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Abstract. Given a family of convex low degree polynomial curve segments with fixed end points, subregions of the plane are characterized in which additional points can be interpolated by at least one member of the family. Specifically, interpolation bounds and regions for quadratic and convex cubic planar curve segments are exhibited.

§1. Introduction

A planar polynomial curve of parametric degree d is uniquely determined by $n = \binom{d+2}{2} - 2$ data in general position. However, given n data points in general position, there may not exist an interpolating polynomial curve. Whether it exists depends on the relative position of the points. Thus the question posed at the outset of this paper is not just how many points can be interpolated by a low degree curve but also *where* these points have to lie with respect to each other. For example, as users of spline functions, we may be tempted to claim that a parabolic arc is uniquely specified by three points. This is incorrect, since it ignores the freedom of parametrization. Approaching the question algebraically, we find that while five points determine a conic, four suffice to pin down a parabolic arc. Yet also this answer is incomplete, as the fourth point has to lie in a specific region with respect to the other points in order to be matched. In the cubic case, De Boor, Höllig, and Sabin [1] show that at times but not always, six data, location, tangent direction and curvature at the end points, can be interpolated by a cubic curve segment. In general, restrictions of the implicit form of the curve to ensure that the curve segment is real valued and has one component are nonlinear and make the problem challenging.

Given the frequent use of quadratic and cubic curve segments in geometric modeling, understanding their basic interpolation properties is fundamental. Applications arise for example from the need of constraint solvers to completely characterize the space of all solutions to a given set of constraints

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on its design primitives (see e.g. [2]). Typically, one needs to know what data can be matched by *one* curve segment. This is in contrast to the well-known problem of spline interpolation. Here the issue of understanding the interpolation properties of curves is usually avoided by using more degrees of freedom than data: it is customary to use one curve segment per data point, additional degrees of freedom to join the pieces smoothly, and some rule to discard the remaining, nonlinear degrees of freedom. An example of the latter is the choice of chord length parametrization for splines.

In this paper we consider the case of polynomial quadratic segments and look at their convex cubic cousins where a direct, symbolic treatment of the question is no longer easy because it requires the solution of two cubic equations in two unknowns. The key to the investigation is to treat the curves as bivariate maps $c(t,v)$, with one parameter, t, fixing the parametrization, while the other, v, serves to traverse the particular curve. The two parameters, can and are in the course of the proofs treated symmetrically. As befits the notion of an interpolation region, the answer is given graphically in the spirit of [3] and [4]. We may choose a convenient coordinate system since the results are affinely invariant; that is, the fixed coordinate system can be mapped to the coordinate system of interest and the interpolation regions are the affine image of the regions for the fixed coordinate system. Thus we choose the first point of the segment to be $(0,1)$ and the last to be $(1,0)$. Also by affine mapping, we can choose the parameter values v at the end points to be 0 and 1 respectively. Additional points through with the curve segment should pass for a parameter value $v=t$ in the open interval $(0..1)$ are called *intermediate points.* To exhibit symmetries in the problem I use

$$u := 1 - v, \qquad s := 1 - t.$$

In this paper, (a,b) denotes a vector, $(a..b)$ an open interval and $[a..b]$ a closed interval.

§2. Interpolation Regions for Quadratic Plane Curves

Consider a family of curve segments, $c(t,v)$, whose members are indexed by the parameter $t \in (0..1)$ and are quadratic in the parameter $v \in [0..1]$. With the appropriate choice of coordinate system, c has the Bernstein-Bézier representation,

$$c(t,v) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^2 + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} 2uv + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v^2, \quad v \in [0,1].$$

To complete the choice of coordinate system we choose the origin to be the first intermediate point. This excludes the trivial case of an intermediate point on the line segment from $(0,1)$ to $(1,0)$, which corresponds to the line segment itself as the region of interpolation. The first lemma characterizes parabolic interpolation of one intermediate point.

Lemma 2.1. *A member of the family $c(t,v)$ interpolates the origin if and only if*

$$b_1(t) = -\frac{t}{2s}, \qquad b_2(t) = -\frac{s}{2t}.$$

Proof: Let t be the value of v at which the quadratic $c(t,\cdot)$ interpolates the origin. Then $t(2sb_1(t)+t)=0$ and $s(2tb_2(t)+s)=0$ and the claim follows, because neither s nor t is zero. ■

We observe that c is a bivariate rational map of degree $2,1$ in the numerator and $0,1$ in the denominator. Since the parameters t and v are still free, we may attempt to interpolate of a fourth point (x,y) by solving the equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v^2 - \frac{t}{s}uv \\ u^2 - \frac{s}{t}uv \end{pmatrix}.$$

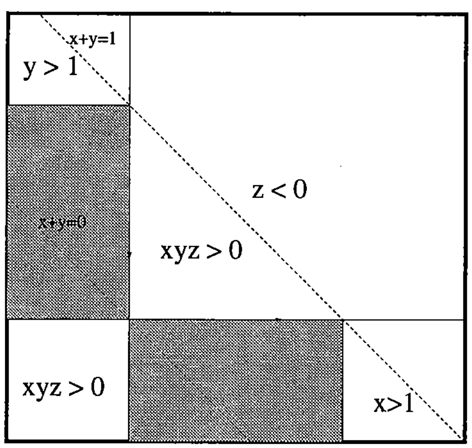

Fig. 2.1. A quadratic curve segment interpolating $(0,1)$, $(0,0)$ and $(1,0)$ in order can interpolate the point (x,y) if and only if (x,y) falls into the shaded region.

The following proposition summarizes parabolic interpolation of two intermediate points.

Proposition 2.2. *Let $c : (0..1) \times [0..1] \mapsto \mathbb{R}^2$ be a family of parabolic curve segments with*

$$c(t,0) = (0,1), \quad c(t,t) = (0,0), \quad c(t,1) = (1,0).$$

Then a member $c(t^*,\cdot)$ of the family can interpolate a point (x,y) if and only if $x < 0$ and $0 < y < 1$ or, symmetrically, $y < 0$ and $0 < x < 1$. If it exists, the curve $c(t^*,\cdot)$ is unique.

Proof: Since s and t are both nonzero, the interpolation constraint is equivalent to

$$xs = v^2 s - tuv = v(s-u) \qquad yt = u^2 t - suv = u(t-v).$$

Since $t, v \in (0..1)$, $xs > vs - vu$ for $x > 1$. Therefore $x < 1$ and, by the symmetric argument, $y < 1$. Moreover, with $\sigma := \operatorname{sgn}(x+y)$ and

$$z := 1 - x - y,$$

the two linear-quadratic equations in two unknowns have the solution $v^*, t^* \in (0,1)$ if and only if

$$v^* = \begin{cases} \frac{x+\sigma\sqrt{-xyz}}{x+y}, & \text{if } x \neq -y \\ \frac{1-y}{2}, & \text{if } x = -y \end{cases} \qquad t^* = \begin{cases} \frac{xz+\sigma\sqrt{-xyz}}{(x+y)z}, & \text{if } x \neq -y \\ \frac{1+y}{2}, & \text{if } x = -y \end{cases}.$$

Since the second intermediate point is to be real,

$$xyz < 0 \qquad z > 0$$

must hold and, conversely, this restriction result in valid t^* and v^*. ■

The choice of $z = 1 - x - y$ allows us to interpret x, y and z as barycentric coordinates of the triangle. We take this point of view as we characterize the interpolation regions of rational quadratics with positive weights. After a rational linear reparametrization, which does not affect the shape of the curve, any conic has the form

$$c(t,v) := \frac{\begin{pmatrix}0\\1\end{pmatrix} u^2 + w \begin{pmatrix}b_1(t)\\b_2(t)\end{pmatrix} 2uv + \begin{pmatrix}1\\0\end{pmatrix} v^2}{u^2 + w * 2uv + v^2}, \quad v \in [0,1].$$

We will only consider conics restricted by the requirement $w > 0$ so that the denominator $q := q(v,w) := u^2 + w * 2uv + v^2 > 0$.

The following corollary characterizes conic interpolation of two intermediate points for positive weights.

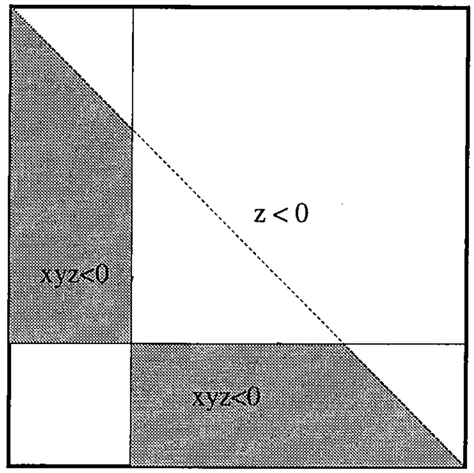

Fig. 2.2. A rational quadratic curve segment with positive weights interpolating the three vertices of the central triangle can interpolate a second intermediate point only in the shaded regions.

Corollary 2.3. *Let $w > 0$ and $c : (0..1) \times [0..1] \mapsto \mathbb{R}^2$, be a family of conic curve segments such that*

$$c(t,0) = (0,1), \quad c(t,t) = (0,0), \quad c(t,1) = (1,0).$$

Then a member $c(t^*, \cdot)$ of the family can interpolate a second intermediate point (x, y) if and only if $xyz < 0$ and $z := 1 - x - y > 0$.

Proof: If $c(t,t)$ is the origin, then

$$b_1(t) = -\frac{t}{2sw}, \qquad b_2(t) = -\frac{s}{2tw}.$$

To interpolate the second intermediate point

$$q\begin{pmatrix} xs \\ yt \end{pmatrix} = \begin{pmatrix} v^2 s - tuv \\ u^2 t - suv \end{pmatrix}$$

has to hold. By choice of w, we may scale $x' := qx$ and $y' := qy$ to satisfy the constraints $x' < 1$, $y' < 1$ of Proposition 2.2. Conversely, by the same argument as in the polynomial case, $x'y'z' < 0$ and $z' > 0$ has to hold. ■

§3. Interpolation Regions for Convex Cubic Plane Curves

We now consider the 2-parameter family of cubic plane curve segments, $c(t,v)$ that interpolate given locations and tangent directions at the end points and are convex. As before, we characterize the regions where one, respectively two intermediate points can be interpolated. Even for the restricted cubics, the problem is considerably harder than in the quadratic case, since a direct solution of the second intermediate point requires finding the joint roots of two bivariate cubic rational functions which in general can not be done symbolically.

Using the intersection point of the tangents emanating from the points $(0,1)$ and $(1,0)$ respectively as the third point necessary to specify a coordinate system, we may assume without loss of generality that

$$c(t,v) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^3 + \begin{pmatrix} 0 \\ b_1(t) \end{pmatrix} 3u^2v + \begin{pmatrix} b_2(t) \\ 0 \end{pmatrix} 3uv^2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v^3.$$

Any other convex configuration of end points and finite tangent intercept is affinely related to the above choice. As before, we start with the interpolation of just one intermediate point and characterize cubic convex interpolation of one intermediate point.

Lemma 3.1. *A member of the family $c(t,v)$ interpolates the origin if and only if*

$$b_2(t) = \frac{x - t^3}{3st^2}, \qquad b_1(t) = \frac{y - s^3}{3s^2t}.$$

Proof: If a curve from the family interpolates (x,y) at $v = t$, then $t^2(3sb_2 + t) = x$ and $s^2(3tb_1 + s) = y$. ■

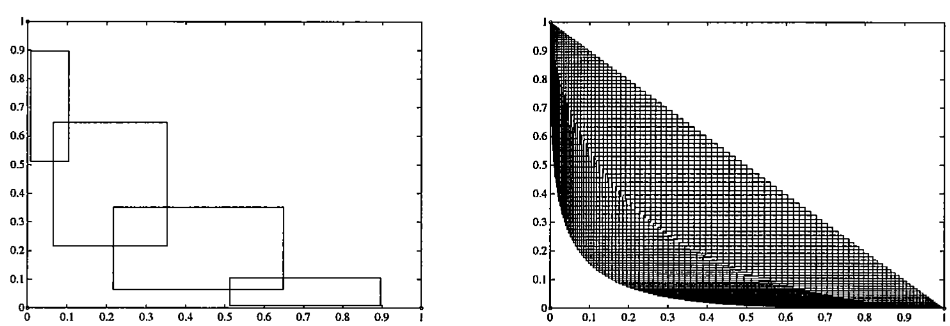

Fig. 3.1: (Left) Regions of points reachable at time $t = [.2, .4, .6, .8]$. (Right) The union of the reachable regions for $0 \le t \le 1$ is the region in which an intermediate point can be interpolated. The line $x + y = 1$ is excluded, because it does not have the correct tangent direction.

Since $0 < b_1(t) < 1$ and $0 < b_2(t) < 1$, t is contrained via

$$t^3 \le x \le 3st^2 + t^3, \quad s^3 \le y \le 3s^2t + s^3.$$

One can interpret the inequalities as yielding for a given parameter value t a $3st^2 \times 3s^2t$ rectangle of points that can be interpolated by curves from the family. We can determine the minimal parameter value $\underline{t}$ and the maximal parameter value $\bar{t}$

$$\underline{t} := \max\{1 - y^{\frac{1}{3}}, r_1\}, \quad \bar{t} := \min\{x^{\frac{1}{3}}, 1 - r_2\}$$

where r_1 is the least real root in $[0..1]$ of $-2z^3 + 3z^2 - x$ and r_2 is the least real root in $[0..1]$ of $-2z^3 + 3z^2 - y$. We note that, if the curve $c(\underline{t}, \cdot)$ interpolates the y-component of the interpolation point with the (least possible) parameter value $v = s^3$, then $b_1 = 0$ and hence $c(\underline{t}, \cdot) = (\,*, u^3\,)$. This implies that $c(\underline{t}, \cdot)$ reaches each y-level with the least possible parameter value.

The next lemma establishes monotonicity of the control polygon with respect to t.

Lemma 3.2. *With increasing t, $b_1(t)$ increases and $b_2(t)$ decreases.*

Proof: The change of the x-component of $c(t, v)$ and the change of b_2 with the parameter t is governed by

$$\frac{db_2(t)}{dt} = -\frac{3t}{(3ts^2)^2}(t^3 + x(2 - 3t))$$

The roots of $t^3 + x(2 - 3t)$ are

$$r_1 = \frac{x}{a} + a, \quad r_{2,3} = \pm\frac{\sqrt{-3}}{2}(\frac{-x}{a} + a) - \frac{r_1}{2}, \quad \text{where} \quad a^3 := x(-1 + \sqrt{1 - x}).$$

For $x \in [0, 1]$, $a \le 0$, and therefore $r_1 < 0$, while r_2 and r_3 are complex conjugate. That is, $\frac{db_2(t)}{dt}$ is of one sign for permissible values of x. Since $\frac{db_2}{dt}(\frac{1}{2}) < 0$, the value of b_2 decreases with increasing t. Differentiation of b_2 with respect to s yields the same result. Since $\frac{db_2}{ds} = -\frac{db_2}{dt}$, the value of b_1 increases with increasing t. ■

We now consider the interpolation region for the second intermediate point. We will show that the interpolation region forms a simply connected region, shaped like a *double crescent* and bounded by a small number of specific curves from the cubic family of interpolants.

Following the direct approach of Section 2, one might attempt to set up and solve two additional interpolation equations. However, since the polynomials involved are cubic in the two parameters t and v, these equations are not symbolically solvable. The key to characterizing the interpolation region is an analysis of the regularity of the bivariate map c.

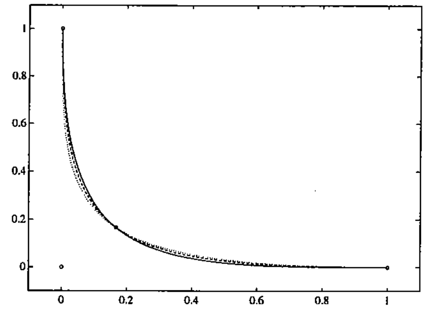

Fig. 3.2: Interpolation region of a second intermediate point of a convex cubic passing through [1/6, 1/6].

Lemma 3.3. *The rank of the Jacobian $Dc(t,v)$ on the segment $(0,1)$, (x,y), respectively on the segment (x,y), $(1,0)$, depends only on t and not on v. On either segment there are at most four real values of t such that $|Dc| = 0$. These are the roots of*

$$f(t) := t^4 - 2t^3(1+x-y) + t^2(1+5x-y) - 4tx + x(1-y).$$

Proof: We compute

$$\frac{\partial c(t,v)}{\partial v} = \frac{1}{(1-t)^2t^2}\begin{pmatrix} sv(2t^3 - 3t^2v - 2x) \\ tu(2s^3 - 3s^2u - 2y) \end{pmatrix}$$
$$\frac{\partial c(t,v)}{\partial t} = \frac{(1-v)v}{(1-t)^2t^2}\begin{pmatrix} \frac{v}{t}(t^3 - 3tx + 2x) \\ \frac{u}{s}(s^3 - 3sy + 2y) \end{pmatrix}$$

to get

$$|Dc(t,v)| = \frac{\partial c(t,v)}{\partial v} \times \frac{\partial c(t,v)}{\partial t} = k * f(t)$$

where $k := (v-t)\frac{3(1-v)^2v^2}{(1-t)^4t^4}$ is of one sign for any segment from one end point to the first intermediate point. ■

The next theorem characterizes convex cubic interpolation of two intermediate points.

Theorem 3.4. *Let $c : (0..1)\times[0..1] \mapsto \mathbb{R}^2$ be a family of cubic curve segments that interpolate*

$$c(t,0) = (0,1), \quad c(t,t) = (x,y), \quad c(t,1) = (1,0),$$

and have tangent directions $(0,-1)$ and $(1,0)$ at $v=0$ and $v=1$ respectively. Then the region in which a second intermediate point can be interpolated is bounded by the six curves

$$c(\underline{t},\cdot), c(t_1,\cdot), \ldots, c(t_4,\cdot), c(\overline{t},\cdot),$$

where $t_1, \ldots, t_4$ are roots of $f(t)$.

Proof: As a subset of the interpolation region for one intermediate point, the interpolation region for two intermediate points is bounded. Consider a point on the boundary corresponding to the parameters t', v'. Then $c(t', v')$ is an extreme point of the curve $c(\cdot, v')$, i.e. either $t' \in \{\underline{t}, \overline{t}\}$ or $c(\cdot, v')$ is singular at t' and hence t' is one of t_1 through t_4. Thus any point on the boundary of the region belongs to one of the six curves. ■

The detailed example in the next section illustrates that indeed any of the singular and extremal curves can be part of the boundary of the region. We state as a *conjecture*, supported by numerous examples, that members of the family of cubic convex curves that interpolate one intermediate curve can not intersect transversally but only by complete overlap. This would imply uniqueness of the interpolation curve to two intermediate points.

We conclude this section with the observation that the iso-v curves are continuous. Since they connect at $v = 0$ and $v = 1$, they must fill the region in between with a continuum of curves. This proves the following corollary.

Corollary 3.5. *The region $c([\underline{t}..\overline{t}], [0..1])$ is simply connected.*

§4. A Cubic Example.

If $x = y$, then the bounds on $b_1(t)$ and $b_2(t)$ imply $0 \le x \le \frac{1}{2}$ (in fact $x \ge \frac{1}{5}$ must hold) and the four roots of $|Dc|$ away from the end points and the first intermediate point are

$$r_{1..4} := \frac{1}{2}(1 \pm \sqrt{1 - 8x \pm 4\sqrt{x(5x-1)}}).$$

Let $\hat{x} \approx 0.278067$ be the upper bound for x, namely the third power of the only root of $1 - 3t^2 + t^3$ in $[0..1]$. Since $1 - 8x + 4\sqrt{x(5x-1)}$ vanishes at $x = \frac{1}{4}$ and $1 - 8x - 4\sqrt{x(5x-1)}$ is either negative or imaginary, $|Dc|$ has either

$$\begin{cases} 0 & \text{real roots if } x < \frac{1}{4} \\ 1 & \text{real roots if } x = \frac{1}{4} \\ 2 & \text{real roots if } \hat{x} \ge x > \frac{1}{4}. \end{cases}$$

If $x < \frac{1}{4}$, Theorem 3.4 implies that $c(\underline{t}, \cdot)$ and $c(\overline{t}, \cdot)$ bound the double crescent. If $\hat{x} > x > \frac{1}{4}$, the two real solutions are $\frac{1}{2}(1 \pm \sqrt{1 - 8x + 4\sqrt{x(5x-1)}})$ and the bounding curves are singular curves that change with x. The pattern for curves of fixed t parameter crossing one another as the first intermediate point is moved from $x = y = \frac{1}{5}$ to $x = y = \frac{1}{2}$ is laid out below, followed by graphs of the singular and extremal curves in a small subsection of the crescent.

$x = y$	a	b	c	d	e	f
$\mid$	a	b	d	c	e	f
$\mid$	a	d	b	e	c	f
$\mid$	a	d	e	b	c	f
$\mid$	a	e	d	c	b	f
$\mid$	e	a	d	c	f	b
$\mid$	e	d	a	f	c	b
$\mid$	e	d	f	a	c	b
$\mid$	e	f	d	c	a	b
$\downarrow$	f	e	d	c	b	a

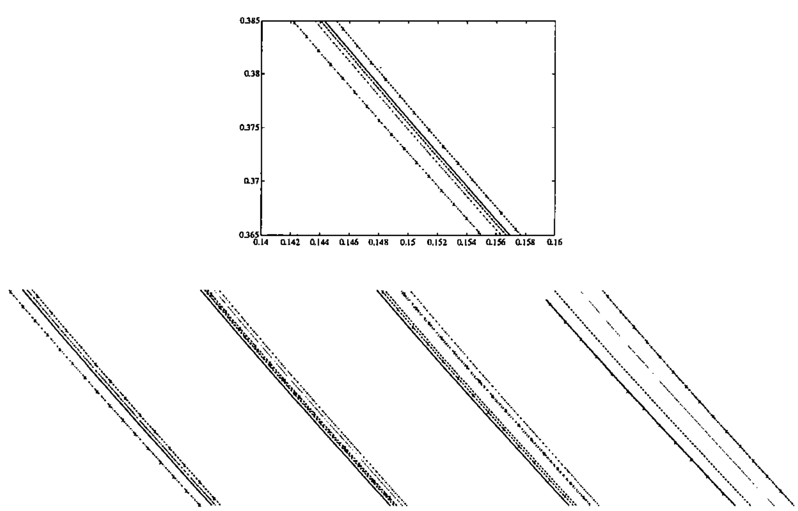

Fig. 4.1. Reversal of the ordering of the curves in the crescent as $x = y$ increases.

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