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Abstract. The convergence rate of a numerical procedure based on *Schwarz Alternating Method (SAM)* for solving elliptic boundary value problems (BVPs) depends on the selection of the so called *interface conditions* applied on the interior boundaries of the overlapping subdomains. It has been observed that the weighted mixed interface conditions ($g(u) = \omega u + (1-\omega)\frac{\partial u}{\partial n}$), controlled by the parameter ω, can optimize SAM's convergence rate. In this paper, we present a matrix formulation of this method based on finite difference approximation of the BVP, review its known computational behavior in terms of the parameter $\alpha = \phi(\omega, h)$, where h is the discretization parameter and ϕ is a derivable relation, and obtain analytically explicit and implicit expressions for the optimum α. Moreover, we consider a parameterized SAM where the parameter ω or α is assumed to be different in each overlapping area. For this SAM and the one-dimensional elliptic model BVPs, we determine analytically the optimal values of α_i. Furthermore, we extend some of these results to two-dimensional elliptic problems.

Key words. elliptic partial differential equations, Schwarz alternating method, Jacobi, Gauss-Seidel, SOR iterative methods

AMS subject classifications. 65N35, 65N05, 65F10

1. Introduction.

Numerical realizations of the classical mathematical approach *Schwarz Alternating Method* (*SAM*) [23] have been recently explored as parallel computational frameworks for the solution of boundary value problems (BVPs). These methods are based on a decomposition of the BVP domain into ovelapping subdomains. The original BVP is reduced to a set of *smaller* BVPs on a number of subdomains with appropriate *interface conditions* on the interior boundaries of the ovelapping areas, whose solutions are coupled through some iterative scheme to produce an approximation of the solution of the original BVP. It is known [2], [10] that under certain conditions the sequence of the solutions of the subproblems converges to the solution of the original problem.

One of the objectives of this research is to study a class of *SAM* whose interface conditions are parameterized and estimate the values of the parameters involved that speed up the convergence of these methods for a class of BVPs. Following, we review some related studies and point out the contributions of the analysis presented in this paper.

In the context of elliptic BVPs the most commonly used *interface conditions* are of Dirichlet type. For this class of numerical SAM several convergence studies exist including the following [15], [17], [21], [22], [19]. In particular, it has been observed [3], [16], [24] that for model problems with Dirichlet interface conditions and a fixed aspect ratio of the overlapping area over the subdomains, the rate of convergence of numerical SAM does not depend on the mesh size. In [25] it is stated that the above property does not hold for mixed interface conditions. However, our investigation has shown that there are one-dimensional (1-D) BVPs where the rate of convergence does not change with the mesh size even for *mixed type interface conditions* with appropriately

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chosen convex combinations of Dirichlet and Neumann boundary conditions.

In [18],convergence results (not explicit formulas) are presented for SAM based on k-way ($k \geq 2$) decompositions of 2-D BVPs with Dirichlet interface conditions and Jacobi and/or Gauss-Seidel inner/outer iterative schemes. It turns out that the regular splitting theory employed in [18] for the classical SAM with Dirichlet interface conditions is not applicable for parameterized SAM with mixed boundary conditions.

The effect of parameterized mixed interface conditions has been considered by a number of researchers [4], [20], [9], [25] and some of the references cited in them. With the exception of [25], these works carry out the SAM analysis at a functional level.

Specifically, [4] deals with 1-D and 2-D BVPs assuming a 2-way domain decomposition, where the values of the approximate solution along the two artificial boundaries are linear combinations of the two previous available ones (iterations). The theoretical and experimental results obtained in [4] for the 1-D case are weaker than the ones presented in this paper. According to this analysis the values of the optimal convergence factor are ranging from 0.339 to 0.887 (third column of Table 1 in [4]). Our analysis has produced a convergence factor of value *zero* (spectral radius of the block Jacobi iteration matrix). In [20] SAM is applied on 2- and 3-way decompositions of 2-D BVPs. Although mixed interface conditions are allowed, they are restricted to cases of Dirichlet/Dirichlet, Dirichlet/Neumann and Neumann/Neumann only. In our analysis general mixed interface conditions without restrictions are assumed.

In [25], it is shown experimentally that an appropriate choice of the parameter ω relating the weights between the Dirichlet and the Neumann conditions allows one to optimize the convergence rates of the numerical SAM based on finite difference discretization of a Poisson type BVP. This study is based on a matrix formulation of the parameterized SAM where the weighted mixed interface conditions are imposed through the parameter $\alpha = \phi(\omega, h)$ with h being the discretization parameter. In this paper, we derive the relation ϕ and obtain analytically explicit and implicit expression for the parameter α.

In [9], a multi-parameter SAM is formulated in which the mixed weighted interface conditions are controlled by a different parameter (ω_i) in the i-th overlapping area. In this paper we formulate a multi-parameter *SAM* at the matrix level where the parameters α_i are used to impose mixed interface conditions. In [9], Fourier analysis is applied to determine the values of ω_i parameters that make the convergence factor of SAM be zero. In our analysis we were able to determine analytically the optimal values of α_i's for 1-D BVPs, which minimize the spectral radius of the block Jacobi iteration matrix associated with the *enhanced* SAM matrix. Finally, we extend the formulation of multi-parameterized SAM and some of the corresponding 1-D results for 2-D elliptic BVPs.

This paper is organized as follows. In Section 2 we provide the matrix formulation of the one-parameter SAM for 1-D elliptic BVPs and study its convergence based on the Jacobi iteration. This analysis is reduced to calculating the spectral radius of the Jacobi iteration matrix corresponding to the Schwarz enhanced matrix [24]. The optimal value of the parameter α is determined so that the Jacobi spectral radius is minimized. In Section 3, we present a matrix formulation of a multi-parameterized numerical SAM whose mixed interface conditions in each subdomain are controlled by different parameters. The values of these parameters are determined so that the spectral radius of the Jacobi iteration matrix of the enhanced multi-parameterized SAM is as small as possible. In addition, in Section 4, we list some numerical data that indicate that the one-parameter SAM is faster than SAM but slower than the

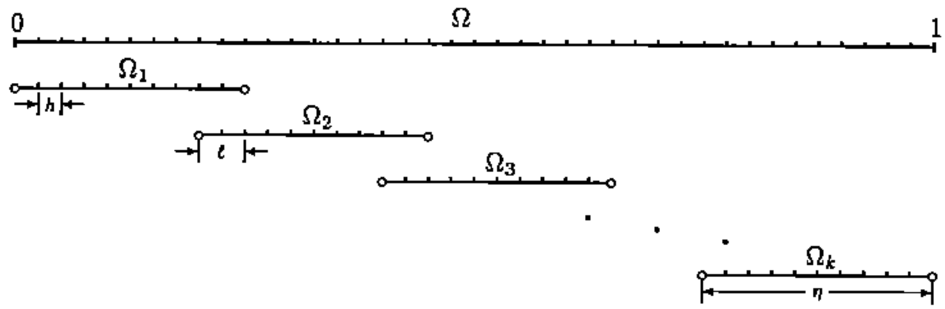

FIG. 1. *1-D overlapping domain splitting.*

multi-parameter SAM. Finally in Section 5, we extend the multi-parameter SAM to 2-D elliptic BVPs and derive implicit formulas for the optimal convergence of the Jacobi iteration based multi-parameter SAM. These results are supported by some numerical experiments.

2. One-Parameter SAM (1PSAM).

We consider the two-point BVP

$$Lu \equiv -u''(t) + q\,u(t) = f(t),\ t \in (0,1),\quad Bu \equiv u(0) = a_0,\ Bu \equiv u(1) = a_1 \tag{1}$$

with $q \geq 0$ being a constant and formulate a numerical instance of SAM based on a k-way splitting of the unit interval and finite difference discretizations of the local BVP over each subdomain with mixed interface conditions

$$g(u) = \omega u + (1-\omega)\frac{\partial u}{\partial n} \tag{2}$$

on the interior boundaries.

Let $T_j(a,b,c)$ denote the tridiagonal $j \times j$ matrix whose diagonal entries are b except that its first and last diagonal elements are a and c, respectively, i.e.,

$$T_j(a,b,c) = \begin{bmatrix} a & -1 & 0 & 0 & \cdots & 0 \\ -1 & b & -1 & 0 & \cdots & 0 \\ 0 & -1 & b & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & b & -1 \\ 0 & 0 & \cdots & 0 & -1 & c \end{bmatrix}_{j\times j}. \tag{3}$$

Let us use $T_j(x)$ to denote the tridiagonal matrix $T_j(x,x,x)$, i.e.,

$$T_j(x) \equiv T_j(x,x,x). \tag{4}$$

The discretization of the BVP (1) by a second order central divided difference discretization scheme with a uniform grid of mesh size h yields the linear system

$$T_n(\beta)\,x = f, \tag{5}$$

where

$$\beta = 2 + qh^2, \quad q \geq 0. \tag{6}$$

Following the matrix formulation of SAM in [25], we split the domain $(0,1)$ into k (≥ 2) overlapping subdomains as shown in Figure 1. Furthermore, we denote by ℓ the length of the overlap and η the length of each subdomain. Provided $n+1 = \frac{1}{h}$, we let $l+1 = \frac{\ell}{h}$ and $m+1 = \frac{\eta}{h}$ which implies the relation $n = mk - l(k-1)$. We assume that $l < \frac{m-1}{2}$ so that no three subdomains can have a common overlap. The open circled points in Figure 1 represent the interior boundaries of the subdomains on which we force the solutions of the local BVP to satisfy the parameterized mixed interface conditions (2) with

$$\omega = \frac{1-\alpha}{1-\alpha+\alpha h}, \quad 0 \leq \alpha < 1. \tag{7}$$

The derivation of (7) is not included in [25]; thus we give it in the following statement.

PROPOSITION 2.1. *Consider the 1-D two-point BVP*

$$-u''(t) + q\,u(t) = f(t), \quad t \in (\tau_1, \tau_2) \tag{8}$$

under the mixed boundary conditions

$$\begin{aligned} &\omega_1 u|_{t=\tau_1} + (1-\omega_1)\frac{\partial u}{\partial n}|_{t=\tau_1} = U_1, \\ &\omega_2 u|_{t=\tau_2} + (1-\omega_2)\frac{\partial u}{\partial n}|_{t=\tau_2} = U_2, \end{aligned} \tag{9}$$

where $0 < \omega_i \leq 1$, $i = 1, 2$, and $\frac{\partial u}{\partial n}|_{t=x}$ is the outwardly directed normal derivative to the boundary at a point $t = x$. If one discretizes the continuous problem (8)-(9) by using a uniform grid of mesh size $h (= \frac{\tau_2 - \tau_1}{m+1})$ and uses finite differences as follows

$$\begin{aligned} u''(t) &\approx \frac{u(t-h) - 2\,u(t) + u(t+h)}{h^2}, \\ \frac{\partial u}{\partial n}|_{t=\tau_1} &\approx \frac{u(\tau_1) - u(\tau_1+h)}{h}, \\ \frac{\partial u}{\partial n}|_{t=\tau_2} &\approx \frac{u(\tau_2) - u(\tau_2-h)}{h}, \end{aligned} \tag{10}$$

then the resulting linear system is given by the following matrix equation

$$\begin{bmatrix} \beta-\alpha_1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \beta & -1 & 0 & \cdots & 0 \\ 0 & -1 & \beta & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \beta & -1 \\ 0 & 0 & \cdots & 0 & -1 & \beta-\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} = \begin{bmatrix} h^2 f_1 + K_1 U_1 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{m-1} \\ h^2 f_m + K_2 U_2 \end{bmatrix}$$

where

$$\begin{aligned} \beta &= 2 + q\,h^2, \\ t_i &= \tau_1 + i\,h, \quad f_i = f(t_i), \quad i = 0, 1, \cdots, m+1, \\ \alpha_i &= \frac{1-\omega_i}{1-\omega_i+\omega_i h}, \quad K_i = \frac{h}{1-\omega_i+\omega_i h}, \quad i = 1, 2. \end{aligned}$$

(Remark 2.1 : Note that (7) is equivalent to the pair of relationships listed below

$$0 < \omega \leq 1 \quad \textit{and} \quad \alpha = \frac{1-\omega}{1-\omega+\omega h}.)$$

The proof of Proposition 2.1 can be found in Proposition 1.1 of [11] (see also [12]).

2.1. Convergence Analysis. For easy exposition of the convergence analysis of the SAM, we consider the case of a 3-way ($k = 3$) splitting of the BVP domain. The treatment of the general case is straightforward. For this particular case, the corresponding discrete equation to BVP (5) is given by the block matrix equation

$$T_n x = \begin{bmatrix} T_{m-l} & -F & 0 & 0 & 0 \\ -E & T_l & -F & 0 & 0 \\ 0 & -E & T_{m-2l} & -F & 0 \\ 0 & 0 & -E & T_l & -F \\ 0 & 0 & 0 & -E & T_{m-l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = f \tag{11}$$

where T_j denotes the tridiagonal matrix defined in (3), (4), i.e.,

$$T_j \equiv T_j(\beta). \tag{12}$$

The matrix E has zero elements everywhere except for the rightmost top element which is 1, and the matrix F has zero elements everywhere except for the leftmost bottom element which is 1. The matrices E and F have compatible sizes with the diagonal blocks in T_n.

Following [25], the corresponding *Generalized Schwarz Enhanced Equation (GSEE)* has the following structure

$$\widetilde{T_n}\widetilde{x} = \begin{bmatrix} T_{m-l} & -F & 0 & 0 & 0 & 0 & 0 \\ -E & B_1 & C_1 & -F & 0 & 0 & 0 \\ -E & C_1' & B_1' & -F & 0 & 0 & 0 \\ 0 & 0 & -E & T_{m-2l} & -F & 0 & 0 \\ 0 & 0 & 0 & -E & B_2 & C_2 & -F \\ 0 & 0 & 0 & -E & C_2' & B_2' & -F \\ 0 & 0 & 0 & 0 & 0 & -E & T_{m-l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \end{bmatrix} = \widetilde{f} \tag{13}$$

where B_i, C_i' are arbitrary matrices with $(B_i - C_i')$ non-singular for $i = 1, 2$, and

$$T_l = B_i + C_i = B_i' + C_i', \quad i = 1, 2.$$

Moreover, we choose the $l \times l$ matrices C_i' and C_i such that all their entries are zero except for an α in the positions $(1,1)$ and (l,l), respectively. It can be shown that for $\beta \geq 2$ the matrix $B_i - C_i' = T_l(\beta-\alpha, \beta, \beta-\alpha)$ is non-singular (see Corollary 1 in [26, p.85]). It turns out that these conditions imply the *equivalence* of the linear systems (11) and (13) ([24, 25]).

One can easily show that the matrix $\widetilde{T_n}$ in (13) can be written in the form

$$\widetilde{T_n} = \begin{bmatrix} T_m(\beta,\beta,\beta-\alpha) & -F' & 0 \\ -E' & T_m(\beta-\alpha,\beta,\beta-\alpha) & -F' \\ 0 & -E' & T_m(\beta-\alpha,\beta,\beta) \end{bmatrix} \tag{14}$$

where E' is the $m \times m$ matrix with zero elements everywhere except for 1 in the position $(1, m-l)$ and $-\alpha$ in the position $(1, m-l+1)$ and F' is the $m \times m$ matrix with zero elements everywhere except for 1 in the position $(m, l+1)$ and $-\alpha$ in the position (m, l). Several splittings can be employed for the matrix T_l. We select the following splitting for the enhanced matrix $\widetilde{T_n}$ in (14)

$$\widetilde{T_n} = M - N$$

$$(15) \qquad = \begin{bmatrix} T_m(\beta,\beta,\beta-\alpha) & 0 & 0 \\ 0 & T_m(\beta-\alpha,\beta,\beta-\alpha) & 0 \\ 0 & 0 & T_m(\beta-\alpha,\beta,\beta) \end{bmatrix} - \begin{bmatrix} 0 & F' & 0 \\ E' & 0 & F' \\ 0 & E' & 0 \end{bmatrix}.$$

The convergence analysis of the *parameterized SAM* based on Jacobi iteration is reduced to calculating the spectral radius of the block Jacobi iteration matrix $\widetilde{J} = M^{-1}N$ of the matrix $\widetilde{T_n}$ in (15). This Jacobi matrix has the form

$$(16) \quad \widetilde{J} = \begin{bmatrix} 0 & T_m^{-1}(\beta,\beta,\beta-\alpha)\,F' & 0 \\ T_m^{-1}(\beta-\alpha,\beta,\beta-\alpha)\,E' & 0 & T_m^{-1}(\beta-\alpha,\beta,\beta-\alpha)\,F' \\ 0 & T_m^{-1}(\beta-\alpha,\beta,\beta)\,E' & 0 \end{bmatrix}.$$

Tang in [25] was able to determine all non-zero eigenvalues of the corresponding block Jacobi iteration matrix in the case of a 3-way decomposition of the domain ($k = 3$) and to show experimentally the relation between the spectral radius of this matrix and the parameter α. He observed *experimentally* that for some value of α the convergence rate of the parameterized SAM was optimized. For the general case $k \geq 4$, he derived a $2(k-1) \times 2(k-1)$ matrix whose eigenvalue spectrum definitely includes all the non-zero eigenvalues of the Jacobi matrix.

In our study we have observed that the block tridiagonal structure of $\widetilde{T_n}$ of (14) implies that $\widetilde{T_n}$ possesses *Young's block property A* (see [26], [28], [1], [8]). Thus, the convergence of the block Jacobi method implies that its Gauss-Seidel counterpart will converge asymptotically twice as fast, while its optimal SOR counterpart will converge much faster. To simplify the presentation we adopt the notation $\rho(A)$ and $\sigma(A)$ for the spectral radius and the spectrum of a matrix A, respectively. The analysis of the SOR method requires some information about the spectrum of the block Jacobi iteration matrix $\widetilde{J}$ in (16). If $\sigma(\widetilde{J})$ is real and $\rho(\widetilde{J}) < 1$, it is well known that the Young's optimal value of the SOR parameter is given by $2/(1+(1-\rho^2(\widetilde{J}))^{\frac{1}{2}})$, (see [27], [28], [26], [1], [8]). Generally, if $\sigma(\widetilde{J})$ is a set of complex numbers satisfying some conditions, the optimal SOR can be found by the Young-Eidson's algorithm (see [29], [28]).

In the following we summarize the observations of [25] in two Lemmas 2.2 and 2.3 and derive the optimal values of the parameter α explicitly for the special cases $k = 2, 3$ and show the conditions that α satisfies in the general case.

LEMMA 2.2. *Consider the block Jacobi iteration matrix $\widetilde{J}$ in (16) and the* 4×4 *matrix*

$$(17) \qquad G_3 = \begin{bmatrix} 0 & g_1 & 0 & 0 \\ g_2 & 0 & 0 & g_3 \\ g_3 & 0 & 0 & g_2 \\ 0 & 0 & g_1 & 0 \end{bmatrix}$$

where

$$g_1 = t^{(1)}_{m-l} - \alpha\, t^{(1)}_{m-l+1}, \quad g_2 = t^{(2)}_{m-l} - \alpha\, t^{(2)}_{m-l+1}, \quad g_3 = t^{(2)}_{l+1} - \alpha\, t^{(2)}_{l}$$

and

$$[t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)}]^T \quad \textit{and} \quad [t_1^{(2)}, t_2^{(2)}, \cdots, t_m^{(2)}]^T$$

are the last columns of $T_m^{-1}(\beta, \beta, \beta - \alpha)$ *and* $T_m^{-1}(\beta - \alpha, \beta, \beta - \alpha)$, *respectively. Then* $\widetilde{J}$ *and* G_3 *have the same spectra except possibly for some zeros, that is, there holds*

$$\sigma(\widetilde{J}) = \sigma(G_3) \cup \{0\}. \tag{18}$$

Proof. First we observe that all row vectors of F' in (16) are zero except the last row of F'. Thus, only the last columns

$$[t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)}]^T \quad \textit{and} \quad [t_1^{(2)}, t_2^{(2)}, \cdots, t_m^{(2)}]^T$$

in $T_m^{-1}(\beta, \beta, \beta - \alpha)$ and $T_m^{-1}(\beta - \alpha, \beta, \beta - \alpha)$ are used when $T_m^{-1}(\beta, \beta, \beta - \alpha)\,F'$ and $T_m^{-1}(\beta - \alpha, \beta, \beta - \alpha)\,F'$ are computed, respectively. Similarly, when $T_m^{-1}(\beta - \alpha, \beta, \beta)\,E'$ and $T_m^{-1}(\beta - \alpha, \beta, \beta - \alpha)\,E'$ are computed, only the first columns in $T_m^{-1}(\beta - \alpha, \beta, \beta)$ and $T_m^{-1}(\beta - \alpha, \beta, \beta - \alpha)$ are used and these columns are given by

$$[t_m^{(1)}, \cdots, t_2^{(1)}, t_1^{(1)}]^T \quad \textit{and} \quad [t_m^{(2)}, \cdots, t_2^{(2)}, t_1^{(2)}]^T,$$

respectively. Since $l < \frac{m-1}{2}$, the matrix $\widetilde{J}$ in (16) has only eight non-zero columns. Let P be the $3\,m \times 3\,m$ permutation matrix that moves these columns, i.e.,

$$m - l,\ m - l + 1,\ m + l,\ m + l + 1,\ 2m - l,\ 2m - l + 1,\ 2m + l,\ 2m + l + 1$$

to the last eight columns in the order $3m - 8 + i,\ i = 1, 2, \cdots, 8$, respectively. Using the permutation matrix P just defined, $\widetilde{J}$ can be transformed to $\widetilde{J}'$ as follows

$$\widetilde{J}' = P^T \widetilde{J} P = \begin{bmatrix} 0 & * \\ 0 & W \end{bmatrix} \tag{19}$$

where the symbol $*$ denotes a possibly non-zero block and

$$W = \begin{bmatrix}
0 & 0 & -\alpha t_{m-l}^{(1)} & t_{m-l}^{(1)} & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha t_{m-l+1}^{(1)} & t_{m-l+1}^{(1)} & 0 & 0 & 0 & 0 \\
t_{m-l+1}^{(2)} & -\alpha t_{m-l+1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_l^{(2)} & t_l^{(2)} \\
t_{m-l}^{(2)} & -\alpha t_{m-l}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{l+1}^{(2)} & t_{l+1}^{(2)} \\
t_{l+1}^{(2)} & -\alpha t_{l+1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l}^{(2)} & t_{m-l}^{(2)} \\
t_l^{(2)} & -\alpha t_l^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l+1}^{(2)} & t_{m-l+1}^{(2)} \\
0 & 0 & 0 & 0 & t_{m-l+1}^{(1)} & -\alpha t_{m-l+1}^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & t_{m-l}^{(1)} & -\alpha t_{m-l+1}^{(1)} & 0 & 0
\end{bmatrix}.$$

Since the matrix W has only four independent columns, a similarity transformation on it yields the matrix G_3 in (17) whose eigenvalues include the four, possibly, non-zero eigenvalues of W (i.e., the only four eigenvalues might be non-zero). Here, we present the derivation of G_3 from W since it is not included in [25]. For this derivation, we let $\widetilde{P}$ be the permutation matrix that moves the columns $1, 4, 5, 8$ to the columns $5, 6, 7, 8$, respectively, and define the matrix

$$Q = diag\left(\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \right).$$

From the definition of $\widetilde{P}$ and Q, we can easily show that

$$W' = \widetilde{P}^T Q^{-1} W Q \widetilde{P} = \begin{bmatrix} 0 & * \\ 0 & G_3 \end{bmatrix}. \tag{20}$$

Then the relation (18) is a direct consequence of (19) and (20). □

It is worth noticing that the roots of the characteristic polynomial of the matrix in (17) are the non-identically zero eigenvalues

$$\pm \sqrt{g_1(g_2 \pm g_3)}. \tag{21}$$

A similar analysis as in the proof of Lemma 2.2 can cover the case k (≥ 3).

LEMMA 2.3. *For k (≥ 3) overlapping subdomains, the non-zero eigenvalues of the Jacobi matrix $\tilde{J}$ are included in those of the following $(k-1)\times(k-1)$ block matrix*

$$G_k = \begin{bmatrix} E & U & 0 & 0 & \cdots & 0 \\ L & D & U & 0 & \cdots & 0 \\ 0 & L & D & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & L & D & U \\ 0 & 0 & \cdots & 0 & L & E^T \end{bmatrix} \tag{22}$$

where

$$E = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} g_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & g_2 \\ g_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ 0 & g_3 \end{bmatrix}.$$

Specifically, the following relation holds

$$\sigma(\widetilde{J}) = \sigma(G_k) \cup \{0\}. \tag{23}$$

Remark 2.2 For $k = 2$, it can be seen that the matrix G_2 is given by

$$G_2 = \begin{bmatrix} 0 & g_1 \\ g_1 & 0 \end{bmatrix}. \tag{24}$$

2.2. Determination of the Optimal Parameter. In this Section we determine *analytically* the *exact value* of the parameter α that minimizes the spectral radius of the block Jacobi iteration matrix of the GSEE matrix. Specifically, we derive explicitly the optimal value of α for the cases $k = 2$ and $k = 3$ for which the spectral radius of the Jacobi matrix turns out to be zero. In general, for k (≥ 3) overlapping subdomains, we present two coupled equations whose roots definitely include all the non-zero eigenvalues of the block Jacobi iteration matrix $\widetilde{J}$. These equations can be used to estimate the optimal value of α numerically.

Specifically, we address the open problem of determining analytically the exact optimal values of α. This problem can be formulated as follows.

> **Problem 1:** *Determine the value of α for which the spectral radius of the block Jacobi iteration matrix of the GSEE is as small as possible.*

For the determination of the optimal α, we obtain analytic expressions for g_1, g_2, g_3 which turn out to be expressions in $[t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)}]^T$ and $[t_1^{(2)}, t_2^{(2)}, \cdots, t_m^{(2)}]^T$. A technique for computing these vectors is suggested in [25] (see also [5]). Moreover, they can be derived from the analysis of a more general case which is formulated and treated in Section 3. The following lemma states the analytic expressions of g_1, g_2, g_3 while its proof contains an outline of their derivation.

LEMMA 2.4. *Let $\theta = \operatorname{arccosh}(\frac{\beta}{2}) \geq 0$, where β is defined in (6). If $\beta > 2$, then we have*

$$g_1 = \frac{\alpha \sinh((m-l+1)\theta) - \sinh((m-l)\theta)}{\alpha \sinh(m\theta) - \sinh((m+1)\theta)},$$
$$g_2 = \frac{\frac{1}{2}\sinh((m-l)\theta)\,(\alpha - e^{\theta})\,(\alpha - e^{-\theta})}{(\alpha \sinh(\frac{m-1}{2}\theta) - \sinh(\frac{m+1}{2}\theta))\,(\alpha \cosh(\frac{m-1}{2}\theta) - \cosh(\frac{m+1}{2}\theta))}, \tag{25}$$
$$g_3 = \frac{(\alpha \sinh(\frac{l-1}{2}\theta) - \sinh(\frac{l+1}{2}\theta))\,(\alpha \cosh(\frac{l-1}{2}\theta) - \cosh(\frac{l+1}{2}\theta))}{(\alpha \sinh(\frac{m-1}{2}\theta) - \sinh(\frac{m+1}{2}\theta))\,(\alpha \cosh(\frac{m-1}{2}\theta) - \cosh(\frac{m+1}{2}\theta))}.$$

On the other hand, if $\beta = 2$, then we have

$$g_1 = \frac{(m-l+1)\alpha - (m-l)}{(m)\,\alpha - (m+1)},$$
$$g_2 = \frac{(m-l)\,(\alpha - 1)}{(m-1)\,\alpha - (m+1)}, \tag{26}$$
$$g_3 = \frac{(l-1)\,\alpha - (l+1)}{(m-1)\,\alpha - (m+1)}.$$

Proof. Since $[t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)}]^T$ is the last column of $T_m^{-1}(\beta, \beta, \beta - \alpha)$, its components satisfy the following system of equations

$$\begin{array}{rcrcrcll}
 & & \beta\, t_1^{(1)} & - & t_2^{(1)} & = & 0, & \\
-t_{p-1}^{(1)} & + & \beta\, t_p^{(1)} & - & t_{p+1}^{(1)} & = & 0, & p = 2, \cdots, m-1, \\
-t_{m-1}^{(1)} & + & (\beta - \alpha)\, t_m^{(1)} & & & = & 1, &
\end{array}$$

which can be transformed into

$$\begin{array}{rcrcrcll}
 & & (\beta - \alpha)\ \delta_1 & - & \delta_2 & = & 1, & \\
-\delta_{p-1} & + & \beta\ \ \delta_p & - & \delta_{p+1} & = & 0, & p = 2, \cdots, m-1, \\
-\delta_{m-1} & + & (\beta - 0)\ \delta_m & & & = & 0, &
\end{array}$$

by substituting δ_{m-p+1} for $t_p^{(1)}$. From the result in Proposition 3.6, we obtain

$$\delta_p = \begin{cases} \dfrac{\sinh((m-p+1)\theta) - 0\,\sinh((m-p)\theta)}{\sinh((m+1)\theta) - (\alpha+0)\,\sinh(m\theta) + \alpha\,0\,\sinh((m-1)\theta)} & for\ \ \beta > 2 \\[2ex] \dfrac{(m-p+1) - 0\,(m-p)}{(m+1) - (\alpha+0)\,m + \alpha\,0\,(m-1)} & for\ \ \beta = 2, \end{cases}$$

for $p = 1, 2, \cdots, m$, where $\theta = \operatorname{arccosh}(\frac{\beta}{2})$. Considering the case of $\beta > 2$, we have

$$t_p^{(1)} = \delta_{m-p+1} = \frac{\sinh(p\theta)}{\sinh((m+1)\theta) - \alpha\,\sinh(m\theta)} \tag{27}$$

for $p = 1, 2, \cdots, m$. Similarly we find that

$$t_p^{(2)} = \frac{\sinh(p\theta) - \alpha\,\sinh((p-1)\theta)}{\sinh((m+1)\theta) - 2\,\alpha\,\sinh(m\theta) + \alpha^2 \sinh((m-1)\theta)} \tag{28}$$

for $p = 1, 2, \cdots, m$. From the expressions in (27), (28), we obtain

$$g_1 = t^{(1)}_{m-l} - \alpha t^{(1)}_{m-l+1} = \frac{\sinh((m-l)\theta) - \alpha \sinh((m-l+1)\theta)}{\sinh((m+1)\theta) - \alpha \sinh(m\theta)},$$

$$g_2 = t^{(2)}_{m-l} - \alpha t^{(2)}_{m-l+1}$$
$$= \frac{\sinh((m-l)\theta) - \alpha\,(\sinh((m-l-1)\theta) + \sinh((m-l+1)\theta)) + \alpha^2 \sinh((m-l)\theta))}{\sinh((m+1)\theta) - 2\alpha \sinh(m\theta) + \alpha^2 \sinh((m-1)\theta)},$$

$$g_3 = t^{(2)}_{l+1} - \alpha t^{(2)}_{l} = \frac{\sinh((l+1)\theta) - 2\alpha \sinh(l\theta) + \alpha^2 \sinh((l-1)\theta)}{\sinh((m+1)\theta) - 2\alpha \sinh(m\theta) + \alpha^2 \sinh((m-1)\theta)}.$$

The numerator and the denominator in g_2 and g_3 are factored using the identities $\sinh(A) = 2\sinh(\frac{A}{2})\cosh(\frac{A}{2})$ and $\sinh(A) + \sinh(B) = 2\sinh(\frac{A+B}{2})\cosh(\frac{A-B}{2})$, and (25) are obtained. For the case of $\beta = 2$ we can take similar steps as above. □

Having obtained explicit expressions for g_1, g_2, g_3, we determine in Theorem 2.6 the value of α for which the spectral radius of the block Jacobi iteration matrix $\widetilde{J}$ becomes as small as possible. In the proof of the theorem, we refer to Proposition 2.5 which uses the matrix polynomial theory (see [6]) to solve a system of difference equations with vectors as unknowns and matrices as coefficients. Similar techniques are also used in [24], [11], [12], [13].

PROPOSITION 2.5. *Let G_k ($k \geq 3$) be the $(k-1) \times (k-1)$ block matrix*

$$G_k = \begin{bmatrix} E & U & 0 & 0 & \cdots & 0 \\ L & D & U & 0 & \cdots & 0 \\ 0 & L & D & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & L & D & U \\ 0 & 0 & \cdots & 0 & L & E^T \end{bmatrix}, \tag{29}$$

where

$$E = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} g_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & g_2 \\ g_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ 0 & g_3 \end{bmatrix}.$$

Assume $g_1 g_2 g_3 \neq 0$, then the eigenvalues λ of the matrix G_k, different from 0 and $\pm(g_2 \pm g_3)$, satisfy the following equation

$$g_3^2\lambda(\zeta_1^k - \zeta_2^k) + (g_2^2 g_3 - g_3^3 - g_1^2 g_3)(\zeta_1^{k-1} - \zeta_2^{k-1}) + (g_1 - g_2)^2\lambda(\zeta_1^{k-2} - \zeta_2^{k-2}) = 0 \tag{30}$$

where ζ_1 and ζ_2 are the two roots of the equation

$$(g_3\lambda)\zeta^2 - (\lambda^2 + g_3^2 - g_2^2)\zeta + (g_3\lambda) = 0. \tag{31}$$

The proof of Proposition 2.5 is very technical and can be found in Proposition 1.2 of [11] (see also [12]).

THEOREM 2.6. *For $k = 2, 3$, the optimal value, of α ($\widehat{\alpha}$) that minimizes $\rho(\widetilde{J}) = \rho(\widetilde{J}(\alpha))$ is given by the expressions*

$$\widehat{\alpha} = \begin{cases} \dfrac{\sinh((m-l)\theta)}{\sinh((m-l+1)\theta)}, & \beta > 2, \\[2ex] \dfrac{m-l}{m-l+1}, & \beta = 2, \end{cases} \tag{32}$$

where $\theta = \operatorname{arccosh}(\frac{\beta}{2}) > 0$, β is defined in (6) and m is an integer such that $h(m+1)$ is the length of each subdomain (see page 4).

For $k \geq 4$, except for some trivial cases, the optimal value of α ($\widehat{\alpha}$) that minimizes $\rho(\widetilde{J}) = \rho(\widetilde{J}(\alpha))$ is the one that minimizes the largest of the moduli of the (non-identically zero) roots λ of the equation

$$g_3^2\lambda \sum_{i=0}^{\left[\frac{k-1}{2}\right]} S_{k-2i-1} + (g_2^2 g_3 - g_3^3 - g_1^2 g_3) \sum_{i=0}^{\left[\frac{k-2}{2}\right]} S_{k-2i-2} + (g_1 - g_2)^2 \lambda \sum_{i=0}^{\left[\frac{k-3}{2}\right]} S_{k-2i-3} = 0 \tag{33}$$

where $[x]$ is the largest integer not exceeding x and S_i is given recursively by

$$\begin{array}{l} S_0 = 2,\ S_1 = (\lambda^2 + g_3^2 - g_2^2)/g_3\lambda, \\ S_i - S_1 S_{i-1} + S_{i-2} = 0, \quad i = 2, 3, \cdots, k-1. \end{array} \tag{34}$$

Proof. For $k = 2$, we have $\sigma(\widetilde{J}) = \sigma(G_2) \cup \{0\}$, where G_2 is the matrix in (24). The eigenvalues of G_2 are given by $\pm|g_1|$. So, $\rho(\widetilde{J})$ can be made zero if and only if $g_1 = 0$. The latter condition holds if and only if α is given by (32).

For $k = 3$, we have from Tang's result in (21) that $\rho(\widetilde{J})$ is given by

$$\rho(\widetilde{J}) = \rho(G_3) = \max\left(\sqrt{|g_1(g_2 + g_3)|}, \sqrt{|g_1(g_2 - g_3)|}\right). \tag{35}$$

We note that $g_2 + g_3$ and $g_2 - g_3$ cannot be made simultaneously zero since then $g_3 = 0$ implies $\alpha > 1$. So, $\rho(\widetilde{J})$ in (35) can be minimized, in fact it can be made zero, if and only if $g_1 = 0$. Therefore the optimal value of α is that of case $k = 2$ in (32).

For $k \geq 4$, by virtue of Lemma 2.3, it is

$$\rho(\widetilde{J}) = \rho(G_k).$$

For $\alpha \in [0, 1)$, we have $g_3 \neq 0$. Therefore, for $g_1 g_2 \neq 0$ and $\lambda \neq \pm(g_2 \pm g_3)$, all the assumptions of Proposition 2.5 are satisfied. Consequently, the eigenvalues of G_k of interest are obtained from the solution of the system of equations (30) and (31). Now, (31) will be satisfied with $\zeta = \zeta_i$, $i = 1, 2$. So, we substitute, successively, ζ_1 and ζ_2 for ζ in (31), multiply then the first resulting equation by ζ_1^{i-2} and the second one by ζ_2^{i-2} and add the two new equations together. Then, substituting $S_i = \zeta_1^i + \zeta_2^i$, $i = 1, 2, \cdots, k-1$, with $S_1 = \zeta_1 + \zeta_2 = (\lambda^2 + g_3^2 - g_2^2)/(g_3\lambda)$, and $S_0 = 2$ we obtain (34). By virtue of the assumption $\lambda \neq \pm(g_2 \pm g_3)$, it is implied that $\zeta_1 \neq \zeta_2$. Hence, dividing (30) through by $\zeta_1 - \zeta_2$ and using (34), we obtain (33). □

Remark 2.3 The solutions of (33) are, possibly, the non-zero eigenvalues of $\widetilde{J}$. So, to solve our problem for $k \geq 4$, we have to solve numerically the equation (33) in λ. After eliminating the denominators that appear in (33), it becomes a polynomial equation of degree $2(k-1)$ that contains only even powers of λ. Since its coefficients are functions of α, the optimal value of α, in this present general case, can only be found computationally by considering a range of values of it in $[0, 1)$.

Remark 2.4 The trivial cases ($g_1 g_2 \neq 0$ and $\lambda \neq \pm(g_2 \pm g_3)$), not examined in the theorem, give essentially similar coupled equations to (33), (34).

Remark 2.5 The characteristic polynomial of the matrix G_k is given by the system of the two coupled equations (33), (34). Even for $k = 2, 3$, these polynomials are recovered from these two equations. For instance, the corresponding characteristic polynomials for $k = 4, 5$ are

$$\lambda^6 - (2g_1g_2 + g_2^2)\lambda^4 + (g_1^2 g_2^2 + 2g_1 g_2^3 - 2g_1 g_2 g_3^2)\lambda^2 + (2g_1^2 g_2^2 g_3^2 - g_1^2 g_2^4 - g_1^2 g_3^4)$$

and

$$\begin{aligned}&\lambda^8-(2g_2^2+2g_1g_2)\lambda^6+(g_2^4+4g_1g_2^3+g_1^2g_2^2-g_2^2g_3^2-2g_1g_2g_3^2)\lambda^4\\&+(4g_1g_2^3g_3^2+2g_1^2g_2^2g_3^2-2g_1g_2g_3^4-2g_1g_2^5-2g_1^2g_2^4)\lambda^2\\&+(g_1^2g_2^6+3g_1^2g_2^2g_3^4-g_1^2g_3^6-3g_1^2g_2^4g_3^2),\end{aligned}$$

respectively.

3. Multi-Parameter SAM (MPSAM). In this section we consider again the two-point BVP in (1) and assume the decomposition for the boundary value domain defined in the previous section. We formulate a *Multi-Parameterized SAM* based on finite difference discretization and Jacobi type iteration scheme and assume the coupling (2) with different ω_i's in the interior boundary between the subdomains Ω_i and Ω_{i+1}. Note that if $\omega_i = \omega$, $i = 1, 2, \cdots, k-1$, then the present multi-parameter case reduces to the one-parameter case considered in Section 2. After formulating the multi-parameterized SAM, we solve the following open problem:

> **Problem 2:** *Determine the values of α_i's for which the spectral radius of the block Jacobi iteration matrix of the GSEE is as small as possible.*

3.1. Formulation of the Multi-Parameterized SAM. We observe that there are many ways of splitting the matrix T_l in (11). Here, we *choose* the matrices B_i, B_i', , C_i, C_i' in (13) in order to define the multi-parameterized SAM. For this formulation, we introduce a set of $k-1$ parameters $\alpha_i, i = 1, 2, \cdots, k-1$, such that each α_i is associated with ω_i. As in the case of the 1PSAM, we establish the following relationship (see Proposition 2.1) between ω_i in (2) and α_i

$$\omega_i = \frac{1-\alpha_i}{1-\alpha_i+\alpha_i h}, \quad i = 1, 2, \cdots, k-1,$$

where h is the grid size and $0 \le \alpha_i < 1$. Let C_i' and C_i be $l \times l$ matrices with zero elements everywhere except for an α_i in the position $(1, 1)$ and (l, l), respectively. Moreover, we define E_i' to be the $m \times m$ matrix with zero elements everywhere except for 1 in the position $(1, m-l)$ and $-\alpha_i$ in the position $(1, m-l+1)$ and F_i' to be the $m \times m$ matrix with zero elements everywhere except for 1 in the position $(m, l+1)$ and $-\alpha_i$ in the position (m, l). Then, the matrix $\widetilde{T_n} (\equiv \widetilde{T_n(\beta)})$ in (13) can be written in the form

$$\widetilde{T_n(\beta)} = \begin{bmatrix} T_m(\beta-\alpha_0, \beta, \beta-\alpha_1) & -F_1' & 0 \\ -E_1' & T_m(\beta-\alpha_1, \beta, \beta-\alpha_2) & -F_2' \\ 0 & -E_2' & T_m(\beta-\alpha_2, \beta, \beta-\alpha_3) \end{bmatrix}$$

where $\alpha_0 = \alpha_3 = 0$. If the number of subdomains k is more than 3, the matrix $\widetilde{T_n}$ is a block $k \times k$ matrix of the form

$$\widetilde{T_n} = \begin{bmatrix} S_1(\beta) & -F_1' & 0 & 0 & \cdots & 0 & 0 \\ -E_1' & S_2(\beta) & -F_2' & 0 & \cdots & 0 & 0 \\ 0 & -E_2' & S_3(\beta) & -F_3' & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -E_{k-2}' & S_{k-1}(\beta) & -F_{k-1}' \\ 0 & 0 & \cdots & 0 & 0 & -E_{k-1}' & S_k(\beta) \end{bmatrix} \tag{36}$$

where

$$S_i(\beta) \equiv T_m(\beta - \alpha_{i-1}, \beta, \beta - \alpha_i), \quad i = 1, 2 \cdots, k, \tag{37}$$

and $\alpha_0 = \alpha_k = 0$. Then, the *multi-parameterized SAM* for $\widetilde{T_n(\beta)}$ is *defined* as

$$\widetilde{T_n(\beta)} = S_{km} - B_{km} \tag{38}$$

where

$$S_{km} = diag(S_1(\beta), S_2(\beta), \cdots, S_k(\beta)) \textit{ and} \tag{39}$$

$$B_{km} = \begin{bmatrix} 0 & F_1' & 0 & 0 & \cdots & 0 & 0 \\ E_1' & 0 & F_2' & 0 & \cdots & 0 & 0 \\ 0 & E_2' & 0 & F_3' & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & E_{k-2}' & 0 & F_{k-1}' \\ 0 & 0 & \cdots & 0 & 0 & E_{k-1}' & 0 \end{bmatrix}. \tag{40}$$

3.2. Convergence Analysis. The convergence analysis of the Jacobi based multi-parameterized SAM is again reduced to calculating the spectral radius of the block Jacobi matrix $\widetilde{J} = M^{-1}N$ of $\widetilde{T_n}$ in (38). The $k \times k$ block-Jacobi matrix $\widetilde{J}$ is given by

$$\widetilde{J} = \begin{bmatrix} 0 & S_1{}^{-1}F_1' & 0 & 0 & \cdots & 0 & 0 \\ S_2{}^{-1}E_1' & 0 & S_2{}^{-1}F_2' & 0 & \cdots & 0 & 0 \\ 0 & S_3{}^{-1}E_2' & 0 & S_3{}^{-1}F_3' & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & S_{k-1}{}^{-1}E_{k-2}' & 0 & S_{k-1}{}^{-1}F_{k-1}' \\ 0 & 0 & \cdots & 0 & 0 & S_k{}^{-1}E_{k-1}' & 0 \end{bmatrix}. \tag{41}$$

where $S_i \equiv S_i(\beta),\ i = 1, 2, \cdots, k$.

In the following analysis we find matrices of smaller orders whose eigenvalues include the non-zero eigenvalues of the block Jacobi matrix $\widetilde{J}$ in (41).

LEMMA 3.1. *Let*

$$[\delta_1^{i,j}, \delta_2^{i,j}, \cdots, \delta_m^{i,j}]^T \tag{42}$$

denote the first column of the matrix $T_m^{-1}(\beta - \alpha_i, \beta, \beta - \alpha_j)$ *and* W *be the* $4(k-1) \times 4(k-1)$ *matrix*

$$W = \begin{bmatrix} 0 & X_{10} & 0 & 0 & 0 & 0 & \cdots & 0 \\ X_{12} & 0 & 0 & Y_{21} & 0 & 0 & \cdots & 0 \\ Y_{12} & 0 & 0 & X_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & X_{23} & 0 & 0 & Y_{32} & \cdots & 0 \\ 0 & 0 & Y_{23} & 0 & 0 & X_{32} & \cdots & 0 \\ & & & & & & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & X_{k-2,k-1} & 0 & 0 & Y_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & Y_{k-2,k-1} & 0 & 0 & X_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 & X_{k-1,k} & 0 \end{bmatrix}$$

with

$$X_{i,i+1} = \begin{bmatrix} \delta_l^{i,i+1} & -\alpha_i \delta_l^{i,i+1} \\ \delta_{l+1}^{i,i+1} & -\alpha_i \delta_{l+1}^{i,i+1} \end{bmatrix}, \quad X_{i,i-1} = \begin{bmatrix} -\alpha_i \delta_{l+1}^{i,i-1} & \delta_{l+1}^{i,i-1} \\ -\alpha_i \delta_l^{i,i-1} & \delta_l^{i,i-1} \end{bmatrix},$$

for $i = 1, 2, \cdots, k-1$, *and*

$$Y_{i,i+1} = \begin{bmatrix} \delta_{m-l}^{i,i+1} & -\alpha_i \delta_{m-l}^{i,i+1} \\ \delta_{m-l+1}^{i,i+1} & -\alpha_i \delta_{m-l+1}^{i,i+1} \end{bmatrix}, \quad Y_{i+1,i} = \begin{bmatrix} -\alpha_{i+1} \delta_{m-l+1}^{i+1,i} & \delta_{m-l+1}^{i+1,i} \\ -\alpha_{i+1} \delta_{m-l}^{i+1,i} & \delta_{m-l}^{i+1,i} \end{bmatrix},$$

for $i = 1, 2, \cdots, k-2$. *Then, the eigenvalues of* W *include the non-zero eigenvalues of the block Jacobi matrix* $\widetilde{J}$ *in (41), i.e.,*

$$\sigma(\widetilde{J}) = \sigma(W) \cup \{0\}. \tag{43}$$

Proof. We observe that all the rows of E'_{i-1} are zero except for the first one. Hence only the first column in $S_i^{-1} = T_m^{-1}(\beta - \alpha_{i-1}, \beta, \beta - \alpha_i)$ is used in computing $S_i^{-1} E'_{i-1}$, $i = 2, 3, \cdots, k$ and the vector in (42) satisfies the system of equations

$$\begin{array}{rcrcl} & & (\beta - \alpha_i)\ \delta_1^{i,j} & - & \delta_2^{i,j} = 1, \\ -\delta_{p-1}^{i,j} & + & \beta\ \delta_p^{i,j} & - & \delta_{p+1}^{i,j} = 0, \quad p = 2, \cdots, m-1, \\ -\delta_{m-1}^{i,j} & + & (\beta - \alpha_j)\ \delta_m^{i,j} & & = 0. \end{array} \tag{44}$$

With this notation and the definition of the matrix E'_{i-1}, we can see that all column vectors in the matrix $S_i^{-1} E'_{i-1} = T_m^{-1}(\beta - \alpha_{i-1}, \beta, \beta - \alpha_i)\, E'_{i-1}$ are zero except for the $(m-l)$-th and $(m-l+1)$-st ones which are given by

$$[\delta_1^{i-1,i}, \delta_2^{i-1,i}, \cdots, \delta_m^{i-1,i}]^T \quad and \quad -\alpha_{i-1}\, [\delta_1^{i-1,i}, \delta_2^{i-1,i}, \cdots, \delta_m^{i-1,i}]^T,$$

respectively. Similarly, all columns in the matrix $S_i^{-1} F'_i = T_m^{-1}(\beta - \alpha_{i-1}, \beta, \beta - \alpha_i)\, F'_i$ are zero except for the $(l+1)$-st and l-th ones which are given by

$$[\delta_m^{i,i-1}, \cdots, \delta_2^{i,i-1}, \delta_1^{i,i-1}]^T \quad and \quad -\alpha_i\, [\delta_m^{i,i-1}, \cdots, \delta_2^{i,i-1}, \delta_1^{i,i-1}]^T,$$

respectively. Note that $[\delta_m^{j,i}, \cdots, \delta_2^{j,i}, \delta_1^{j,i}]^T$ is the last column of $T_m^{-1}(\beta - \alpha_i, \beta, \beta - \alpha_j)$. Hence the matrices $S_i^{-1} E'_{i-1}$ and $S_i^{-1} F'_i$ have the following forms

$$S_i^{-1} E'_{i-1} = \begin{bmatrix} 0 & \cdots\cdots & 0 & \delta_1^{i-1,i} & -\alpha_{i-1}\delta_1^{i-1,i} & 0 & \cdots & 0 \\ 0 & \cdots\cdots & 0 & \delta_2^{i-1,i} & -\alpha_{i-1}\delta_2^{i-1,i} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots\cdots & 0 & \delta_m^{i-1,i} & -\alpha_{i-1}\delta_m^{i-1,i} & 0 & \cdots & 0 \end{bmatrix}_{m \times m},$$

$$S_i^{-1} F'_i = \begin{bmatrix} 0 & \cdots & 0 & -\alpha_i\, \delta_m^{i,i-1} & \delta_m^{i,i-1} & 0 & \cdots\cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_i\, \delta_2^{i,i-1} & \delta_2^{i,i-1} & 0 & \cdots\cdots & 0 \\ 0 & \cdots & 0 & -\alpha_i\, \delta_1^{i,i-1} & \delta_1^{i,i-1} & 0 & \cdots\cdots & 0 \end{bmatrix}_{m \times m}.$$

Therefore, considering $l < \frac{m-1}{2}$, the matrix $\widetilde{J}$ in (41) has exactly $4(k-1)$ non-zero columns. Let P be the $k\,m \times k\,m$ permutation matrix that moves the columns

$im-l,\ im-l+1,\ im+l,\ im+l+1$ to the columns $km-4(k-1)+4(i-1)+j$, $j = 1,2,3,4$, respectively, for each $i = 1,2,\cdots,k-1$. Then, $\widetilde{J}$ can be transformed to $\widetilde{J}'$ as follows

$$\widetilde{J}' = P^T \widetilde{J} P = \begin{bmatrix} 0 & * \\ 0 & W \end{bmatrix} \tag{45}$$

and the result in (43) is an immediate consequence of (45). □

The following lemma shows that there is a still smaller matrix whose eigenvalues include the non-zero eigenvalues of the block Jacobi iteration matrix $\widetilde{J}$ in (41).

LEMMA 3.2. *The eigenvalues of the matrix G_k include the non-zero eigenvalues of the matrix $\widetilde{J}$, i.e.,*

$$\sigma(\widetilde{J}) = \sigma(G_k) \cup \{0\} \tag{46}$$

where

$$G_k = \begin{bmatrix} 0 & x_{10} & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_{12} & 0 & 0 & y_{21} & 0 & 0 & \cdots & 0 \\ y_{12} & 0 & 0 & x_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{23} & 0 & 0 & y_{32} & \cdots & 0 \\ 0 & 0 & y_{23} & 0 & 0 & x_{32} & \cdots & 0 \\ & & & & & & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & x_{k-2,k-1} & 0 & 0 & y_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & y_{k-2,k-1} & 0 & 0 & x_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 & x_{k-1,k} & 0 \end{bmatrix}_{2(k-1)\times 2(k-1)} \tag{47}$$

and the entries of G_k are

$$x_{ij} = \delta_{l+1}^{i,j} - \alpha_i \delta_l^{i,j}, \quad y_{ij} = \delta_{m-l}^{i,j} - \alpha_j \delta_{m-l+1}^{i,j}. \tag{48}$$

Proof. We define the non-singular matrix

$$Q = diag(Q_1, Q_1^T, Q_2, Q_2^T, \cdots, Q_{k-1}, Q_{k-1}^T),$$

where

$$Q_i = \begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix}, \quad {Q_i}^T = \begin{bmatrix} 1 & 0 \\ \alpha_i & 1 \end{bmatrix}, \quad {Q_i}^{-1} = \begin{bmatrix} 1 & -\alpha_i \\ 0 & 1 \end{bmatrix}, \quad {Q_i}^{-T} = \begin{bmatrix} 1 & 0 \\ -\alpha_i & 1 \end{bmatrix}.$$

Using the matrices Q and Q^{-1}, W can be transformed into W' via the transformation

$$W' = Q^{-1} W Q = \begin{bmatrix} 0 & X'_{10} & 0 & 0 & 0 & 0 & \cdots & 0 \\ X'_{12} & 0 & 0 & Y'_{21} & 0 & 0 & \cdots & 0 \\ Y'_{12} & 0 & 0 & X'_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & X'_{23} & 0 & 0 & Y'_{32} & \cdots & 0 \\ 0 & 0 & Y'_{23} & 0 & 0 & X'_{32} & \cdots & 0 \\ & & & & & & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & X'_{k-2,k-1} & 0 & 0 & Y'_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & Y'_{k-2,k-1} & 0 & 0 & X'_{k-1,k-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 & X'_{k-1,k} & 0 \end{bmatrix}$$

where

$$X'_{i,i+1} = Q_i^{-T} X_{i,i+1} Q_i = \begin{bmatrix} \delta_l^{i,i+1} & 0 \\ \delta_{l+1}^{i,i+1} - \alpha_i \delta_l^{i,i+1} & 0 \end{bmatrix},$$

$$X'_{i,i-1} = Q_i^{-1} X_{i,i-1} Q_i^T = \begin{bmatrix} 0 & \delta_{l+1}^{i,i-1} - \alpha_i \delta_l^{i,i-1} \\ 0 & \delta_l^{i,i-1} \end{bmatrix},$$

for $i = 1, 2, \cdots, k-1$, and

$$Y'_{i,i+1} = Q_{i+1}^{-1} Y_{i,i+1} Q_i = \begin{bmatrix} \delta_{m-l}^{i,i+1} - \alpha_{i+1} \delta_{m-l+1}^{i,i+1} & 0 \\ \delta_{m-l+1}^{i,i+1} & 0 \end{bmatrix},$$

$$Y'_{i+1,i} = Q_i^{-T} Y_{i+1,i} Q_{i+1}^T = \begin{bmatrix} 0 & \delta_{m-l+1}^{i+1,i} \\ 0 & \delta_{m-l}^{i+1,i} - \alpha_i \delta_{m-l+1}^{i+1,i} \end{bmatrix},$$

for $i = 1, 2, \cdots, k-2$. Thus W and W' have the same eigenvalue spectra, i.e.,

$$\sigma(W) = \sigma(W'). \tag{49}$$

We now observe that except for $2(k-1)$ columns, all other columns of the matrix W' are zero vectors. Let $\widetilde{P}$ be the $4\,(k-1) \times 4\,(k-1)$ permutation matrix that moves the columns $4(i-1)+1,\ 4(i-1)+4$ to the columns $2(k-1)+2(i-1)+1,\ 2(k-1)+2(i-1)+2$, respectively, for each $i = 1, 2, \cdots, k-1$. Then, W' can be transformed to W'' as follows $W'' = \widetilde{P}^T W' \widetilde{P} = \begin{bmatrix} 0 & * \\ 0 & G_k \end{bmatrix}$. Thus, $\sigma(G_k)$ definitely includes the non-zero eigenvalues of the matrix W'', i.e.,

$$\sigma(W'') = \sigma(G_k) \cup \{0\}. \tag{50}$$

The relations (43), (49) and (50) imply the conclusion of the lemma. □

3.3. Determination of Optimal Multi-Parameter Sets. Having obtained the matrix G_k in (47) we can show that there is a choice of its elements $x_{i,i+1} = 0$, $i = 1, 2, \cdots, k-1$, that makes all its eigenvalues equal to zero. This is given in the following lemma.

LEMMA 3.3. *If $x_{i,i+1} = 0, i = 1, 2, \cdots, k-1$, then $\det(G_k - \lambda I) = \lambda^{2(k-1)}$ and all the eigenvalues of the matrix G_k are zero.*

Proof. The assertion is proved by induction. It is easily seen from (47) that the lemma holds true for $k = 2$, since we have $G_2 = \begin{bmatrix} 0 & x_{10} \\ 0 & 0 \end{bmatrix}$. Assume that the lemma holds true for any $k \geq 2$. Then, the choice $x_{i,i+1} = 0$, $i = 1, \cdots, k-1$, forces G_k to have all its eigenvalues zero, i.e., $\det(G_k - \lambda I) = \lambda^{2(k-1)}$. Choose $x_{k,k+1} = 0$. Then, the characteristic polynomial for G_{k+1} is

$$\det(G_{k+1} - \lambda I) = \det \left(\left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & G_k - \lambda I & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 0 & y_{k,k-1} \\ 0 & \cdots & 0 \;\; y_{k-1,k} & 0 & -\lambda & x_{k,k-1} \\ 0 & \cdots & 0 \;\; 0 & 0 & 0 & -\lambda \end{array} \right] \right) \tag{51}$$

$$= \det(G_k - \lambda I)(-\lambda)^2 = \lambda^{2(k-1)}\,\lambda^2 = \lambda^{2k}.$$

Thus the lemma holds true for $k+1$, which concludes the proof of lemma. □

Notice that there are other choices of the $x'_{i,j}s$ that make $\rho(G_k)=0$.

LEMMA 3.4. *If* $x_{i,i-1}=0, i=1,2,\cdots,k-1$, *then* $\det(G_k-\lambda I)=\lambda^{2(k-1)}$ *which implies that all the eigenvalues of the matrix* G_k *are zero.*

For the proof see Lemma 1.7 of [11] (and also [12]).

Moreover, Lemmas 3.3 and 3.4 allow us to prove a more general result.

LEMMA 3.5. *If for any* $j=0,1,...,k-1$ *we have*

$$x_{i,i-1}=0,\quad i=1,2,\cdots,j, \tag{52}$$
$$x_{i,i+1}=0,\quad i=j+1,\cdots,k-1, \tag{53}$$

then the $\det(G_k-\lambda I)=\lambda^{2(k-1)}$ *and all the eigenvalues of the matrix* G_k *are zero.*

Proof. Using condition (52), Lemma 3.4 can be applied to the $2j\times 2j$ principal submatrix G_{j+1} of G_k to give

$$\det(G_{j+1}-\lambda I)=\lambda^{2j}. \tag{54}$$

Then, using the series of relationships in (51) with the conditions (54) and (53), we can easily obtain

$$\det(G_k-\lambda I)=\lambda^{2(k-1)}.$$
□

The following proposition provides the expressions of $\delta_p^{i,j}$ in (44), which in turn help us to derive those of $x_{i,i-1}, x_{i,i+1}$ in Lemma 3.5.

PROPOSITION 3.6. *The solution* $[\delta_1,\delta_2,\cdots,\delta_m]^T$ *of the system of equations*

$$\begin{array}{rcccccll} & & (\beta-\alpha_1) & \delta_1 & - & \delta_2 & = 1, & \\ -\delta_{p-1} & + & \beta & \delta_p & - & \delta_{p+1} & = 0, & p=2,\cdots,m-1, \\ -\delta_{m-1} & + & (\beta-\alpha_2) & \delta_m & & & = 0, & \end{array}$$

where $0\le\alpha_i<1,\ i=1,2$, *and* $\beta\ge 2$ *is given by*

$$\delta_p=\begin{cases}\dfrac{\sinh((m-p+1)\theta)-\alpha_2\sinh((m-p)\theta)}{\sinh((m+1)\theta)-(\alpha_1+\alpha_2)\sinh(m\theta)+\alpha_1\alpha_2\sinh((m-1)\theta)} & \text{for}\quad \beta>2\\[2ex] \dfrac{(m-p+1)-\alpha_2(m-p)}{(m+1)-(\alpha_1+\alpha_2)\,m-\alpha_1\alpha_2\,(m-1)} & \text{for}\quad \beta=2,\end{cases}$$

where $\theta=\operatorname{arccosh}(\frac{\beta}{2})$.

The proof of Proposition 3.6 is rather lengthy and can be found in Proposition 1.3 of [11] (see also [12]).

Based on the above lemmas and proposition, the following theorem holds.

THEOREM 3.7. *Let* $\theta=\operatorname{arccosh}(\frac{\beta}{2})$ *with* $\beta=2+qh^2$ *as defined in (6) and the* *values* α_i, $i = 0, 1, \cdots, k$, *be defined as follows:*

For $q > 0$ $(i.e., \theta > 0)$:

$$\begin{aligned}
\alpha_0 &= 0,\\
\alpha_i &= \frac{\sinh((m-l)\theta)-\alpha_{i-1}\sinh((m-l-1)\theta)}{\sinh((m-l+1)\theta)-\alpha_{i-1}\sinh((m-l)\theta)}, \quad i = 1, 2, \cdots, j,\\
\alpha_i &= \frac{\sinh((m-l)\theta)-\alpha_{i+1}\sinh((m-l-1)\theta)}{\sinh((m-l+1)\theta)-\alpha_{i+1}\sinh((m-l)\theta)}, \quad i = j+1, \cdots, k-1,\\
\alpha_k &= 0.
\end{aligned}$$

For $q = 0$ $(i.e., \theta = 0)$:

$$\begin{aligned}
\alpha_0 &= 0,\\
\alpha_i &= \frac{(m-l)-\alpha_{i-1}(m-l-1)}{(m-l+1)-\alpha_{i-1}(m-l)}, \quad i = 1, 2, \cdots, j,\\
\alpha_i &= \frac{(m-l)-\alpha_{i+1}(m-l-1)}{(m-l+1)-\alpha_{i+1}(m-l)}, \quad i = j+1, \cdots, k-1,\\
\alpha_k &= 0,
\end{aligned}$$

for any $j = 0, 1, ..., k-1$. *Then,* $\rho(G_k)$ *is zero which implies that the spectral radius of the block Jacobi matrix* $\widetilde{J}$ *in (41) is zero too.*

Proof. From Proposition 3.6, we have that

$$\delta_p^{i,j} = \begin{cases} \dfrac{\sinh((m-p+1)\theta)-\alpha_j\sinh((m-p)\theta)}{\sinh((m+1)\theta)-(\alpha_i+\alpha_j)\sinh(m\theta)+\alpha_i\alpha_j\sinh((m-1)\theta)} & , \theta > 0\\[2ex] \dfrac{(m-p+1)-\alpha_j(m-p)}{(m+1)-(\alpha_i+\alpha_j)m+\alpha_i\alpha_j(m-1)} & , \theta = 0. \end{cases}$$

Note that the case $\theta = 0$ can be obtained from the case $\theta > 0$ and a limiting process argument allowing $\theta \to 0^+$. The definitions of x_{ij} in (48) and of α_i give

$$\begin{aligned}
x_{i,i-1} &= \delta_{l+1}^{i,i-1} - \alpha_i\delta_l^{i,i-1}\\
&= \begin{cases} \dfrac{(\sinh((m-l)\theta)-\alpha_{i-1}\sinh((m-l-1)\theta)) - \alpha_i(\sinh((m-l+1)\theta)-\alpha_{i-1}\sinh((m-l)\theta))}{\sinh((m+1)\theta)-(\alpha_i+\alpha_{i-1})\sinh(m\theta)+\alpha_i\alpha_{i-1}\sinh((m-1)\theta)} & , \theta > 0\\[2ex] \dfrac{((m-l)-\alpha_{i-1}(m-l-1)) - \alpha_i((m-l+1)-\alpha_{i-1}(m-l))}{(m+1)-(\alpha_i+\alpha_{i-1})m+\alpha_i\alpha_{i-1}(m-1)} & , \theta = 0 \end{cases}\\
&= 0.
\end{aligned}$$

for $i = 1, 2, \cdots, j$. Similarly, we can obtain that $x_{i,i+1} = 0$, for $i = j+1, \cdots, k-1$. Since, the conditions of Lemma 3.5 are satisfied, all the eigenvalues of the matrix G_k are zero. Hence, by virtue of (46), the conclusion of the statement follows. □

4. Numerical Experiments. In this section we attempt to measure experimentally the convergence factor of the *Classical SAM (SAM)*, *One-Parameter SAM (1PSAM)*, and *Multi-Parameterized SAM (MPSAM)* methods for different domain splittings. First, we have verified the numerical results presented in [25] for the two-point Poisson type BVP used in this study and our implementation of *1PSAM*. Second, we have applied *1PSAM* to the following Helmholtz type BVP

$$u''(t) - 4u = 4\cosh(1),\ t \in (0,1),\ \ u(0) = 0,\ u(1) = 0 \tag{55}$$

whose solution is $u(t) = \cosh(2t - 1.0) - \cosh(1.0)$.

In all the experiments, the vector with all its components -0.25 was used as initial guess of the solution vector. The value -0.25 is midway the two extreme values of the function $u(t)$. The convergence factor r_p is computed as the p-th root of the relative ℓ_2-norm of the residual of the corresponding system of equations after p iterations, i.e.,

$$r_p = \sqrt[p]{||Ax^{(p)} - f||_2/||Ax^{(0)} - f||_2}. \tag{56}$$

In Table 1 we show the convergence factor of SAM computed after 3, 4 and 8 iterations for different domain splittings, overlaps, and local grid sizes. The results indicate slow convergence.

In Table 2 we present the convergence factor for the $1PSAM$ method. It is worth recalling that in [25], the optimal value of the parameter of this method for $k = 3$ was found experimentally. In Section 2, we found the simple equations (33), (34) that the optimal values of $1PSS$ satisfy for any value of k. In the case of $k = 2$ and 3, the formulas can be solved explicitly while for $k \geq 4$ we solve them numerically. Table 2 indicates the computed single parameter value and the convergence factor r_k of the method computed after k iterations where k is the number of subdomains. Notice that in case $k = 3$ our theoretical value of α coincides with the numerical one computed in [25].

It is worth noticing that our experiments indicate that $MPSAM$ computes solutions whose relative residual in the ℓ_2-norm is 2×10^{-15} after k iterations, where k is the number of subdomains. This is consistent for all k tried up to $k = 64$. Table 3 gives the exact parameters predicted by the theory in the previous sections. Clearly, $MPSAM$ achieves a rapid convergence within a very small number of iterations. The convergence rate is very sensitive to the computed optimal value of parameter α_i's and the symmetric choice of them reduces the error propagation when we compute the optimal value of parameters α_i's.

The data obtained suggest that $1PSAM$ is faster than SAM but slower than $MPSAM$.

5. Multi-Parameter SAM for Two-Dimensional Problems. The basic analysis of the parameterized SAM with the one parameter case for 2-D problems was presented in [25]. In this section, we develop a similar analysis for 2-D problems using a set of parameters α_i, $i = 1, 2, \cdots, k-1$, with k being the number of subdomains and attempt to attack *Problem 2* which was completely solved in the 1-D case as was seen in Section 3. However, in the 2-D case, it is an open problem even when only one parameter, i.e., $\alpha_i = \alpha, i = 1, 2, \cdots, k-1$, is used.

Consider the 2-D BVP

$$Lu \equiv -\nabla^2 u(x) + q\, u(x) = f(x), \quad x \in \Omega, \quad u(x)|_\Gamma = g(x) \tag{57}$$

where Γ is the boundary of $\Omega \equiv (0,1) \times (0,1)$ and $q \geq 0$ is a constant. We formulate a SAM based on a k-way splitting of the domain Ω, i.e., we decompose our domain into k overlapping subdomains Ω_i along the x_1-axis and make a strip-type decomposition of the rectangular domain Ω (for instance, see Figure 2). Next we apply the mixed interface conditions (2) on the two interior boundaries between subdomains Ω_i and Ω_{i+1}. Let ℓ be the length of the overlap in x_1-direction and η be the length of each subdomain in the same direction. Figure 2 depicts such a 2-way splitting of the unit square Ω.

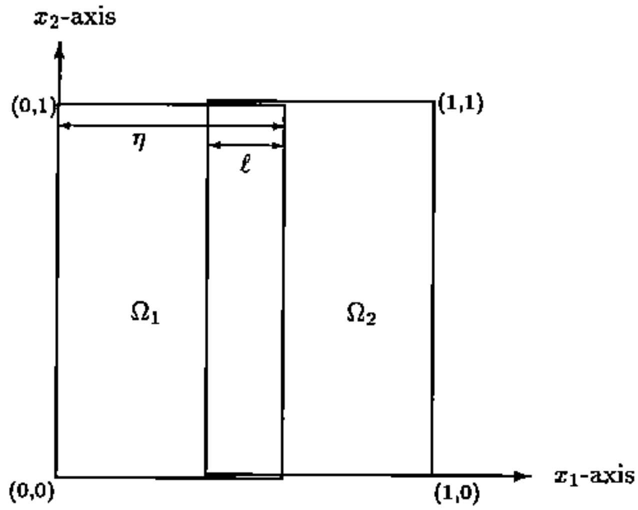

FIG. 2. *A 2-way splitting of the unit square Ω.*

5.1. Formulation of the Multi-Parameter SAM for Two-Dimensional Problems. To begin our analysis we use a 5-point finite difference discretization scheme with uniform grid of mesh size $h = \frac{1}{n+1}$ and discretize the BVP in (57) to obtain a linear system of the form

$$A\,x = f. \tag{58}$$

The natural ordering of the nodes is adopted starting from the origin and going in the x_2-direction first so that the resulting matrix A can be partitioned into block matrices corresponding to the subdomains, respectively. Using tensor product notation (see [7], and [14] in which tensor products in connection with BVP's were used for the first time), the matrix A in (58) can be written as

$$A = T_n(\beta) \otimes I_n + I_n \otimes T_n(2) \tag{59}$$

where $\beta = 2 + qh^2$ and the $T_j(x)$ is defined in (4).

Define $l + 1 = \frac{\ell}{h}$ and $m + 1 = \frac{\eta}{h}$ so that $n = mk - l(k-1)$ and $l < \frac{m-1}{2}$. As in the 1-D case in Section 3, the 2-D MPSAM transforms the matrix A in (59) into the corresponding *Schwarz Enhanced Matrix* $\widetilde{A}$ with parameters α_i. Specifically we have

$$\widetilde{A} = \widetilde{T_n(\beta)} \otimes I_n + I_{km} \otimes T_n(2). \tag{60}$$

Based on (60), the *multi-parameterized SAM* for $\widetilde{A}$ in (60) is *defined* as

$$\widetilde{A} = M - N \tag{61}$$

with

$$M = S_{km} \otimes I_n + I_{km} \otimes T_n(2), \quad N = B_{km} \otimes I_n, \tag{62}$$

where S_{km} and B_{km} were defined in formulas (39) and (40), respectively.

5.2. Convergence Analysis. The convergence analysis in the present case is reduced to determining the spectral radius of the block Jacobi matrix

$$\widetilde{J} = M^{-1}N \tag{63}$$

of $\widetilde{A}$ in (61). To begin our analysis, we state and prove two lemmas.

LEMMA 5.1. *Let A and B be $m\times m$ and $n \times n$ matrices, respectively. Then, there exists an $(mn) \times (mn)$ permutation matrix P such that $P(A\otimes B)P^{-1} = B \otimes A$.*

Proof. The permutation matrix P is the matrix that moves the rows $(i-1)n+j$ to the rows $(j-1)m+i$, for every $i = 1,2,\cdots,m$ and for every $j = 1,2,\cdots,n$. □

LEMMA 5.2. *The matrix $\widetilde{J} = M^{-1}N$ in (63) is similar to the matrix*

$$diag(J_1, J_2, \cdots, J_n) \tag{64}$$

where

$$\begin{array}{l} J_i = (diag(S_1(\beta+\gamma_i), S_2(\beta+\gamma_i), \cdots, S_k(\beta+\gamma_i)))^{-1}\ B_{km}, \\ \gamma_i = 2 + 2\ \cos(\frac{i\pi}{n+1}) \end{array} \tag{65}$$

for $i = 1,2,\cdots,n$, where $S_j(x)$, $j = 1,2,\cdots,k$, are defined in (37).

Proof. Let X_n be the $n \times n$ orthogonal matrix whose columns are the eigenvectors of the matrix $T_n(2)$. Since the eigenvalues of the matrix $T_n(2)$ are known to be $\gamma_i = 2+2\cos(\frac{i\pi}{n+1})$, $i = 1,2,\cdots,n$, we can write

$$X_n^T\ T_n(2)\ X_n = D_n \equiv diag(\gamma_1, \gamma_2, \cdots, \gamma_n). \tag{66}$$

Let I_{km} be the identity matrix of order $km(= k\times m)$ and let $X = I_{km}\otimes X_n$, then its inverse is given by $X^{-1} = I_{km}\otimes X_n^T$. Using X, we can construct a new matrix $\widetilde{J}'$, which is similar to the matrix $\widetilde{J}$, as follows

$$\widetilde{J}' = X^{-1}\widetilde{J}X = X^{-1}M^{-1}NX = (X^{-1}MX)^{-1}(X^{-1}NX).$$

However, if we replace X and M by their tensor product representations and perform simple operations, we obtain

$$\begin{aligned} X^{-1}MX &= (I_{km}\otimes X_n^T)\ (S_{km}\otimes I_n)\ (I_{km}\otimes X_n) \\ &\quad +(I_{km}\otimes X_n^T)\ (I_{km}\otimes T_n(2))\ (I_{km}\otimes X_n) \\ &= S_{km}\otimes (X_n^T\ I_n\ X_n) + I_{km}\otimes (X_n^T\ T_n(2)\ X_n) \\ &= S_{km}\otimes I_n + I_{km}\otimes D_n. \end{aligned}$$

Similarly,

$$\begin{aligned} X^{-1}NX &= (I_{km}\otimes X_n^T)\ (B_{km}\otimes I_n)\ (I_{km}\otimes X_n) \\ &= ((I_{km})B_{km}(I_{km}))\otimes (X_n^T\ I_n\ X_n). \\ &= B_{km}\otimes I_n. \end{aligned}$$

By Lemma 5.1, there exists a permutation matrix P such that

$$\begin{aligned} \widetilde{J}'' &= P\widetilde{J}'P^{-1} \\ &= P(X^{-1}MX)^{-1}(X^{-1}NX)P^{-1} \\ &= (P(S_{km}\otimes I_n)P^{-1} + P(I_{km}\otimes D_n)P^{-1})^{-1}\ P(B_{km}\otimes I_n)P^{-1} \\ &= (I_n\otimes S_{km} + D_n\otimes I_{km})^{-1}\ (I_n\otimes B_{km}). \end{aligned}$$

On the other hand we have

$$
\begin{aligned}
&I_n \otimes S_{km} + D_n \otimes I_{km} \\
&= I_n \otimes diag(S_1(\beta), S_2(\beta), \cdots, S_k(\beta)) + diag(\gamma_1, \gamma_2, \cdots, \gamma_n) \otimes I_k \otimes I_m \\
&= diag(\ diag(S_1(\beta+\gamma_1), S_2(\beta+\gamma_1), \cdots, S_k(\beta+\gamma_1)), \\
&\ diag(S_1(\beta+\gamma_2), S_2(\beta+\gamma_2), \cdots, S_k(\beta+\gamma_2)), \\
&\ \cdots\cdots, \\
&\ diag(S_1(\beta+\gamma_n), S_2(\beta+\gamma_n), \cdots, S_k(\beta+\gamma_n))).
\end{aligned}
$$

So the result (64)–(65) follows immediately. $\square$

From Lemma 5.2, we see that each submatrix J_i in (65) has the same form as the Jacobi matrix (41) in the 1-D case in (3). All submatrices in (64) are related to the same set of parameters α_i, $i = 1, 2, \cdots, k-1$. However, the entries of any submatrix in (64) are different from those of the other submatrices. It follows that in order to minimize the spectral radius of the matrix in (64) we must find the set of α_i's which minimizes the maximum of all spectral radii of the submatrices in (64).

5.3. On the Determination of the Optimal Parameters. From Lemma 5.2 and Lemma 3.2, we know that the spectral radius of $\widetilde{J}$ in (63) is

$$\rho(\widetilde{J}) = \max(\rho(G_k^1), \rho(G_k^2), \cdots, \rho(G_k^n)) \tag{67}$$

where G_k^i, $i = 1, 2, \cdots, n$, is the matrix G_k in (47) with

$$
\begin{aligned}
&x_{ij} = \delta_{l+1}^{i,j} - \alpha_i \delta_l^{i,j}, \quad y_{ij} = \delta_{m-l}^{i,j} - \alpha_j \delta_{m-l+1}^{i,j}, \\
&\delta_p^{i,j} = \frac{\sinh((m-p+1)\theta_i) - \alpha_j \sinh((m-p)\theta_i)}{\sinh((m+1)\theta_i) - (\alpha_i+\alpha_j)\sinh(m\theta_i) + \alpha_i\alpha_j \sinh((m-1)\theta_i)}, \\
&\theta_i = \operatorname{arccosh}(\tfrac{\beta+\gamma_i}{2}).
\end{aligned} \tag{68}
$$

Note that $\rho(G_k^i)$ is a function of $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$. Our goal is to determine the optimal values of $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ which minimize the spectral radius $\rho(\widetilde{J})$ in (67).

In the case of two subdomains ($k = 2$), our MPSAM case is reduced to the 1PSAM case with $\alpha_1 = \alpha$. From Remark 2.2 and Lemma 2.4, we know that the spectral radius of G_2^i, $i = 1, 2, \cdots, n$, is given by

$$\rho(G_2^i(\alpha)) = |\varphi_i(\alpha)| \tag{69}$$

with $\alpha = \alpha_1$ and that

$$\varphi_i(\alpha) = \frac{\sinh((m-l)\theta_i) - \alpha\sinh((m-l+1)\theta_i)}{\sinh((m+1)\theta_i) - \alpha\sinh(m\theta_i)}, \tag{70}$$

where $\theta_i = \operatorname{arccosh}(\frac{\beta+\gamma_i}{2})$.

In the following lemmas, the properties of the functions $\varphi_i(\alpha)$, $i = 1, 2, \cdots, n$, are investigated. Their proofs as well as that of 5.5 are rather technical and the interested reader is referred to Lemmas 2.3-2.6 and Proposition 2.1 of [11].

LEMMA 5.3. *Each function φ_i, $i = 1, 2, \cdots, n$, is strictly decreasing in the interval $(0,1)$ with $\varphi_i(0) > 0$ and $\varphi_i(1) < 0$. Therefore each equation $\varphi_i(\alpha) = 0$, $i = 1, 2, \cdots, n$, has a unique solution, say α_i, in the interval $(0,1)$.*

LEMMA 5.4. *If one defines $\varphi_i^+(\alpha) = \varphi_n(\alpha) + \varphi_i(\alpha)$ for $\alpha \in [0,1]$, $i = 1, 2, \cdots, n-1$, then each function φ_i^+ is strictly decreasing in the interval $(0,1)$ with $\varphi_i^+(0) > 0$*

and $\varphi_i^+(1) < 0$. *Therefore each equation* $\varphi_i^+(\alpha) = 0,\ i = 1,2,\cdots,n-1$, *has a unique solution, say* α_i^+, *in the interval* $(0,1)$.

PROPOSITION 5.5. *If* $\varepsilon \in (0,1)$ *is fixed, the functions*

$$\phi_1(\theta) = \frac{\cosh(\varepsilon\theta)}{\cosh(\theta)} \quad and \quad \phi_2(\theta) = \frac{\sinh(\varepsilon\theta)}{\sinh(\theta)}$$

are strictly decreasing functions of $\theta \in (0,\infty)$.

LEMMA 5.6. *If one defines* $\varphi_i^-(\alpha) = \varphi_n(\alpha) - \varphi_i(\alpha)$ *for* $\alpha \in [0,1]$, $i = 1,2,\cdots,n-1$, *then we have* $\varphi_i^-(0) > 0$ *and* $\varphi_i^-(1) < 0$ *and each equation* $\varphi_i^-(\alpha) = 0$, $i = 1,2,\cdots,n-1$, *has a unique solution, say* α_i^-, *in the interval* $(0,1)$.

LEMMA 5.7. *If* $\alpha_i, \alpha_i^+, \alpha_i^-$, $i = 1,2,\cdots,n-1$, *are defined to be the solutions of the equations* $\varphi_i(\alpha) = 0$, $\varphi_i^+(\alpha) = 0$, $\varphi_i^-(\alpha) = 0$, *respectively, then we have*

$$\alpha_i < \alpha_{i+1} \quad and \quad \alpha_i < \alpha_i^+ < \alpha_n < \alpha_i^-,$$

for each $i = 1,2,\cdots,n-1$.

Based on the statements so far we can state and prove the following theorem.

THEOREM 5.8. *Let* α_i^+ *be the solution of the equation* $\varphi_i^+(\alpha) = 0$, $i = 1,2,\cdots,n-1$, *in* $(0,1)$. *Then the optimal value* α^o *of* α, *which minimizes the spectral radius of the matrix* $\rho(\widetilde{J}) = \rho(\widetilde{J}(\alpha))$ *in (63), is given by*

$$\alpha^o = \min\{\alpha_i^+ : i = 1,2,\cdots,n-1\}.$$

Proof. Let i_0 be the index so that $\alpha^o = \alpha_{i_0}^+$. We will show that

$$\rho(G_2^n(\alpha^o)) \le \max\left(\rho(G_2^1(\alpha)), \rho(G_2^2(\alpha)), \cdots, \rho(G_2^n(\alpha))\right), \tag{71}$$

for each of the three cases $\alpha \in [0,\alpha_{i_0}^+)$, $\alpha \in (\alpha_{i_0}^+, 1]$ and $\alpha = \alpha_{i_0}^+$. Then, our assertion follows from (67) and (71). *Case 1)* : $\alpha \in [0,\alpha_{i_0}^+)$. We have

$$\alpha < \alpha_i^+,\ i = 1,2,\cdots,n-1, \tag{72}$$

because $\alpha_{i_0}^+ \le \alpha_i^+$, and hence by Lemma 5.7 we have

$$\alpha < \alpha_i^-,\ i = 1,2,\cdots,n-1. \tag{73}$$

By Lemmas 5.4 and 5.6, (72) and (73) we have that $\varphi_i^+(\alpha) > 0$ *and* $\varphi_i^-(\alpha) > 0$, $i = 1,2,\cdots,n-1$, i.e., $|\varphi_n(\alpha)| > |\varphi_i(\alpha)|$, $i = 1,2,\cdots,n-1$, Consequently, we have

$$|\varphi_n(\alpha)| = \max(|\varphi_1(\alpha)|, |\varphi_2(\alpha)|, \cdots, |\varphi_n(\alpha)|). \tag{74}$$

On the other hand, by Lemma 5.7 we know $\alpha_{i_0}^+ < \alpha_n$ and hence by Lemma 5.3 we obtain

$$|\varphi_n(\alpha_{i_0}^+)| < |\varphi_n(\alpha)|. \tag{75}$$

From (74) and (75), we conclude that

$$|\varphi_n(\alpha_{i_0}^+)| \le \max(|\varphi_1(\alpha)|, |\varphi_2(\alpha)|, \cdots, |\varphi_n(\alpha)|),$$

which implies (71) by (69).

Case 2) : $\alpha \in (\alpha_{i_0}^+, 1]$. By Lemma 5.3

$$\varphi_{i_0}(\alpha) < \varphi_{i_0}(\alpha_{i_0}^+). \tag{76}$$

Since $\alpha_{i_0} < \alpha_{i_0}^+$ by Lemma 5.6, we have, by Lemma 5.3,

$$\varphi_{i_0}(\alpha_{i_0}^+) < 0. \tag{77}$$

From (76) and (77), we obtain

$$|\varphi_{i_0}(\alpha_{i_0}^+)| < |\varphi_{i_0}(\alpha)|. \tag{78}$$

Since $\varphi_n(\alpha_{i_0}^+) + \varphi_{i_0}(\alpha_{i_0}^+) = 0$ by the definition of $\alpha_{i_0}^+$, we have

$$|\varphi_{i_0}(\alpha_{i_0}^+)| = |\varphi_n(\alpha_{i_0}^+)|. \tag{79}$$

Then (78) and (79) imply $|\varphi_n(\alpha_{i_0}^+)| < |\varphi_{i_0}(\alpha)|$. Since $1 \le i_0 \le n-1$, we can write

$$|\varphi_n(\alpha_{i_0}^+)| \le \max(|\varphi_1(\alpha)|, |\varphi_2(\alpha)|, \cdots, |\varphi_n(\alpha)|),$$

which implies (71) by (69).

Case 3) : $\alpha = \alpha_{i_0}^+$. It is obvious that

$$|\varphi_n(\alpha_{i_0}^+)| \le \max(|\varphi_1(\alpha)|, |\varphi_2(\alpha)|, \cdots, |\varphi_n(\alpha)|),$$

which implies (71) by (69). □

For the case of three subdomains ($k = 3$), we can compute the spectral radii of the matrices G_3^i, $i = 1, 2, \cdots, n$, in (67) from the expression of G_3 in Lemma 3.2 as follows

$$\rho(G_3^i) = \max\left(\sqrt{\left| x_{10}x_{12} + x_{21}x_{23} \pm \sqrt{x_{10}^2 x_{12}^2 + x_{21}^2 x_{23}^2 + 2x_{10}x_{23}(2y_{12}y_{21} - x_{12}x_{21})} \right|}\right) \tag{80}$$

where x_{ij}, y_{ij}, $\delta_p^{i,j}$, θ_i are given in the beginning of this section. Since $\delta_p^{i,j}$ is a function of the parameters α_1, α_2, so is G_3^i. It seems very difficult to determine analytically the optimal values of α_1, α_2 which minimize the spectral radius

$$\rho(\widetilde{J}) = \max(\rho(G_3^1), \rho(G_3^2), \cdots, \rho(G_3^n))$$

of $\widetilde{J}$ in (63) even if we only consider just the one-parameter case, i.e., $\alpha_1 = \alpha_2$. In this case the expression (80) reduces to $\rho(G_3^i) = \max\left(\sqrt{|g_1(g_2 \pm g_3)|}\right)$ (see (35)), where g_1, g_2, g_3 are defined in (25) with $\theta(= \theta_i) = \operatorname{arccosh}(\frac{\beta+\gamma_i}{2})$. This is something we intend to investigate further in the near future.

5.4. Numerical Experiment. In this section, we present a numerical experiment for the case of $k = 2$, in order to confirm the analysis in Theorem 5.8. For this, consider the following model problem

$$\begin{aligned} -\nabla^2 u(x,y) &= 0, \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ u(x,y) &= f(x,y), \quad (x,y) \in \Gamma, \end{aligned} \tag{81}$$

where Γ is the boundary of Ω, with solution

$$u(x,y) = \sin(2\pi x)\cos(2\pi y).$$

We report on experiment for the problem (81) using $k = 2, m = 6, l = 1$ and $n = 11$. Figure 3 shows the spectral radii of the eleven submatrices $J_i(\alpha)$ in (64). Using Theorem 5.8, the optimal value of α can be calculated numerically as $\alpha = 0.654$. Figure 4 shows the number of the block Jacobi iterations required to reduce the ℓ_2-norm of the residual by a factor of 10^{-4}. Figure 5 shows the ratio of the ℓ_2-norm of the residual relative to its initial norm after five block Jacobi iterations. Figures 4 and 5, show that the smallest number of iterations and the smallest relative ℓ_2-norm of the residual are achieved near the value $\alpha = 0.654$ confirming our theoretical analysis.

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TABLE 1

The classical SAM is applied to the BVP (55) for $k = 3,4,8$ domain splittings, $m = 10$ and 20 local grids and minimum (1) and half ($[m-1]/2$) overlap. Columns 3 to 5 display the convergence factors r_p (56), with $p = k$, after k iterations.

Local grid	*Overlap*	*Convergence factor*		
m	l	k=3	k=4	k=8
10	1	0.55	0.57	0.71
10	4	0.63	0.63	0.75
20	1	0.46	0.51	0.69
20	9	0.62	0.63	0.75

TABLE 2

The 1PSAM is applied to the BVP (55) for the input parameters defined in Table 1. The parameter α is computed as the numerical solution of equations (33) and (34). The convergence factor r_k and the value of the optimum parameter α are displayed in columns 3 to 5.

Local grid	*Overlap*	*Convergence factor (value of α)*		
m	l	k=3=*iters*	k=4=*iters*	k=8=*iters*
10	1	1.4E–5 (0.887)	1.5E–1 (0.893)	4.8E–1 (0.925)
10	4	1.3E–5 (0.844)	2.1E–1 (0.851)	5.2E–1 (0.907)
20	1	1.3E–5 (0.943)	1.4E–1 (0.947)	4.2E–1 (0.963)
20	9	1.3E–5 (0.909)	2.1E–1 (0.914)	5.0E–1 (0.955)

TABLE 3

The MPSAM is applied to the BVP (55) for the input parameters defined in Table 1. The relative residual in ℓ_2-norm is 2E–15 after k iterations. The parameters α_i are selected to be symmetric, thus only $[k-1]/2+1$ are depicted. The convergence factor r_k and half of the parameters α_i are displayed in columns 3 to 5.

		Convergence factor (values of α_i's)		
Local grid	*Overlap*	$k = 3 = iters$	$k = 4 = iters$	$k = 8 = iters$
m	l	r_3 (α_1)	r_4 (α_1, α_2)	r_8 ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$)
10	1	1.3E-5 (0.887)	1.8E-4 (0.892, 0.932)	1.3E-2 (0.898, 0.943, 0.958, 0.965)
10	4	1.3E-5 (0.844)	1.8E-4 (0.848, 0.906)	1.3E-2 (0.855, 0.918, 0.939, 0.949)
20	1	1.3E-5 (0.943)	1.8E-4 (0.946, 0.967)	1.3E-2 (0.949, 0.972, 0.980, 0.983)
20	9	1.3E-5 (0.909)	1.8E-4 (0.912, 0.947)	1.3E-2 (0.915, 0.954, 0.966, 0.972)

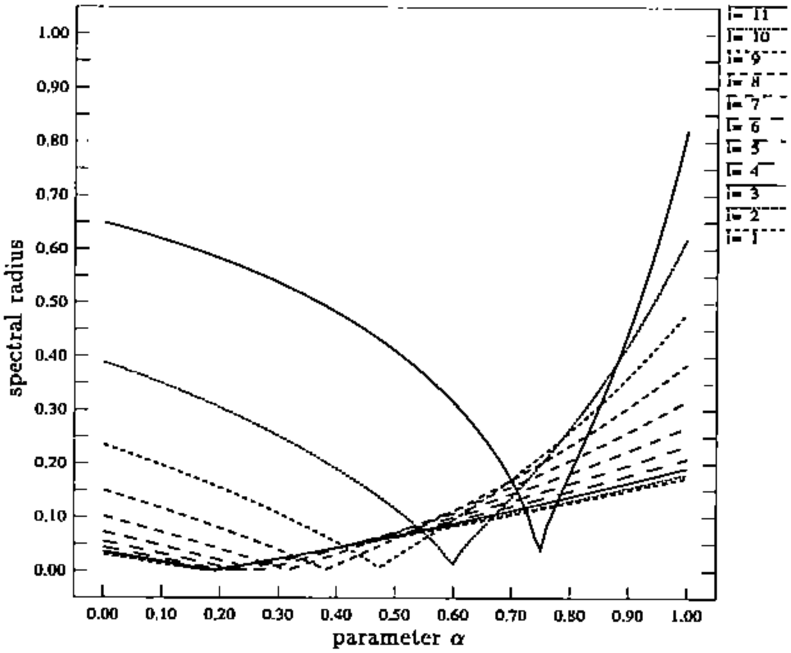

FIG. 3. *Depicts the numerical behavior of the spectral radii of the blocks $\widetilde{J}_i(\alpha)$ in the Jacobi iteration matrix $\widetilde{J}(\alpha)$ of 1PSAM when it is applied to the 2-D model problem (81) with $k = 2$, $m = 6$, and overlap $l = 1$.*

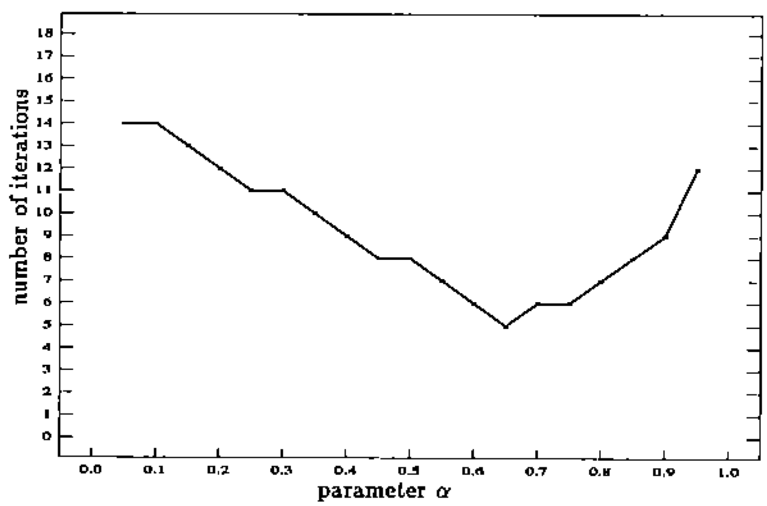

FIG. 4. *Displays the number of iterations versus the parameter α required to reduce the ℓ_2-norm of the initial residual by a factor 10^{-4} for the 1PSAM considered in Figure 3.*

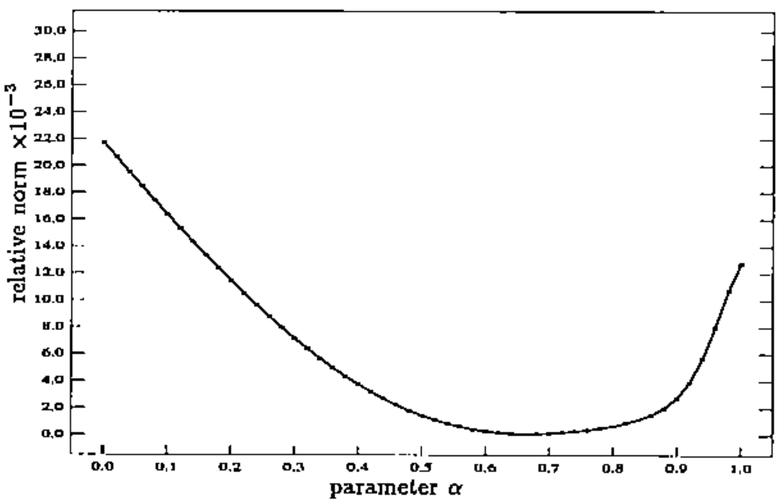

FIG. 5. *The 1PSAM is applied to BVP (81) for $m = 6$, $k = 2$, and overlap $l = 1$. The graph displays the relative ℓ_2-norm of the residual versus the parameter α after five block Jacobi iterations.*