slice in succession.

For the first slice, the shear and effective normal side forces, X and E' , are the boundary forces, X_0 and E'_0 ,

respectively. The location of the line of effective thrust at the left side of the first slice, y'_t , coincides with the effective normal boundary force E'_0 . When E'_0 is absent, y'_t is at the base of the first slice's shear surface.

The calculated values of $X + \Delta X$, $E' + \Delta E'$, and $y'_t + \Delta y'_t$ become the values of X , E' , and y'_t for the next slice, respectively. The procedure is repeated until all slices are exhausted. The values of $\Delta E'$ for each slice are obtained from expression (23).

The position of the line of thrust provides an opportunity to judge the reasonableness of a solution. There are an infinite number of choices for values of ΔX which will satisfy complete equilibrium. If the choice used produces a solution having a line of thrust totally within the sliding mass implying no tension within the soil mass, the solution should be reasonable.

Another criterion used to establish reasonableness of a solution satisfying complete equilibrium is whether or not excessive interslice shear stresses exist. The solution will not be acceptable if the interslice shear stress exceeds the strength of the soil at any vertical face. Satisfaction of this criterion is not generally sufficient. The most critical internal shear stress generally occurs on an inclined plane (Seed and Sultan, 1967). If the shear strength mobilized along a vertical plane is well below the available shear strength, an adequate solution can be assumed (Whitman and Bailey, 1967).

Bishop (1955) had found that although an infinite number of ΔX values will satisfy complete equilibrium, the variation in values of the factor of safety is insignificant. Thus verification of a reasonable line of thrust or satisfactory interslice shear stress is not always justified, considering the effort often required to achieve thorough satisfaction of all criteria.

When $\Delta X = 0$ for all slices, expression (13) reduces to the expression for the Simplified Bishop Method (10). Equation (14a) is satisfied but (15) generally is not. Therefore complete equilibrium is generally not attained. However the value of the factor of safety is reasonable and conservative compared with solutions in complete equilibrium (Wright, Kulhawy, and Duncan, 1973). The conservatism of the Simplified Bishop Method is of little importance to economic considerations of design.

Nonveiller (1965) extended Bishop's approach to consider shear surfaces of general shape. The solution reduces to Bishop's when applied to circular shear surfaces.

Overall moment equilibrium is established by summing moments about an arbitrary point. The following expression is obtained

$$\sum_1^n \Delta W x_w - \sum_1^n \Delta S_r x_s - \sum_1^n \Delta N' x_n = 0 \quad (27)$$

where x_w , x_s , and x_n are the lever arms of the weight, shear resistance, and effective normal force of each slice respectively (Figure 8).

FIGURE 8 - LEVER ARMS OF NONVEILLER EXTENSION OF BISHOPS APPROACH.

Substituting equation (12), the expression for shear resistance, and solving for the factor of safety the following expression is obtained,

$$F = \frac{\sum_{i=1}^n (\Delta C'_a + \Delta N' \tan \phi'_a) x_s}{\sum_{i=1}^n \Delta W x_w - \sum_{i=1}^n \Delta N' x_n} \quad (28)$$

Substitution of expression (20) for the effective normal force derived by summing forces vertically, the following expression for the factor of safety is obtained,

$$F = \frac{\sum_{i=1}^n \frac{\Delta C'_a \cos \alpha + (\Delta W + \Delta X) \tan \phi'_a}{\cos \alpha (1 + \tan \phi'_a \tan \alpha / F)} x_s}{\sum_{i=1}^n \Delta W x_w - \sum_{i=1}^n \frac{\Delta W + \Delta X - \Delta C'_a \sin \alpha / F}{\cos \alpha (1 + \tan \phi'_a \tan \alpha / F)} x_n} \quad (29)$$

An acceptable solution must satisfy complete equilibrium, expressions (22a) and (24), and also criteria previously mentioned with regard to the line of effective thrust and interslice stress.

Janbu's Generalized Procedure of Slices

Janbu (1954, 1957, and 1973) developed a method of analysing the stability of slopes assuming shear surfaces of general shape. The procedure, is based upon differential equations, which govern moment and force equilibrium of the mass above an assumed shear surface.

Moment equilibrium is considered about the midpoint of the base of a slice of infinitesimal thickness (Figure 9).

FIGURE 9 - INFINITESIMAL SLICE AND FORCE SYSTEM.

The weight of the slice, dW , and the effective normal force, dN' , acting on its base are both assumed to act through the midpoint of the base. Thus their contribution to the resulting moment equilibrium equation is eliminated. An equation in terms of differentials similar to equation (25), which was derived for a slice of finite thickness, is obtained,

$$E'[(y'_t - y) - \frac{dy}{2}] - (E' + dE')[(y'_t + dy'_t) - (y + dy) + \frac{dy}{2}] + X \frac{dx}{2} + (X + dX) \frac{dx}{2} = 0 \quad (30)$$

where dx is the width of the slice, dE' and dX are changes across the slice of the values for the internal forces E' and X , respectively, and dy and dy'_t are the changes across the slice of the functions defining the shear surface, $y(x)$, and the line of effective thrust, $y'_t(x)$, respectively.

Simplifying equation (30), it becomes,

$$E' \frac{dy'_t}{dx} + \frac{dE'}{dx} [y'_t + dy'_t - y - \frac{dy}{2}] + dX \frac{dx}{2} - X = 0 \quad (31)$$

As dx approaches zero, this equation approaches the following limit,

$$(y'_t - y) \frac{dE'}{dx} + E' \frac{dy'_t}{dx} - X = 0 \quad (32)$$

The differential equation governing force equilibrium is obtained from equations satisfying vertical and horizontal force equilibrium of an infinitesimal slice (Figure 9),

$$dW + dX - dN' \cos \alpha - dS_r \sin \alpha = 0 \quad (33a)$$

$$dE' + dN' \sin \alpha - dS_r \cos \alpha = 0 \quad (33b)$$

where dS_r is the resisting shear force at the base of the slice.

Eliminating dN from both of these expressions, the following differential expression is arrived at,

$$dE' = \frac{dS_r}{\cos \alpha} - (dW + dX) \tan \alpha \quad (34)$$

Considering vertical force equilibrium, an expression for the effective normal stress σ'_n acting on the base of an infinitesimal slice can be arrived at,

$$\sigma'_n = \frac{dW}{dx} + \frac{dX}{dx} - s_r \tan \alpha \quad (35)$$

where $\sigma'_n = \frac{dN'}{dx}$ and s_r is the shear strength mobilized,

$$s_r = \frac{dS_r}{dx} = \frac{c'_a + \sigma'_n \tan \phi'_a}{F} \quad (36)$$

Substituting the expression for the effective normal stress (35) into the expression for the shear strength required to be mobilized for limiting equilibrium (36), and solving for the required shear strength,

$$s_r = \frac{c'_a + \left(\frac{dW}{dx} + \frac{dX}{dx} \right) \tan \phi'_a}{F (1 + \tan \alpha \tan \phi'_a / F)} \quad (37)$$

Substitution of this expression for the shear strength into expression (34) yields the differential equation governing force equilibrium,

$$dE' = \frac{c'_a + \left(\frac{dW}{dx} + \frac{dX}{dx}\right) \tan \phi'_a}{F (1 + \tan \alpha \tan \phi'_a / F)} \frac{dx}{\cos^2 \alpha} - \left(\frac{dW}{dx} + \frac{dX}{dx}\right) \tan \alpha dx \quad (38)$$

Satisfaction of overall horizontal equilibrium and boundary conditions requires that,

$$\int_0^n dE = E'_0 - E'_n \quad (39)$$

Therefore, stability requires that,

$$\int_0^n \left[\frac{c'_a - \left(\frac{dW}{dx} + \frac{dX}{dx}\right) \tan \phi'_a}{F (1 + \tan \phi'_a \tan \alpha / F)} \frac{dx}{\cos^2 \alpha} - \left(\frac{dW}{dx} + \frac{dX}{dx}\right) \tan \alpha dx \right] = E'_0 - E'_n \quad (40)$$

where E'_n is usually zero.

From this expression the factor of safety can be solved for, using finite difference approximation of the integrals,

$$F = \frac{\sum_1^n \frac{c'_a \Delta x + (\Delta W + \Delta X) \tan \phi'_a}{\cos^2 \alpha (1 + \tan \alpha \tan \phi'_a / F)}}{\sum_1^n [\Delta W + \Delta X] \tan \alpha + E'_0 - E'_n} \quad (41)$$

The solution for the value of the factor of safety is as follows. An initial value of the factor of safety is obtained from expression (41) assuming that ΔX is equal to zero for each slice. Since the factor of safety is implicitly expressed, a trial and error or iterative approach is used.

By assuming a line of effective thrust and using the initial calculated value of the factor of safety, equations

(32) and (38) can be used to determine approximate values for ΔX . Recalculation of the value of the factor of safety, using equation (41) and the approximate values of ΔX , should produce a more accurate result. The process is repeated until the value of the factor of safety converges to a reasonable degree of accuracy.

The procedure will generally converge within a reasonable number of iterations. However, occasionally the solution will not converge. This could be a serious inconvenience. The only information available in this instance would be the initial value of the factor of safety calculated using the assumption that $\Delta X = 0$. When the solution does converge, the procedure as outlined is more efficient than other methods considering side forces, because the solution arrived at is based upon an assumed reasonable line of effective thrust.

Morgenstern and Price Method

Morgenstern and Price (1965) approached the problem of slope stability analysis in a somewhat different manner, developing a procedure capable of analysing shear surfaces of general shape. The approach also requires formulation of differential equations, governing equilibrium of the mass above an assumed shear surface. The differential equation governing moment equilibrium is identical to that derived earlier for Janbu's generalized procedure of slices method,

$$(y'_t - y) \frac{dE}{dx} + E' \frac{dy'_t}{dx} - X = 0 \quad (32)$$

Force equilibrium of individual slices is satisfied for directions normal and parallel to their bases (Figure 9), rather than in vertical and horizontal directions as considered by Janbu's generalized procedure of slices,

$$dN' - dW \cos \alpha - dX \cos \alpha + dE' \sin \alpha = 0 \quad (42a)$$

$$dS_r - dE' \cos \alpha - dX \sin \alpha - dW \sin \alpha = 0 \quad (42b)$$

The resisting shear force, dS_r , can be expressed in terms of the following Mohr-Coulomb strength relationship containing the definition of the factor of safety with respect to shear strength,

$$dS_r = \frac{1}{F} (c'_a dx / \cos \alpha + dN' \tan \phi'_a) \quad (43)$$

By substituting this expression into equation (42b), solving for dN' , and substituting the resulting expression for dN' into equation (42a), the following differential equation is arrived at,

$$\begin{aligned} \frac{c'_a}{F} [1 + (\frac{dy}{dx})^2] + \frac{\tan \phi'_a}{F} [\frac{dW}{dx} + \frac{dX}{dx} - \frac{dE'}{dx} \frac{dy}{dx}] \\ = \frac{dE'}{dx} + \frac{dX}{dx} \frac{dy}{dx} + \frac{dW}{dx} \frac{dy}{dx} \end{aligned} \quad (44)$$

where $\frac{dy}{dx} = \tan \alpha$.

Equations (32) and (44) are the two governing differential equations satisfying moment and force equilibrium for infinitesimal slices. They contain three unknown functions

E' , X , and y'_t for the internal effective normal and shear forces and the line of effective thrust, respectively, requiring additional assumptions.

Morgenstern and Price (1965) assumed a relationship between the internal forces,

$$X = \lambda f(x) E' \quad (45)$$

where λ is a scaling factor, and $f(x)$ is a function of position, which defines the variational relationship of the internal shear force, X , with the effective normal internal force, E' . The function f is arbitrarily selected, but is acceptable only when the values of the solved unknowns are reasonable.

To arrive at a solution, equations (32) and (44) are integrated, over the range of x defining the potential sliding mass, to test for satisfaction of boundary conditions. To simplify integration, the mass is divided into slices of finite thickness. Each slice is linearized; the base of each slice is linear; the ground surface at the top of each slice and interfaces between different soil types within each slice are linear, so that the weight varies linearly across each slice; and the function f is assumed to vary linearly across each slice. The slice divisions are treated as positions of discontinuity. To integrate over the total mass, integration is performed over the individual slices, in turn.

Examining a typical linear slice, the following is observed. The shear surface can be expressed as a linear equation,

$$y = A x + B \quad (46)$$

where A and B are the slope and intercept of the slice base. The first derivative of the weight with respect to position within the slice can be expressed as,

$$\frac{dW}{dx} = p x + q \quad (47)$$

where p and q are the slope and intercept of the linear relationship. The function f can be expressed as

$$f = k x + m \quad (48)$$

where k and m are the slope and intercept of a linear relationship of the ratio of internal forces within a slice.

Substituting equations (45), (46), (47), and (48) into equation (44), the following equation can be arrived at,

$$(K x + L) \frac{dE'}{dx} + K E' = N x + P \quad (49)$$

where

$$K = - \lambda k (\tan \phi'_a / F - A) \quad (49a)$$

$$L = - \lambda m (\tan \phi'_a / F - A) + 1 + A \tan \phi'_a / F \quad (49b)$$

$$N = p (\tan \phi'_a / F - A) \quad (49c)$$

$$P = \frac{c'_a}{F} (1 + A^2) + q (\tan \phi'_a / F - A) \quad (49d)$$

If equation (49) is integrated over a linear slice,

then,

$$E' = [E'_1 L + \frac{N x^2}{2} + P x] / (L + N x) \quad (50)$$

where E'_1 is the value of E' at the left edge of the slice, and x is a position along the slice measured from its left edge.

Starting at the beginning of the shear surface, with E'_1 equal to the known boundary value, E_0 , which is usually zero, the value of E' at the right edge of the first slice can be calculated with equation (50). This value, in turn, is used as the value of E'_1 for the next slice. The integration is continued in this manner, until all the slices have been exhausted. Then, if the boundary conditions are satisfied, E' , at the end of the shear surface will equal the known boundary value of E'_n , which is generally zero.

If complete equilibrium is to be satisfied, boundary conditions with respect to internal moments must also be satisfied. Equation (32) can be integrated over each linear slice, providing an expression for the internal moment.

$$M = E' (y'_t - y) = \int_{x_0}^x (X - E' \frac{dy}{dx}) dx \quad (51)$$

The boundary conditions with respect to internal moments are satisfied when the integral over the entire sliding mass is equal to the known boundary value, M_n , which is usually zero.

If the boundary conditions are satisfied for equation (51), then the same equation can be used to determine values of y'_t to judge the reasonableness of the solution.

The solution requires assuming the function f and an initial set of values for F and λ . Then integration is performed across all of the slices, slice by slice, to determine E_n and M_n at the end of the shear surface. If the boundary conditions are not satisfied, then, by an iterative approach (Morgenstern and Price, 1967), new values of λ and F are chosen until the boundary conditions are satisfied.

When this is accomplished, reasonableness of the calculated line of effective thrust and of the magnitude of interslice shear forces is examined. If either is judged unsatisfactory, the entire procedure must be repeated, beginning with the selection of a new function f , until a satisfactory solution is arrived at.

Spencer's Method

Spencer (1967) developed a procedure satisfying complete equilibrium, assuming a circular shear surface and parallel resultant interslice side forces. The method is easily extended to assumed shear surfaces of general shape, and to other assumptions with regard to the slice side forces (Wright, 1969). The effective normal and shear side forces acting on both sides of each slice are replaced with statically equivalent resultant side forces, Q , (Figure 10). Each is assumed to act concurrently, through the midpoint of the base of each respective slice, with the other slice forces; the weight, ΔW , the effective normal force, $\Delta N'$, and the resisting shear force, ΔS_r .

FIGURE 10 - SLICE AND RESULTANT OF THE SIDE FORCES.

Summation of forces, normal and tangent to the base of each slice, provide two equations of force equilibrium,

$$\Delta N' - \Delta W \cos \alpha + Q \sin (\alpha - \theta) = 0 \quad (52a)$$

$$\Delta S_r - \Delta W \sin \alpha - Q \cos (\alpha - \theta) = 0 \quad (52b)$$

where θ is the inclination of the resultant side force for each slice. From equation (52a) an expression for the effective normal force acting on the base of each slice is obtained,

$$\Delta N' = \Delta W \sin \alpha + Q \sin (\alpha - \theta) \quad (53)$$

Substituting this expression into that for the resisting shear force at the base of each slice (12), and then substituting the expression obtained into equation (52b), the resultant side forces, Q , can be solved for,

$$Q = \frac{(\Delta C'_a + \Delta W \cos \alpha \tan \phi'_a) / F - \Delta W \sin \alpha}{\cos (\alpha - \theta) (1 + \tan \phi'_a \tan (\alpha - \theta) / F)} \quad (54)$$

If the overall moment produced about an arbitrary point by the external forces is zero, then the overall moment of the internal forces must also be zero. Then,

$$\sum_{i=1}^n Q R \cos (\alpha - \theta) = 0 \quad (55)$$

where R is the distance from the point, about which moments are summed, to the center of each slice.

Also, if overall force equilibrium is to be satisfied, the summation of internal forces in two mutually exclusive

directions must be zero. For horizontal and vertical directions

$$\sum_{i=1}^n Q \cos \theta = 0 \quad (56a)$$

$$\sum_{i=1}^n Q \sin \theta = 0 \quad (56b)$$

If the resultant side forces are assumed to be parallel; e.g., $\theta_i = \text{constant}$, equations (56) become identical and can be expressed as,

$$\sum_{i=1}^n Q = 0 \quad (57)$$

The inclination of the resultant side forces could also be expressed as

$$\theta_i = \theta f(x) \quad (58)$$

where θ is a scaling angle of inclination and $f(x)$ is some distribution defining how θ_i varies with position x (Wright, 1969). Parallel side forces occur when $f(x) = 1$ for the range of x .

The parallel resultant side force assumption is equivalent to the Morgenstern and Price Method (1965) when $f(x) = 1$. Then Spencer's $\tan \theta$ is equivalent to Morgenstern's and Price's λ (Spencer, 1973).

Unique values for the factor of safety, F , and the scaling inclination of the resultant side forces, θ , can be solved for, if equations (55) and (56) are satisfied;

equations (55) and (57) apply if $f(x) = 1$ for the range of x .

Reasonableness of the solution can be judged by the position of the line of thrust and the magnitude of the interslice shear stresses. Both are obtained from the moment equilibrium equations for the individual slices (25).

Spencer's Method requires less computation time than the Morgenstern and Price Method and accomplishes the same task of attaining complete equilibrium, using an assumed relationship between the internal forces. Both require attention to solved unknowns to check for their reasonableness.

Carter's Method

Carter (1971) demonstrated that if ΔX is assumed to be zero for each slice, the value of the factor of safety, determined by satisfying overall moment equilibrium and vertical force equilibrium of the individual slices, varies with the position of the point about which overall moment equilibrium is satisfied. Nonveiller (1965) had not indicated this behavior, when he extended Bishop's rigorous method to surfaces of general shape and suggested simplifying the calculations by assuming $\Delta X = 0$. Overall moment equilibrium had been satisfied about an arbitrary point.

By summing moments about an arbitrary point O (Figure 11), the following expression for moment equilibrium is obtained,

FIGURE 11 - SLICE FORCES REFERENCED TO AN ARBITRARY POINT.

$$\sum_1^n [\Delta N' (\bar{y} \sin \alpha - \bar{x} \cos \alpha) + W \bar{x} - \Delta S_r (\bar{x} \sin \alpha + \bar{y} \cos \alpha)] = 0 \quad (59)$$

where x and y are the horizontal and vertical distances from point 0 to the midpoint of the base of each slice.

Expression (12) for the resisting shear force, ΔS_r , is substituted into equation (59). Then by substituting expression (20) for the effective normal force, $\Delta N'$, as determined by satisfying vertical force equilibrium for each slice, rearranging like terms, simplifying, and transposing the ΔX term, an expression containing $n+1$ unknowns, F and n values for ΔX , is arrived at.

$$\sum_1^n \bar{y} \frac{\Delta C'_a / \cos \alpha + \Delta W (\tan \phi'_a - F \tan \alpha)}{F + \tan \alpha \tan \phi'_a} = \sum_1^n \bar{y} \Delta X \left[\frac{F \tan \alpha - \tan \phi'_a}{F + \tan \alpha \tan \phi'_a} - \frac{\bar{x}}{\bar{y}} \right] \quad (60)$$

If the assumption, $\Delta X = 0$, is made, the expression reduces to,

$$\sum_1^n \bar{y} \frac{\Delta C'_a / \cos \alpha + W (\tan \phi'_a - F \tan \alpha)}{F + \tan \alpha \tan \phi'_a} = 0 \quad (61)$$

The factor of safety is implicitly expressed. Although its value is independent of the horizontal position of the point about which moments are summed to achieve overall moment

equilibrium, it is dependent upon the vertical position of that point.

Carter (1971) found that as $\bar{y}$ goes to infinity, the value of the factor of safety approaches a minimum asymptotically. At infinity, $\bar{y}$ can be taken out of the summation,

$$\sum_{1}^n \frac{\Delta C'_a / \cos \alpha + \Delta W (\tan \phi'_a - F \tan \alpha)}{F + \tan \alpha \tan \phi'_a} = 0 \quad (62)$$

This expression satisfies vertical equilibrium of the individual slices and overall horizontal force equilibrium. It does not satisfy moment equilibrium of the individual slices. The value of the factor of safety is conservative with respect to reasonable solutions obtained by more accurate methods satisfying complete equilibrium. The method enables evaluation of general shear surfaces with a minimum amount of computation time.

METHOD CHOSEN FOR COMPUTER PROGRAM

Of the limiting equilibrium models discussed, those which divide the mass above the assumed shear surface into slices are most ideally suited to being programmed to solve general slope stability problems. The Ordinary Method of Slices is restricted to circular-shaped shear surfaces, and may provide a very conservative evaluation of the stability of a slope. For these two reasons, it was not considered appropriate for the program developed. The Simplified Bishop method, although of satisfactory accuracy, is also limited to circular shear surfaces, and therefore was not considered.

Although the Rigorous Bishop method (1955) also evaluated only circular shear surfaces, the extension of Bishop's work to consideration of general shear surfaces by Nonveiller (1965) gives the type of model that is desired. Difficulty is encountered, however, when selecting values of ΔX for each slice to satisfy the conditions required for limiting equilibrium. After selecting a satisfactory set of values for ΔX , a reasonable line of effective thrust and reasonable interslice shear stresses must be verified. If not, a new set of ΔX values, satisfying limiting equilibrium conditions, must be selected, and the cycle is repeated.

The process of obtaining a satisfactory solution can become lengthy, since there is no systematic procedure of selecting better values of ΔX for each additional cycle of calculations.

If ΔX is assumed to equal zero for each slice, as Nonveiller (1965) suggested as a simplified solution, complete equilibrium is not, in general, satisfied. The value of the factor of safety will vary with the position of the point about which overall moment equilibrium is taken. It is known that for homogeneous problems evaluated assuming circular shear surfaces, satisfactory solutions for the value of the factor of safety are obtained if the overall moment equilibrium is taken about the center of the circular surface. However for irregularly shaped shear surfaces, there is insufficient experience to draw such a conclusion.

Janbu's generalized procedure of slices satisfies complete equilibrium, and is based upon an assumed reasonable line of effective thrust. The only criterion which requires checking is that for excessive interslice shear stresses. If the interslice shear stresses are not satisfactory, a different line of effective thrust must be assumed. In addition, the solution may involve difficulties in convergence upon a unique value for the factor of safety.

The Morgenstern and Price method (1965) offers no advantage over the use of the Spencer method as Wright (1969)

has modified it. The Spencer method is simpler and requires significantly less calculation. The Morgenstern and Price method requires numerical integration over each slice, which is a lengthy procedure. Both methods have the same advantage over that presented by Nonveiller (1965), because numerical procedures can be used to arrive at an acceptable side force condition, assuming a variational relationship of position between the interslice shear forces, $f(x)$. Reasonableness of the line of effective thrust and the magnitude of the interslice shear stresses still require checking, and if not satisfactory, a different variational relationship of position between the interslice forces, $f(x)$, must be assumed, until an acceptable solution is obtained. This can sometimes require substantial effort.

Carter's method (1971), capable of considering assumed shear surfaces of general shape, does not satisfy complete equilibrium, but does provide values of factors of safety, which are reasonably accurate. The result is always conservative. The amount of calculation is minimal, and is comparable to that of the Simplified Bishop method. There is no need to check the reasonableness of the solution, as the solution is conservative with respect to more accurate methods. For use in routine slope stability analysis, the method is practical and has therefore been incorporated within the program developed.

Carter's method, derived earlier for fundamental forces, is expanded here to consider additional forces,

which are frequently encountered in slope stability problems. The forces acting on a typical slice are shown in Figure 12. The geometry of each slice is described by its height, h , measured along its centerline, its width, Δx , and by the inclination of its base and top, α and β , respectively.

Overall moment equilibrium about an arbitrary point, O , is expressed as,

$$\sum_1^n [(\Delta N' + \Delta U_\alpha) (\bar{y} \sin \alpha - \bar{x} \cos \alpha) + \Delta U_\beta (\bar{x} \cos \beta - \bar{a} \sin \beta) + \Delta Q (\bar{x} \cos \delta - \bar{a} \sin \delta) + \Delta W (1 - k_v) \bar{x} + k_h W (\bar{y} - h_{eq}) - \Delta S_r (\bar{x} \sin \alpha + \bar{y} \cos \alpha)] = 0 \quad (63)$$

where ΔU_α and ΔU_β are the resultant forces of pressures produced by water at the base and top of each slice, respectively; ΔQ is the resultant force of a uniformly distributed surcharge boundary load acting upon the top of each slice with δ as its angle of inclination; k_v and k_h are vertical and horizontal earthquake coefficients which relate the magnitude of the earthquake force to the weight of each slice; h_{eq} is the distance from the base of each slice to the horizontal component of the earthquake force; and $\bar{a} = \bar{y} - h$.

Vertical force equilibrium for each slice is expressed as,

$$(\Delta N' + \Delta U_\alpha) \cos \alpha + \Delta S_r \sin \alpha + \Delta X - \Delta W (1 - k_v) - \Delta U_\beta \cos \beta - \Delta Q \cos \delta = 0 \quad (64)$$

FIGURE 12 - FORCES CONSIDERED FOR COMPUTER MODEL.

Expression (12) for the resisting shear force, ΔS_r , is substituted into equations (63) and (64). By solving for the effective normal force, $\Delta N'$, in equation (64), and substituting the resulting expression into equation (63), rearranging like terms, simplifying, and transposing a term, the following equation containing $n+1$ unknowns, F and the ΔX values, is obtained.

$$\sum_{i=1}^n \bar{Y} \frac{A_1 - F A_2}{F + A_3} = \sum_{i=1}^n \bar{Y} \Delta X \left[\frac{\tan \alpha - \tan \phi'_a / F}{1 + \tan \alpha \tan \phi'_a / F} - \frac{\bar{X}}{\bar{Y}} \right] \quad (65)$$

where,

$$\begin{aligned} A_1 = & \frac{\Delta C'_a}{\cos \alpha} + \tan \phi'_a \left\{ \Delta W \left[-k_h \left(1 - \frac{h}{\bar{Y}} \right) \tan \alpha + (1 - k_v) \right. \right. \\ & \left. \left. - \frac{\Delta U_g}{\cos \alpha} + \Delta U_\beta \left(\cos \beta + \frac{\bar{a}}{\bar{Y}} \tan \alpha \sin \beta \right) \right. \right. \\ & \left. \left. + \Delta Q \left(\cos \delta + \frac{\bar{a}}{\bar{Y}} \tan \alpha \sin \delta \right) \right\} \end{aligned}$$

$$\begin{aligned} A_2 = & \Delta W \left\{ (1 - k_v) \tan \alpha + k_h \left(1 - \frac{h}{\bar{Y}} \right) \right\} \\ & + \Delta U_\beta \left(\tan \alpha \cos \beta - \frac{\bar{a}}{\bar{Y}} \sin \beta \right) \\ & + \Delta Q \left(\tan \alpha \cos \delta - \frac{\bar{a}}{\bar{Y}} \sin \delta \right) \end{aligned}$$

$$A_3 = \tan \alpha \tan \phi'_a$$

By using the Bishop simplifying assumption, $\Delta X = 0$, expression (65) becomes,

$$\sum_{i=1}^n \bar{Y} \frac{A_1 - F A_2}{F + A_3} = 0$$

(66)

As the y coordinate of the point, about which overall moment equilibrium is satisfied, approaches infinity, the relative differences between the $\bar{y}$ of each slice become insignificant and $\bar{y}$ can be considered identical for each slice. Considering this limiting condition, $\bar{y}$ is taken out of the summation so equation (66) becomes,

$$\sum_{i=1}^n \frac{A_1 - F A_2}{F + A_3} = 0 \quad (67)$$

Also as $\bar{y}$ approaches infinity, the values of $h_{eq}/\bar{y}$ and $\bar{a}/\bar{y}$ approach zero and unity, respectively. Then the terms A_1 and A_2 become,

$$A_1 = C'_a / \cos \alpha + \tan \phi'_a [W (1 - k_v - k_h \tan \alpha)$$

$$- U_\alpha / \cos \alpha + U_\beta (\cos \beta + \tan \alpha \sin \beta)$$

$$+ Q (\cos \delta + \tan \alpha \sin \delta)]$$

$$A_2 = W [1 - k_v) \tan \alpha + k_h]$$

$$+ U_\beta (\tan \alpha \cos \beta - \sin \beta)$$

$$+ Q (\tan \alpha \cos \delta - \sin \delta)$$

Equation (67) forms the working model programmed for the computer program developed herein.

PROGRAM STABL

The actual calculation of the value of the factor of safety, although time consuming, comprises a relatively small portion of the program developed. Most of the programming effort was directed at handling the geometry of a general slope stability problem. A description of the program's capabilities, and instruction with regard to its use, is reported elsewhere (Siegel, 1975). A discussion of how the program performs its functions is presented below. A listing of the program is provided in the Appendix A of this volume.

Division of Mass into Slices above the Assumed Shear Surface

The mass above the trial shear surface is divided into slices with vertical interfaces. The interfaces occur at all specified coordinate points defining geometry within the sliding mass or the assumed shear surface (Figure 13). If a water surface has been specified, the sliding mass is further divided at intersections of the water surface with the trial shear surface, and also at intersections of the water surface with subsurface boundaries within the sliding mass. Slice divisions are also located at the intersections of subsurface boundaries with the trial shear surface.

FIGURE 13- LOCATIONS OF SLICE DIVISIONS.

Further divisions of the sliding mass into slices is not required, because doing so does not increase the accuracy by Carter's method of slices.

After the sliding mass has been divided into slices as outlined previously, the following characteristics of each slice can be noted. The top and bottom are single straight line segments. If more than one soil type exists within a slice, the interfaces are also single straight line segments. And if a water surface transverses a slice, the surface is also a single straight line segment. If a boundary surcharge load exists at the top of a slice, the load extends the full width of the slice, and its magnitude and direction is constant. The sliding mass has been sliced such that each slice is linear with respect to its geometry and boundary loading.

Total Weight Calculation for Each Slice

Recognizing that each slice is a trapezoid with vertical parallel sides, the calculation of the total weight for each slice is simple. The end slices constitute a special case of a trapezoid, when one of the vertical parallel sides has no length (Figure 14), i.e., a triangle.

If a slice is subdivided by a subsurface boundary, the division is linear and extends across the entire width of the slice establishing subsections of the slice, which are also trapezoids with parallel vertical sides (Figure 15).

FIGURE 14 - GEOMETRY OF SLICES.

FIGURE I5 - HETEROGENEOUS SLICES PARTIALLY SUBMERGED.

The total weight is calculated separately for each subsection, if the slice is subdivided. If not, the total weight for the slice is calculated as if the slice consisted of a single subdivision equivalent to the entire slice. Each subsection contains one soil type. If a subsection is totally above or totally below the water surface, the total weight of a subsection is equal to the product of the area of the subsection and the respective moist or saturated unit weight. Units are consistent, since the weight is in terms of a unit thickness normal to the plane containing the problem's profile. By multiplying the difference in elevation of the top and bottom boundaries of the subsection, measured at the slice centerline, by the width of the slice, Δx , the area of a subsection can be calculated. The elevation difference measured at the slice centerline is the average of the length of the two parallel sides of a trapezoid.

If a subsection is partially submerged, the subsection is divided into submerged and drained portions. As a result of the method by which the sliding mass was divided into slices, the water surface will only intersect a subsurface boundary at a face of the slice (Figure 15(a)). However, the water surface could coincide with a subsurface boundary, making the subsection above entirely drained, and the one below entirely saturated (Figure 15(b)). Since the water surface will only intersect a subsurface boundary at the

face of the slice, both the saturated and unsaturated portions of a partially submerged subsection are trapezoids. The total weight of each portion will be the product of the areas and their respective unit weights. The areas are calculated as described before.

After the weights of the individual subsections are calculated, they are summed to obtain the total weight of the slice.

Water Forces

For problems when the slope is totally or partially submerged, each individual slice, which is totally submerged, has a force, ΔU_{β} , acting upon the soil-water interface (Figure 16). The average hydrostatic pressure, acting upon the top of each submerged slice, is equal to the product of the unit weight of water and the average depth of water above the slice, which is measured at the slice centerline. The magnitude of the force ΔU_{β} is the product of the average hydrostatic pressure and the true length of the top of the slice. The force acts normal to the soil-water interface. Its position is not required for the factor of safety calculations.

The pore pressure acting on the base of each slice is defined in a number of ways. First, if a saturation line has been specified, the pore pressure at any point beneath it may be related to it. The pore pressure at a point beneath the saturation line may be approximated by assuming

FIGURE 16 - WATER FORCES ACTING UPON SUBMERGED SLICES.

that it is equal to the product of the unit weight of water and the vertical distance measured from that point to the line of saturation.

In general, the procedure is conservative and reasonably accurate (Figure 17). However, accuracy decreases as the slope of the saturation line increases. The result can be unconservative in the region beneath a surface of seepage.

Second, the pore pressure may also be defined as a ratio of the overburden pressure, $r_u \sigma_o$, where r_u is a homogeneous pore pressure parameter (Bishop and Morgenstern, 1960) and σ_o is the vertical overburden pressure. Finally, the pore pressure may also be defined as a constant value throughout a zone. A combination of any two or all three of these means of defining the pore pressure may also be used.

The water force, ΔU_α , acting at the base of a slice, is then the product of the average pore pressure, determined at the midpoint of the base, and the true length of the base. Its direction is normal to the base of the slice, and again, its position is not required for the factor of safety calculation.

If a pseudo-static earthquake load is applied to a slice, the pore pressure at the base of the slice will decrease or increase, depending upon the direction of the applied force. The magnitude of the water force, ΔU_α , is decreased by an amount equal to the component of the earthquake force normal to the base of the slice, if the

FIGURE 17 - APPROXIMATION OF HEAD FOR PORE PRESSURE CALCULATION.

earthquake load is directed away from the base. If the earthquake load is toward the base of the slice, ΔU_{α} is increased. The force ΔU_{α} is allowed to be negative, but a minimum value, corresponding to the product of the cavitation pressure and the true length of the base, is maintained.

SEARCHING TECHNIQUES

The number of slopes having subsurface profiles for which an engineer can intuitively select the most critical shear surfaces for analysis are few. For cases of slope profiles containing a layer or seam of weak strength, it may be obvious that the critical surface will pass through the layer. However, the position and orientation of the critical shear surface within a weak layer may not be obvious unless the layer is thin. Even then, it may be neither obvious where the critical shear surface enters and exits the layer, nor what path it might take beyond these points. Quite a few trial shear surfaces may have to be analysed before an engineer can be reasonably assured of locating the critical surface.

Charts have been prepared for locating the critical shear surface of circular shape for slopes having horizontal toes and crests and composed of homogeneous and isotropic material (Taylor, 1937; Morgenstern and Price, 1965; and Spencer, 1967). Their usefulness is confined to problems, which can be idealized, as such, without gross assumptions.

To reduce the amount of time required to prepare data defining a large number of trial shear surfaces, three trial shear surface generators have been developed for use

with program STABL. The first, named CIRCLE, generates surfaces of circular shape. This generator is good for slopes comprised of roughly homogeneous and isotropic material. The second, named BLOCK generates trial failure surfaces of a sliding block character, and is most useful when a well defined weak layer exists. Finally, RANDOM generates surfaces of general irregular shape. It is most useful for heterogeneous soil profiles where both the location and shape of the critical surface are uncertain. Each generator uses pseudo-random techniques for generating the surfaces.

Random Numbers

Before explaining how each trial failure surface generator works, the means of obtaining random numbers, which are required by each generator, is discussed.

Pseudo-random numbers are generated in an exact deterministic process by which a number is generated from its predecessors in a series. The random number generator used for program STABL must be able to generate real numbers having magnitudes within the range between 0 and 1. To be random, the numbers generated must be uniformly distributed within this range. A number generated by such a process is not truly random, but for most practical purposes it may be assumed so.

Most computer systems have a library function for generating random numbers. Control Data Corporation has

such a function, named RANF(X), in its 6000 series system library. This function is referenced within program STABL. Relatively simple random number generators can be written for any system if not readily available (Ralson and Wilf, 1962).

Circular Surface Generation

A circular-shaped surface, composed of a series of straight line segments of equal length, is generated from left to right. The length of the line segments is specified. The surface initiates from a point on the ground surface.

To begin, a line segment is projected from the initiation point in a direction chosen between two limits. The direction limits are either specified or automatically calculated by the program. In the latter case the clockwise direction limit, α_2 , is -45° with respect to the horizontal, and the counterclockwise direction limit, α_1 , is 5° less than the inclination of the ground surface to the right of the initiation point (Figure 18).

The selection of the inclination, θ , at which the initial line segment of a surface projects, is determined randomly. A random number, R , having a value between 0 and 1, is generated and squared. The squared random number and the difference in the inclination of the two direction limits, $\alpha_1 - \alpha_2$, are multiplied. The result is added to the

FIGURE I8-SELECTING THE INITIAL LINE SEGMENT.

inclination of the clockwise direction limit, α_2 , to obtain the angle of inclination of the initial line segment,

$$\theta = \alpha_2 + (\alpha_1 - \alpha_2) R^2 \quad (68)$$

Squaring the value of the random number generated, before multiplying it with the absolute value of the difference of inclination of the two direction limits, $\alpha_1 - \alpha_2$, introduces a bias in the random direction selection process. Although selection of any inclination of the initial line segment within the two limits is possible, it is more likely that the initial line segment be located nearer the clockwise limit.

Figure 19 indicates the probability that an initial line segment will be oriented within any of ten equal ranges of direction, which subdivide the complete range of orientations. If the process of selecting an initial line segment were repeated one hundred times, it could be expected that for approximately 32 times the initial line segment would be oriented within the subdivision next to the clockwise limit, while only 5 times would it be within the subdivision next to the counterclockwise limit.

The primary reason for introducing the bias is to obtain a good distribution of completed surfaces. When the initial line segment was given an equal likelihood of being oriented in any direction within the direction limits, poor distributions of completed surfaces were obtained.

FIGURE 19 - PROBABILITY OF THE LOCATION OF THE INITIAL LINE SEGMENT.

After establishing the initial line segment of a trial failure surface, the remaining portion of the circular-shaped surface is randomly located, within some constraints. Although the exact location where the critical shear surface exits the top of the slope is not known, a zone where it probably will exit, can be assumed. Termination limits are established at the ground surface, so that only surfaces of interest are considered.

The initial line segment is assumed as the chord of a family of circles (Figure 20). Only a subset of this family of circles is of interest. The circles having centers below the initial line segment are not considered for obvious reasons. Of the remaining circular surfaces, only those which lie within the zone bounded by the circular surfaces passing through the two termination limits are of interest.

For a circular-shaped surface composed of straight line segments of equal length, T , each successive line segment is deflected by a constant angle, $\Delta\theta$, with respect to the line segment preceding it. The two limiting deflection angles, $\Delta\theta_{\min}$ and $\Delta\theta_{\max}$ (Figure 20), which bound all the circular-shaped surfaces of interest are back-calculated (Figure 21),

$$\Delta\theta = 2 \sin^{-1} (T/2r) \quad (69)$$

where r is the radius of the circle. The same procedure is repeated for the other termination limit.

FIGURE 20-FAMILY OF CIRCLES HAVING INITIAL LINE SEGMENT AS A CHORD.

FIGURE 21 - BACK - CALCULATING A DEFLECTION LIMIT FOR CIRCULAR SURFACE.

Using the two deflection angles back-calculated as the deflection limits, a deflection angle, $\Delta\theta'$, is randomly chosen from within the range defined in a fashion similar to selection of the direction of the initial line segment,

$$\Delta\theta' = (\Delta\theta_{\max} - \Delta\theta_{\min}) R^2 \quad (70)$$

The orientation, θ , of each succeeding line segment is equal to the inclination of the preceding line segment, plus the randomly selected deflection angle,

$$\theta_i = \theta_{i-1} + \Delta\theta' \quad (71)$$

There are a number of circumstances when the procedure generates unsatisfactory circular-shaped surfaces, and adjustments to the surface generation parameters, α_1 , α_2 , $\Delta\theta_{\max}$, and $\Delta\theta_{\min}$, become necessary.

The minimum value of $\Delta\theta_{\min}$ allowed is zero, i.e., a planar shear surface. If $\Delta\theta_{\min}$ is less than zero, it is adjusted to equal zero. Circular-shaped shear surfaces which are concave downward (Figure 22(a)) are not considered desirable. If the value of $\Delta\theta_{\max}$ is also less than zero, all surfaces would be concave downward, and the inclination of the initial line segment must then be changed. If this is the case, the clockwise direction limit, α_1 , for selection of the new initial line segment is adjusted to the inclination of the current initial line segment. A new inclination for the initial line segment is randomly selected between α_2 and the new α_1 using equation (68).

FIGURE 22 - LIMITATIONS IMPOSED ON DEFLECTION ANGLE.

If the new inclination of the initial line segment is also unsatisfactory, $\Delta\theta_{\max}$ is less than zero, the process of modifying α_1 and selecting a new inclination for the initial line segment is repeated, until a positive value for $\Delta\theta_{\max}$ is detected.

The maximum value of $\Delta\theta_{\max}$ permitted is arbitrarily set at 45° , the rationale being that a surface composed of line segments deflected at larger angles would be too crude an approximation of what is intended to be a smooth shear surface (Figure 22(b)). If $\Delta\theta_{\max}$ is greater than 45° , it is adjusted to equal this value. If the value of $\Delta\theta_{\min}$ is also greater than this value, the inclination of the initial line segment must be changed. In this case, the counter-clockwise direction limit, α_2 , for selecting the new initial line segment is adjusted to the inclination of the current initial line segment, similar to the process seen before.

Another adjustment which is sometimes required occurs when the shear surface overturns (Figure 23(a)). If for an initial line segment, the center of the circle passing through the left termination limit is below an elevation $T/2$ above the ground surface at that termination limit, $\Delta\theta_{\max}$ is adjusted to the deflection angle corresponding to a circle having a center at that elevation (Figure 23(b)).

If the center of a circle passing through the right termination limit is also below an elevation $T/2$ above the ground surface at that termination limit, no satisfactory

(a)

(b)

FIGURE 23 - LIMITATIONS IMPOSED ON INCLINATION OF FINAL LINE SEGMENT.

surface can be generated with the initial line segment, inclined as it is. A new inclination is selected randomly for it, using equation (68), after adjusting the clockwise direction limit, α_2 , to the present inclination of the initial line segment. This may have to be repeated until a satisfactory inclination for the initial line segment is selected.

If a limitation is imposed with regard to a maximum depth at which a surface is desired, the value of $\Delta\theta_{\min}$ can be adjusted to accommodate this limitation, if necessary. If the minimum angle of deflection meeting this limitation is greater than $\Delta\theta_{\max}$, then a new inclination is randomly selected, modifying α_2 first as before.

One final form of adjustment of the surface generation parameters for circular-shaped surfaces is encountered with respect to surface generation boundaries which can be established. Line segment boundaries are established with the intention that no shear surface should pass through them. Each individual boundary is designated such that if intersected, an attempt is made to complete a surface using the same inclination of the initial line segment, either above or below the boundary.

If an intersected boundary is designated as an upward deflecting boundary, a surface having a deflection angle equal to $\Delta\theta_{\max}$ is checked for possible intersection with the same boundary. If it does intersect, a new inclination of

the initial line segment is selected without modification to the direction angle limits α_1 or α_2 . If it does not intersect, $\Delta\theta_{\min}$ is changed to the deflection angle of the surface which originally intersected the boundary, and a new deflection angle is randomly selected. The proceeding is repeated if necessary until no intersection with any boundary occurs.

General Irregular Shear Surface Generation

The inclination of the initial line segment for a general irregular shear surface is selected in the same manner as described for the circular-shaped shear surface. However, the direction of each subsequent line segment is determined independently of the line segments preceding it.

The counterclockwise limit for the direction of a line segment, $\Delta\theta_1$, is normally deflected 45° counterclockwise from the projection of the preceding line segment (Figure 24). If for a particular line segment the orientation of the counterclockwise direction limit is greater than 90° (measured with respect to the horizontal), the inclination of the counterclockwise direction limit, θ_1 , is adjusted to 90° .

The clockwise deflection limit for the direction of a line segment, $\Delta\theta_2$, is randomly selected for each shear surface generated,

$$\Delta\theta_2 = R^2 45^\circ \quad (72)$$

FIGURE 24 - DIRECTION LIMITS FOR SUCCESSIVE LINE SEGMENTS OF IRREGULAR SURFACE.

If for a particular line segment the inclination of the clockwise direction limit, θ_2 , is less than -45° , it is set at this value.

The inclination of a line segment is then established by,

$$\theta = \theta_2 + (\theta_1 - \theta_2) R^{(1+R)} \quad (73)$$

where the angular extent of permissible directions, $\theta_1 - \theta_2$, is multiplied by a random number raised to a random power between 1 and 2 and added to the inclination of the clockwise direction limit, θ_2 .

The above procedure, although somewhat arbitrary, does produce reasonable irregular surfaces of random shape and position. Reverse curvature of a shear surface is possible, but the frequency of occurrence is small. Unless very short line segments are used, "kinkyness" of the resulting generated shear surfaces is not a problem.

The other restrictions imposed upon the circular-shaped surfaces are also imposed for irregular-shaped surfaces, but in a somewhat different manner. If a surface should intersect the ground surface short of the left termination limit, it is rejected, and a replacement surface is generated.

If a surface should intersect the ground surface beyond the right termination limit, or if an uncompleted surface extends beyond this limit, the possibility of successfully completing the surface without rejecting the entire

surface is checked. First, the violating line segment is rotated to the inclination of the counterclockwise direction limit, θ_1 , which was used to select it. If at this inclination the surface intersects the ground surface beyond the right termination limit, or it extends uncompleted beyond this limit, the surface is rejected, and a replacement is generated.

If a line segment extends below the maximum depth permitted, it is rotated to its counterclockwise direction limit. If it still extends below the depth limit, the entire surface is rejected, and a replacement is generated. If not, θ_2 is adjusted to the old inclination of the line segment, and a new inclination is selected as before. The procedure is repeated as necessary.

When a surface generation boundary is intersected, the possibility of rotating the intersecting line segment clockwise or counterclockwise (the direction depending upon the designation of the boundary) to clear the boundary is checked. If it can clear, a new direction is selected for the line segment in a fashion similar to that cited before.

Searching for the Critical Circular or General Shear Surfaces

Trial shear surfaces are generated from a number of initiation points with equal horizontal spacing along the ground surface. For example, Figure 25 shows ten circular-shaped trial shear surfaces generated from each of ten initiation points; a total of one hundred surfaces. As the

FIGURE 25 - 100 GENERATED CIRCULAR - SHAPED SURFACES.

surfaces are generated, their respective factors of safety are evaluated, and the ten most critical are identified (Figure 26).

The ten most critical surfaces will in most cases form a distinctive critical zone. Occasionally, more than one critical zone can be distinguished. In this case each zone could be considered individually.

The compactness of the critical zone, its location within the zone containing all the surfaces generated, and the magnitude of the range of values for the factors of safety of the ten most critical surfaces, indicate the likelihood that a shear surface exists with a factor of safety significantly lower than any calculated. If the critical zone is narrow, if non-critical surfaces have been generated on both sides of the critical zone, and if the values of the ten most critical surfaces are nearly the same, the defined factor of safety of the most critical surface generated could be assumed, with reasonable confidence. On the other hand, if the critical zone is wide, and/or the more critical surfaces lie along one edge of the zone containing all the surfaces generated, and/or the magnitude of the range of factor of safety values for the critical surfaces is large, the possibility of a shear surface with a significantly smaller value for the factor of safety may require generation of additional surfaces.

FIGURE 26-10 MOST CRITICAL SURFACES OF THOSE IN FIGURE 25.

If more trial shear surfaces are required, the limitations can be revised to restrict the additional surfaces to a particular zone of interest. For example, additional shear surfaces generated for the problem shown in Figure 26 should all initiate from the trough of the bench on the slope. They should also be required to terminate at least 15 ft behind the crest.

In this particular problem, the lower subsurface interface is defined by upward deflecting surface generation boundaries representing a competent slightly folded bedrock surface. Figure 25 demonstrates the control these boundaries have on surface generation. If additional surfaces were to be generated, a supplemental downward deflecting surface generation boundary could be defined. It would extend from the ground surface, somewhere on the face of the slope, to a point above the critical zone sufficient to prevent generation of trial shear surfaces through noncritical zones.

Sliding Block Surface Generation

If a zone of weakness within the mass of a slope is obvious, or if a critical zone has been fairly well defined by previous usage of the circular or irregular surface generators, it is often advantageous to use a third generator.

A series of boxes, a minimum of two, is located along the path of the zone to be investigated. The boxes are parallelograms with vertical sides. They can be specified to bracket the zone to be investigated (Figure 27).

Intensive Search of Critical Zone Previously
Defined by CIRCLE or RANDOM

FIGURE 27-SLIDING BLOCK GENERATOR USING MORE
THAN TWO BOXES.

A point is randomly selected from within each box. Any point within a particular box has an equal likelihood of being selected. The points are then connected in sequence with straight line segments (Figure 27).

If two boxes are specified within the weak layer of the problem shown in Figure 28, and a point is randomly selected from each, the resulting line segment forming the base of the central block has random length, orientation, and position.

Next, the active and passive portions of the trial shear surface are generated (Figure 29). The active and passive portions are composed of line segments of equal specified length. The inclination of each line segment for the active portion of the shear surface is selected in a biased random fashion between direction limits inclined at 45° and 90° ,

$$\theta = 45^\circ + 45^\circ R^2 \quad (74)$$

For the passive portion of the shear surface, the inclination of each line segment is selected in a biased random fashion between direction limits inclined at -45° and 0° ,

$$\theta = -45^\circ + 45^\circ R^2 \quad (75)$$

The same judgement, as discussed for critical shear surface searching with the circular and irregular-shaped shear surface generators, can also be applied to this

FIGURE 28 - SIMPLE SLIDING BLOCK PROBLEM.

FIGURE 29- GENERATION OF ACTIVE AND PASSIVE PORTIONS OF SLIDING SURFACE.

generator, with regard to the probable existence of a significantly more critical shear surface than those generated.

COMPARISON OF CARTER'S METHOD WITH SPENCER'S
METHOD ASSUMING PARALLEL SIDE FORCES

A comparison was performed to assess the magnitude of error resulting from use of the $\Delta X=0$ assumption. The comparison is of limited value, but it does demonstrate the nature of differences in values of the factor of safety to be expected. The Spencer method assuming parallel side forces was used as a standard. Even though the method assuming parallel side forces does not, in general, produce a reasonable line of effective thrust, it does, however, satisfy the equilibrium conditions. Spencer (1973) has shown that solutions having an acceptable line of effective thrust do not yield values for the factor of safety significantly differently from those obtained when assuming parallel side forces.

Combinations of the available shear strength intercept, c'_a , the available shear strength angle, ϕ'_a , the homogeneous pore pressure parameter, r_u , and the slope inclination, β , resulting in values for the factor of safety of unity, were selected from charts prepared by Spencer (1967). A height of slope equal to 100 ft and a moist unit weight for the soil of 100 pcf were assumed. Spencer's charts are based upon the critical circular shear surface passing through the toe of simple slopes composed of homogeneous

isotropic material and having a horizontal crest. Using the selected combinations of geometry and soil parameters, values of the factor of safety (Table 2) were calculated for the critical surfaces using Carter's method.

Curves fitted to the data found in Table 2 for the 1.5:1 and 2:1 slopes are shown in Figure 30. Each curve represents the variation of percent difference of the value of the factor of safety found while comparing Carter's method to Spencer's, as the combination of the shear strength parameters c'_a and ϕ'_a vary. The curves are plotted with respect to ϕ'_a , but it is implied that as the magnitude of ϕ'_a increases, the value of c'_a decreases in a complimentary manner to maintain a value of the factor of safety of unity, based upon Spencer's charts.

When $\phi'_a=0$, the value of c'_a required to maintain a slope with a factor of safety of unity is identical for the two methods. For this case, ΔX for each slice does not contribute frictional resistance to the resisting shear force at the base of each slice. When $c'_a=0$, the value of ϕ'_a required to maintain a slope with a factor of safety of unity is also identical for the two methods. For this case, the critical shear surface is at the face of the slope. ΔX is equal to zero for each slice, and the side forces are parallel, thus the agreement between the two methods. Differences occur when both c'_a and ϕ'_a are non-zero.

Table 2

Factors of Safety Calculated Using Carter's Method*

Slope	ϕ'_a (deg)	$r_u = 0.5$		$r_u = 0.25$		$r_u = 0$	
		c'_a (psf)	F	c'_a (psf)	F	c'_a (psf)	F
1.5:1	40	450	.864	70	.959	--	--
	30	700	.885	290	.917	45	.976
	20	950	.907	610	.916	345	.939
	10	1250	.958	1040	.955	845	.950
2:1	40	175	.903	--	--	--	--
	30	425	.892	85	.955	--	--
	20	725	.892	385	.915	140	.943
	10	1075	.919	850	.917	650	.923
3:1	40	--	--	--	--	--	--
	30	100	.947	--	--	--	--
	20	400	.904	110	.973	--	--
	10	850	.897	580	.900	370	.920
4:1	40	--	--	--	--	--	--
	30	--	--	--	--	--	--
	20	190	.942	--	--	--	--
	10	680	.903	380	.910	180	.951

* Height of each slope equal to 100 ft, and moist unit weight equal to 100 pcf.

FIGURE 30- COMPARISON OF CARTER'S METHOD TO SPENCER'S METHOD.

If ϕ'_a , for a slope having a factor of safety of unity when $c'_a = 0$, is reduced, while c'_a is increased to maintain a factor of safety of unity, the critical shear surface will curve (Figure 31). When this happens, ΔX for each slice becomes non-zero, and an error is introduced. As the value of ϕ'_a approaches zero, the frictional resistance induced by ΔX of each slice becomes insignificant.

Considering the uncertainties involved in slope stability analysis, the error which can be expected, generally less than 10%, is not unreasonable. Consideration of a more accurate model is more an economic question of with respect to the additional computer time and effort.

FIGURE 31- CURVATURE OF SLOPE COMPOSED OF $c'-\phi$ SOIL.

SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

The primary objective of this research was the development of a computer program capable of handling the general slope stability problem. The resulting program can analyse slope profiles having multiple slope ground surfaces, and containing any arrangement of subsurface soil types having differing soil properties. Pore pressure may be related to a steady state flow domain, related to the overburden, or specified within zones. Uniform boundary surcharge loads can be specified upon the ground surface. Pseudo-static earthquake loads, having vertical and horizontal components, are related to the weight of the mass above the assumed shear surface.

Trial shear surface generators have been developed enabling random generation of surfaces of circular, general, or specified character. Each generator provides means for generation of only shear surfaces of particular interest.

The program STABL, written in Fortran IV source language, is routinely used on a Control Data Corporation 6500 computer at Purdue University. With full plotting capabilities 66K octal of core storage is required for execution. Program length is 6000+ statements including comments.

Although the program is very versatile, and efficient with respect to computing time, it is rather bulky. For many problems only a fraction of its capability will be needed. The entire program must be loaded into a computer's memory core, denying possible use of a large portion of the core to other programs. It is therefore suggested that a number of smaller, more specialized, and simpler programs be developed from STABL for the more efficient solution of some practical problems.

Comparisons with solutions satisfying complete equilibrium, and requiring reasonable lines of effective thrust and interslice shear stress, should be made to more accurately assess the importance of the side force assumption.

The method of slices is a very versatile limiting equilibrium model. It has potential applications other than slope stability analysis. For example, this work could also be extended to lateral earth pressure problems, retaining wall design, and bearing capacity analysis.

To investigate the impact of input variables, such as soil properties, geometry, pore pressure distributions, boundary loads, and earthquake loadings, on the solution of slope stability problems, a parametric or sensitivity study should be performed on STABL. Information resulting from such a study would enable identification of elements of a problem which require special attention.

LIST OF REFERENCES

LIST OF REFERENCES

- Bishop, A. W. (1955). "The Use of the Slip Circle in the Stability Analysis of Slopes," Geotechnique, Vol. 5, No. 1, March, pp. 7-17.
- Bishop, A. W. and Morgenstern, N. (1960). "Stability Coefficients for Earth Slopes," Geotechnique, Vol. 10, No. 4, Dec., pp. 129-150.
- Boussinesq, J. (1885). "Application des Potentiels à l'Étude de l'Équilibre et du Mouvement des Solides Elastiques," Gauthier-Villars, Paris.
- Carter, R. K. (1971). "Computer Oriented Slope Stability Analysis by Method of Slices," MSCE Thesis, Purdue University, West Lafayette, Indiana.
- Coulomb, C. A. (1776). "Essai sur une Application des Règles des Maximis et Minimis à Quelques Problèmes de Statique Relatifs à L'Architecture," Histoire et Memoires de L'Académie Royal des Sciences, Vol. 7, Paris.
- Fellenius, W. (1927). "Erdstatische Berichnungen mit Reibung und Kohäsion," Wilhelm Ernst and Sohn KG, Berlin.
- Frohlich, O. K. (1953). "The Factor of Safety With Respect to Sliding of a Mass of Soil Along the Arc of a Logarithmic Spiral," Proceedings, of the Third International Conference on Soil Mechanics and Foundation Engineering, Switzerland, Vol. 2, Aug., pp. 230-233.
- Frohlich, O. K. (1955). "General Theory on Stability of Slopes," Geotechnique, Vol. 3, No. 1, March, pp. 37-54.
- Hansen, J. B. (1953). Earth Pressure Calculation, Danish Technical Press, Copenhagen.
- Harr, M. E. (1966). Foundations of Theoretical Soil Mechanics, McGraw-Hill, New York.
- Janbu, N. (1954). "Stability Analysis of Slopes with Dimensionless Parameters," Harvard Soil Mechanics Series, No. 46, January.

- Janbu, N. (1957). "Earth Pressures and Bearing Capacity Calculations by Generalized Procedure of Slices," Proceedings, of the Fourth International Conference on Soil Mechanics and Foundation Engineering, London, Vol. 2, pp. 207-212.
- Janbu, N. (1973). "Slope Stability Computations," Embankment-Dam Engineering, R. C. Hirschfeld and S. J. Poulos, ed., Casagrande Volume, John Wiley & Sons, New York, pp. 47-86.
- Morgenstern, N. R. and Price, V. E. (1965). "The Analysis of the Stability of General Slip Surfaces," Geotechnique, Vol. 15, No. 1, March, pp. 79-93.
- Morgenstern, N. R. and Price, V. E. (1967). "A Numerical Method for Solving the Equations of Stability of General Slip Surfaces," The Computer Journal, Great Britain, Vol. 9, No. 4, Feb., pp. 388-393.
- Nonveiller, E. (1965). "The Stability Analysis of Slopes with a Slip Surface of General Shape," Proceedings, of the Sixth International Conference on Soil Mechanics and Foundation Engineering, Montreal, Vol. 2, pp. 522-525.
- Ralston, A. and Wilf, S., ed. (1962). Mathematical Methods for Digital Computers, John Wiley & Sons, New York.
- Rankine, W. J. M. (1857). "On the Stability of Loose Earth," Philosophical Transactions of the Royal Society, Vol. 147, London.
- Romani, F., Lovell, Jr., C. W., and Harr, M. E. (1972). "Influence of Progressive Failure on Slope Stability," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 98, No. SM11, Nov., pp. 1209-1223.
- Seed, H. B. and Sultan, H. A. (1967). "Stability Analyses for a Sloping Core Embankment," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 93, No. SM4, July, pp. 69-83.
- Siegel, R. A. (1975). "STABL User Manual, a Description of the Use of STABL's Programmed Solution for the General Slope Stability Problem," Technical Report (to be Published), Joint Highway Research Project, Purdue University, West Lafayette, Indiana, April.
- Spencer, E. (1967). "A Method of Analysis of the Stability of Embankments Assuming Parallel Inter-Slice Forces," Geotechnique, Vol. 17, No. 1, March, pp. 11-26.

- Spencer, E. (1973). "Thrust Line Criterion in Embankment Stability Analysis," Geotechnique, Vol. 23, No. 1, March, pp. 85-100.
- Taylor, D. W. (1937). "Stability of Earth Slopes," Journal of the Boston Society of Civil Engineers, Vol. 24, No. 3, July.
- Whitman, R. V. and Baily, W. A. (1967). "Use of Computers for Slope Stability Analysis," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 93, No. SM4, July, 475-498.
- Whitman, R. V. and Moore, P. J. (1963). "Thoughts Concerning the Mechanics of Slope Stability Analysis," Proceedings of the Second PanAmerican Conference on Soil Mechanics and Foundation Engineering, Brazil, Vol. 1, pp. 391-411.
- Wright, S. G. (1969). "A Study of Slope Stability and the Undrained Shear Strength of Clay Shales," Ph.D. Thesis, University of California, at Berkeley, California.
- Wright, S. G., Kulhawy, F. H., and Duncan, J. M. (1973). "Accuracy of Equilibrium Slope Stability Analysis," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 99, No. SM10, Oct., pp. 783-791.

APPENDIX A

LISTING OF PROGRAM STABL

This Appendix is not included in this copy of this Report because of its size (98 pages). Any person receiving a copy of this Report will be provided a copy of the Listing without charge upon written request to the Joint Highway Research Project, Civil Engineering Building, Purdue University, West Lafayette, Indiana, 47907. The request should specify the "Listing of Program Stabl" for Report JHRP-75-8. The several subroutines of the Program are listed in the Table of Contents of this Report.

STABL USER MANUAL

A STABL USER MANUAL has also been prepared and is available as a separate publication with title as above, Publication JHRP-75-9, June 1975. Availability is from the Joint Highway Project, address as above. The cost is \$5/copy.

COVER DESIGN BY ALDO GIORGINI