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Regular Algebraic Curve Segments(I) –Definitions and Characteristics

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Abstract

In this paper (part one of a trilogy), we introduce the concept of a discriminating family of regular algebraic curves (real, nonsingular and connected). Several discriminating families are obtained by different characterizations of the zero contours of the Bernstein-Bezier (BB) form of bivariate polynomials over the plane triangle and quadrilateral. Algorithms for the graphics display of these regular curve families are also provided.

1 Introduction

We consider regular algebraic curves defined by a real polynomial equation $f(x, y) = 0$ in Bernstein-Bezier (BB) form over a plane triangle or a quadrilateral. By regular, we mean that the real curve is continuous and smooth in the given domain. Notwithstanding the overwhelming popularity of parametric curves in computer aided geometric design (CAGD), one is increasingly aware that curves defined by implicit equations are more suitable for certain CAGD operations than parametric curves [1]. The primary drawback for the widespread use of the implicit algebraic curves is that the real curve may have singularities (cuspidal cubic), and may be disconnected (e.g. hyperbola) in a given region of the plane. In this paper (part I of III), we focus on isolating a regular piece of algebraic curve that is defined on a given triangle or a given quadrilateral in the BB (Bernstein Bezier) form. We introduce the concept of a discriminating family of regular algebraic curve segments (real, nonsingular and connected). Several discriminating families are obtained by different characterizations of the Bernstein-Bezier (BB) form of the implicitly defined bivariate polynomials over the plane triangle and the quadrilateral. In parts II and III of this trilogy of papers, we consider the problems of interpolation and approximation by splines of regular algebraic curve segments[15] and their applications in scattered and dense data fitting[16].

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The main difficulties of dealing with real algebraic curves are problems of curve singularities and discontinuities. In attempts to overcome these difficulties, Sederberg [12] characterized the coefficients of the BB (Bernstein-Bézier) form of an implicitly defined bivariate polynomial on a triangle in such a way that if the coefficients on the lines that parallel to one side, say L, of the triangle all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the algebraic curve segment at most once. In [13], Sederberg, Zhao and Zundel give another similar characterization which guarantees the single sheeted property of their TPAC by requiring that $\beta_{i0} \geq 0$, that $\beta_{0i}, \beta_{m-1,i} \leq 0$ and that the directional derivative of PAC (piecewise algebraic curves) with respect to any direction $s = \alpha u$ be non-zero within the triangle domain, here β_{ij} denotes the Bézier coefficient. Papers [6, 7] of Paluszny and Patterson constructs C^1 and C^2 continuous cubic algebraic splines by using the cubic $F(\alpha_0, \alpha_1, \alpha_2) = \beta_{201}\alpha_0^2\alpha_2 + \beta_{102}\alpha_0\alpha_2^2 - \beta_{120}\alpha_0\alpha_1^2 - \beta_{021}\alpha_1^2\alpha_2 + \beta_{111}\alpha_0\alpha_1\alpha_2$ with $\beta_{201} > 0$, $\beta_{102} > 0$, $\beta_{120} > 0$, $\beta_{021} > 0$, and $(\alpha_0, \alpha_1, \alpha_2)$ being barycentric coordinates. All the above characterizations dealing with BB triangles can be regarded as special cases of [2] in which the coefficients of the BB form has a one-time sign change. The present paper is an extension of our paper [2] for dealing with the BB form on the triangle and the quadrilateral. For the BB form on the quadrilateral, a characterization for single sheeted purpose is given in [8] and similar to Sederberg's in [13]. In particular, if the coefficients increase or decrease monotonically in x or y direction, then any line that is parallel to x or y axis will intersect the curve at most once. This is again a special case of our characterization in this paper.

2 Preliminary and Discriminating Families

Let $p_i = (x_i, y_i)^T \in I\!R^2$ for $i = 0, 1, \ldots, k$. Then $[p_0p_1 \ldots p_k]$ denotes the *closed convex hull* of $\{p_i\}_{i=0}^k$. That is, $[p_0p_1 \ldots p_k] = \{p \in I\!R^2 : p = \sum_{i=0}^k \alpha_i p_i, 0 \leq \alpha_i \leq 1, \sum_{i=0}^k \alpha_i = 1\}$. If $0 < \alpha_i < 1$, then the set defined is the *open convex hull* of $\{p_i\}_{i=0}^k$, denoted by $(p_0p_1 \ldots p_k)$. If $\alpha_i \in (-\infty, \infty)$, then the set defined is the *affine hull* of $\{p_i\}_{i=0}^k$, denoted by $\langle p_0p_1, \ldots p_k \rangle$. If $k = 2$ and p_0, p_1 and p_2 are affine independent, then $[p_0p_1p_2]$ is a triangle and $(\alpha_0, \alpha_1, \alpha_2)^T$ is known as barycentric coordinate which relates with $(x, y)^T$ by

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} y_1 - y_2, & x_2 - x_1 \\ y_2 - y_0, & x_0 - x_2 \end{bmatrix} \begin{bmatrix} x - x_2 \\ y - y_2 \end{bmatrix} / \Delta(p_0, p_1, p_2) \tag{2.1}$$

with $\Delta(p_0, p_1, p_2) = det \begin{bmatrix} p_0 & p_1 & p_2 \\ 1 & 1 & 1 \end{bmatrix}$. On a triangle, the algebraic curve will be defined by the zero contour of $F(\alpha_0, \alpha_1, \alpha_2) = \sum_{i+j+k=n} \beta_{ijk} B_{ijk}^n(\alpha_0, \alpha_1, \alpha_2)$ with $B_{ijk}^n(\alpha_0, \alpha_1, \alpha_2) = \frac{n!}{i!j!k!}\alpha_0^i\alpha_1^j\alpha_2^k$. If $k = 3$ and any three of $p_i (i = 0, 1, 2, 3)$ are affine independent, then $[p_0p_1p_2p_3]$ is a quadrilateral. We shall assume p_0, p_1, p_3 and p_2 are clockwise, and map $[p_0p_1p_2p_3]$ to the unit square $S = [0, 1] \times [0, 1]$ in the uv-plane by

$$p = (p_0 + p_3 - p_1 - p_2)uv + (p_2 - p_0)u + (p_1 - p_0)v + p_0 \tag{2.2}$$

If $p_0 + p_3 = p_1 + p_2$, i.e., $[p_0p_1p_2p_3]$ is a parallelogram, (2.2) is linear. On a quadrilateral, the algebraic curve will be defined by the zero contour of $G(u, v) = \sum_{i=0}^m \sum_{j=0}^n \beta_{ij} B_i^m(u) B_j^n(v)$ with $B_i^n(s) = C_n^i s^i (1 - s)^{n-i}$.

The following lemma is important in proving the regularity of the algebraic curve segments.

Lemma 2.1. *Let* $f(s) = \sum_{i=0}^{n} \beta_i B_i^n(s)$. *If there exist an integer* $k (0 < k < n)$ *such that* $\beta_i \geq 0$ *for* $i = 0, \cdots, k-1$ *and* $\beta_i \leq 0$ *for* $i = k+1, \cdots, n$, *and there is at least one strict inequality in each set of the inequalities, then for any real* $\alpha > 0$, *the function* $f_\alpha(s) = \sum_{i=0}^{n} \beta_i [B_i^n(s)]^\alpha$ *has only one zero in the interval* $(0,1)$.

Proof. If $\alpha = 1$, the lemma follows from the *variation diminishing property*(see [5], pp54). Since $f_\alpha(s) = (1-s)^{n\alpha} \sum_{i=0}^{n} c_i x^i$ with $c_i = (C_n^i)^\alpha \beta_i$, $x = \left(\frac{s}{1-s}\right)^\alpha \in (0, \infty)$. Under the assumption of the lemma, the sequence $c_0, c_1, \cdots, c_n$ has a one time sign change. It follows from Descartes sign rule that the function $\sum_{i=0}^{n} c_i x^i$ has only one zero in $(0, \infty)$. Therefore, $f_\alpha(s)$ has only one zero in $(0,1)$. ◇

Lemma 2.2. $\sum_{i=0}^{n} \beta_i [B_i^n(s)]^\alpha \geq 0$ *(or* > 0*) on* $[0,1]$, *if and only if* $\sum_{i=0}^{n} \beta_i (C_n^i)^{\alpha-1} B_i^n(s) \geq 0$ *(or* > 0*) on* $[0,1]$, *where* $\alpha > 0$.

Proof. Let $t = \frac{s^\alpha}{s^\alpha + (1-s)^\alpha}$. Then for $s \in [0,1]$ we have $t \in [0,1]$ and

$$\begin{aligned}
\sum_{i=0}^{n} \beta_i [B_i^n(s)]^\alpha &= [s^\alpha + (1-s)^\alpha]^n \sum_{i=0}^{n} \beta_i (C_n^i)^\alpha \frac{s^{i\alpha}}{[s^\alpha + (1-s)^\alpha]^i} \frac{(1-s)^{(n-i)\alpha}}{[s^\alpha + (1-s)^\alpha]^{n-i}} \\
&= [s^\alpha + (1-s)^\alpha]^n \sum_{i=0}^{n} \beta_i (C_n^i)^{\alpha-1} B_i^n(t)
\end{aligned}$$

The lemma follows directly from this equality.

Definition 2.1. *For a given triangle or quadrilateral* R, *let* R_1 *and* R_2 *be two closed boundaries of* R *and let* $D = \{A_s(x,y) = \gamma(x,y) - s\delta(x,y) = 0 : s \in [0,1]\}$ *be an algebraic curve family with* s *as a parameter and* $\delta(x,y) > 0$ *on* $R \setminus \{R_1, R_2\}$ *such that*

1. $R_1 \cap R_2 = \emptyset$.
2. *Each curve in* D *passes through* R_1 *and* R_2.
3. *Each curve in* D *is regular in the interior of* R.
4. *For* $\forall p \in R \setminus \{R_1, R_2\}$, *there exists one and only one* $s \in [0,1]$ *such that* $A_s(p) = 0$.

Then we say D *is a discriminating family on* R, *denoted by* $D(R, R_1, R_2)$.

Figure 2.1 shows four examples of discriminating families.

Definition 2.2. *For a given discriminating family* $D(R, R_1, R_2)$, *let* $f(x,y)$ *be a bivariate polynomial (or* C^1 *continuous function on* $R \setminus \{R_1, R_2\}$*). If the curve* $f(x,y) = 0$ *intersects with each curve in* $D(R, R_1, R_2)$ *only once in the interior of* R, *we say the curve* $f = 0$ *is regular(or smooth) with respect to* $D(R, R_1, R_2)$ *(concisely stated as being* $D(R, R_1, R_2)$ *regular)).*

Here we need to make the meaning of *intersect once* precise. Let $p^* = (x^*, y^*)^T$ be a point on the curve $A_s(x,y) = 0$. Since the curve is regular, we can represent it locally as $y = h(x)$(or $x = g(y)$) in the neighborhood of p^*. The term *intersect once* means x^* is a single zero of $f(x, h(x))$.

It is easy to show that a $D(R, R_1, R_2)$ regular (or $D(R, R_1, R_2)$ smooth) curve $f = 0$ is regular (or smooth). In fact, if $\nabla f = 0$ at a point $(x^*, y^*)^T$ on the curve, then x^* will be a double zero of $f(x, h(x))$ since $\frac{d}{dx} f(x^*, h(x^*)) = [1, h'(x^*)] \nabla f = 0$, here $h(x)$ is the same as

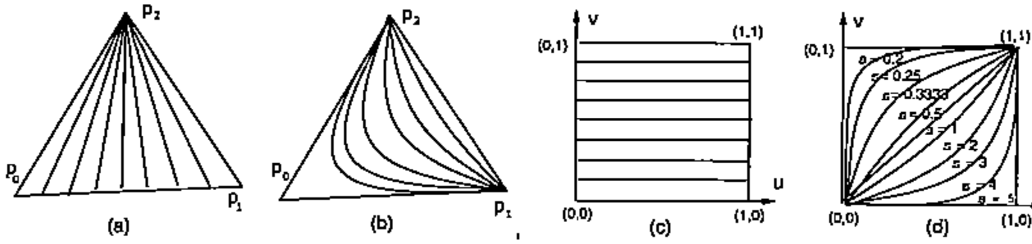

Figure 2.1: (a) The Lines D_1; (b) The Quadratic Family D_2; (c) The Lines D_3; (d) The Hyperbolic Family D_4.

the function defined in the above paragraph. Therefore $D(R, R_1, R_2)$ regular is a sufficient condition of regularity.

3 Regular Curve Segments

Now we give some discriminating families and characterize the regular curve segments defined on a triangle or a quadrilateral by these families.

Theorem 3.1. *For the triangle* $[p_0p_1p_2]$, *let* $D_1([p_0p_1p_2], p_2, [p_0p_1]) = \{\alpha_1 - s(\alpha_0 + \alpha_1) = 0 : s \in [0,1]\}$, *then* $D_1([p_0p_1p_2], p_2, [p_0p_1])$ *is a discriminating family. Let* $F(\alpha_0, \alpha_1, \alpha_2) = \sum_{i+j+k=n} \beta_{ijk} B^n_{ijk}(\alpha_0, \alpha_1, \alpha_2)$ *and* $B_k(s) = \sum_{i+j=n-k} \beta_{ijk} B^{n-k}_j(s)$, *then if there exists an integer* $l (0 < l < n)$ *such that* $B_k(s) \geq 0$ *for all* $s \in [0,1]$ *and* $k < l$; $B_k(s) \leq 0$ *for all* $s \in [0,1]$ *and* $k > l$ *and* $\sum_{k=0}^{l-1} B_k(s) > 0$, $\sum_{k=l+1}^{n} B_k(s) < 0$ *on* $(0,1)$, *then the curve* $F(\alpha_0, \alpha_1, \alpha_2) = 0$ *is* $D_1([p_0p_1p_2], p_2, [p_0p_1])$ *regular.*

Proof. $D_1([p_0p_1p_2], p_2, [p_0p_1])$ consists of straight lines(see Figure 2.1(a)) in the triangle $[p_0p_1p_2]$ that connect the point p_2 and all the points on $[p_0p_1]$. Hence it obviously is a discriminating family. For a given $s \in [0,1]$, let

$$\begin{cases} s = \dfrac{\alpha_1}{\alpha_0 + \alpha_1} \\ t = \alpha_2 \end{cases} \tag{3.1}$$

That is,

$$\begin{cases} \alpha_0 = (1-t)(1-s) \\ \alpha_1 = (1-t)s \\ \alpha_2 = t \end{cases} \tag{3.2}$$

Then

$$\begin{aligned} F(\alpha_0, \alpha_1, \alpha_2) &= \textstyle\sum_{k=0}^{n} \sum_{i+j=n-k} \beta_{ijk} \frac{n!}{i!j!k!} \alpha_0^i \alpha_1^j \alpha_2^k \\ &= \textstyle\sum_{k=0}^{n} B^n_k(t) \sum_{i+j=n-k} \beta_{ijk} B^{n-k}_j(s) \\ &= \textstyle\sum_{k=0}^{n} B_k(s) B^n_k(t) \end{aligned}$$

For a given s, since $B_k(s) \geq 0$ for $k < l$, $B_k(s) \leq 0$ for $k > l$ and there is a strict inequality in each of them, $F(\alpha_0, \alpha_1, \alpha_2) = 0$ has exactly one root $t \in (0,1)$ by Lemma 2.1. This (s,t) will gives us unique $(\alpha_0, \alpha_1, \alpha_2)$ by (3.2). ◇

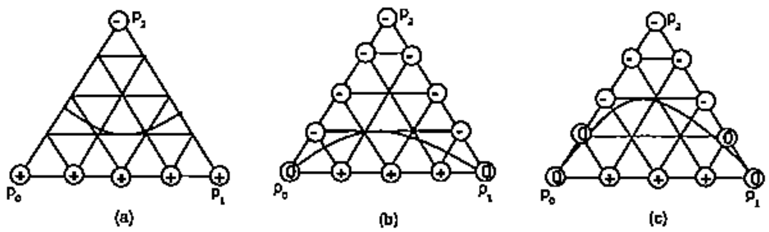

Figure 3.1: The $D_1([p_0p_1p_2], p_2, [p_0p_1])$ regular curves

Note that a sufficient condition for a Bernstein polynomial $\sum_{i=0}^{n}\beta_i B_i^n(s)$ to be nonnegative on $[0,1]$ is that the coefficients β_i are nonnegative. Hence we have the following corollary:

Corollary 3.1(see Theorem 3.1 of [2]). *For the triangle $[p_0p_1p_2]$, let $F(\alpha_0,\alpha_1,\alpha_2) = \sum_{i+j+k=n}\beta_{ijk}B_{ijk}^n(\alpha_0,\alpha_1,\alpha_2)$, then if there exists an integer $l(0<l<n)$ such that $\beta_{ijk}\geq 0$,* for *$k<l$ and $\beta_{ijk}\leq 0$* for *$k>l$ and there is at least one strict inequality in each set of the inequalities, then the curve $F(\alpha_0,\alpha_1,\alpha_2)=0$ is $D_1([p_0p_1p_2],p_2,[p_0p_1])$ regular.*

The $D_1([p_0p_1p_2],p_2,[p_0p_1])$ regular curves are between the point p_2 and the line segment $[p_0p_1]$ and away from p_2(if $\beta_{00n}\neq 0$) and the open line (p_0p_1)(see Figure 3.1(a)). They can pass through p_0 and/or p_1(see Figure 3.1(b)) and furthermore, the curves can be tangent with the line $[p_0p_2]$ and/or the line $[p_1p_2]$ at p_0 and/or p_1(see Figure 3.1(c)).

Theorem 3.2. *Let $D_2([p_0p_1p_2],p_1,p_2) = \{(1-s)\alpha_1\alpha_2 - s\alpha_0^2 = 0 : s\in[0,1]\}$, then $D_2([p_0p_1p_2],p_1,p_2)$ is a discriminating family. Let $F(\alpha_0,\alpha_1,\alpha_2)=\sum_{2i+j+k=2m}\beta_{2i,jk}B_{2i,jk}^{2m}$ $(\alpha_0,\alpha_1,\alpha_2)$ and $B_n(s)=\sum_{j-k=2n}\frac{(m-n)!(n+k)!(m-k)!}{(2m-j-k)!j!k!}\beta_{2m-j-k,jk}B_{n+k}^{m+n}(s)$, then if there exists an integer $l(-m<l<m)$ such that $B_n(s)\leq 0$ for all $s\in[0,1]$ and $n<l$; $B_n(s)\geq 0$ for all $s\in[0,1]$ and $n>l$; and $\sum_{n=-m}^{l-1}B_n(s)<0$, $\sum_{n=l+1}^{m}B_n(s)>0$ on $(0,1)$, then the curve $F(\alpha_0,\alpha_1,\alpha_2)=0$ is $D_2([p_0p_1p_2],p_1,p_2)$ regular.*

Proof. $D_2([p_0p_1p_2],p_1,p_2)$ consists of quadratics(see Figure 2.1(b)). It is easy to see that it is a discriminating family. Now we prove the regularity of the curve defined in the theorem. The case $s=0$, when the curve in $D_2([p_0p_1p_2],p_1,p_2)$ degenerates to straight lines $\alpha_1=0$ and $\alpha_2=0$, need to be considered separately. For instance, if the coefficients $\beta_{2i,0,2m-2i}, i=0,1,\cdots,m$ has a one time sign change, we let $\alpha_1=0$, and then

$$F(\alpha_0,\alpha_1,\alpha_2)=\sum_{2i+k=2m}\beta_{2i,0,k}B_{2i,0,k}^{2m}(\alpha_0,0,1-\alpha_0)=\sum_{i=0}^{m}\beta_{2i,0,2m-2i}B_{2i}^{2m}(\alpha_0)$$

Hence the equation $F(\alpha_0,\alpha_1,\alpha_2)=F(\alpha_0,0,1-\alpha_0)=0$ with α_0 as unknown has one root in $[0,1]$. The case that $\beta_{2i,2m-2i,0}, i=0,1,\cdots,m$ has a one time sign change is similar. Now suppose $s\in(0,1]$. let

$$\begin{cases} s = \dfrac{\alpha_1\alpha_2}{\alpha_0^2+\alpha_1\alpha_2} \\ t = \dfrac{\alpha_2}{\alpha_1+\alpha_2} \end{cases} \tag{3.3}$$

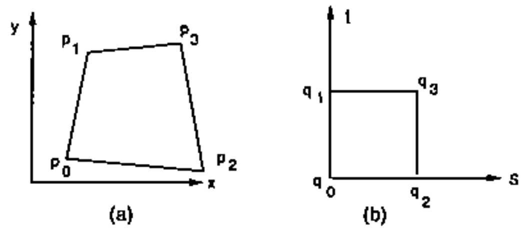

Figure 3.2: Quadrilateral and Unit Square

Then it follows from the second equality of (3.3) that $\alpha_1 = (1-\alpha_0)(1-t)$, $\alpha_2 = t(1-\alpha_0)$. Substituting these into the first equality of (3.3), we have $(1-s)(1-t)t(1-\alpha_0)^2 = s\alpha_0^2$. From which we get

$$\begin{cases} \alpha_0 = \dfrac{\sqrt{(1-s)(1-t)t}}{\sqrt{s}+\sqrt{(1-s)(1-t)t}} \\ \alpha_1 = \dfrac{\sqrt{s(1-t)^2}}{\sqrt{s}+\sqrt{(1-s)(1-t)t}} \\ \alpha_2 = \dfrac{\sqrt{st^2}}{\sqrt{s}+\sqrt{(1-s)(1-t)t}} \end{cases} \tag{3.4}$$

For $j-k=2n$(that is $j+k=2n+2k$),

$$\begin{aligned} B^{2m}_{2m-j-k,jk}(\alpha_0,\alpha_1,\alpha_2) &= \frac{(2m)!}{(2m-j-k)!j!k!}\alpha_0^{2(m-n-k)}\alpha_1^j\alpha_2^k \\ &= \frac{(2m)!}{(2m-j-k)!j!k!}\frac{s^{n+k}(1-s)^{m-n-k}t^{m-n}(1-t)^{m+n}}{(\sqrt{s}+\sqrt{(1-s)(1-t)t})^{2m}} \\ &= \frac{(m-n)!(n+k)!(m-k)!}{(2m-j-k)!j!k!}\frac{B^{m+n}_{n+k}(s)B^{2m}_{m-n}(t)}{(\sqrt{s}+\sqrt{(1-s)(1-t)t})^{2m}} \end{aligned}$$

Then

$$\begin{aligned} F(\alpha_0,\alpha_1,\alpha_2) &= \sum_{n=-m}^{m}\sum_{j-k=2n}\beta_{2m-j-k,jk}B^{2m}_{2m-j-k,jk}(\alpha_0,\alpha_1,\alpha_2) \\ &= \frac{1}{(\sqrt{s}+\sqrt{(1-s)(1-t)t})^{2m}}\sum_{n=-m}^{m}B_n(s)B^{2m}_{m-n}(t) \end{aligned} \tag{3.5}$$

It follows from the assumption of the theorem that, for a given s, $B_n(s) \le 0$ for $n < l$, $B_n(s) \ge 0$ for $n > l$ and there is a strict inequality in each of them. Then by Lemma 2.1 $F(\alpha_0,\alpha_1,\alpha_2) = 0$ has exactly one root $t \in (0,1)$ for any given $s \in (0,1]$. This (s,t) will gives us the unique $(\alpha_0,\alpha_1,\alpha_2)$ by (3.4). ◇

Just as Corollary 3.1 from Theorem 3.1, we have similarl corollaries from Theorem 3.2 and the Theorems 3.3, 3.4 below. We do not repeat them here.

Now we consider the algebraic curve $f(x,y) = 0$ defined on a four sided polygon $[p_0p_1p_2p_3]$ (see Figure 3.2). We shall characterize the coefficients (or weights) β_{ij} such that the curve $G(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n}\beta_{ij}B_i^m(u)B_j^n(v) = 0$ in the unit square S is a regular curve segment. This curve segment will be transformed to the given quadrilateral $[p_0p_1p_2p_3]$ (see Figure 3.2).

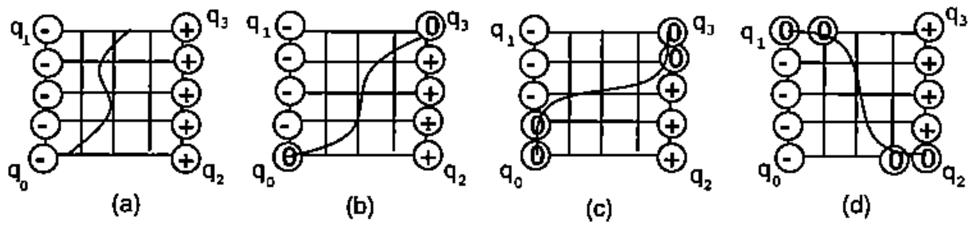

Figure 3.3: The D_3 regular curves

Theorem 3.3. *Let $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3]) = \{v = s : s \in [0,1]\}$, then $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3])$ is a discriminating family(see Figure 2.1(c)). Let $G(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n}\beta_{ij} B_i^m(u)B_j^n(v)$ and $B_i(s) = \sum_{j=0}^{n}\beta_{ij}B_j^n(s)$. If there exists an integer l $(0 < l < m)$ such that $B_i(s) \le 0$ for all $s \in [0,1]$ and $0 \le i < l$; $B_i(s) \ge 0$ for all $s \in [0,1]$ and $l < i \le m$ and $\sum_{i=0}^{l-1} B_i(s) < 0$, $\sum_{i=l+1}^{m} B_i(s) > 0$ on $(0,1)$, then the curve $G(u,v) = 0$ is $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3])$ regular.*

Similar conclusions holds for the $D_3([p_0p_1p_2p_3],[p_0p_2],[p_1p_3])$ discriminating family. The proof of the theorem is similar to the triangle case (see the proof of Theorem 3.1, we do not repeat it here).

The $D_3([p_0p_1p_2p_3],[p_0p_1],[p_2p_3])$ regular curves are between the line $[q_0q_1]$ and the line $[q_2q_3]$ (see Figure 3.3(a)). They can pass through q_0 or q_1 or q_2 or q_3(see Figure 3.3(b))and furthermore, the curves can be tangent with the edges at the vertices(see Figure 3.3(c–d)). The above theorem implies that if the weights have a one time sign change in the u or v direction, then curve inside the unit square is regular. The next theorem says that this conclusion is true, if the weights have one time sign change in the diagonal direction.

Theorem 3.4. *Let $D_4([p_0p_1p_2p_3],p_0,p_3) = \{(1-s)u(1-v)-s(1-u)v = 0 : s \in [0,1]\}$, then $D_4([p_0p_1p_2p_3],p_0,p_3)$ is a discriminating family. Let $B_k(s) = \sum_{i=0}^{m} C_m^i C_n^{k-i}\beta_{i,k-i}B_{n+2i-k}^{m+n}(s)$. If there exists an integer l $(0 < l < m+n)$ such that $B_k(s) \le 0$ for all $s \in [0,1]$ and $0 \le k < l$; $B_k(s) \ge 0$ for all $s \in [0,1]$ and $l < k \le m+n$, and $\sum_{k=0}^{l-1} B_k(s) < 0$, $\sum_{k=l+1}^{m+n} B_k(s) > 0$ on $(0,1)$, then the curve $G(u,v) = 0$ is $D_4([p_0p_1p_2p_3],p_0,p_3)$ regular in the unit square.*

Proof. $D_4([p_0p_1p_2p_3],p_0,p_3)$ consists of a family of hyperbolas (see Figure 2.1(d)). It is indeed a discriminating family. We now claim that every member in the family of hyperbolas will intersect the curve $G(u,v) = 0$ once and only once. When $s = 0$ or $s = 1$, the curve in that family degenerates to the boundary of the unit square, and our claim is trivially true. Now we suppose $s \in (0,1)$. Let

$$\begin{cases} s = \dfrac{u(1-v)}{u(1-v)+(1-u)v} \\ t = \dfrac{uv}{uv+(1-u)(1-v)} \end{cases} \tag{3.6}$$

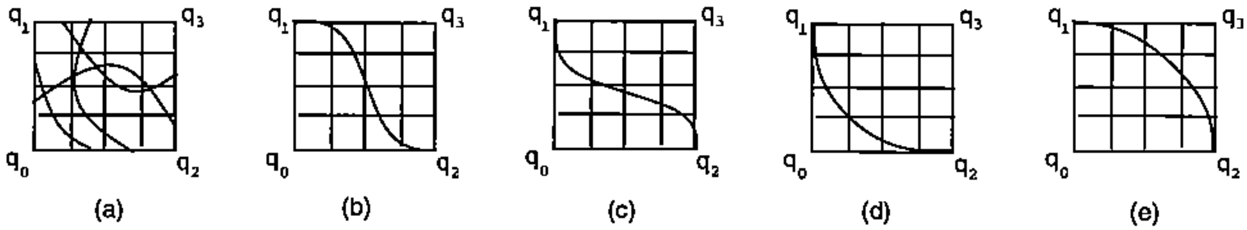

Figure 3.4: The $D_4([p_0p_1p_2p_3], p_0, p_3)$ regular curves

where $t \in (0,1)$. It is not difficult to show, from (3.6), that

$$\left\{ \begin{array}{rcl} u & = & \dfrac{\sqrt{st}}{\sqrt{st} + \sqrt{(1-s)(1-t)}} \\ v & = & \dfrac{\sqrt{(1-s)t}}{\sqrt{(1-s)t} + \sqrt{s(1-t)}} \end{array} \right. \tag{3.7}$$

Hence

$$\begin{aligned} G(u,v) &= \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{ij} B_i^m(u) B_j^n(v) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\beta_{ij} C_m^i C_n^j \sqrt{B_{n+i-j}^{m+n}(s)} \sqrt{B_{i+j}^{m+n}(t)}}{\sqrt{C_{m+n}^{i+j} C_{m+n}^{n+i-j}} (\sqrt{st} + \sqrt{(1-s)(1-t)})^m (\sqrt{(1-s)t} + \sqrt{s(1-t)})^n} \\ &= \sum_{k=0}^{m+n} \sum_{i=0}^{m} \frac{C_m^i C_n^{k-i} \beta_{i,k-i} \sqrt{B_{n+2i-k}^{m+n}(s)} \sqrt{B_k^{m+n}(t)}}{\sqrt{C_{m+n}^k C_{m+n}^{n+2i-k}} (\sqrt{st} + \sqrt{(1-s)(1-t)})^m (\sqrt{(1-s)t} + \sqrt{s(1-t)})^n} \\ &= \frac{1}{(\sqrt{st} + \sqrt{(1-s)(1-t)})^m (\sqrt{(1-s)t} + \sqrt{s(1-t)})^n} \sum_{k=0}^{m+n} \beta_k(s) \sqrt{B_k^{m+n}(t)} \end{aligned}$$

where $\beta_{ij} = 0$ for $j < 0$ or $j > n$ $\beta_k(s) = \sum_{i=0}^{m} C_m^i C_n^{k-i} \beta_{i,k-i} \sqrt{B_{n+2i-k}^{m+n}(s)} / \sqrt{C_{m+n}^k C_{m+n}^{n+2i-k}}$. Hence $G(u,v) = 0$ is equivalent to $\sum_{k=0}^{m+n} \beta_k(s) \sqrt{B_k^{m+n}(t)} = 0$. Under the assumptions of this theorem and Lemma 2.2, $\beta_k(s) \leq 0$ for $k = 0, 1, \ldots, l-1$, $\beta_k(s) \geq 0$ for $k = l+1, \ldots, m+n$ there exist strict inequalities for each set of inequalities. It follows from Lemma 2.1 that $G(u,v) = 0$ has a single root t in $(0,1)$ for any given $s \in (0,1)$. Hence the hyperbolic in D_4 has only one intersection with the curve $G(u,v) = 0$ in the unit square. The intersection point is defined by (3.7). Therefore the curve $G(u,v) = 0$ in the unit square is regular. ◇

The $D_4([p_0p_1p_2p_3], p_0, p_3)$ regular curves are between the point q_0 and point q_3 and away from q_0 and q_3(see Figure 3.4(a)). They can pass through q_1 or q_2 and can be tangent with the edges at the vertices(see Figure 3.4(b–e)).

4 Display of Regular Algebraic Curves

Displaying parametric curves is undoubtedly easier than implicit algebraic curves. Here we show that fast display (graphing) algorithms exist for our regular curve families. For general degree curves these algorithms depend on a root finding routine for real coefficients polynomials equation. If the degree of the polynomial is less than 5 (these are the most useful and important cases in CAGD), closed form solutions exist.

For the $D_1([p_0p_1p_2], p_2, [p_0p_1])$ regular curve defined by Theorem 3.1, we can evaluate the curve as follows: For a given $s \in [0,1]$, determine t as in the proof of Theorem 3.1, then use (3.2) to compute $(\alpha_0(s,t), \alpha_1(s,t), \alpha_2(s,t))$. To generate an ordered sequence of points on the curve for computer graphics display first choose a sequence $\{s_i\}$ $(0 = s_0 < s_1 < \cdots < s_l = 1)$, then compute $\{t_i\}$ and finally the points $\{(\alpha_0(s_i,t_i), \alpha_1(s_i,t_i), \alpha_2(s_i,t_i)\}$. Connecting these points by lines will give a piecewise linear approximation of the curves. For the $D_2([p_0p_1p_2], p_1, p_2)$ regular curve defined by Theorem 3.2, formula (3.4) gives a closed form for evaluating $(\alpha_0, \alpha_1, \alpha_2)^T$, where $s \in [0,1]$ is given arbitrarily and t is determined as in the proof of Theorem 3.2. For the $D_4([p_0p_1p_2p_3], p_0, p_3)$ regular curve defined by Theorem 3.4, we use (3.7) to evaluate the curve. The display of the $D_3([p_0p_1p_2p_3], [p_0p_1], [p_2p_3])$ regular curves defined by Theorem 3.3 are similar.

References

[1] Bajaj, C., (1993), "The Emergence of Algebraic Curves and Surfaces in Geometric Design" *Directions in Geoemtric Computing* ed. R. Martin, Information Geometers Press, 1 - 29.

[2] Bajaj, C., and Xu,Guoliang(1993), A-Splines: Local Interpolation and Approximation using C^k-Continuous Piecewise Real Algebraic Curves, Purdue University, Computer Science Technical Report, CAPO-92-44, 1992.

[3] Curry,H.B. and Schoenberg, I.J.(1966), On Pólya frequency functions IV: the fundamental spline functions and their limits, *J. d'Analyse Math.* 17, 71–107.

[4] deBoor, C.(1987), *A Practical Guide to Splines*, Springer Verlag, New York.

[5] Farin, G.(1990), *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide, Second Edition*, Academic Press Inc.

[6] Paluszny, M., Patterson, R. R. (1993) "A Family of Tangent Continuous Algebraic Splines" *ACM Transaction on Graphics* 12(3), 209-232.

[7] Paluszny, M., Patterson, R. R.(1992) Curvature Continuous Cubic Algebraic Splines, Proc. of SPIE, *Curves & Surfaces in Computer Vision and Graphics III*, Vol. 1830, 48-57.

[8] Patrikalakis N. M., Kriezis G. A.(1989), Representation of piecewise continuous algebraic surfaces in terms of B-splines, *The Visual Computer*, 5:360–374.

[9] Patterson, R. R.(1987), Parametric cubics as algebraic curves *Computer Aided Geometric Design* 5(1988) 139-159

[10] Sahney B. N.(eds)(1978), Polynomial and Spline-Approximation, D. Reidel Publishing Company.

[11] Sederberg, T.(1984), Planar piecewise algebraic curves, *Computer Aided Geometric Design*, 1(1984) 241-255.

[12] Sederberg, T., Anderson, D., Goldman, R.(1985), Implicitization, Inversion, and Intersection of Planar Rational Cubic Curves, *Computer Vision, Graphics and Image Processing*, 31, 89 - 102.

[13] Sederberg, T., Zhao, J., and Zundel, A.(1988), Approximate Parametrization of Algebraic Curves In *Theory and Practice of Geometric Modeling*, eds. W. Strasser and H-P. Seidel, Springer Verlag, 33-54.

[14] Singh S. P., Burry J. W. H. and Watson B.(ed)(1983), Approximation Theory and Spline Functions, D. Reidel Publishing Company.

[15] Xu, G. and C.L. Bajaj(1995), Regular Algebraic Curve Segments(II)–Interpolation and Approximation by Their Reduced Form, submitted.

[16] Xu, G. and C.L. Bajaj(1995), Regular Algebraic Curve Segments(III)–Data Fitting, submitted.