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**BIQUARTIC C1 SPLINE SURFACES
OVER IRREGULAR MESHES**

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Biquartic C^1 spline surfaces over irregular meshes

by

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Abstract

Extending and complementing the definition of C^1 surface splines in [Peters '93], the biquartic surface splines improve continuity and shape properties in the case where the user chooses to model entirely with four-sided patches. The splines generalize the construction of C^1 surfaces from tensor-product B-spline control points to a wider class of control meshes capable of outlining arbitrary free-form surfaces in space. Irregular meshes with non quadrilateral cells and more or fewer than four cells meeting at a point are allowed so that arbitrary free-form surfaces with or without boundary can be modeled in the same conceptual frame work as tensor-product B-splines. That is, the mesh points serve as control points of a smooth piecewise polynomial surface representation that is local and evaluates by averaging.

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1. Introduction

This paper extends and complements the definition of surface splines in [Peters '93] improving continuity and shape properties in the case where the user chooses to model entirely with four-sided patches rather than with biquadratics and three-sided cubics.

The representation is motivated by the fact that tensor-product B-splines, though widely used to represent functions and surfaces, can only model a small subclass of surfaces that arise in geometric modeling, because every interior point of the B-spline control mesh outlining the surface must be surrounded by exactly four quadrilateral cells. Thus the surface pieces must form a regular, checker board arrangement that allows only deformations of the plane and deformations of the torus to be modeled without singularity. Even for such topologically restricted objects, the rigid structure prevents a natural modeling of frequent features such as suitcase corners or house corners, where three or five quadrilaterals meet. In practice, patch trimming and certain singular parametrizations are used to overcome the restrictions. However, by destroying the consistent B-spline framework, these techniques create a number of special cases and difficult problems such as desingularization and the need for smoothly joining trimmed surfaces.

The biquartic C^1 surface splines developed in this paper extend C^1 tensor-product splines, removing the regularity restrictions while preserving the desirable properties of B-splines, such as a low degree polynomial or rational representation of maximal smoothness and a geometrically intuitive variation of the surface in terms of the coefficients as evidenced by a local convex hull property. (A more detailed list of properties of surface splines is compiled at the end of this section.) As with the C^1 surface splines defined in [Peters '93], shape handles in the form of blend ratios are associated with each edge allowing for local control of the change of the normal and the curvature across that edge.

Related work. Building on work by [Sabin '83] and [Goodman '91], [Höllig, Mögerle '89] pioneered the idea of geometrically continuous spline spaces. Explicit representations of such G-spline surfaces in terms of say the Bernstein-Bézier form require the solution of a large linear, irregularly sparse system of equations to match data. Newer work ([Piah '91], [Lee, Majid '91] and [Reif '93]), improves by exhibiting explicit formulas for the coefficients of the surface patches in terms of the input mesh. Still, the size of the support of these formulas makes it non trivial to predict the shape of the resulting surface. The formulas for biquartic G-splines, in contrast, are simple enough to read of, say the convex hull property. – Surface splines are related to the subdivision algorithms [Doo '78] and [Catmull '78] in that their first construction step subdivides the input mesh and creates a refined mesh. Ffs s are based on reparametrization and geometric smoothness as surveyed in [Gregory '90], but differ from approaches like [Sarraga '87] and [Hahn '89] in that no constraint systems have to be solved at run time to enforce patch to patch smoothness. B-patches ([Seidel '91], [Dahmen, Micchelli, Seidel '92]) and S-patches [Loop, DeRose '90] provide alternative approaches to defining smooth surfaces over irregular meshes at the cost of a presently nonstandard patch representation.

Structure of the paper. Section 2 defines the biquartic C^1 surface splines, Section 3 establishes their continuity, vector space and shape properties and Section 4 lists extensions and shows examples. This section concludes with a list of *properties of surface splines* (cf. [Peters '93]).

- *free-form modeling capability.* There are no restrictions on the number of cells meeting at a mesh point or the number of edges to a mesh cell. Mesh cells need not be planar.
- *built-in smoothness (unless explicitly reduced) and local smoothness preserving editability.* For given connectivity and shape parameters, the surface spline form a vector space of geometrically smooth surface parametrizations. In order to manipulate the surface spline, it suffices to add, subtract or move the mesh points locally.
- *low degree parametrization.* The surface is parametrized by low degree polynomial patches. The representation can be extended to rational patches by using a fourth coordinate.
- *simple interpolation.* Mesh points and normals can be interpolated without solving a system of constraints.
- *evaluation by averaging.* The coefficients of the parametrization in Bernstein-Bézier form can be obtained by applying averaging masks to the input mesh. (The Bernstein-Bézier form in turn is evaluated by averaging.) Thus the algorithm is local and can be interpreted as a rule for cutting an input polytope such that the limit polytope is the spline surface.
- *convex hull property.* The surface lies locally and globally in the convex hull of the input mesh. In other words, every point on the surface can be computed as an average of the mesh points with coefficients that are positive and sum to one.
- *intuitive shape parameters.* The averaging process is geometrically intuitive. Parameters analogous to knot distances govern the depth of the cuts into the polytope outlined by the control mesh. (In concave regions the complement of the polytope is cut.) Smaller cuts result in a surface that follows the input mesh more closely and changes the normal direction more rapidly across the boundary. In the limit this allows adjusting the built-in smoothness, e.g. reducing it to continuity for zero cuts. Discontinuity can be achieved by a change of mesh connectivity.
- *taut interpolation of the control mesh for zero blend ratios.* Cuts of zero depth result in a singular parametrization at the mesh points analogous to singularities of a spline with repeated knots. The continuity of the surface is reduced, but in return the edges of the input mesh are interpolated and the surface is taut, e.g. planar when the mesh cell is planar.

In general any two of the following three properties can be achieved using C^1 surface splines.

	convex hull property	
tangent continuity		interpolation

2. Biquartic C^1 surface splines defined

This section gives an algorithm for generating the Bernstein-Bézier coefficients of a biquartic C^1 surface from an irregular mesh of points. Analogous to tensor-product splines, the biquartic C^1 surface spline is defined by

- mesh points \mathcal{M} ,
- mesh connectivity \uparrow ,
- blend ratios (knot spacings) a_i ,

and an evaluation algorithm. Since the evaluation of polynomials in Bernstein-Bézier form is well understood, we express the surface spline in terms of the Bernstein-Bézier patches. A biquartic patch p_i in tensor-product Bernstein-Bézier form is a map

$$[0, 1] \times [0, 1] \mapsto \mathbb{R}^3$$

$$(u, v) \mapsto p_i := \sum_{k,l=0..4} b_{kl,i} (1-u)^{4-k} u^k (1-v)^{4-l} v^l$$

where $b_{kl,i} \in \mathbb{R}^3$, $k, l \in \{0, 1, 2, 3, 4\}$ are the Bernstein-Bézier coefficients (see [Boehm, Farin, Kahmann '84] or [Farin '90] for a general treatment of the Bernstein-Bézier form). The process of deriving the coefficients from the mesh points consists of the following steps.

Algorithm $(\mathcal{M}, \uparrow, a_{ij}) \mapsto b_{kl,i}$.

- 1 Midpoint mesh refinement.
- 2 Generation of intermediate control points.
- 3 Generation of the Bernstein-Bézier coefficients.

The *input* to the algorithm is any mesh of points such that at most two cells abut along any edge. The mesh cells need not be planar, and there is no constraint on the number of cells meeting at a vertex. The mesh may model bivariate surfaces with or without boundary and of arbitrary topological genus. To achieve the design intent, each mesh cell should be convex in the sense that there exists a projection of the cell vertices into a plane such that none of the projected vertices lies in the convex hull of the other projected vertices. For example, facets whose boundary representation have inner loops should be broken up before using them as cells, since the surface is generated by averaging and would cover the intended holes. The input *parameters* are as follows. Associated with each pair, cell and cell vertex, are two scalar weights $0 < a_i < 1$, $a_i^* := 1/2$, $i = 1, 2$, called *blend ratios*. (Default settings of a parameter are indicated by the superscript *.) Geometrically, smaller ratios result in a surface that follows the input mesh more closely and changes the normal direction more rapidly close to the mesh edges. The blend ratios are similar to relative knot spacings in that zero ratios cause the mesh edges to be interpolated and the smoothness to be reduced. The blend ratios of each cell may be modified independently of each other and of those in other cells. Other parameters, α and β are associated with each mesh point and determine the width of the tangent plane and the relative position of the corresponding point on the surface.

The *output* is a C^1 surface that follows the outline of the input mesh and consists of no more than $2e$ biquartic pieces, where e is the number of edges of the input mesh.

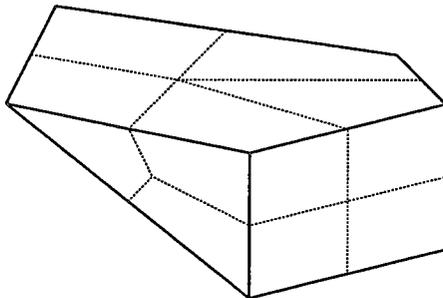


Figure 2.1: Midpoint mesh refinement.

1. Midpoint mesh refinement. Inserting additional points at centroid of each cell and the midpoint of each edge, we obtain a refined mesh of quadrilateral subcells and two types of points of the refined mesh: vertices of type M that correspond to edge midpoints and therefore have exactly four neighbors, and vertices of type P that correspond to original mesh points and centroids and therefore are surrounded by vertices of type M .

2. Generation of intermediate control points. For each quadrilateral subcell, define a point C_i subject to the blend ratios $0 < a_{i1}, a_{i2} < 1$ as follows. Let V_1, V_2, V_3, V_4 be the vertices of the i th subcell joining at $P = V_1$. Then

$$C_i := (1 - a_{i1})(1 - a_{i2})V_1 + (1 - a_{i1})a_{i2}V_2 + a_{i1}a_{i2}V_3 + (1 - a_{i2})a_{i1}V_4$$

$$P' := (1 - \beta)P + \beta \sum_{l=1}^n C_l/n \quad \beta^* := 1/8$$

$$C'_i := P' + \frac{\alpha}{n} \sum_{k=1}^n \cos\left(\frac{2\pi}{n}k\right)C_{i+k} \quad \alpha^* := \beta$$

$$\tilde{b}_{1,i} := (C'_i + C'_{i+1})/2$$

$$\tilde{b}_{2,i} := (C_i + C_{i+1})/2$$

Proposition 3.2 shows that all points generated so far are convex combinations of the input mesh points if $\alpha \leq \beta$. Choosing $\beta = 1$ yields a surface that interpolates the mesh point. To reduce inflections the C_i should subsequently be moved into the direction of the average of the normals at the vertices.

For the last step consider a mesh point P and its n neighbors M_i , $i = 1..n$. as illustrated in Figure 2.2. Let p_i , $i = 1..n$ be the biquartic patches in tensor-product Bernstein-Bézier form that surround a common point $p_i(0, 0) = b_{00,i}$. The patches are connected so that $p_i(0, t) = p_{i+1}(t, 0)$ by setting

$$b_{0j,i} = b_{j0,i+1}.$$

To obtain tangent continuity along the common boundary, denote by $D_2 p_i := \frac{\partial p_i}{\partial u}(t, 0)$ the versal derivative along the boundary and by $D_1 p_i$ and $D_2 p_{i+1}$ the transversal derivatives

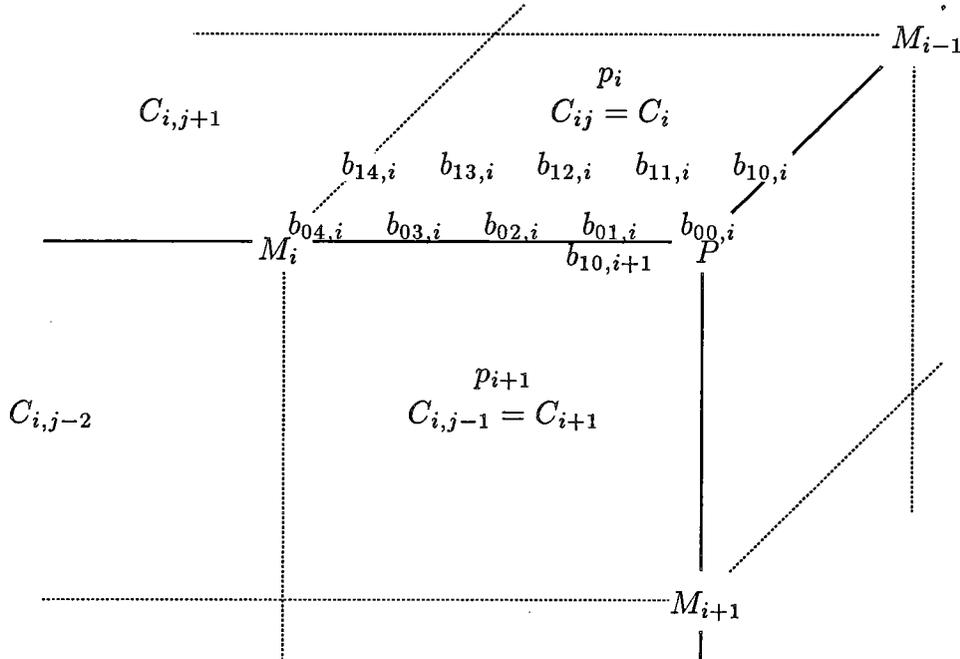


Figure 2.2: Naming conventions for intermediate and Bernstein-Bézier coefficients.

along the boundary. The coefficients of the patches are determined in Step 3 to enforce

$$2c(1-t)^2 D_2 p_i = D_1 p_i + D_2 p_{i+1}, \quad c := \cos(2\pi/n).$$

3. Generation of the Bernstein-Bézier coefficients. Let C_{ij} , $j = 1..4$ be the intermediate control points surrounding M_i . Set

$$\begin{aligned} b_{00,i} &:= P' \\ b_{01,i} &:= (3\tilde{b}_{1,i} + b_{00,i})/4 \\ b_{02,i} &:= (3\tilde{b}_{2,i} + 3\tilde{b}_{1,i})/6 \\ b_{03,i} &:= (3\tilde{b}_{2,i} + b_{04,i})/4 \\ b_{04,i} &:= \sum_{l=1}^4 C_{il}/4 \\ b_{11,i} &:= \frac{3c}{8}C_i + \frac{6-3c}{8}C'_i + b_{00,i}/4 \\ b_{12,i} &:= \frac{c}{16}C_{i,j+1} + \frac{8-c}{16}C_i + C'_i/2 - \gamma(C_i - C_{i+1} + C'_i - C'_{i+1}), \quad \gamma^* := 1/8 \\ b_{21,i+1} &:= \frac{c}{16}C_{i,j-2} + \frac{8-c}{16}C_{i+1} + C'_{i+1}/2 + \gamma(C_i - C_{i+1} + C'_i - C'_{i+1}) \\ b_{13,i} &:= (3C_i + b_{04,i})/4 \\ b_{22,i} &:= C_i \end{aligned}$$

All remaining coefficients of the patch p_i are determined by the analogous construction at the other subcell vertex of type P . Tangent continuity of the resulting surface is proved in Theorem 3.1.

3. Smoothness and shape properties

We now show tangent continuity of the resulting surfaces. Linearity of differentiation then implies that surface splines with the equal blend ratios and connectivity form a vector space.

Theorem 3.1. *The biquartic surface splines form a C^1 surface.*

Proof Consider Figure 2.2. Since D_2p_i is a univariate polynomial of degree 2 and D_1p_i and D_2p_{i+1} are of degree 4,

$$2c(1-t)^2 D_2p_i = D_1p_i + D_2p_{i+1}$$

holds when the five coefficients of the quartic polynomials on either side of the equation agree, i.e. if

$$\frac{3}{4}2c(\tilde{b}_{1,i} - b_{00,i}) = (b_{01,i-1} + b_{01,i+1} - 2b_{00,i}) \quad (3.0)$$

$$\frac{3}{4}2c2(\tilde{b}_{2,i} - \tilde{b}_{1,i}) = 4(b_{11,i} + b_{11,i+1} - 2b_{01,i}) \quad (3.1)$$

$$\frac{3}{4}2c(b_{04,i} - \tilde{b}_{2,i}) = 6(b_{12,i} + b_{21,i+1} - 2b_{02,i}) \quad (3.2)$$

$$0 = 4(b_{13,i} + b_{31,i+1} - 2b_{03,i}) \quad (3.3)$$

$$0 = b_{14,i} + b_{41,i+1} - 2b_{04,i} \quad (3.4)$$

In the following, we check by substitution of the formulas from Section 2, that each of the equalities hold. Equation (3.0) follows from

$$\begin{aligned} & C'_{i-1} - 2cC'_i + C'_{i+1} - 2(1-c)b_{00,i} \\ &= \frac{2\alpha}{n} \sum_{i=1}^n C_i \left(\cos\left(\frac{2\pi}{n}(i-1)\right) + \cos\left(\frac{2\pi}{n}(i+1)\right) - 2\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}i\right) \right) = 0 \end{aligned}$$

Equation (3.1) can be rewritten and verified as

$$b_{11,i} + b_{11,i+1} = \frac{3c}{8}(C_i + C_{i+1} - C'_i - C'_{i+1}) + 3(C'_i + C'_{i+1})/4 + 2b_{00,i}/4$$

and Equation (3.2) as

$$\begin{aligned} b_{12,i} + b_{21,i+1} &= \frac{c}{16} \left(\sum C_l - 2C_i - 2C_{i+1} \right) + \tilde{b}_{2,i} + \tilde{b}_{1,i} \\ &= \frac{c}{16}(C_{j+1} + C_{j-2}) + \frac{8-c}{16}(C_i + C_{i+1}) + (C'_i + C'_{i+1})/2. \end{aligned}$$

For the last two equations, let the i th patch at $b_{00,i}$ be the j th patch joining at $b_{04,i}$ as illustrated in Figure 2.2. Then (3.3) equals

$$b_{13,i} + b_{31,i+1} - 2b_{03,i} = 3(C_i + C_{i+1})/4 - 6b_{04,i}/4$$

and (3.4) reads

$$\begin{aligned} (b_{14,i} - b_{04,i}) + (b_{41,i+1} - b_{04,i}) &= \frac{3}{4}(\tilde{b}_{2,j+1} - b_{04,i}) + \frac{3}{4}(\tilde{b}_{2,j-1} - b_{04,i}) \\ &= \frac{3}{16}(2C_j + 2C_{j+1} - \sum_l C_l + 2C_{j-1} + 2C_{j-2} - \sum_l C_l) \\ &= 0. \end{aligned}$$

■

Proposition 3.2. *If $1/5 \leq \alpha \leq (1 - \beta)/2$, then the coefficients $b_{kl,i}$ are local convex combinations of vertices of the input mesh points, except possibly $b_{12,i}$ and $b_{21,i}$ and then only if $n = 3$ and the blend ratios are small.*

Proof Since $\beta > 0$,

$$C'_i = \beta P + \frac{1}{n} \sum_{k=1}^n (1 - \beta + 2\alpha \cos(\frac{2\pi}{n}k)) C_{i+k}$$

is a nonnegative combination of the C_{i+k} if $\alpha \leq (1 - \beta)/2$. If $c := \cos(\frac{2\pi}{n}) > 0$, then inspection of $b_{11,i}$ and $b_{12,i}$ shows that all Bernstein-Bézier coefficients are convex combinations of the intermediate mesh points. If P has $n = 3$ neighbors, then $C'_i := b_{00,i} + \alpha(C_i - b_{00,i})$ and hence

$$b_{11,i} = \frac{-3}{16}C_i + \frac{15}{16}C'_i + b_{00,i}/4 = \frac{15\alpha - 3}{16}C_i + \frac{4 + 15(1 - \alpha)}{16}b_{00,i}/4.$$

For $\alpha \geq 1/5$ all weights are nonnegative. However, $b_{12,i}$ has an uncanceled negative component, $-C_{i,j+1}/32$. ■

If a violation of the convex hull property does occur, it is small compared to the distance between mesh points. To see this, we consider a point P with $n = 3$ neighbors. Without loss of generality, we may assume P to be the origin. The negative weight in $b_{12,i}$ has maximal influence if all the blend ratios at P are zero. Therefore we consider zero blend ratios and expand p_i along the ray $v = ru$ close to the origin:

$$u^3((\frac{1}{4}r - \frac{9}{32})P_i r^2 + (\frac{1}{4} - \frac{9}{32}r)P_{i+1}) + u^4((\frac{-3}{32}r + \frac{9}{16})P_i r^2 + (\frac{-3}{32} + \frac{9}{16}r)P_{i+1}r) + o(u^5)$$

On minimizing the factor multiplying P_i ,

$$\min_{u \in [0, \epsilon], r \in [0, 1]} u^3 r^2 \left(\frac{1}{4}r - \frac{9}{32} + u \left(\frac{-3}{32}r + \frac{9}{16} \right) \right),$$

we find that the minimum value to be

ϵ	min. value
1/8	$471 * 10^{-7}$
1/16	$92 * 10^{-7}$
1/32	$10 * 10^{-7}$
1/64	$1 * 10^{-7}$

That is, depending on which ϵ gives a good approximation, the surface exits the local convex hull by roughly 10^{-6} times the distance between P and its furthest neighboring original mesh point.

Proposition 3.3. *If all blend ratios a_{ij} are zero, then the input mesh is interpolated.*

Proof Let P_i be the i th original mesh point neighbor of P . Then

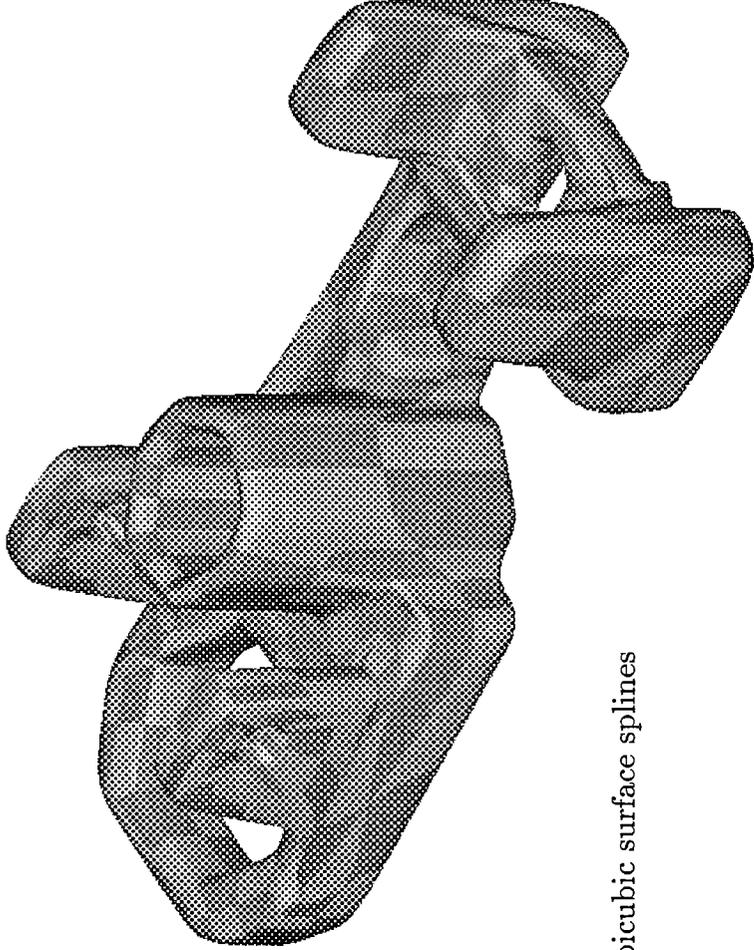
$$\begin{aligned} C_i &= P' = C'_i = \tilde{b}_{1,i} = \tilde{b}_{2,i} = P, \\ b_{00,i} &= b_{01,i} = b_{02,i} = b_{11,i} = b_{22,i} = P, \\ b_{03,i} &= b_{13,i} = (7P + P_i)/8, \\ b_{04,i} &= (P + P_i)/2, \\ b_{12,i} &= P + \frac{c}{16}(P_i - P) \end{aligned}$$

■

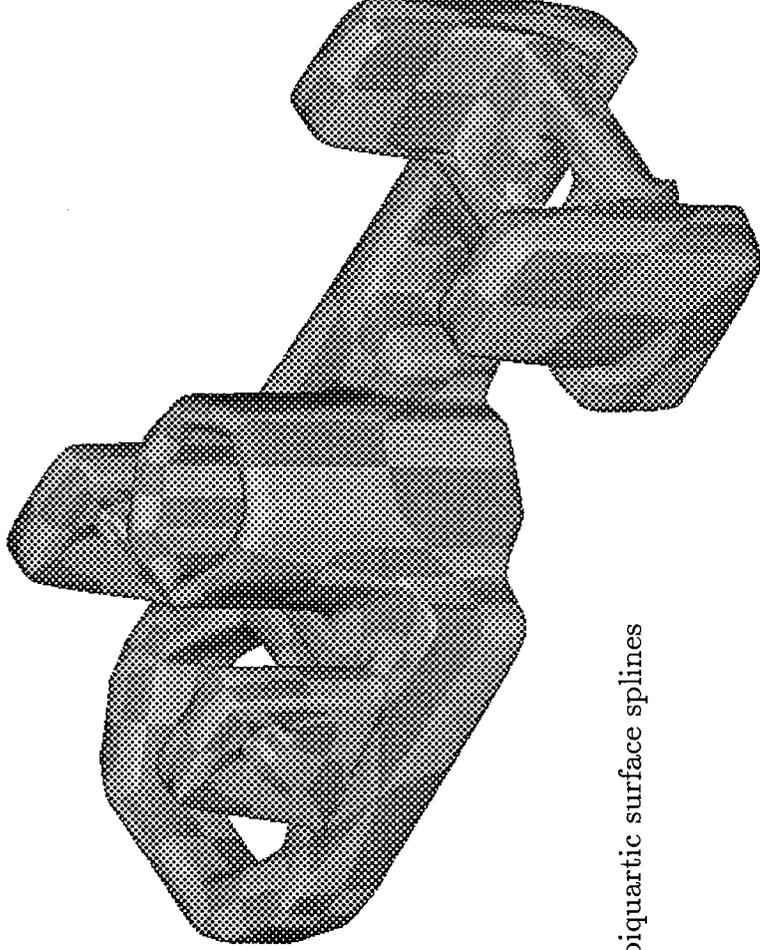
4. Remarks and observations

The contribution of this paper is to give simple formulas for the Bernstein-Bézier coefficients of a C^1 surface spline based entirely on four-sided patches. The key to this construction is the particular choice of cubic boundary curves, and a use of degrees of freedom that yields a Bernstein-Bézier net as a refined mesh by averaging the original mesh points. The biquartic construction improves on an obvious biquintic construction. The latter is useful, if all cells of the input mesh are four-sided to start with, since then the mesh need not be refined.

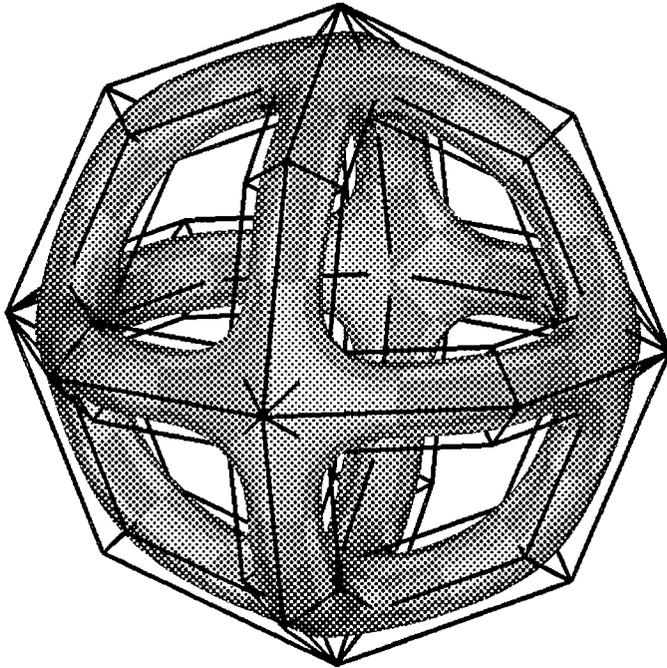
Comparing the biquartic splines with the cubic surface splines defined in [Peters '93], we find that the number of Bernstein-Bézier coefficients per subcell is 25 in either case. Moreover, both surface constructions can be joined to biquadratic patches wherever the mesh is regular. However, the biquadratic-cubic combination defined in [Peters '93] is superior in that it always satisfies the convex hull property. Also, combining the biquartic construction with biquadratic patches requires a transition layer of bicubic patches. The following first pair of figures compares the shape of the bicubic surface splines defined in [Peters '93] with the present biquartic surface splines.



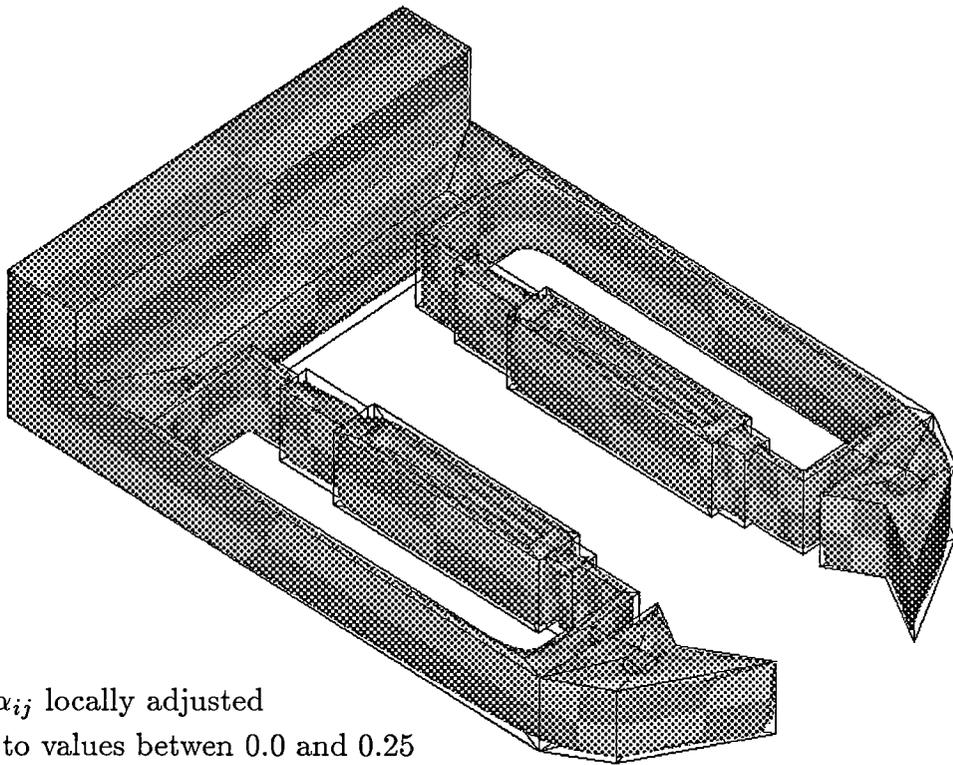
bicubic surface splines



biquartic surface splines



all $\alpha_{ij} = 0.5$



α_{ij} locally adjusted
to values between 0.0 and 0.25

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