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Jörg Peters

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Jorg Peters

Department of Computer Sciences
Purdue University
West Lafayette, IN 47907-1398

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Decompositions of the identity and the construction of smooth surfaces

by

Jörg Peters†
jorg@cs.purdue.edu

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Abstract

Geometric modelers typically define surfaces as images of closed polygonal regions under polynomial or rational maps, called patches. The images, also called patches, do not overlap but join along curves in $\mathbb{R}^3$. Differential topologists define surfaces as domains of invertible maps, also called patches, from $\mathbb{R}^3$ to open sets in $\mathbb{R}^2$. The patches cover the surface by overlapping in open subsets. This paper develops a surface model that reconciles the apparent discrepancy between the constructive and the analytic approach by defining and characterizing maps that link the domains and ranges of the various types of patches. Of particular interest are families of maps whose composition matches the Taylor expansion of the identity map. Such families are named decompositions of the identity and its members roots of the identity.

† Department of Computer Science, Purdue University, W-Lafayette IN 47907-1398
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1. Introduction: Synthetic vs. analytic definition of surfaces

A popular approach to modeling surfaces on the computer is to assemble them from polynomial or rational patches* $p_k : \Delta_k \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ that map polygonal regions Δ_k to surface pieces in $\mathbb{R}^3$ (cf. Figure 1, left). Two pieces are joined by mapping one domain boundary of each to the same curve in $\mathbb{R}^3$. A globally consistent association of domain boundaries and curves then results in a quilt-like definition of the surface. The corresponding adjacency information is recorded as part of the so called boundary representation, short brep, of the surface. Standard data structures exist to maintain and modify such a brep (see e.g. [Mäntylä '88]).

In contrast, differential geometry conceptually starts with an existing surface viewed as a locally two-dimensional subset of $\mathbb{R}^3$ and considers as patches maps q_k from the surface to $\mathbb{R}^2$. The surface is defined by requiring that there be a collection of invertible patches that cover the surface; that is, for each surface point there exists an open neighborhood that is in the domain of some patch. Thus we have a local parametrization of the surface. To make this parametrization consistent, the surface definition requires additionally that if the domains of two patches q_l and q_{l-1} overlap, then the composite map $q_l \circ q_{l-1}^{-1}$ is invertible (cf [Hirsch '88, p11–12]). The patches together with their domains each form a chart and combine into an atlas of the surface.

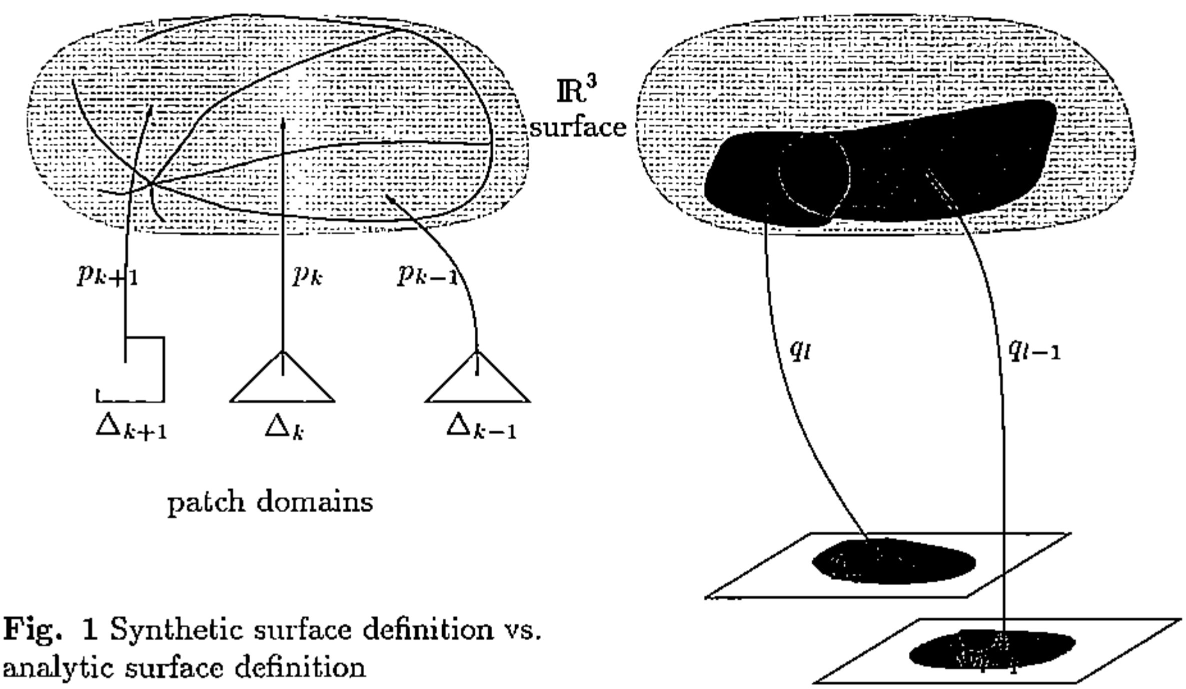

Fig. 1 Synthetic surface definition vs. analytic surface definition

The main inconsistency between the synthetic surface definition of the designer and the analytic surface definition of the geometer concerns the overlap of the sets involved. The modeler's patches map closed sets to closed sets that overlap in at most one-dimensional

* The word patch customarily describes both the map and the image of the map.

sets, whereas the inverses of the geometer's patches map open sets to open sets that overlap in open sets homeomorphic to a disk. The latter is convenient for checking the smoothness of the surface. If $q_l \circ q_{l-1}^{-1}$ is C^r as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ for all overlapping charts then the surface is a C^r manifold. In contrast, in the synthetic h model,, a first order smooth join between the patches is characterized as agreement of tangent planes (or their normals) along the common boundary of two patches. To capture the notion of tangent continuity algebraically we need for $l \in \{k-1, k\}$ the derivative $D_i p_l$ of the patch p_l in the direction of the ith Cartesian coordinate vector e_i and the domain edge E_l that is mapped under p_l to the common curve. With the cross product $\wedge$, tangent continuity between patch p_{k-1} and patch p_k can then be expressed as

$$\frac{D_1 p_k \wedge D_2 p_k}{\|D_1 p_k \wedge D_2 p_k\|}(E_k) = \frac{D_1 p_{k-1} \wedge D_2 p_{k-1}}{\|D_1 p_{k-1} \wedge D_2 p_{k-1}\|}(E_{k-1}).$$

If the patches and hence $D_i p_k$ and $D_i p_{k-1}$ are not known but need to be determined subject to matching data, then this characterization of smoothness is so nonlinear as to be impractical. Unless one patch is already known, the equivalent coplanarity condition $0 = \det[D_1 p_k(E_k), D_2 p_k(E_k), D_1 p_{k-1}(E_{k-1})]$ where we assume that e_1 is a direction transversal to E_{k-1}, is also not very useful. Fortunately, it is not difficult to show (see e.g. [Peters '90a]) that the agreement of normals can alternatively be written as continuity after reparametrization:

$$D_1 p_k(E_k) = D_1[p_{k-1} \circ \phi_{k,k-1}](E_k), \tag{1.1}$$

where $\phi_{k,k-1}$ is a reparametrization, mapping $\mathbb{R}^2$ to $\mathbb{R}^2$ (the map $\phi_{k,k-1}$ will be discussed in detail in the later sections). Now, if the reparametrization is fixed and the patches are polynomial, the continuity conditions are linear in the coefficients. Many surface constructions can therefore be characterized by their choice of reparametrization [Peters '90b].

To distinguish this geometric notion of continuity after reparametrization from the familiar smooth join of two function pieces characterized by the agreement of the derivatives across the common edge, say $D_1 p_k = D_1 p_{k-1}$ and $D_2 p_k = D_2 p_{k-1}$, the geometric modeling community uses the terms visual continuity, short VC^1 (see e.g. [Piper '87]) and geometric continuity, abbreviated either as GC^1 (see e.g. [Gregory '89]) or as G^1 (see e.g. [Höllig, '89]). The latter abbreviation now seems to be standard. The intended result of joining the patches in a G^r fashion is a C^r surface. But, except for Hahn's work [Hahn '89], there do not seem to be any attempts at reconciling the two notions. It may be for that reason that some geometric modelers also speak of G^r surfaces. One of the benefits of the joint synthetic and analytic model developed below is that it allows interpreting such surfaces as C^r surfaces.

The remainder of the paper is organized as follows. Section 2 reconciles the two surface definitions. Section 3 motivates assumptions that should be placed on the surface model in the context of geometric modeling. Of particular interest for constructing surfaces are maps that can be used to surround smoothly a point by three or more patches since without such connecting maps, we can only build surface strips. Sections 4 through 6 characterize suitable maps. Section 4 develops linear roots of the identity, while Section 5 and 6 characterize higher-oder decompositions. The characterizations are used in Section 7 to derive a lower bound on the degree of curvature continuous piecewise polynomial surfaces.

2. Synthetic and analytic surface definition reconciled

We reconcile the synthetic with the analytic surface model by forming a joint model. To this end consider again the left display of Figure 1. The synthetic model is incomplete since it does not provide for a domain of the reparametrization $\phi_{k,k-1}$ which maps from a neighborhood of E_k, excluding the interior of Δ_k, to Δ_{k-1}. This shortcoming can be remedied, in an *ad hoc* fashion, by adding a region adjacent to Δ_k to the model and tracing the direction of differentiation from the neighborhood of the domain of one patch to its neighbor under the map $\phi_{k,k-1}$ (see for example [Hahn '89, Def. 3.1], [Peters '92]). Note that it does not help to simply enlarge the patch domains. This does not provide the required open region of overlap homeomorphic to the disk, because the Taylor expansions of the two patches generally differ.

Extending the model as above works to explain smoothness under reparametrization but is awkward to deal with. The model is unsymmetric with one patch domain dominating, requires a cumbersome definition of the domain and action of the reparametrization $\phi_{k,k-1}$ and does not tie in with the analytic model. Hence we develop an alternative joint analytic-synthetic model below.

Figure 2.1 gives the gist of how the two surface definitions can be joined in a two-level model. In particular, we can define π_v to be a piecewise C^r map with pieces $\pi_{v,k}$ mapping from a wedge of the open set Ω_v into the domain of the kth patch p_k. Thus we have as a typical patch in the sense of the analytic surface definition the piecewise C^r map q_v defined by

$$q_v^{-1}(\Omega_{v,k}) := p_k \circ \pi_{v,k}(\Omega_{v,k})$$

where $\circ$ denotes composition, and as the reparametrization required in the synthetic model

$$\phi_{v,k-1,k} := \pi_{v,k} \circ \pi_{v,k-1}^{-1}.$$

We note that $\Omega_v = \bigcup_k \Omega_{v,k}$ is open even though $\Omega_{v,k}$ is not and that the definition of $\phi_{v,k-1,k}$ requires a larger domain than just $\Delta_{v,k-1}$. This is expressed in Figure 2.1 by the partial boundary curves emanating from the vertices of the domain. The patches p_k of the synthetic definition are preserved and first order continuity under the reparametrization π_v along the edge E_k can be expressed as

$$D_{E_k^\perp}[p_k \circ \pi_{v,k}](E_k) = D_{E_k^\perp}[p_{k-1} \circ \pi_{v,k-1}](E_k). \tag{2.1}$$

A close look shows that the joint model largely agrees with Hahn's second definition of geometric patch complexes implicit in [Hahn '89, Theorem 7.3]. Hahn however, uses neighborhoods of the patch domains Δ_k as his starting point, and maps these to a joint domain; that is, his maps π_k have the opposite orientation. To construct this joint vertex domain

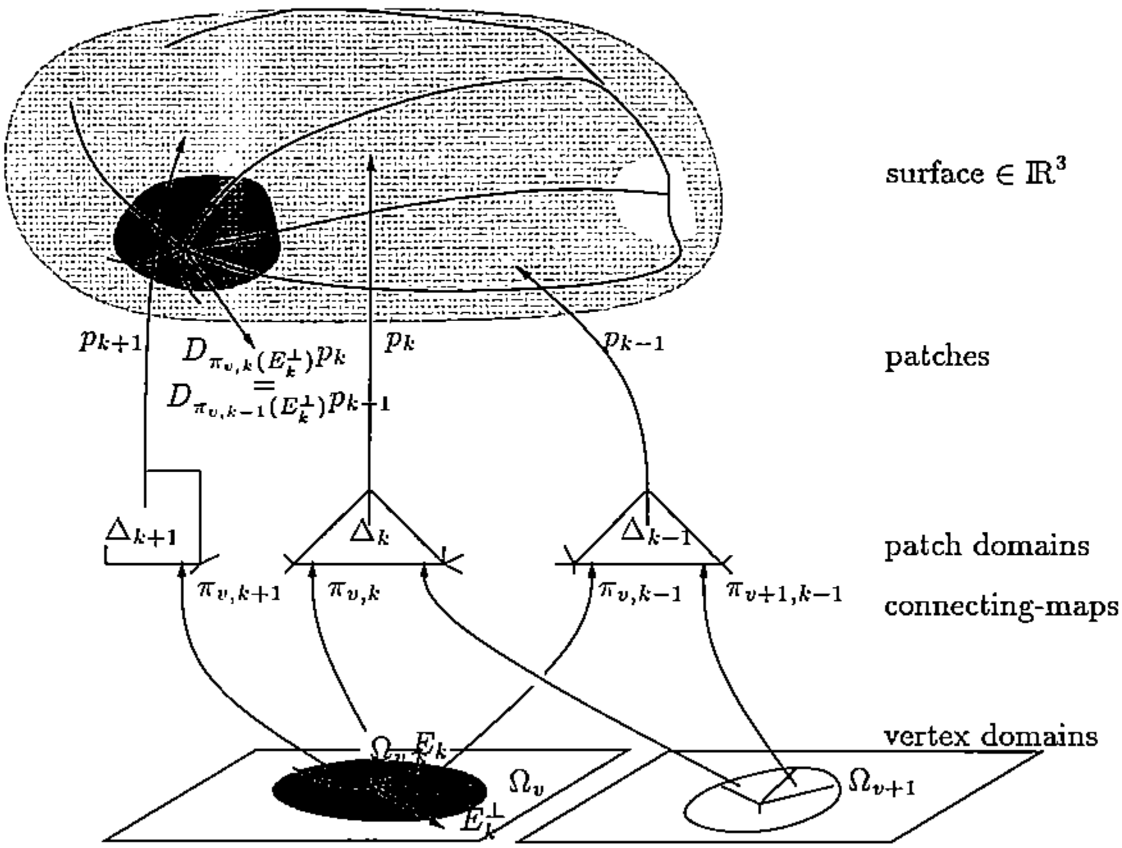

Fig. 2.1 Joint analytic-synthetic surface model.

without overlap or gap, a number of conditions have to be placed on the maps π_k. In contrast, starting with the vertex domains and considering a single piecewise map π, these condition are naturally satisfied. Further advantages of the point of view summarized in Figure 2.1 are as follows. The model is symmetric in that both patches are reparametrized (cf. Equation 2.1 compared to Equation 1.1). The constraint that the Taylor expansion of the composition of all reparametrizations $\phi_{k,k-1}$ equals the expansion of the identity map up to rth order follows directly from $\phi_{v,k} := \pi_{v,k} \circ \pi_{v,k-1}^{-1}$ whereas it requires a theorem in Hahn's approach [Hahn '89, Theorem 7.1]. The joint model allows for singular and unfaithful parametrizations. These can be ruled out by the same constraints on tangent sectors that are *required* to make Hahn's model consistent.

We will now refine the joint analytic-synthetic model to create an atlas. While it is possible to extend the vertex-neighborhoods to overlap on the interior $p_k(\Omega_k)$ and $p_{k-1}(\Omega_{k-1})$, we instead refine the model to capture the construction sequence common to most surfacing techniques. The detailed model is illustrated in Figure 2.2.

1. The first step is to join patches locally around a vertex to form a smooth piecewise map defined up to the rth order Taylor term, i.e. to determine for $k = 1..n$ the maps the rth order expansion of $p_k \circ \pi_{v,k}(\Omega_{v,k})$.
2. The second step is to connect these vertex neighborhoods along edges with a Hermite

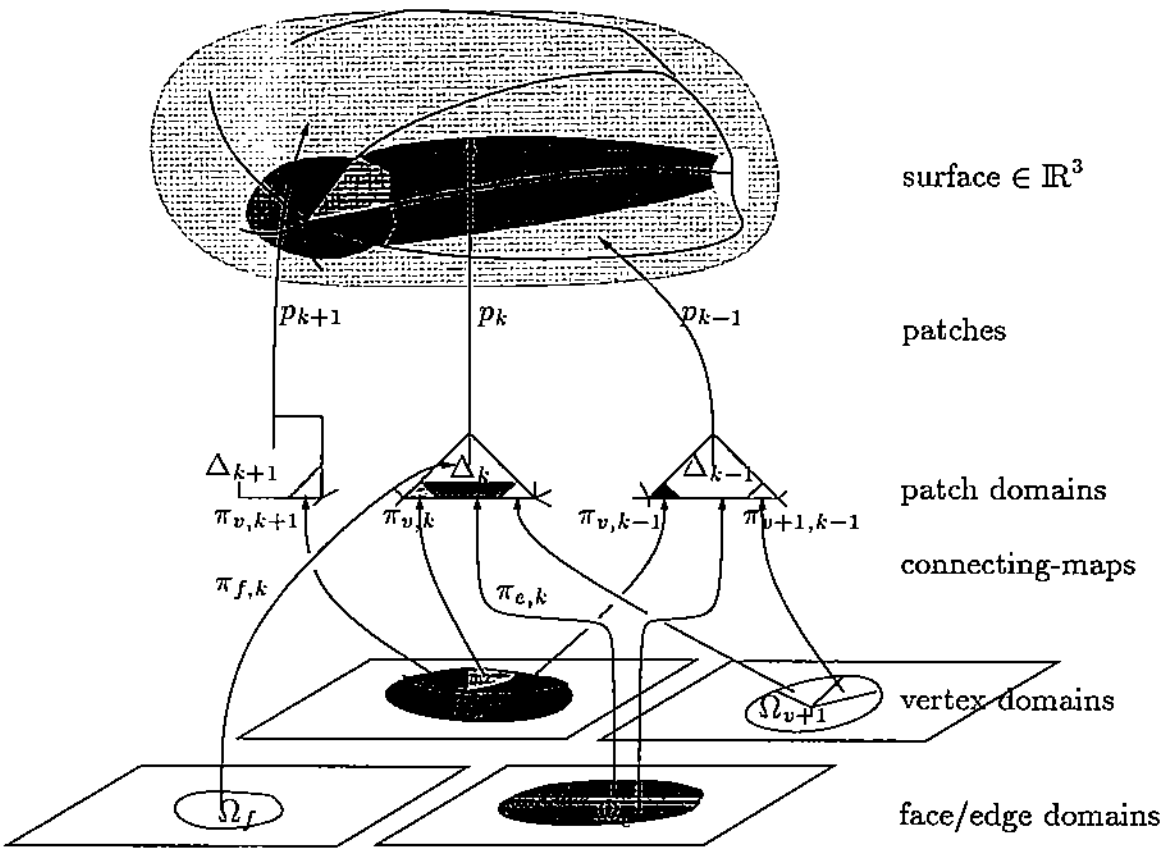

Fig. 2.2 A detailed model of piecewise parametric surface construction and decomposition

interpolant to the expansions computed earlier. That is, for $l = k, k-1$, the rth order expansion perpendicular to the edge of $p_l \circ \pi_{e,l}(\Omega_{e,l})$ is computed.

3. The final step completes the definition of each patch by determining the Hermite interpolant $p_k \circ \pi_{k,f}(\Omega_{k,f})$ to the expansions along the boundaries.

Even though the images of the partial maps may have boundaries, they overlap if they overlap also in open sets homeomorphic to disks as required. In fact within each surface piece the overlap is C^∞.

3. Roots of the identity and the degree of smooth surfaces

In the previous section, we observed that the definition $\phi_{v,k,k-1} := \pi_{v,k} \circ \pi^{-1}_{v,k-1}$ for the pieces of the C^r map π_v implies that the composition of all reparametrizations $\phi_{v,k,k-1}$ at a vertex v matches the expansion of the identity map up to rth order. Hence we call $\phi_{v,k,k-1}$, $k = 1..n$ modulo n, *roots of the identity.*

The expansion of the roots strongly influences the degree of the corresponding surface construction: in step 2 of the construction sequence the reparametrizations $\pi_{e,l}$, $l = k-1, k$, that connect two patches across an edge e from v to v', Hermite-interpolate the expansions of $\pi_{v,l}$ and $\pi_{v',l}$ up to rth order in order to construct rth order smooth surfaces. A typical condition for the first-order smooth match of the corresponding patches $p := p_{e,k-1}$ and $q := p_{e,k}$ is (cf. [Gregory '89])

$$\lambda D_1\gamma = D_2p + D_2q \tag{G$_1$}$$

where D_2p and D_2q are trasnsversal derivatives in the direction of the second coordinate vector, $\lambda := D_1D_2\pi^{[1]}_{e,l}$ the first component of the mixed derivative of the reparamtrization and $\gamma(u) = p(u,0) = q(u,0)$ is the boundary curve common to both patches. If γ is a polynomial of degree d_γ and λ is a polynomial of degree d_λ, then p and q must be represented as polynomials of degree $d_\gamma + d_\lambda$. Similarly the second-order conditions symmetric in p and q with $\mu := D_1D_2\pi^{[2]}_{e,l} = 1$,

$$D_2^2p - D_2^2q - \lambda D_1D_2p + \lambda D_1D_2q = \frac{1}{2}(D_2q - D_2p)D_2^2\phi^{[2]}, \tag{G$_2$}$$

imply that $-\lambda D_1D_2p + \lambda D_1D_2q$ and hence $D_2^2p - D_2^2q$ must have a representation of degree $d_\gamma + 2d_\lambda - 2$, i.e. p and q of degree $d_\gamma + 2d_\lambda$.

Choosing $\pi_{e,l}$ of degree $2r$ allows us to write down a Hermite interpolant without knowledge of the expansions. However, often we can choose the expansions of the roots of the identity at the end points to reduce the degree of the Hermite interpolant. For example, for a tensor-product arrangement of the patches, the expansions can be chosen to allow the identity map as the Hermite interpolant. The overall degree of the surface can then be $O(r)$ whereas, if the degree of $\pi_{e,l}$ is $O(r)$ then the overall degree of the surface will be $O(r^2)$. This motivates the study of the constraints on roots of the identity in the remaining sections of this paper.

4. Localness and coordinate systems

In this and the remaining sections, we concentrate on piecewise C^r maps π_v whose pieces $\pi_{v,k}$, $k = 1..n$, map a wedge $\Omega_{v,k}$ of the neighborhood of the vertex v to the neighborhood of a cornerpoint of the polygonal patch domain $\Delta_k \in \mathbb{R}^2$. Since the vertex is fixed we drop the subscript v indicating the vertex and count the other subscript k modulo n.

To define a piecewise surface, two types of data are required: the connectivity of the patches, and geometric data determining the position of the surface in space. For geometric design it is desirable to have the average of two G^r-connected patch complexes with the same connectivity be again a G^r-connected patch complex. That is, G^r-connected patch complexes with the same connectivity should form a vector space. For this it is necessary and sufficient to choose the reparametrizations π independently of the geometric data. Sufficiency is due to the linearity of differentiation (cf. [Peters '93]) while necessity follows from the fact that already the average of two piecewise C^1 curves joined with different reparametrizations is in general not a C^1 curve under any reparametrization. Having the reparametrization depend only on the connectivity of the patches has the additional advantage of splitting the modeling problem into a connectivity problem solved by the reparametrizations π_v and a geometry problem solved by the patches p_k. We therefore proceed under the stipulation that the reparametrizations π depend only on the connectivity of the patches and not on the geometry.

A second requirement for effective geometric modeling is the ability to keep the construction local so that changes in the connectivity (or, for the patches p_k, in the geometry) do not necessarily propagate across the whole surface. Thus, if we construct a directed acyclic reachability graph whose edges are the edges connecting, say a vertex v of the piecewise surface with its neighbor vertices, then π_v should depend on the valence of the nodes in the graph only up to a certain fixed depth δ. Similarly edge and face reparametrizations should only depend on the data close by. In other words, the reparametrizations π depend only on the local connectivity of the patches.

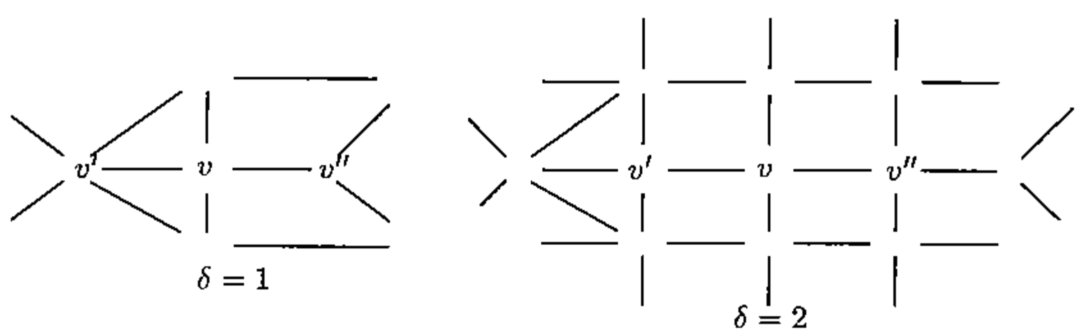

Fig. 4: The connectivity in a δ-neighborhood of v is uniform. The connectivity in a δ neighborhood of v' and v'' differs.

In the subsequent sections, we characterize the maps π_k and $\phi_{k-1,k}$ for C^r constructions by exhibiting constraints on their Taylor-expansion. For this it suffices to look at maps whose Jacobian depends on a $\delta = 1$ neighborhood; that is, it depends up to first order only on the number of neighbors at vertex. This can be seen as follows. If $\delta = 1$, then all maps joining at the vertex are treated equally, because the only information defining them is that n maps join. If $\delta > 1$, say $\delta = 2$, then we can always pick a mesh so that the neighbors $v_1, \ldots, v_n$ are indistinguishable as in the case $\delta = 1$ and we should just as well use equal roots of the identity. Figure 4 illustrates this for the case $n = 4$. Section 4 shows that dependence on a $\delta = 1$ neighborhood at a vertex implies that the Jacobians $D\pi_k(o)$ for $k = 1..n$ differ only by a rotation. As Lemma 5.1 shows, this determines the linear part of π_k (and hence the linear part of $\phi_{k-1,k}$) completely.

When we work only locally at a vertex, we may assume without loss of generality, that π_k maps the edges of the wedge Ω_k to the Cartesian coordinate vectors e_1 and e_2 respectively. Treating Δ_k as if it has a right angle is convenient so that we do not have to distinguish between different patch domains. Clearly not every patch domain has all right angles. So once we relate the local expansion along an edge by Hermite interpolation, the expansion is appropriately transformed by imposing a joint coordinate system.

5. Linear factors of the identity

This section characterizes the linear part of any map π_k. Let E_{k-1} and E_k be the two edges of Ω_k at o (in clockwise order), θ_k the angle formed by the two edges and e_1 and e_2 be the two edges of Δ_k under π_k. Then the linear part of π_k can be factored into a rotation R_k that maps E_{k-1} to e_1, followed by a skew transformation $\tilde{\pi}_k$ which maps E_k to e_2 while leaving the first edge unchanged.

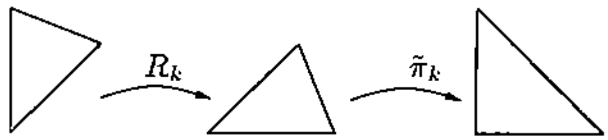

Fig. 5.1: Decomposition of π_k

Observation 5.1. *If $\sum_{k=1}^n \theta_k = 2\pi$, then $D\pi_k = \tilde{\pi}_k R_k$ where*

$$R_k := \begin{bmatrix} \cos(\sigma_k) & \sin(\sigma_k) \\ -\sin(\sigma_k) & \cos(\sigma_k) \end{bmatrix}, \qquad \tilde{\pi}_k := \begin{bmatrix} 1 & -c_k/s_k \\ 0 & 1/s_k \end{bmatrix},$$

and $c_k := \cos(\theta_k), s_k := \sin(\theta_k), \sigma_k := \sum_{l=1}^{k-1} \theta_l$.

With $R_{k,k-1} := R_k R_{k-1}^{-1}$, we can rewrite the reparametrization

$$\phi_{k-1,k} = \pi_k \circ \pi_{k-1}^{-1} = \tilde{\pi}_k R_{k,k-1} \tilde{\pi}_{k-1}^{-1} = \begin{bmatrix} 1 & -c_k/s_k \\ 0 & 1/s_k \end{bmatrix} \begin{bmatrix} c_{k-1} & s_{k-1} \\ -s_{k-1} & c_{k-1} \end{bmatrix} \begin{bmatrix} 1 & c_{k-1} \\ 0 & s_{k-1} \end{bmatrix}$$
$$= \begin{bmatrix} c_{k-1} + c_k s_{k-1}/s_k & 1 \\ -s_{k-1}/s_k & 0 \end{bmatrix}$$

Lemma 5.2. *The linear components of $\phi_{k-1,k} := \pi_k \circ \pi_{k-1}^{-1}$, $k = 1..n$ form a decomposition of the linear part of the identity.*

Proof By the chain rule and because $\sum_{k=1}^n \theta_k = 2\pi$,

$$\begin{aligned} D(\circ_{l=1}^n \phi_l) &= \prod_{l=1}^n D\phi_l \\ &= \tilde{\pi}_n R_{n,n-1} \tilde{\pi}_{n-1}^{-1} \tilde{\pi}_{n-1} R_{n-1,n-2} \cdot \ldots \cdot \tilde{\pi}_0^{-1} \\ &= \tilde{\pi}_n R_{n,n-1} R_{n-1,n-2}, \cdot \ldots \cdot R_{1,n} \tilde{\pi}_n^{-1} \\ &= I = D \text{ id} \end{aligned}$$

where I is the identity matrix as required. ■

Definition 5.3. *The members of a family of reparametrizations $\phi_{k,k-1} := \phi_k \circ \phi_{k-1} = \pi R \pi^{-1}$, $\quad k = 1..n(mod n)$ that do not depend on the index are called* uniform roots *of the* *identity. If* $(D\phi(o))^j(x) \neq (D\phi(o))^l(x)$ *for* $j \neq l$, $x > o$ *then the uniform roots are said to* not overlap.

Lemma 5.4. *Uniform roots of the identity without overlap have the Jacobian*

$$D\phi(o) = \begin{bmatrix} 2c & 1 \\ -1 & 0 \end{bmatrix}$$

where $c := \cos(2\pi/n)$.

Proof Since $(D\pi(o))^n = I$, $D\pi(o)$ is a unitary matrix. The non-overlap condition rules out rotations other than by $2\pi/n$. ■

Note that while the linear part of the maps π_k and $\phi_{k-1,k}$ is now characterized in the uniform case, all higher order terms are unrestricted. The following sections consider subclasses of these first-order uniform maps constrained by higher smoothness of the piecewise map π.

6. Higher-order roots of the identity at a point with four neighbors

In this section we consider a vertex surrounded by $n = 4$ patches. We assume that the reparametrizations π_k depend up to first order only on the number of neighbors at vertex, i.e. are first-order uniform. Higher-order terms may differ depending on the index k.

$$\tilde{\pi}_k^{-1}(u,v) := \begin{bmatrix} u + a_{1k}u^2 + b_{1k}uv + c_{1k}v^2 \\ v + a_{2k}u^2 + b_{2k}uv + c_{2k}v^2 \end{bmatrix} + O(u^3, u^2v, uv^2, v^3)$$

for some scalar constants a_i, b_i, c_i. Expanding the identity map id $= \tilde{\pi}_k^{-1} \circ \tilde{\pi}_k$, we obtain

$$\tilde{\pi}_k(u,v) = \begin{bmatrix} u - a_{1k}u^2 - b_{1k}uv - c_{1k}v^2 \\ v - a_{2k}u^2 - b_{2k}uv - c_{2k}v^2 \end{bmatrix} + O(u^3, u^2v, uv^2, v^3).$$

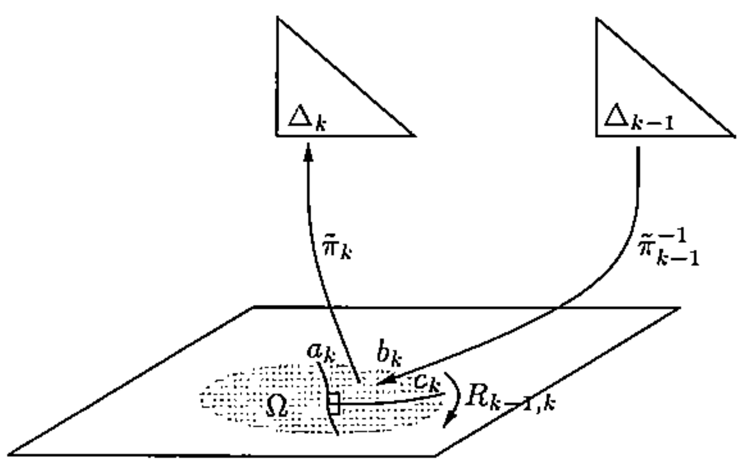

Fig. 6: Decomposition of $\phi_{k-1,k}$

Lemma 6.1. *If $n = 4$ and the reparametrizations π depend up first order only on the number of neighbors at vertex then $D_1 D_1 \phi_k = D_1 D_1 \phi_{k+2}$. If additionally*

$$D_1^m D_j^n D_2 \phi^{[2]}_{k-1,k} = 0 \text{ for } m, n \geq 0, \tag{6.2}$$

then $D_2 D_1 \phi^{[1]}_k = D_2 D_1 \phi^{[1]}_{k+2}$.

Proof Consider Figure 6. Set $U = \tilde{\pi}^{[2]}_{k-1}(u,v)$ and $V = -\tilde{\pi}^{[1]}_{k-1}(u,v)$, where $\tilde{\pi}^{[i]}$ denotes the ith component of $\tilde{\pi}$. Since the boundary curves of the vertex domain have to match and $\tilde{\pi}^{-1}_{k-1}(0,v) = \tilde{\pi}^{-1}_k(u,0)$, we have

$$c_{2,k-1} = a_{1k}, \quad c_{1,k-1} = -a_{2k}.$$

Substituting also $c_{2k} = a_{1,k+1}$ and $c_{1k} = -a_{2,k+1}$, the Taylor expansion of $\phi_{k-1,k}(u,v) = \pi_k \circ R_{k-1,k} \circ \pi^{-1}_{k-1}(u,v) = \phi_{k-1,k}(U,V)$ is

$$\begin{bmatrix} v + (a_{2,k-1} + a_{2,k+1})u^2 + (b_{1k} + b_{2,k-1})uv \\ -u - (a_{1,k-1} + a_{1,k+1})u^2 + (b_{2k} - b_{1,k-1})uv \end{bmatrix} + O(u^3, u^2 v, uv^2, v^3).$$

Since $\phi_{k+3} = \phi_{k-1}$ for $n = 4$, $a_{2,k-1} + a_{2,k+1} = a_{2,k+1} + a_{2,k+3}$ and hence $D_1 D_1 \phi_k = D_1 D_1 \phi_{k+2}$. Constraint (6.2) implies $D_2 D_1 \phi^{[2]}_k = b_{2k} - b_{1,k-1} = 0$ and hence $D_2 D_1 \phi^{[1]}_k = b_{2,k+1} + b_{2,k-1} = D_2 D_1 \phi^{[1]}_{k+2}$. ■

The vanishing of $D_1^m D_j^n D_2 \phi^{[2]}_{k-1,k}$ corresponds to a symmetry across edges that helps keep the number of constraints low. Under this condition Lemma 6.1 asserts that the mixed derivatives of the reparameterization $\phi_{k,k-1}$ cannot be chosen independently but must agree in alternate pairs. This in turn implies that the polynomial λ in equation G_1 in Section 3 cannot be chosen linear; thus p and q satisfying the G_2 constraints must have a representation of at least degree $d_\gamma + 4$. If the common boundary curve is quartic, this yields curvature continuous surfaces of degree eight or higher.

The statement can be generalized as follows. If $n = 4$, the reparametrizations π depend up to first order only on the number of neighbors at vertex, and if the terms of the Taylor-expansion beyond the linear term are zero up to $r-1$st order, then for the homogenous polynomial $[a_l(u,v)]^{r-1} := \sum_{i+j=r-1} a_{lj} u^i v^j$ and $a_l \in \mathbb{R}^{r-1}$, $b_l \in \mathbb{R}$,

$$\phi_l := \phi_{l-1,l} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + u \begin{bmatrix} [a_l(u,v)]^{r-1} \\ b_l u^{r-1} \end{bmatrix} + h.o.t.$$

Proposition 6.3. *If $n = 4$ and the reparametrizations π depend up to first order only on the number of neighbors n at vertex, and if the terms of the Taylor-expansion beyond the linear term are zero up to $r-1$st order, then the Taylor expansion of $\circ_{l=1}^4 \phi_l$ agrees with the Taylor expansion of the identity up to order r if and only if*

$$a_{l,j} = (-1)^{r-1} a_{l+2,j}, \quad b_l = (-1)^{r-1} b_{l+2}, \quad \text{for } l \in \{0,1\}, j \in \{1, \ldots, r-1\}.$$

Proof Consider patches p_0, p_1, p_2, p_3 with u and v the parameters of p_0. To indicate truncation of the Taylor expansion after the rth order term, we use $\approx$ in place of $=$ and compute

$$\begin{aligned}
u_1 &\approx v + u[a_0(u,v)]^{r-1}\\
v_1 &\approx -u + b_0 u^r\\
u_2 &\approx v_1 + v[a_1(v,-u)]^{r-1}\\
v_2 &\approx -u_1 + b_1 v^r\\
u_3 &\approx v_2 - u[a_2(-u,-v)]^{r-1}\\
v_3 &\approx -u_2 + b_2(-u)^r\\
u_0 &\approx v_3 - v[a_3(-v,u)]^{r-1}\\
v_0 &\approx -u_3 + b_3(-v)^r
\end{aligned}$$

Since $u = u_0$ and $v = v_0$, the last equation implies

$$\begin{aligned}
0 &= -v[a_3(-v,u)]^{r-1} + b_2(-u)^r - v[a_1(v,-u)]^{r-1} - b_0 u^r,\\
0 &= b_3(-v)^r + u[a_2(-u,-v)]^{r-1} - b_1 v^r + u[a_0(u,v)]^{r-1}.
\end{aligned}$$

Setting the coefficients of the monomials to zero proves the proposition. ■

Lemma 6.1 is a special case of Proposition 6.3 for $r = 2$. Since the proof of the proposition involves only the rth order Taylor-expansion of the reparametrizations, it holds also if $\tilde{\pi}_k$ is a piecewise C^r map. If the pieces result from subdividing $\tilde{\pi}_k$ (see for example the construction in [Piper '87]) then the pieces inherit the restrictions.

Corollary 6.5. *If $n = 8$ and each pair of maps $\tilde{\pi}_{2k}$, $\tilde{\pi}_{2k+1}$, $k = 1..4$, is obtained by subdividing one smooth map, then Proposition 6.3 applies to the even numbered maps.*

7. Uniform roots of the identity

In this section we consider the vertex neighborhood of a vertex v with an unrestricted number of neighbors n but a uniform decomposition characterized by $\tilde{\pi}_k = \tilde{\pi}_{k-1}$ for $k = 1..n$ modulo n. That is, the maps π_k differ only by a rotation. (Earlier, we assumed that the linear part of the maps differs just by a rotation.)

We restate without proof Theorem 4 of [Peters '94].

Theorem 7.2. *If the reparametrizations π depend up to first order only on the number of neighbors at vertex, then*

$$\begin{aligned}
\phi :=& \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}\\
&+ \frac{1}{2}\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ b_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_1\\
&+ \frac{1}{2}\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_2\\
&+ h.o.t.
\end{aligned}$$

(C.2) holds for any choice of the scalar constants a_1, a_2, b_1 if $n \neq 3$. If $n = 3$, then $a_1 = a_2 = -2b_1$ is necessary and sufficient.

We con lentity.

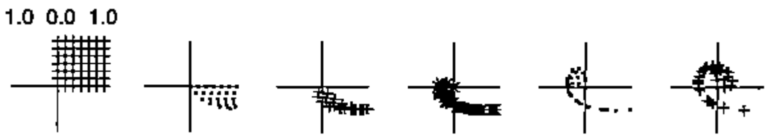

Figure 7.1: Five applications of a second-order uniform root of the identity corresponding to $n = 5$ and $a_1 = 1 = a_2$ and $b_1 = 0$, on equally spaced points in the positive quadrant.

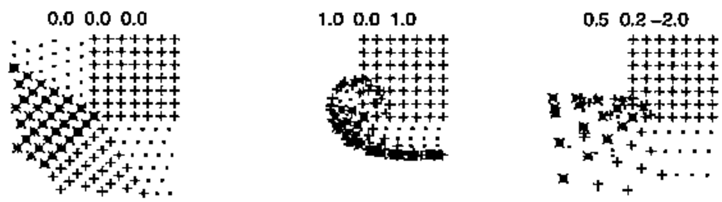

Figure 7.2: Action of three second-order uniform fifth roots of the identity on the positive quadrant.

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Jörg Peters
Department of Computer Science
Purdue University
W-Lafayette IN 47907-1398
USA
e-mail: jorg@cs.purdue.edu
http://www.cs.purdue.edu/people/jorg