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Abstract

To blend between simple implicit surfaces, such as the sphere, the cone, the cylinder and the torus, we propose to locally employ the zero set of a serendipitous trivariate box spline. This box spline is defined by seven directions that form a regular partition of space into tetrahedra. The resulting blend surface is curvature continuous. An approximate parametrization of the piecewise implicit surface of degree four is obtained by subdivision and sign comparison.

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§1. Introduction

Geometric shapes with a dual representation are essential building blocks of many geometric modeling systems. Planar facets, the sphere, the cone, the cylinder and the torus have both a simple parametric representation that allows for easy display and an implicit equation that facilitates point classification. Attempts to enrich either type of representation beyond the basic shapes to encompass, say surfaces that smoothly join the basic primitives lead to a number of challenges. For example, extending the parametric representation leads to the problem of surface-surface intersection for point classification. More complex implicit surfaces on the other hand are difficult to trace out and usually have to be approximated by a piecewise linear parametrization (see e.g. [Bl88]).

Blending, i.e. the smooth join of several primitives, is one of the classical problems of modeling with implicit representations [HH87, Hol87, LHH90, Ko89]. Our approach to blending is to rerepresent the implicit primitives locally as the zero set of a trivariate spline and to apply any Boolean operation (join, intersection, etc.) between the primitives to the coefficient space of the spline representations. Since the result is a spline over a uniform grid, smoothness is guaranteed. Specifically, we replace a box encompassing the region of contact of the primitives by a 7-direction box spline that partitions the box into subboxes and the subboxes into tetrahedra of one type. The resulting spline is curvature continuous and of total degree four, i.e. the spline is a polynomial of degree four on each tetrahedron. The level surfaces taken from each tetrahedron are therefore curvature continuous and of algebraic degree four, the minimal degree for representing the torus.

The paper is structured as follows. After a review of related literature in the second section, and of challenges when modeling with implicits in the third section, we review box splines with unit directions, followed by the definition of the 7-direction box spline, its evaluation and reproducing properties. The paper concludes with a set of examples.

§2. More earlier work

There are currently two main approaches to defining the individual pieces of a function whose zero set represents a surface. The first is to locally generate a shell-like structure consisting of polygonal cells that follow the outlines of and enclose the intended surface. This approach is illustrated by the recent work in [Sed90], [Dah89], [Guo93], [DT93], [BCX94] and [MD94]. Creating the appropriate shell structure can be challenging and sometimes leads to special cases, say in the case of coplanar pieces.

The alternative, regular lattice approach goes back to spatial enumeration and octtrees. It consists of defining a function on a regular, global lattice. A straightforward choice of such a function to allow for smooth shapes is, for example, a piecewise triquadratic, C^1 tensor-product spline ([MW91]). Working with a regular lattice has the advantage that non rectilinear features of the surface do not require special treatment. However, the representation is not very efficient, because potentially n^3 values at the lattice points have to be stored, compared to $O(n^2)$ plus the shell structure in the case of a surface-following implicit approach.

§3. Two challenges of implicit modeling

3.1. Single, multiple and unconnectedness. The real zero set of a trivariate function consist in general of several connected components. The components may touch one another yielding a non-unique surface continuation, and they may contain handles and holes that do not reflect the design intent [Guo93]. A guarantee of existence and uniqueness, i.e the identification of a singly connected sheet in the region of interest, is therefore a main concern when choosing an implicit surface representation [BCX94]. Constraining the implicit representation to be monotone is only a partial solution, because such nonlinear constraints imply loss of the linear vector space property, i.e. averages of the constrained representation may not satisfy the constraints. The algebraic degree of the surface pieces is of interest in this context, because a higher degree allows for more complex zero sheets. Triquadratic C^1 splines of total polynomial degree six are therefore less desirable than the C^2 box spline of total degree four that we are about to develop.

3.2 Cube ambiguity. As is well-known from the marching cubes algorithm, specifying values at the vertices of a cube and invoking the intermediate value theorem for continuous functions to deduce the connectivity of the zero sheet is not a well-defined operation. Already looking at one cube-face with the sign pattern $\pm\mp$ allows for two different piecewise linear zero sheet approximations attached to the midpoints of edges with changing sign pattern: one with two sheets running from the upper left to the lower right and the other with two sheets running from the upper right to the lower left. For uniqueness of the zero sheet, it is desirable to have tetrahedral regions associated with the regular mesh, because a tetrahedron defines a unique connectivity of the midpoint based zero sheet for any given plus-minus pattern at the vertices: the least degree zero sheet separating positive and negative values at the midpoint of the corresponding edge is either empty, a triangle or a bilinear quadrilateral patch.

§4. Box splines with unit directions

Box splines are defined by a set of n directions which determine both the support of the piecewise polynomials and their continuity properties. For the purpose of this paper it suffices to look at box splines whose direction set contains all unit vectors of the domain of the box spline. Examples of this class of box splines are all univariate B-splines over uniform knot sequences, the bivariate Zwart-Powell element and the 7-direction box spline which is the subject of this paper. For a detailed discussion of box splines in a general setting it useful to consult [BHR94].

We define the box spline with respect to the m unit vectors of $\mathbb{R}^m$ as the characteristic function of the unit cube spanned by the vectors. This box spline has degree zero and is discontinuous. To obtain a box spline with a larger direction set, the lower order box spline is successively convolved in each of the remaining directions, thus increasing the degree and the support but not necessarily the continuity. The degree increases by one with each direction, and the support grows by forming the Minkowski sum of the previous support with the unit cube shifted in the direction. To determine the continuity of the resulting box spline we need to determine the number a, which counts the minimal number of directions that need to be removed from the direction set to obtain a reduced set that does not span $\mathbb{R}^m$. Then the continuity class is C^{a-2}. The zero set of such a function is of the same smoothness class since the expansions of the derivatives agree in all directions.

Example 1. The univariate uniform cubic spline has the direction set { 1, 1, 1, 1 } (cf. Figure 5.1, right). Hence $m = 1$, $n = 4$, and $a = 4$ since all elements of the set have to be removed to make it nonspanning. The degree is therefore $n - 1 = 3$ and the continuity is of order $a - 2 = 2$.

Fig. 5.1. Uniform univariate splines.

Example 2. The bivariate Zwart Powell element is the box spline with the direction set $\{(1,0),(0,1),\,(1,1),(-1,1)\}$. We have $m = 2$, $n = 4$, $a = 3$ and hence the degree of the element is 2 and the order of continuity is 1. Figure 5.2 shows the support of box splines for a given set of directions.

Example 3. The continuity of the box spline with the direction set $\{(1,0),(1,.5),(1,1),(.5,1),(0,1)\}$ is a maximal $4-2$. However, the resulting tesselation of the domain consists of both triangles and quadrilaterals which makes it difficult to represent the polynomial pieces in standard form.

§5. The 7-direction box spline

The Zwart-Powell element is special among the low degree box splines defined over the plane, in that it has maximal smoothness $n - 1$ and is piecewise polynomial over a regular triangulation. The 7-direction box spline is a similar

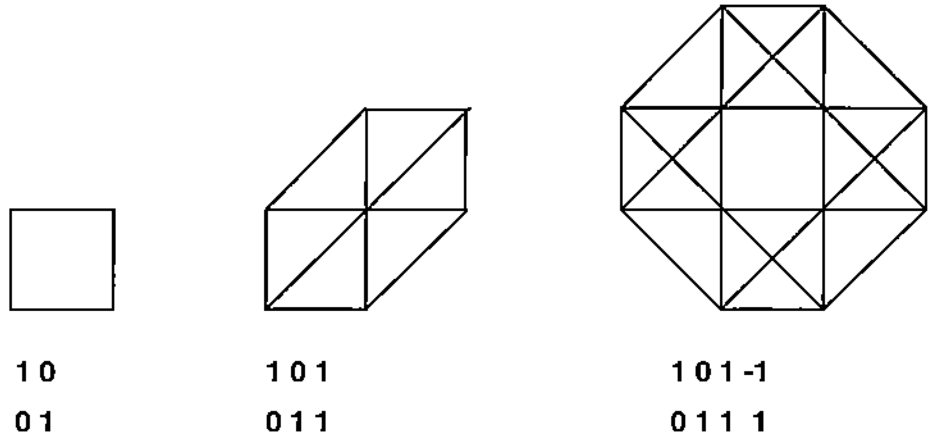

Fig. 5.2. The support regions of some bivariate box splines.

serendipity element among the trivariate box splines. The seven directions

$$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}, \begin{matrix} -1 \\ 1 \\ 1 \end{matrix}, \begin{matrix} 1 \\ -1 \\ 1 \end{matrix}, \begin{matrix} -1 \\ -1 \\ 1 \end{matrix}$$

cut $\mathbb{R}^3$ into a symmetric regular arrangement of tetrahedra.

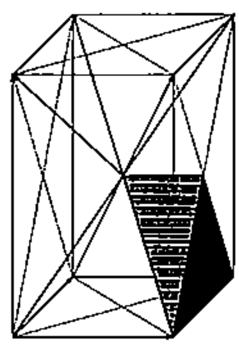

Fig. 6.1. The 7 directions of the box spline and its domain tetrahedra.

The simple count yields $m = 3$, $n = 7$ and $a = 4$. Thus the polynomial piece defined over each tetrahedron is of degree four and the spline formed as a linear combination of box splines is C^2. The support has the shape of an octahedron, consisting of 5 cubes along the diagonals.

§6. Reproduction of basic implicit shapes

To model blends with the zero set of the 7-direction box spline, we replace a cube enclosing the contact region of the implicit primitives by a cube containing the spline on a regular tetrahedral lattice. Each lattice point has a real number associated with it, the box spline coefficient. We may interpret this in a way similar to (cf. [WPW]) as creating a field that defines a function. Even though the 7-direction box spline is itself only of degree four, [BHR94, (59) Propositon, p 53] assures us that its linear combinations can reproduce, i.e. model exactly, all trivariate cubics and some additional surfaces of degree four. The actual coefficients are computed from Marsden's identity [BHR94, (11) p 67]. For example, when representing the cylinder

$$(x-y)^2 + z^2 = 1$$

we need to associate with the 3D lattice point (i,j,k) the value

$$(i^2+i-\frac{1}{6}) - 2(ij+i/2+j/2+\frac{1}{4}) + (j^2+j-\frac{1}{6}) + (k^2+5k+\frac{35}{6}) - 1$$
$$= i^2+j^2-2ij+k^2+5k+4.$$

§7. Approximate parametrization of the zero set

A unique approximate evaluation of the surface is achieved as follows. Consider the tesselation of each cube into tetrahedra and associate the average of the values at the vertices with the center of each cube and cube face. This allows the construction of a continuous piecewise linear approximation to the zero sheet. For each edge whose endpoints have an opposite sign, mark the midpoint. Each tetrahedron has either zero, three or four marked edge-midpoints. Correspondingly, we add no, one or two (coplanar) triangles connecting the midpoints to a list of triangles. The union of the triangles in the list then form the surface approximation.

The surface approximation is refined by averaging according to the subdivision rules of the 7-direction box spline. That is, each value is replicated over a cube of half the edge length and then the values on this refined lattice are averaged consecutively in each of the four diagonal directions of the box spline. The quadratic convergence of the coefficients of the box spline to the surface [BHR94, (30)Theorem] assures us that the sequence of linear approximations converges quadratically to the surface. That is, we need not evaluate the box spline explicitly ever, but rather evaluate approximately up to a tolerance defined by the output requirements. The simple averaging of the coefficients also guarantees stability of evaluation to high accuracy.

The evaluation by averaging implies also that features are smoothed out at each step. This has the desirable effect that no additional features are introduced and hence the shape of the surface can be inferred from the first approximation. In particular, no additional zero sheets at a given location can be generated so that the surface is single-sheeted if the first approximation is single-sheeted.

§8. Fixed grid implicits and general purpose modeling

While the generation of smooth surfaces as the zero set of a piecewise polynomial function on a fixed-grid is conceptually simple and capable of modeling free-form objects (see e.g. [MW91], [Pet95]), the simplicity of the approach is offset by a lack of efficiency. The efficiency of the global approach can be improved by maintaining only the lattice points close to the intended surface, but it is unclear how one can get a viable general purpose modeling paradigm that supports generic operations like rotation for arbitrary shapes, because a rotated regular lattice will not match up with the original.

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Examples of page 8 (Rows are numbered, columns are lettered.)

1a-c Evaluation by subdivision: zero, one and two subdivision applied to a cylinder intersection. The evaluation process is presently not fast: generating the second subdivision took 3 minutes on a Sun Sparc 5.

2a-c The blending process: two (Inventor) primitives, their union and their smooth blend. The highlighted region is the surface represented by the 7-direction box spline.

3a-c Various implicit primitives blended.

4a-c Join of two tori: the torus primitive, the box spline blend and the complete blended object.