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Abstract

In this paper (part two of the trilogy) we introduce three classes of reduced form regular algebraic curves that are defined in the part one paper. The approximated curves classes by the reduced form curve classes are described and the interpolation and approximation problems of the curves in the approximated curves classes by the curves in the reduced form curve classes are solved. Explicit formulas and error bounds of the curve interpolations and approximations are also given.

1 Introduction

In the first part of this trilogy of papers[8], we introduced the concept of a discriminating family of curves by which regular algebraic curve segments are isolated. Using different discriminating families, several characterizations of the Bernstein-Bezier (BB) form of the implicitly defined real bivariate polynomials over the plane triangle and the quadrilateral are given, so that the zero contours of the polynomials define smooth and single sheeted real algebraic (called regular) curve segments. In this part of the trilogy of papers, we shall use the reduced form regular algebraic curve segments to interpolate the given data and to approximate a given function. By reduced form, we mean that the most coefficients of the BB form of the curve are zero, which make the interpolation and approximation problem by the zero contour of the bivariate polynomial as easy as the problem of one variable polynomials or rational functions.

It is well known that data interpolation and function approximation by one variable polynomial or rational function is a classical and very well developed area in the fields of approximation theory as well as CAGD. In contrast to this, interpolation and approximation by the zero sets of bivariate polynomials is rather new [1], [2], [6], [4], [5], and [7] and relatively less results are available at the present time. After overcoming the difficulties of having singularities and discontinuity properties of the implicitly defined curves in part I of this trilogy of papers, our main results in part II are the existence and uniqueness of the

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interpolation and approximation solutions, and explicit error bounds on the approximation. The reduced form real algebraic curves we use in this paper have the following advantages:

1. They are non-singular and without discontinuities

2. The curve (that is the polynomial coefficients) can be generated as easily as the classical polynomial or rational function.

3. Unlike the polynomial rational functions, the curves are always located in the given domain. Hence the approximation errors are controllable within the domain of interest. This is significant and in contrast to the Runge phenomenon of classical polynomial interpolation.

In section 2 of this paper, a concept of transversal family corresponding to the concept of discriminating family of paper I is introduced and some transversal families are given. This concept is important because a pair of discriminating family and transversal family serves as an xy coordinate system within the domain of interest In section 3, we introduce three reduced form curve classes which are subset of the regular curve segments introduced in [8]. Then in section 4, we find the corresponding three classes of smooth curves which can be approximated by the reduced form curves. In section 5, we reduce the interpolation and approximation problems to the classical ones, and hence every problem proposed above has rational polynomial-like solutions. Section 6 gives explicit formulas for the solution of the interpolation and approximation problems. The last section consider the error bounds of the approximants.

2 Definitions and Preliminaries

We introduced the concept of discriminating family in the earlier paper [8]. Corresponding to this concept we now introduce the concept of a transversal family.

Definition 2.1. *Let $D(R, R_1, R_2)$ be a given discriminating family, and $T(R, R_1', R_2') = \{T_t(x,y) = \mu(x,y) - t\nu(x,y) = 0 : \; t \in (-\infty, \infty)\}$ be an algebraic curve family with t being a linear parameter, $\nu(x,y) > 0$ on $R \setminus \{R_1, R_2\}$ and R_1' and R_2' being two open(no end points) boundaries of R. If*

1. $\partial R \setminus (R_1 \cup R_2) = R_1' \cup R_2'$ and $R_1' \cap R_2' = \emptyset$.

2. Each curve in T passes through R_1' and R_2'.

3. Each curve in T is $D(R, R_1, R_2)$ regular.

4. For $\forall p \in R \setminus \{R_1, R_2\}$, there exists one and only one $t \in (-\infty, \infty)$ such that $T_t(p) = 0$. Then we say $T(R, R_1', R_2')$ is a transversal family of $D(R, R_1, R_2)$, where ∂ stands for the boundary.

The following are transversal families corresponding to the discriminating families D_1, D_3 and D_4 introduced in [8]:

(A) $T_1([p_0p_1p_2], (p_0p_2), (p_1p_2)) = \{B_1^2(\alpha_2)t - B_2^2(\alpha_2) + B_0^2(\alpha_2) = 0 : t \in (-\infty, \infty)\}$ is a transversal family of $D_1([p_0p_1p_2], p_2, [p_0p_1])$(see Figure 2.1(a)).

(B) $T_3([p_0p_1p_2p_3], (p_0p_2), (p_1p_3)) = \{B_1^2(u)t - B_2^2(u) + B_0^2(u) = 0 : t \in (-\infty, \infty)\}$ is a transversal family of $D_3([p_0p_1p_2p_3], [p_0p_1], [p_2p_3])$(see Figure 2.1(b)).

(C) $T_4([p_0p_1p_2p_3], (p_0p_1] \cup [p_1p_3), (p_0p_2] \cup [p_2p_3)) = \{[u(1-v) + (1-u)v]t - (1-u-v) = 0 : \; t \in (-\infty, \infty)\}$ is a transversal family of $D_4([p_0p_1p_2p_3], p_0, p_3)$(see Figure 2.1(c)).

For the discriminating family $D_1([p_0p_1p_2], p_1, p_2)$, $T_2([p_0p_1p_2], (p_0p_1), (p_0p_2)) = \{(\alpha_0^2 + \alpha_1\alpha_2)t - \alpha_2^2 + \alpha_1^2 = 0 : t \in (-\infty, \infty)\}$ is one of its transversal families. However, we do

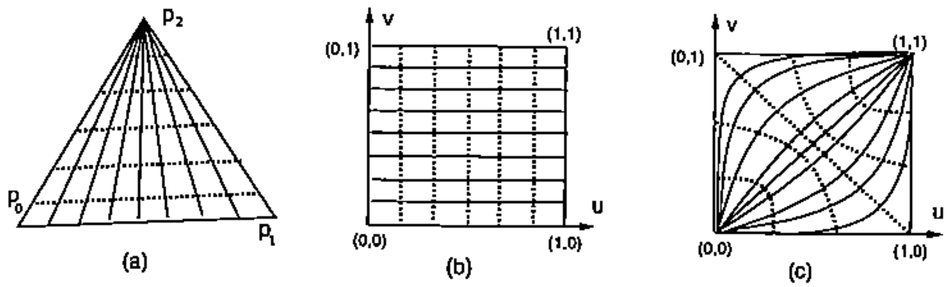

Figure 2.1: Discriminating and transversal families.

not use the D_2 regular reduced form curves, since the D_1 D_3 and D_4 regular reduced form curves provide enough choice in the applications.

Given a discriminating family $D = \{A_s(x,y) = 0\}$ and its transversal family $T = \{B_t(x,y) = 0\}$, we are given in fact a map between $R \setminus \{R_1, R_2\}$ in xy-plane and the strip $[0,1] \times (-\infty, \infty)$ in the st-plane. Since s and t are linear parameters in $A_s(x,y) = 0$ and $B_t(x,y) = 0$, respectively, they can be written as

$$M(D,T) : \begin{cases} s = \alpha(x,y) = \frac{\gamma(x,y)}{\delta(x,y)} \\ t = \beta(x,y) = \frac{\mu(x,y)}{\nu(x,y)} \end{cases} \tag{2.1}$$

where α and β are well defined rational functions on $R \setminus \{R_1, R_2\}$. For our three pairs of families we have the following transform:

(A). $M(D_1([p_0p_1p_2], p_2, [p_0p_1]), T_1([p_0p_1p_2), (p_0p_2), (p_1p_2)))$ is given by

$$\begin{cases} s = \dfrac{\alpha_1}{\alpha_0 + \alpha_1} \\ t = \dfrac{B_2^2(\alpha_2) - B_0^2(\alpha_2)}{B_1^2(\alpha_2)} \end{cases} \tag{2.2}$$

and its inverse $M(\ D_1([p_0p_1p_2], p_2, [p_0p_1]), T_1([p_0p_1p_2), (p_0p_2), (p_1p_2)))^{-1}$ is given by

$$\begin{cases} \alpha_0(s,t) = (1-s)\left(1 - \dfrac{1}{\sqrt{1+t^2}+1-t}\right) \\ \alpha_1(s,t) = s\left(1 - \dfrac{1}{\sqrt{1+t^2}+1-t}\right) \\ \alpha_2(s,t) = \dfrac{1}{\sqrt{1+t^2}+1-t} \end{cases} \tag{2.3}$$

(B). $M(D_3([p_0p_1p_2p_3], [p_0p_1], [p_2p_3]), T_3([p_0p_1p_2p_3], (p_0p_2), (p_1p_3)))$:

$$\begin{cases} s = v \\ t = \dfrac{B_2^2(u) - B_0^2(u)}{B_1^2(u)} \end{cases} \tag{2.4}$$

and $M(D_3([p_0p_1p_2p_3], [p_0p_1], [p_2p_3]), T_3([p_0p_1p_2p_3], (p_0p_2), (p_1p_3)))^{-1}$ is given by

$$\begin{cases} u(s,t) = \dfrac{1}{\sqrt{1+t^2}+1-t} \\ v(s,t) = s \end{cases} \tag{2.5}$$

(C). $M(D_4([p_0p_1p_2p_3],p_0,p_3),T_4([p_0p_1p_2p_3],(p_0p_1]\cup[p_1p_3),(p_0p_2]\cup[p_2p_3)))$:

$$\begin{cases} s &= \dfrac{u(1-v)}{u(1-v)+(1-u)v} \\ t &= \dfrac{1-u-v}{u(1-v)+(1-u)v} \end{cases} \tag{2.6}$$

$M(D_4([p_0p_1p_2p_3],p_0,p_3),T_4([p_0p_1p_2p_3],(p_0p_1]\cup[p_1p_3),(p_0p_2]\cup[p_2p_3)))^{-1}$ is

$$\begin{cases} u(s,t) &= \dfrac{2s}{t+2s+\sqrt{t^2+4s(1-s)}} \\ v(s,t) &= \dfrac{2(1-s)}{t+2(1-s)+\sqrt{t^2+4s(1-s)}} \end{cases} \tag{2.7}$$

where (u,v) are defined by the limit when $s=0$ or $s=1$. That is

$$\begin{cases} u(0,t) &= 0 \\ v(0,t) &= \dfrac{1}{t+1}, \end{cases} \quad \text{if } t\geq 0; \quad \begin{cases} u(0,t) &= \dfrac{t}{t-1} \\ v(0,t) &= 1, \end{cases} \quad \text{if } t<0 \tag{2.8}$$

and

$$\begin{cases} u(1,t) &= \dfrac{1}{t+1} \\ v(1,t) &= 0, \end{cases} \quad \text{if } t\geq 0; \quad \begin{cases} u(1,t) &= 1 \\ v(1,t) &= \dfrac{t}{t-1}, \end{cases} \quad \text{if } t<0 \tag{2.9}$$

3 Reduced Form Algebraic Curve

The reduced form algebraic curves are a special form of the regular algebraic curve segments discussed in [8]. In this special form, we take most of the BB-form coefficients to be zero and arrange the nonzero coefficients on the horizontal lines(see the dots in Figure 3.1(a)) or on the vertical lines(see the the dots in Figure 3.1(b)) or on the diagonal lines(see the dots in Figure 3.1(c)). That is, we define three reduced form curve classes: horizontal form, vertical form and diagonal form.

A. Horizontal Form HT_m. This class is a subset of $D_1([p_0p_1p_2],p_2,[p_0p_1])$ regular curves defined by:

$$\begin{aligned} HT_m = \{ \quad & F(\alpha_0,\alpha_1,\alpha_2)=0 : (\alpha_0,\alpha_1,\alpha_2)^T\in[p_0p_1p_2]\setminus\{p_2,[p_0p_1]\}, \\ & F=\frac{2}{m+1}\sum_{i=0}^{m}\beta_iB^{m+1}_{m-i,i,1}(\alpha_0,\alpha_1,\alpha_2)-\frac{2}{m(m+1)}\sum_{i=0}^{m-1}w_iB^{m+1}_{m-1-i,i,2}(\alpha_0,\alpha_1,\alpha_2) \\ & +\sum_{i=0}^{m+1}w_i'B^{m+1}_{m+1-i,i,0}(\alpha_0,\alpha_1,\alpha_2),\ \sum_{i=0}^{m-1}w_iB_i^{m-1}>0;\ w_i'=\sum_{j=i-2}^{i}\frac{C^j_{n-1}C_2^{i-j}w_j}{C^i_{n+1}}\ \} \end{aligned}$$

Where w_i' are given by degree elevation formula so that $\sum_{i=0}^{m-1}w_iB_i^{m-1}=\sum_{i=0}^{m+1}w_i'B_i^{m+1}$. Then by Theorem 3.1 of [8], we know that $F(\alpha_0,\alpha_1,\alpha_2)=0$ is a $D_1([p_0p_1p_2],p_2,[p_0p_1])$ regular curve in the triangle $[p_0p_1p_2]$ for any β_i, $i=0,1,\ldots,m$. The curves in HT_m are between p_2 and $[p_0p_1]$ and away from them(see the curve in Figure 3.1(a)).

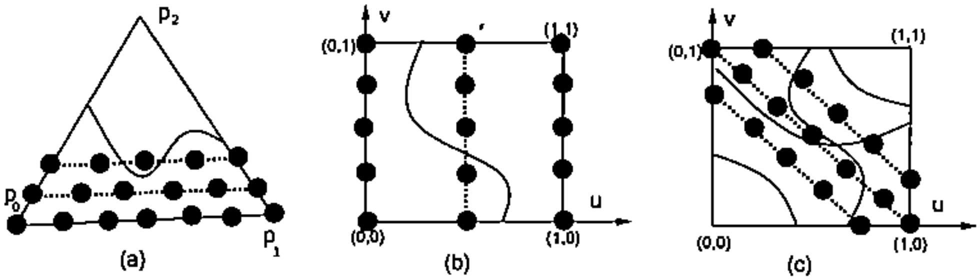

Figure 3.1: Reduced form algebraic curves

B. Vertical Form VS_m. This class of curves is a subset of $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3])$ regular curves defined by:

$$VS_m = \{ \quad F(u,v)=0 : 0 < u < 1, \;\; 0 \le v \le 1, \; F(u,v) = B_1^2(u)\sum_{i=0}^{m}\beta_i B_i^m(v)$$
$$+[B_0^2(u)-B_2^2(u)]\sum_{i=0}^{m-1} w_i B_i^{m-1}(v), \quad \sum_{i=0}^{m-1} w_i B_i^{m-1} > 0 \;\; \}$$

It follows from Theorem 3.3 of [8], the curve $F(u,v)=0$ is $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3])$ regular. The curves in this set are between $[p_0p_1]$ and $[p_2p_3]$ and away from them(see the curve in Figure 3.1(a)).

C. Diagonal Form DS_m. This class of curves is a subset of $D_4([p_0p_1p_2p_3],p_0,p_3)$ regular curves defined by:

$$DS_m = \{ \quad F(u,v)=0 : (u,v)^T \ne (0,0)^T,(1,1)^T, \;\; F(u,v) = \sum_{i=0}^{m}\gamma_i\beta_i B_i^m(u)B_{m-i}^m(v)$$
$$-\sum_{i=0}^{m-1}\delta_i w_i B_i^m(u)B_{m-i-1}^m(v) + \sum_{i=0}^{m-1}\eta_i w_i B_{i+1}^m(u)B_{m-i}^m(v), \; \sum_{i=0}^{m-1} w_i B_i^{m-1} > 0 \;\}$$

where $\gamma_i = 1/C_m^i$, $\delta_i = C_{m-1}^i/(C_m^{i+1}C_m^i)$, $\eta_i = C_{m-1}^i/(C_m^{i+1}C_m^{m-i})$. By Theorem 3.4 of [8], the curve $F=0$ is $D_4([p_0p_1p_2p_3],p_0,p_3)$ regular. The curves in DS_m are between p_0 and p_3 and away from these two points(see the curves in Figure 3.1(c)).

In the definition of these curves families, the parameters β_i are free but $\sum_{i=0}^{m-1} w_i B_i^{m-1}(s) > 0$ on $[0,1]$. If we cannot guarantee this condition in the interpolation and approximation problems, we can simply take $w_i = 1$. However, we then lose $m-1$ degrees of freedom.

4 Approximation of Curve Classes

We have defined three classes of regular algebraic curves in §3. Each of them have different features from the other. These different features make each of them suitable for approximating different curve classes. In this section, we define these curve classes. We assume that the curves we consider are always smooth. That is we always assume the function $f(x,y)$ is C^1 continuous in some region R in which the curve $f(x,y)=0$ lies.

Let $D(R, R_1, R_2)$ be a given discriminating family. Then the approximation class corresponding to $D(R, R_1, R_2)$ is the collection of curves that intersects each curve in $D(R, R_1, R_2)$ once and only once. Denote this class by $A(R, R_1, R_2)$. The curve in $A(R, R_1, R_2)$ has the following features:

1. It is smooth in $R \setminus \{R_1, R_2\}$.
2. It is away from R_1 and R_2.
3. If the point p in $R \setminus \{R_1, R_2\}$ is on the curve $a(x, y) = 0 \in A(R, R_1, R_2)$ and the curve $d(x, y) = 0 \in D(R, R_1, R_2)$, then the two curves are not tangent at p.

For a given curve $a(x, y) = 0$ in $A(R, R_1, R_2)$ there may be infinitely many functions $a(x, y)$ that have the same zero contour. We shall show that all these functions can be represented by one single function $f \in C^1[0, 1]$. We say $a(x, y)$ and $f(s)$ are equivalent in defining the curve. Here $C^1[0, 1]$ is the set of all C^1 continuous functions on $[0, 1]$.

Theorem 4.1. *Let $D(R, R_1, R_2) = \{s = \alpha(x, y) : s \in [0, 1]\}$ be a discriminating family. $T(R,\ R_1',\ R_2') = \{t = \beta(x, y) : t \in (\infty, \infty)\}$ be a transversal family of $D(R, R_1, R_2)$. Then $a(x, y) = 0 \in A(R, R_1, R_2)$ if and only if there exists an unique $f \in C^1[0, 1]$ such that the curve $a(x, y) = 0$ is equivalently defined by $\beta(x, y) - f(\alpha(x, y)) = 0$.*

Proof. Let $a(x, y) = 0 \in A(R, R_1, R_2)$. Then by the definition of $A(R, R_1, R_2)$, we know that for any $s \in [0, 1]$, the curve $s = \alpha(x, y) \in D(R, R_1, R_2)$ will intersect with $a(x, y) = 0$ only once in $R \setminus \{R_1, R_2\}$. Let $(x(s), y(s))^T$ be the intersection point. Then by implicit function theory, $x(s)$ and $y(s)$ are C^1 continuous functions of s and $(x(s), y(s))^T \notin R_1 \cup R_2$. Let $f(s) = \beta(x(s), y(s))$. Then $f \in C^1[0, 1]$. Now for this f, $\beta(x, y) - f(\alpha(x, y)) = 0$ define the curve $a(x, y) = 0$ since $(x(s), y(s))^T$ satisfies the equation. It is easy to see that this f is unique. Hence the necessary part of the theorem is proved.

Let $f \in C^1[0, 1]$, then $a(x, y) = \beta(x, y) - f(\alpha(x, y))$ is clearly C^1 continuous on $R \setminus \{R_1, R_2\}$. For any given $s \in [0, 1]$ the curve $s = \alpha(x, y)$ intersects with $a(x, y) = 0$ at a point $(x, y)^T$ that satisfies (2.1) with $t = f(s)$. Hence, there exist unique $(x, y)^T \in R \setminus \{R_1, R_2\}$ that satisfy (2.1). Since

$$\nabla a = \nabla \beta - \nabla \alpha f' \tag{4.1}$$

then by the definition of the transversal family, we know that $\nabla a \neq 0$. That is the curve $a(x, y) = 0$ is smooth. Since $\nabla \alpha$ and $\nabla \beta$ are linear independent, (4.1) also implies that ∇a and $\nabla \alpha$ are linear independent. That is the intersection is only once. Therefore $a(x, y) = 0 \in A(R, R_1, R_2)$. $\diamond$

This theorem gives an invertible mapping from $A(R, R_1, R_2)$ to $C^1[0, 1]$ that maps a zero contour $a(x, y) = 0 \in A(R, R_1, R_2)$ to a function $f \in C^1[0, 1]$. We denote this map by $\mathcal{M}(a = 0) = f$.

For the three cases we considered, that is the horizontal form on a triangle, the vertical form on a square and the diagonal form on a square, we denote $A(R, R_1, R_2)$ as HT, VS, and DS respectively.

5 Interpolation and Approximation

We consider now the main problem: interpolation and approximation by the reduced form algebraic curves. By interpolation, we mean that we are given a set of data points, we wish to construct a curve to interpolate (or approximate) these points. For the approximation problem, we are given a smooth curve in $A(R, R_1, R_2)$ and then wish to construct an algebraic curve to approximate it within some eror norm. The interpolation and approximation for a given curve $a(x, y) = 0 \in A(R, R_1, R_2)$ will be realized by the following steps:

1. Find $f \in C^1[0,1]$, such that $\mathcal{M}(a = 0) = f$.
2. Determine the interpolant or approximant $g \in C^1[0,1]$ of f in $C^1[0,1]$.
3. Determine $b = 0 \in A(R, R_1, R_2)$ such that $\{b(x,y) = 0\} = \mathcal{M}^{-1}(b = 0) = g$.

Then $b = 0$ is the approximation of the curve $a = 0$. As we shall see in the following, the problems of determining g in the second step will be led to a rational polynomial problem. That is, determine the coefficients β_i, $i = 0, 1, \ldots, m$ and w_i, $i = 0, 1, \ldots, m-1$, such that

$$\frac{\sum_{i=1}^{m} \beta_i B_i^m(s)}{\sum_{i=0}^{m-1} w_i B_i^{m-1}(s)} = t(s), \tag{5.1}$$

approximately on some points or for all $s \in [0,1]$ and a known function $t(s) \in C^1[0,1]$. Now we show how to obtain the problem (5.1) from our reduced form algebraic curve interpolation and approximation problems.

5.1 Equivalent Problems

A. Horizontal Form HT_m

Using the first equality of map (2.2), curve $f(\alpha_0, \alpha_1, \alpha_2)$ defined in HT_m can be written as $F(\alpha_0, \alpha_1, \alpha_2) = (1 - \alpha_2)^{m-1} G(s, t)$ with

$$G(s,t) = B_1^2(\alpha_2) \sum_{i=0}^{m} \beta_i B_i^m(s) - (B_2^2(\alpha_2) - B_0^2(\alpha_2)) \sum_{i=0}^{m-1} w_i B_i^{m-1}(s). \tag{5.2}$$

Since $0 < \alpha_2 < 1$, the curve $F(\alpha_0, \alpha_1, \alpha_2) = 0$ is equivalently defined by $G(s,t) = 0$. By the second equality of the map (2.2), $G(s,t) = 0$ can be written in the form (5.1) since $B_1^2(\alpha_2) > 0$ on $(0,1)$.

B. Vertical Form VS_m

In VS_m, $G(s,t) = F(u,v)$. Hence $G(s,t) = 0$ can be written in the form (5.1)(see (2.4)).

C. Diagonal Form DS_m

By map (2.6), we can write $F(u,v)$ defined in DS_m as

$$F(u,v) = [u(1-v) + (1-u)v]^{m-1} G(s,t) \tag{5.3}$$

where

$$G(s,t) = [u(1-v) + (1-u)v] \sum_{i=0}^{m} \beta_i B_i^m(s) - (1 - u - v) \sum_{i=0}^{m-1} w_i B_i^{m-1}(s) \tag{5.4}$$

and u and v are defined by (2.7). Since $(u,v) \neq (0,0)$, $(1,1)$, $u(1-v) + (1-u)v > 0$. Hence $F(u,v) = 0$ if only if $G(s,t) = 0$. By the map (2.6), $G(s,t) = 0$ can be written as (5.1).

5.2 Rational Interpolation and Approximation

We have achieved the problem of determining the coefficients β_i and w_i such that (5.1) holds, where $(s,t)^T$ is related to $(x,y)^T$ by the known map (2.1). Hence interpolation and approximation problems can be solved as classical rational ones.

A. Hermite Interpolation. Suppose we are given some points with derivatives in the domain we are considering:

$$(x_j,\ y(x_j),\ y^{(1)}(x_j), \ldots, y^{(k_j)}(y_j)), \quad j = 0,1,\ldots,n \tag{5.5}$$

or $(y_j,\ x(y_j),\ x^{(1)}(y_j), \ldots, x^{(k_j)}(y_j))$, $\quad j = 0,1,\ldots,n$, such that $s_j = \alpha(x_j, y_j)$ are distinct and

$$\sum_{j=0}^{n} k_j + 1 \geq 2m \quad (\text{or } m+1 \text{ if } w_i \text{ are fixed}) \tag{5.6}$$

From the discussion above, s_j distinct means that (x_j, y_j) is separable by the corresponding discriminating family. That is, each curve in the discriminating family contains at most one (x_j, y_j). Then by (2.1) we can compute the points $\{s_j, t(s_j), t^{(1)}(s_j), \ldots, t^{(k_j)}(s_j)\}$, $j = 0,1,\ldots,n$. For example, if (5.5) is given, then

$$\begin{aligned}
s_j &= \alpha(x_j, y(x_j)), \quad t(s_j) = \beta(x_j, y(x_j)), \quad t'(s_j) = \left[\tfrac{\partial\beta}{\partial x} + \tfrac{\partial\beta}{\partial y} y'(x_j)\right] \tfrac{\partial x}{\partial s} \\
t''(s_j) &= \left[\tfrac{\partial^2\beta}{\partial x^2} + 2\tfrac{\partial^2\beta}{\partial x \partial y} y'(x_j) + \tfrac{\partial^2\beta}{\partial y^2}(y'(x_j))^2 + \tfrac{\partial\beta}{\partial y} y''(x_j)\right] \tfrac{\partial x}{\partial s} + \left[\tfrac{\partial\beta}{\partial x} + \tfrac{\partial\beta}{\partial y} y'(x_j)\right] \tfrac{\partial^2 x}{\partial s^2} \\
&\cdots\cdots\cdots
\end{aligned}$$

where $\frac{\partial x}{\partial s}, \frac{\partial^2 x}{\partial s^2}, \ldots$ are determined by differenting $s = \alpha(s, y)$. That is

$$\begin{aligned}
1 &= \left(\tfrac{\partial\alpha}{\partial x} + \tfrac{\partial\alpha}{\partial y} y'(x_j)\right) \tfrac{\partial x}{\partial s} \\
0 &= \left[\tfrac{\partial^2\alpha}{\partial x^2} + 2\tfrac{\partial^2\alpha}{\partial x \partial y} y'(x_j) + \tfrac{\partial^2\alpha}{\partial y^2}(y'(x_j))^2 + \tfrac{\partial\alpha}{\partial y} y''(x_j)\right] \tfrac{\partial x}{\partial s} + \left[\tfrac{\partial\alpha}{\partial x} + \tfrac{\partial\alpha}{\partial y} y'(x_j)\right] \tfrac{\partial^2 x}{\partial s^2} \\
&\cdots\cdots\cdots
\end{aligned}$$

It is clear that $\frac{\partial^k x}{\partial s^k}$ is well defined if $\frac{\partial\alpha}{\partial x} + \frac{\partial\alpha}{\partial y} y'(x_j) \neq 0$. The geometric meaning of this requirement is that the curve constructed is not tangent with the curve in the discriminating family at the interpolating points. Differentiating (5.1) about s, we get a linear system of equations

$$\frac{d^\ell(\sum_{i=0} \beta_i B_i^m(s_j))}{ds^\ell} = \frac{d^\ell(t(s_j) \sum_{i=0}^{m-1} w_i B_i^{m-1}(s_j))}{ds^\ell}; \ \ell = 0,1,\ldots,k_j,\ j = 0,1,\ldots,n$$

with β_i and w_i as unknowns. This is a classical Hermite interpolation problem. Under condition (5.6) it can be solved in the least squares or Chebyshev sense. If we interpolate two points, the highest order smoothness we can achieve is C^{m-1}.

If the points (5.5) come from a curve that is in the corresponding approximated curve class discussed in §4, then the interpolation error formula for the function $t(s)$ can be used. Of course, some good behavior interpolation scheme, such as Fejér interpolation, or interpolation on Chebyshev points, can be used here. In summary, there are plenty of methods and any of the results for rational or polynomial interpolation can be used here.

B. Approximation

Given a curve in the corresponding class defined in §4. Then in each case, the map (2.1) leads us to a function $t = t(s) \in C^1[0,1]$, and the approximation problem is transformed to determine the coefficients, such that (5.1) holds. In any case as discussed before, $t(s)$ is always a well defined C^1 continuous function. So we are led to the classical rational approximation problem, and related methods and results can be used.

Hence we have proved the following theorem:

Theorem 5.1. *The interpolation and approximation problems of the curves in the approximated curve classes by the curves in the reduced form curves classes always have unique solutions.*

6 Explicit Formulae

After the coefficients β_i, $i = 0, 1, \ldots, m$ and w_i, $i = 0, 1, \ldots, m-1$ are determined by solving the equivalent problem (5.1), it becomes necessary to give explicit formulas for the curve $F = 0$ in the three cases we discussed. These formulas are important for the error estimation in the next section as well as serve for evaluating the curve at several points. Let $t_m(s) = \sum_{i=0}^{m} \beta_i B_i^m(s) \ / \ \sum_{i=0}^{m-1} w_i B_i^{m-1}(s)$.

A. Horizontal Form HT_m. From (5.2), it is easy to find that

$$\alpha_i(s) = \alpha_i(s, t_m(s)), \quad i = 0, 1, 2 \tag{6.1}$$

where $\alpha_i(s,t)$ is explicitly defined by (2.3).

B. Vertical Form VS_m. Since $G(s,t)$ is in the same form of (5.2), then we have

$$\left\{ \begin{array}{lcl} u & = & \dfrac{1}{\sqrt{1 + t_m(s)^2} + 1 - t_m(s)} \\ v & = & s \end{array} \right. \tag{6.2}$$

C. Diagonal Form DS_m. From $G(s,t) = 0$, where $G(s,t)$ is defined by (5.4), we can obtain

$$\left\{ \begin{array}{lcl} u(s) & = & u(s, t_m(s)) \\ v(s) & = & v(s, t_m(s)) \end{array} \right. \tag{6.3}$$

where $(u(s,t), v(s,t))^T$ is defined by (2.7).

7 The Errors of Interpolation and Approximation

From the discussion of Section 2 and Section 4, the interpolation and approximation problems by the reduced form algebraic curves lead to the classical rational interpolation and approximation problem. We assume some interpolation and approximation method is used and the error bounded is known. For example, if we do exact polynomial interpolation, the Lagrange remainder formula can be used. If we do Chebyshev approximation, then Jackson theorem [3] for the approximation order can be used. If we use Chebyshev series expansion, then Dini-Lipschitz theorem can be used [3]. Our purpose is to get the error bound of algebraic curve interpolation or approximation from the known error bound for rational interpolation and approximation.

7.1 Errors in the Unit Triangle and Square

Note that our original curves considered are on the triangle or quadrilateral, and are transformed into a strip $S = [0,1] \times (-\infty, \infty)$. The interpolation or approximation is done in S. So we first determine the error in the unit triangle $[(0,0)^T(1,0)^T(0,1)^T]$ or unit square $[(0,0)^T(1,0)^T(0,1)^T(1,1)^T]$ for the given error in S. In S we have the uniform problem (5.1). Let $E_m(s) = t_m(s) - t(s)$ be the error of (5.1). In order to estimate the error in the unit triangle or unit square, we need to treat the difference of functions in the form $\frac{b}{\sqrt{ab+y^2}+b-y}$(see (2.3), (2.5) and (2.7)). Hence we introduce the following lemma:

Lemma 7.1. *Let* $\psi(y,a,b) = \frac{b}{\sqrt{ab+y^2}+b-y}$, $a > 0,\ b > 0$. *Then* $\psi'(y,a,b) > 0$ *achieve its maximal value* $\frac{2}{(\sqrt[3]{a}+\sqrt[3]{b})^3}$ *at* $y^* = \frac{1}{2}(\sqrt[3]{a^2b} - \sqrt[3]{ab^2})$ *and* $\psi'(y,a,b)$ *decrease to zero monotonicitily on* (y^*, ∞) *and* $(-\infty, y^*)$ *when* $y \to \pm\infty$ *and furthermore,*

$$\lim_{y\to-\infty} y^2\psi'(y,a,b) = \frac{1}{2}, \quad \lim_{y\to+\infty} y^2\psi'(y,a,b) = \frac{a}{2b} \tag{7.1}$$

Proof. Since

$$\psi'(y,a,b) = \frac{b(\sqrt{a+y^2}-y)}{\sqrt{a+y^2}(\sqrt{a+y^2}+b-y)^2},$$

we know that $\psi'(y,a,b) > 0$, and $\psi'(y,a,b) \to 0$ as $y \to \pm\infty$, and (7.1) holds. Now we show that y^* is the only zero of $\psi''(y,a,b) = 0$. It follows from $\psi''(y,a,b) = 0$, we obtain

$$ab\left(\sqrt{ab+y^2}+b-y\right) = 2\left(y - \sqrt{ab+y^2}\right)^2 \sqrt{ab+y^2} \tag{7.2}$$

That is

$$(ab+4y^2)\sqrt{ab+y^2} = ab^2 + 3aby + 4y^3 \tag{7.3}$$

Take square, we obtain

$$ab^3 - a^2b^2 + 6ab^2y + 8by^3 = 0, \text{ or } y^3 + \frac{3ab}{4}y + \frac{ab^2-a^2b}{8} = 0 \tag{7.4}$$

Since the discriminate of this equation $\Delta = \left(\frac{ab^2-a^2b}{16}\right)^2 + \left(\frac{ab}{4}\right)^3 = \left(\frac{ab^2+a^2b}{16}\right)^2 > 0$, it has only one real root $\frac{1}{2}\left(\sqrt[3]{a^2b} - \sqrt[3]{ab^2}\right) = y^*$. Now we compute $\psi'(y^*,a,b)$. Since $a + y^{*2} = \frac{1}{4}(\sqrt[3]{a^2b} + \sqrt[3]{ab^2})^2$, from (7.2) we have

$$\psi'(y^*,a,b) = \frac{a^2}{4(\sqrt{a+y^{*2}}-y^*)^3(a+y^{*2})\sqrt{a+y^{*2}}} = \frac{2}{(\sqrt[3]{a}+\sqrt[3]{b})^3}$$

Hence the lemma is proven. $\diamond$

Case 1: In (2.3), (6.1), (2.5) and (6.2), $a = b = 1$, then $y^* = 0$ and $\psi'(y^*, a, b) = \frac{1}{4}$.

Case 2: In the first equation of (2.7) and (6.3), $a = 2(1-s)$, $b = 2s$, then $y^* = \sqrt[3]{s(1-s)^2} - \sqrt[3]{s^2(1-s)}$

$$\psi'(y^*, a, b) = \phi(s), \quad \phi(s) = \frac{1}{(\sqrt[3]{s} + \sqrt[3]{1-s})^3}$$

It is not hard to show that $\frac{1}{4} \leq \phi(s) \leq 1$.

Case 3: In the second equation of (2.7) and (6.3), $a = 2s$, $b = 2(1-s)$, then $y^* = \sqrt[3]{(1-s)s^2} - \sqrt[3]{(1-s)^2 s}$ and $\psi'(y^*, a, b) = \phi(s)$.

Therefore we have the following theorem:

Theorem 7.2. *For a given function $t(s) \in C^1[0,1]$, let $t_m(s)$ be its approximation and $E_m(s)$ be the approximation error. Let $\kappa_0(a,b,c,d) = \max_{y \in [c,d]} \psi'(y,a,b)$, $\kappa_1(a,b) = \frac{1}{(1+|a|)(1+|b|)}$ and $\kappa_2(a,b) = max\{\frac{1}{1+|a|}, \frac{1}{1+|a|}\}$. Then*

(i). In HT and HT_m, let $E\alpha_i(s) = \alpha_i(s,t(s)) - \alpha_i(s,t_m(s))$, we have

$$|E\alpha_0(s)| \leq \kappa_0(1,1,t(s),t_m(s))(1-s)E_m(s) \leq \frac{1-s}{4}E_m(s) \tag{7.5}$$

$$|E\alpha_1(s)| \leq \kappa_0(1,1,t(s),t_m(s))sE_m(s) \leq \frac{s}{4}E_m(s) \tag{7.6}$$

$$|E\alpha_2(s)| \leq \kappa_0(1,1,t(s),t_m(s))E_m(s) \leq \frac{1}{4}E_m(s) \tag{7.7}$$

(ii) In VS and VS_m, let $Eu(s) = u(s,t(s)) - u(s,t_m(s))$, $Ev(s) = v(s,t(s)) - v(s,t_m(s))$,

$$|Eu(s)| \leq \kappa_0(1,1,t(s),t_m(s))E_m(s) \leq \frac{1}{4}E_m(s), \quad |Ev(s)| = 0 \tag{7.8}$$

(iii) In DS and DS_m, if $s \in (0,1)$, we have

$$|Eu(s)| \leq \kappa_0(2(1-s),2s,t(s),t_m(s))E_m(s) \leq E_m(s) \tag{7.9}$$

$$|Ev(s)| \leq \kappa_0(2s,2(1-s),t(s),t_m(s))E_m(s) \leq E_m(s) \tag{7.10}$$

If $s = 0$ or $s = 1$,

$$|Eu(s)|,\ |Ev(s)| \leq \kappa_1(t(s),t_m(s))E_m(s) \leq E_m(s) \ \text{ if } \ t(s)t_m(s) \geq 0 \tag{7.11}$$

$$|Eu(s)|,\ |Ev(s)| \leq \kappa_2(t(s),t_m(s))E_m(s) \leq E_m(s) \ \text{ if } \ t(s)t_m(s) < 0 \tag{7.12}$$

Proof. (7.5)–(7.7) follows from (2.3) and and Lemma 7.1. (7.8) follows from (2.5) and (7.9) and (7.10) follows from (2.7). From (2.8) and (2.9), we get (7.11) and (7.12). $\diamond$

In this theorem, the error in the standard triangle or square are bounded by the error in the strip with a factor κ_0 or κ_1 or κ_2. This factor has compression property. That is, κ_0 and κ_1 go to zero quadratically as $t(s)$ and $t_m(s)$ tend to ∞ while κ_2 goes to zero linearly.

7.2 Errors in the Original Domains

Now we consider the errors in the triangle or quadrilateral. For given a domain R, a discriminating family $D(R, R_1, R_2) = \{s = \alpha(x, y) : s \in [0, 1]\}$. Let $A(R, R_1, R_2)$ be the approximated class. The error between two curves $f = 0$, $g = 0$ in $A(R, R_1, R_2)$ is defined by

$$dis(f, g) = \sup_{s \in [0,1]} ER(f, g, s) \tag{7.13}$$

where

$$ER(f, g, s) = ||p_f(s) - p_g(s)|| \tag{7.14}$$

and $p_f(s)$ is the intersection point of $f = 0$ and $s = \alpha(x, y)$. It is easy to see that $dis(f, g) \geq h(f, g)$ where $h(f, g) = \max\{\sup_{p \in f=0} \inf_{q \in g=0} ||p - q||,\ \sup_{q \in g=0} \inf_{p \in f=0} ||p - q||\}$ is the Hausdorff distance between $f = 0$ and $g = 0$. Now we estimate $dis(f, g)$ for the three cases considered.

A. Error in HT. Let $f = 0 \in HT$, $f_m = 0 \in HT_m$ be the approximation of $f = 0$. It follows from (7.5)–(7.7) that

$$ER(f, f_m, s) \leq \pi(s) E_m(t(s), t_m(s))$$

where

$$\pi(s) = \frac{1}{4}\sqrt{[1 - s, s, 1]T^T T[1 - s, s, 1]^T}, \quad T = \begin{bmatrix} p_0 & p_1 & p_2 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$dis(f, f_m) \leq \max_{s \in [0,1]} \pi(s) \max_{s \in [0,1]} |E_m(t(s), t_m(s))|$$

B. Error in VS. Let $f \in VS$, $f_m \in VS_m$ be an approximation of f. Then the error estimation in the unit square is provided by (7.8). Now by (2.2) of [8], we have

$$ER(f, f_m, s) = \frac{1}{4}||(p_0 + p_3 - p_1 - p_2)s + p_1 - p_0||\ |E_m(t(s), t_m(s))|$$

and then

$$dis(f, f_m) \leq \frac{1}{4}\max\{||p_1 - p_0||, ||p_2 - p_3||\} \max_{s \in [0,1]} |E_m(t(s), t_m(s))|$$

C. Error in DS. Let $f \in DS$, $f_m \in DS_m$ be an approximation of f. Then by (2.2) of [8] and (7.9) we have

$$ER(f, f_m) \leq \frac{1}{4}(||p_0 + p_3 - p_1 - p_2|| + ||p_1 - p_0|| + ||p_2 - p_0||)|E_s(t(s), t_m(s))|$$

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