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**SOME NOTES ON MULTISPLITTING METHODS AND m -STEP
PRECONDITIONERS FOR LINEAR SYSTEMS**

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Some Notes on Multisplitting Methods and m -Step Preconditioners for Linear Systems

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Abstract

To solve the real nonsingular linear system $Ax = b$ on parallel and vector machines, multisplitting methods, m -step preconditioners and m -step additive preconditioners are considered. In this work, in particular: i) We extend the method and the corresponding convergence results in [14]. ii) We determine suitable relaxed m -step preconditioners ([1], [6]) and solve the problem of minimizing the related condition number, with respect to the relaxation (extrapolation) parameter involved. iii) Finally, we extend the theory for determining suitable m -step additive preconditioners [2] and complete the theoretical solution of the problem of determining the optimum SOR-additive iterative method [2] for 2-cyclic positive definite matrices.

Key words and phrases: multisplitting methods, m -step preconditioners, extrapolation method, successive overrelaxation (SOR) method.

AMS (MOS) Subject Classifications: 65F10. CR categories: 5.14.

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1 Introduction

For solving the large nonsingular linear system of equations

$$Ax = b, \quad (1.1)$$

where $A \in \mathbb{R}^{n,n}$, $b \in \mathbb{R}^n$, parallel iterative methods, called multisplitting methods, were introduced in [12]. According to [12], given a multisplitting of A

$$A = M_k - N_k, \quad \det(M_k) \neq 0, \quad k = 1(1)p, \quad (1.2)$$

the corresponding multisplitting method is defined by

$$x^{(m+1)} = \sum_{k=1}^p D_k M_k^{-1} N_k x^{(m)} + \sum_{k=1}^p D_k M_k^{-1} b, \quad m = 0, 1, 2, \dots, \quad (1.3)$$

where D_k is a diagonal matrix, with $D_k \geq 0$, $k = 1(1)p$, and $\sum_{k=1}^p D_k = I$. Setting

$$H = \sum_{k=1}^p D_k M_k^{-1} N_k \quad \text{and} \quad G = \sum_{k=1}^p D_k M_k^{-1}, \quad (1.4)$$

(1.3) takes the form

$$x^{(m+1)} = H x^{(m)} + c, \quad m = 0, 1, 2, \dots, \quad (1.5)$$

where $c = Gb$. Moreover we have

$$H = I - GA. \quad (1.6)$$

According to [18], Thm. 2.6, p. 68, (1.5) is consistent with (1.1). Furthermore (1.5) is completely consistent with (1.1) iff G is nonsingular. From now on we assume that (1.5) is completely consistent with (1.1); hence it is obvious that (1.5) can be obtained using the splitting

$$A = G^{-1} - G^{-1}H. \quad (1.7)$$

It is well known that (1.5) converges to $A^{-1}b$ for any starting vector $x^{(0)}$ iff $\rho(H) < 1$, where $\rho(\cdot)$ denotes spectral radius. Convergence results of (1.5), under various assumptions, can be found in the literature (see, e.g., [4], [5], [7], [8], [11], [12], [14], [16], [17], [19]).

In [1], [6] for the linear system (1.1), where A is positive definite (cf. [18], p. 21) a splitting $A = M - N$, $\det(M) \neq 0$, is considered, where M is positive definite and $\rho(M^{-1}N) < 1$, and the associated preconditioning matrix or m -step preconditioner is defined by

$$M_m = M(I + G + G^2 + \dots + G^{m-1})^{-1}, \quad m > 1, \quad (1.8)$$

where $G = M^{-1}N$. If $A \approx M$, then M_m is an improved approximation to A and is used instead of M for accelerating the rate of convergence of Chebyshev and Conjugate Gradient methods. Also in [2] for the same purpose m -step additive preconditioners are defined, which are connected with the multisplitting method (1.5) for $p = 2$ and $D_1 = D_2 = \frac{1}{2}I$. In particular, in [2] the SOR-additive preconditioner is defined and an optimal value ω_{opt} for the parameter ω of the 2-cyclic SOR-additive iterative method is also determined.

In the present paper we give in Section 2 two theorems concerning the convergence of the method (1.5), when: (i) A in (1.1) satisfies $A^{-1} \geq 0$ and (1.2) are weak regular splittings (cf. [3])

and (ii) A is positive definite and (1.2) are P -regular splittings (see [13]). Also in Section 2 we extend the two-splitting method (method of the arithmetic mean) treated in [14] and prove some theorems which generalize Thms 1, 2, 3 in [14]. In Section 3 we give a method for finding a suitable m -step preconditioner M_m , $m \geq 1$, for system (1.1). The given preconditioner contains a parameter ω and we determine the optimal value of ω so that the condition number of $M_m^{-1}A$ is minimized. We also extend the procedure given in [2] for defining m -step additive preconditioners and give in a theorem sufficient conditions for determining suitable additive preconditioners. Finally, in Section 4 we complete the theoretical solution of the problem of determining the optimal ω of the SOR-additive iterative method studied in [2] to include all possible theoretical cases too.

2 Convergence Results

We consider the linear system (1.1) and the multisplitting method (1.5). Then we obtain the following results which are useful in the sequel (see also Thm 1 (a), (b) in [12] and Thm 1 and Cor 1 in [17]).

Theorem 2.1

Let in (1.1) $A^{-1} \geq 0$ and (1.2) be weak regular splittings of A . Then (1.7) is also a weak regular splitting of A ; hence (1.5) converges ($\rho(H) < 1$).

Proof: It follows from Thm 1 and Cor 1 in [17]. \square

Theorem 2.2

Let A in (1.1) be positive definite, (1.2) be P -regular splittings of A and $D_k = a_k I$ with $a_k \geq 0$, $\sum_{k=1}^p a_k = 1$. Then (1.7) is also a P -regular splitting of A ; hence (1.5) converges.

Proof: From the hypothesis M_k is nonsingular and $M_k + N_k$ is positive real (see [18], Thm 2.9, p. 24), i.e., $M_k + N_k + (M_k + N_k)^T$ is positive definite or equivalently $M_k + M_k^T - A$, $k = 1(1)p$, is positive definite (C^T denotes the transpose of C). Since A is positive definite, according to [18], Thm 5.3, p. 79, it suffices to show that

$$M + M^T - A = \frac{1}{2}[M + N + (M + N)^T]$$

is positive definite, where $M = G^{-1}$, $N = G^{-1}H$ ($A = M - N$), or equivalently that

$$M^{-1}(M + M^T - A)M^{-T} = M^{-T} + M^{-1} - M^{-1}AM^{-T} =: Q$$

is positive definite. Thus we have

$$\begin{aligned} Q &= \sum_{k=1}^p a_k M_k^{-T} + \sum_{k=1}^p a_k M_k^{-1} - \left(\sum_{k=1}^p a_k M_k^{-1} \right) A \left(\sum_{k=1}^p a_k M_k^{-T} \right) \\ &= \sum_{k=1}^p a_k (M_k^{-T} + M_k^{-1} - M_k^{-1} A M_k^{-T}) + \sum_{k=1}^p a_k M_k^{-1} A M_k^{-T} \\ &\quad - \left(\sum_{k=1}^p a_k M_k^{-1} \right) A \left(\sum_{k=1}^p a_k M_k^{-T} \right). \end{aligned}$$

The matrix $S_1 \equiv \sum_{k=1}^p a_k (M_k^{-T} + M_k^{-1} - M_k^{-1} A M_k^{-T}) = \sum_{k=1}^p a_k M_k^{-1} (M_k + M_k^T - A) M_k^{-T}$ is positive definite, since $a_k \geq 0$ and $M_k^{-1} (M_k + M_k^T - A) M_k^{-T}$, $k = 1(1)p$, is positive definite. Moreover, for the symmetric matrix $S_2 \equiv Q - S_1$ we have

$$\begin{aligned}
S_2 &= \sum_{k=1}^p a_k M_k^{-1} A M_k^{-T} - \left(\sum_{k=1}^p a_k M_k^{-1} \right) A \left(\sum_{k=1}^p a_k M_k^{-T} \right) \\
&= \left(\sum_{j=1}^p a_j \right) \left(\sum_{k=1}^p a_k M_k^{-1} A M_k^{-T} \right) - \sum_{k,j=1}^p a_k a_j M_k^{-1} A M_j^{-T} \\
&= \sum_{k,j=1}^p a_k a_j [M_k^{-1} A M_k^{-T} - M_k^{-1} A M_j^{-T}].
\end{aligned}$$

Hence

$$\begin{aligned}
2S_2 &= S_2 + S_2^T \\
&= \sum_{k,j=1}^p a_k a_j (M_k^{-1} A M_k^{-T} - M_k^{-1} A M_j^{-T}) + \sum_{k,j=1}^p a_k a_j (M_k^{-1} A M_k^{-T} - M_j^{-1} A M_k^{-T}) \\
&= \sum_{k,j=1}^p a_k a_j (M_k^{-1} A M_k^{-T} - M_k^{-1} A M_j^{-T} + M_j^{-1} A M_j^{-T} - M_j^{-1} A M_k^{-T}) \\
&= \sum_{k,j=1}^p a_k a_j [(M_k^{-1} - M_j^{-1}) A (M_k^{-1} - M_j^{-1})^T].
\end{aligned}$$

S_2 , as a sum of nonnegative definite matrices, is nonnegative definite. This implies that $Q = S_1 + S_2$ is positive definite and that $A = G^{-1} - G^{-1}H$ is a P -regular splitting of A ; hence $\rho(H) < 1$. \square

Remark: The proof just given parallels that in [12]. However, it is simpler because it is based on Thm 5.3, p. 79 of [18], instead of on the more complicated one used in [12].

In the following an extension of the method of the arithmetic mean of [14] is suggested. Our extension is mainly two-fold: i) Instead of a forward-backward Gauss-Seidel type process, we propose a forward-backward SOR-type one, and ii) Instead of having a 2-processor MIMD machine in mind and after each complete iteration taking the arithmetic mean of the two iterates as the next iteration, which is sent back to the two processors, a $2q$ -processor one is considered and a convex combination of the $2q$ iterates is taken as the next iteration (see, e.g., [19]).

Let $A = D - L - U$, $D = \text{diag}(A)$ and L, U be strictly lower and upper triangular matrices, respectively. Consider the multisplitting of A

$$A = M_k - N_k, \quad \det(M_k) \neq 0, \quad k = 1(1)2q, \quad (2.1)$$

where

$$M_k = \frac{1}{\omega} D + W_k - L, \quad N_k = \left(\frac{1}{\omega} - 1 \right) D + W_k + U, \quad k = 1(1)q, \quad (2.2)$$

$$M_k = \frac{1}{\omega} D + W_k - U, \quad N_k = \left(\frac{1}{\omega} - 1 \right) D + W_k + L, \quad k = q+1(1)2q. \quad (2.3)$$

In (2.2), (2.3) $W_k (> 0)$ is a diagonal matrix with positive diagonal entries and ω a real positive parameter. For the corresponding multisplitting method (1.5), where $p = 2q$ and M_k is given by (2.2), (2.3), $k = 1(1)2q$, we prove the theorems below, which generalize Thms 1, 2, 3 in [14]. We simply mention that in [14], $p = 2$, $\omega = 1$, $W_1 = W_2$, and $D_1 = D_2 = \frac{1}{2}I$.

Theorem 2.3

Let A in (1.1) be irreducibly diagonally dominant L -matrix ([15], p. 23 and [18], p. 42), M_k be given by (2.2), (2.3), $k = 1(1)2q$, with $0 < \omega \leq 1$, and $D_k = a_k I$. Then the multisplitting method (1.5), where $p = 2q$, converges.

Proof: The matrix M_k is nonsingular, since $D > 0$, $W_k > 0$ and $\omega > 0$, $k = 1(1)2q$. According to the hypothesis (see [15], Cor 1, p. 85) A is a nonsingular M -matrix with $A^{-1} > 0$. Obviously M_k is a strictly diagonally dominant L -matrix, $k = 1(1)2q$; hence M_k is an M -matrix and therefore $M_k^{-1} \geq 0$, $k = 1(1)2q$. We also have $N_k \geq 0$, $k = 1(1)2q$. Consequently, (2.1) are regular splittings of A and hence weak regular splittings of A . Now, by Thm 2.1 we have $\rho(H) < 1$. \square

Remark: Thm 2.3 holds true for any choice of the nonnegative diagonal matrices D_k in (1.3) and holds, therefore, also for the multiprocessor model considered in [4].

Theorem 2.4

Let A in (1.1) be a positive real matrix. Let M_k be given by (2.2), (2.3) with $\omega = 1$. Let also $W_k = \rho_k I$, $D_k = a_k I$, $k = 1(1)2q = p$, and

$$\rho_k > \begin{cases} \max\{0, -\frac{\eta_m}{\lambda_m}\} & \text{for } k = 1(1)q \\ \max\{0, -\frac{\theta_m}{\lambda_m}\} & \text{for } k = q + 1(1)2q, \end{cases} \quad (2.4)$$

where λ_m is the smallest eigenvalue of $A + A^T$ and η_m, θ_m are the smallest eigenvalues of the matrices $(D - L)(D - L)^T - UU^T$ and $(D - U)(D - U)^T - LL^T$, respectively. Then the multisplitting method (1.5) converges.

Proof: Since A is positive real, we have that A is nonsingular, $B \equiv A + A^T$ is positive definite and $D > 0$. Consequently M_k is nonsingular, $k = 1(1)2q$, since $\rho_k > 0$. Moreover we have $\lambda_m > 0$. The matrices $C_1 \equiv (D - L)(D - L)^T - UU^T$ and $C_2 \equiv (D - U)(D - U)^T - LL^T$ are symmetric and for any $z \in \mathbb{R}^n$, $z \neq 0$, we have

$$\frac{z^T(\rho_k B + C_1)z}{z^T z} \geq \rho_k \lambda_m + \eta_m, \quad \frac{z^T(\rho_k B + C_2)z}{z^T z} \geq \rho_k \lambda_m + \theta_m. \quad (2.5)$$

Because of (2.4), (2.5) implies that the matrices $\rho_k B + C_1$, $k = 1(1)q$, and $\rho_k B + C_2$, $k = q + 1(1)2q$, are positive definite. Setting $G_k = M_k^{-1} N_k$, $k = 1(1)2q$, it can be shown that

$$\rho_k B + C_1 = M_k(I - G_k G_k^T) M_k^T, \quad k = 1(1)q, \quad (2.6)$$

$$\rho_k B + C_2 = M_k(I - G_k G_k^T) M_k^T, \quad k = q + 1(1)2q. \quad (2.7)$$

From (2.6), (2.7) we have that $I - G_k G_k^T$, $k = 1(1)2q$, are positive definite; hence the eigenvalues of $G_k G_k^T$ belong to $[0, 1)$, $k = 1(1)2q$. Thus we obtain $\|G_k\|_2 = [\rho(G_k G_k^T)]^{1/2} < 1$, $k = 1(1)2q$, and

$$\|H\|_2 = \left\| \sum_{k=1}^{2q} a_k G_k \right\|_2 \leq \sum_{k=1}^{2q} a_k \|G_k\|_2 < \sum_{k=1}^{2q} a_k = 1,$$

implying that the method converges. \square

Remark: From the proof of Thm 2.4 one may observe that the assumption $D_k = a_k I$, $k = 1(1)p$, in Thm 2.4, could be replaced by $\sum_{k=1}^p \|D_k\|_2 \leq 1$. However, it can be shown that $\sum_{k=1}^p \|D_k\|_2 < 1$ is not consistent with $\sum_{k=1}^p D_k = I$ while $\sum_{k=1}^p \|D_k\|_2 = 1$ is equivalent to $D_k = a_k I$, $k = 1(1)p$.

Theorem 2.5

Let A in (1.1) be a positive definite matrix, M_k be given by (2.2), (2.3), $D_k = a_k I$, $p = 2q$, and $0 < \omega < 2$. Then the multisplitting method (1.5) converges.

Proof: In this case we have $U = L^T$ and $A = D - L - L^T$, $D > 0$. The splittings (2.2), (2.3) are P -regular splittings, since M_k is nonsingular and $M_k + N_k + (M_k + N_k)^T = 2(M_k + M_k^T - A) = 2[(\frac{2-\omega}{\omega})D + 2W_k]$, $k = 1(1)2q$. Thus by Thm 2.2 we obtain the desired result. \square

3 m-Step Preconditioners

We consider the linear system (1.1), where A is positive definite. If

$$A = M - N, \quad \det(M) \neq 0, \quad (3.1)$$

then using the iterative method

$$Mx^{(m+1)} = Nx^{(m)} + b, \quad m = 0, 1, 2, \dots,$$

we solve in every iteration a linear system of the form

$$My = c. \quad (3.2)$$

It is known that M is chosen so that it approximates A as well as possible ($A \approx M$) and $\rho(G) < 1$, where $G = M^{-1}N$. Choosing a positive definite M ($A \approx M$) with $\rho(G) < 1$, we can find improved approximations to A using the Neumann expansion (see, e.g., [1], [2], [6])

$$A^{-1} = (I - G)^{-1}M^{-1} = (I + G + G^2 + \dots)M^{-1}. \quad (3.3)$$

Thus we have

$$A \approx M_m = M(I + G + G^2 + \dots + G^{m-1})^{-1}, \quad m \geq 1. \quad (3.4)$$

It can be shown (see Thm 3.1 of [6]), that under the above assumptions M_m is also positive definite and therefore M_m^{-1} is usually used to accelerate the convergence of the Conjugate Gradient method. The matrix M_m is the preconditioning matrix or m -step preconditioner. One comment here: In Thm 1 of [1], it was proved that for m odd the hypothesis “ A and M are positive definite” is sufficient for M_m to be positive definite. However, this hypothesis does not guarantee that M_m will be a better than M approximation to A , since then

$$M_m^{-1}N_m = M_m^{-1}(M_m - A) = I - (I + G + \dots + G^{m-1})(I - G) = G^m.$$

Therefore the condition $\rho(G) < 1$ should be included in our assumptions for all m (odd or even).

Taking into consideration the theory mentioned previously (see also [10]), in order to find suitable m -step preconditioners for (1.1), we can work as follows: We choose some positive definite matrix M and write $A = M - N$. Then $G = M^{-1}N$ has real eigenvalues λ_i such that $\lambda_i < 1$, $i = 1(1)n$. Suppose that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n < 1$. Then, the eigenvalues $\nu_i = 1 - \lambda_i$ of $I - G$, which will be used more often in the sequel, will satisfy

$$0 < \nu_n \leq \dots \leq \nu_2 \leq \nu_1.$$

We consider now the splitting

$$A = \hat{M} - \hat{N}, \quad (3.5)$$

where $\hat{M} = \frac{1}{\omega} \hat{M}$. As is known the splitting (3.5) defines the extrapolated method based on the original splitting. Obviously \hat{M} is positive definite for $\omega > 0$ and it is $\rho(\hat{M}^{-1} \hat{N}) < 1$ iff $0 < \omega < \frac{2}{\nu_1}$. Hence an m -step preconditioner, which is positive definite and approximates A well, is given by

$$\hat{M}_m = \hat{M}(I + \hat{G} + \hat{G}^2 + \dots + \hat{G}^{m-1})^{-1}, \quad m \geq 1, \quad (3.6)$$

where $\hat{G} = \hat{M}^{-1} \hat{N}$ and $\omega \in (0, \frac{2}{\nu_1})$. Certainly \hat{M}_m depends on ω and the problem, as how to choose ω for a fixed m so that the condition number $\kappa(\hat{M}_m^{-1} A)$ of $\hat{M}_m^{-1} A$ is as small as possible, arises. It is easy to show that

$$I - \hat{M}_m^{-1} A = \hat{G}^m = [I - \omega(I - G)]^m; \quad (3.7)$$

hence

$$\kappa(\hat{M}_m^{-1} A) = \frac{\max_i \mu_i^{(m)}}{\min_i \mu_i^{(m)}}, \quad (3.8)$$

where $\mu_i^{(m)}$, $i = 1(1)n$, are the eigenvalues of $\hat{M}_m^{-1} A$. The eigenvalues of \hat{G} are ordered as follows

$$-1 < 1 - \omega\nu_1 \leq 1 - \omega\nu_2 \leq \dots \leq 1 - \omega\nu_n < 1. \quad (3.9)$$

So, because of (3.7), (3.8) becomes

$$\kappa(\hat{M}_m^{-1} A) = \frac{\max_i \{1 - [1 - \omega\nu_i]^m\}}{\min_i \{1 - [1 - \omega\nu_i]^m\}}, \quad m \geq 1. \quad (3.10)$$

It can be shown, as in [1], that for $\omega \in (0, \frac{2}{\nu_1})$

$$\kappa(\hat{M}_m^{-1} A) = \begin{cases} \frac{1 - (1 - \omega\nu_1)^m}{1 - (1 - \omega\nu_n)^m}, & \text{if } m \text{ is odd,} \\ \frac{1 - [\min_i |1 - \omega\nu_i|]^m}{1 - [\max_i |1 - \omega\nu_i|]^m}, & \text{if } m \text{ is even.} \end{cases} \quad (3.11)$$

In the sequel, we solve the problem of determining $\min_{\omega} \kappa(\hat{M}_m^{-1} A)$ completely, first for any even $m \geq 2$ and then for any odd $m \geq 3$. The results are given in Thms 3.2 and 3.4. In these theorems it is assumed that $\nu_n < \nu_1$ (or $\lambda_1 < \lambda_n$), for if $\nu_1 = \nu_n$, then $\kappa(\hat{M}_m^{-1} A) = 1$ for all m and all admissible values of ω .

To derive the optimal results for even $m \geq 2$ first, we introduce the notation " $a \sim b$ " to denote that the expressions a and b are of the same sign and then state and prove the lemma below, a basic key to the proof of two of our main results.

Lemma 3.1

For any even $m \geq 2$ the function

$$\phi_m \equiv \phi_m(x) := \frac{x^{m-1} - x^m}{1 - x^m}, \quad x \in (-1, 1) \quad (3.12)$$

is a strictly increasing function of x in $(-1, 1)$. For any odd $m \geq 3$ the function ϕ_m strictly decreases with $x \in (-1, 0]$ and strictly increases with $x \in [0, 1)$.

Proof: In the case m even, differentiating (3.12) with respect to x we obtain

$$\frac{\partial \phi_m}{\partial x} \sim (m-1) - mx + x^m = (m-1)(1-x) - x(1-x^{m-1}). \quad (3.13)$$

If $x \in (-1, 0]$, the rightmost expression in (3.13) is positive since $1-x > 0$, $-x \geq 0$ and $1-x^{m-1} > 0$, implying that ϕ_m strictly increases in $(-1, 0]$. For $x \in [0, 1)$ let

$$z \equiv z(x) := (m-1) - mx + x^m, \quad x \in [0, 1). \quad (3.14)$$

Then on differentiation we take $\frac{\partial z}{\partial x} = -m(1-x^{m-1}) < 0$ and therefore $z(x)$ strictly decreases in $[0, 1)$ with $\lim_{x \rightarrow 1^-} z(x) = 0$ and $z(0) = m-1 > 0$. Hence $z(x)$ takes on positive values only and by virtue of (3.14) and (3.13) so does $\frac{\partial \phi_m}{\partial x}$. Consequently ϕ_m strictly increases in $[0, 1)$. In the case m odd, the proof is similar and is omitted. \square

In the sequel we state and prove two theorems that solve the problem of determining the optimal extrapolation parameter for all even $m \geq 2$.

Theorem 3.1

Let the eigenvalues ν_i , $i = 1(1)n$, of $I - G$ in (3.7) satisfy

$$0 < \nu_n \leq \dots \leq \nu_2 \leq \nu_1 = 2 - \nu_n \quad (\nu_n < \nu_1). \quad (3.15)$$

Then the condition number $\kappa_m = \kappa_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized with respect to $\omega \in (0, \frac{2}{\nu_1})$ for

$$\omega_{opt} = 1. \quad (3.16)$$

Proof: Let $1 - \nu_i$ and $1 - \nu_{i+1}$, $i \in \{1, 2, \dots, n-1\}$ be the absolutely smallest nonpositive and nonnegative eigenvalues of G , respectively. Two cases are distinguished depending on the sign of $2 - \nu_i - \nu_{i+1}$.

Case I: Let $\nu_i + \nu_{i+1} > 2$. (The subcase $\nu_i + \nu_{i+1} = 2$ can be trivially examined after the analysis is complete.) We subdivide the interval for ω , $(0, \frac{2}{\nu_1})$, into a number of (at most $2n+1$) subintervals. For continuity arguments to apply, all of them are taken to be closed, except the first and the last ones. The subdivision points are

$$\frac{1}{\nu_1}, \frac{2}{\nu_1 + \nu_2}, \frac{1}{\nu_2}, \frac{2}{\nu_2 + \nu_3}, \dots, \frac{1}{\nu_i}, \frac{2}{\nu_i + \nu_{i+1}}, 1, \frac{1}{\nu_{i+1}}, \frac{2}{\nu_{i+1} + \nu_{i+2}}, \dots$$

The last point is either $\frac{1}{\nu_j}$ for some $j \in \{i+1, i+2, \dots, n\}$ iff $\frac{1}{\nu_j} < \frac{2}{\nu_1} \leq \frac{2}{\nu_j + \nu_{j+1}}$ or $\frac{2}{\nu_{j-1} + \nu_j}$ for some $j \in \{i+2, i+3, \dots, n\}$ iff $\frac{2}{\nu_{j-1} + \nu_j} < \frac{2}{\nu_1} \leq \frac{1}{\nu_j}$. Let $I_1, I_2, I_3, \dots, I_{2i}, I_{2i+1}, I_{2i+2}, \dots$ be the successive subintervals of $(0, \frac{2}{\nu_1})$ defined by these points. Let also

$$\lambda_k(\omega) := 1 - \omega \nu_k, \quad k = 1(1)n. \quad (3.17)$$

The ordering of the eigenvalues $\lambda_k(\omega)$ of $\hat{G} \equiv G_\omega$ is that in (3.9). We then claim that: “ $\kappa_m = \kappa_m(\omega)$ is a strictly decreasing function of ω in each subinterval I_ℓ , $\ell = 1(1)2i+1$, and a strictly increasing one in each I_ℓ , $\ell \geq 2i+2$ ”. The proof of our claim will prove (3.16). For this we shall distinguish four cases: (a) $\omega \in I_\ell$, $\ell = 2(2)2i$, (b) $\omega \in I_\ell$, $\ell = 1(2)2i+1$, (c) $\omega \in I_\ell$, $\ell = 2i+2, 2i+4, \dots$, and (d) $\omega \in I_\ell$, $\ell = 2i+3, 2i+5, \dots$. In case (a), $\omega \in [\frac{1}{\nu_k}, \frac{2}{\nu_k + \nu_{k+1}}]$, $k = \ell/2$. It can be readily checked that $\lambda_k(\omega)$ and $\lambda_{k+1}(\omega)$ are, respectively, the absolutely smallest nonpositive and nonnegative eigenvalues of G_ω with $0 \leq -\lambda_k(\omega) \leq \lambda_{k+1}(\omega)$. On the other hand $0 \leq -\lambda_1(\omega) \leq \lambda_n(\omega)$. So, $\kappa_m(\omega)$ will be given by the expression

$$\kappa_m(\omega) = \frac{1 - \lambda_k^m(\omega)}{1 - \lambda_n^m(\omega)}. \quad (3.18)$$

Since m is even, and both $\lambda_k(\omega)$ and $\lambda_n(\omega)$ strictly decrease with ω increasing it is concluded that the numerator and the denominator of the expression in (3.18) decreases and increases, respectively, making $\kappa_m(\omega)$ be a strictly decreasing function of $\omega \in I_\ell$. In case (b), $\omega \in [\frac{2}{\nu_{k-1} + \nu_k}, \frac{1}{\nu_k}]$, $k = \frac{\ell+1}{2}$. (I_1 is open on the left with bound 0 and I_{2i+1} is closed on the right with bound 1.) Now $-\lambda_{k-1}(\omega) \geq \lambda_k(\omega) \geq 0$, so that $\kappa_m(\omega)$ will be given again by (3.18). However, this time both terms of the fraction strictly increase with ω . Thus, differentiating with respect to ω one obtains

$$\begin{aligned} \frac{\partial \kappa_m}{\partial \omega} &\sim (1 - \lambda_n^m(\omega))\nu_k \lambda_k^{m-1}(\omega) - (1 - \lambda_k^m(\omega))\nu_n \lambda_n^{m-1}(\omega) \\ &\sim \frac{\lambda_k^{m-1}(\omega)(1 - \lambda_k(\omega))}{1 - \lambda_k^m(\omega)} - \frac{\lambda_n^{m-1}(\omega)(1 - \lambda_n(\omega))}{1 - \lambda_n^m(\omega)} = \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_n(\omega)), \end{aligned} \quad (3.19)$$

because of $\omega\nu_j = 1 - \lambda_j(\omega)$, $j = k, n$, and in view of (3.12). Since ω varies in $I_{2k-1} \subset (0, 1]$ and $\lambda_k(\omega) \leq \lambda_n(\omega)$ Lemma 3.1 applies, implying that $\frac{\partial \kappa_m}{\partial \omega} \leq 0$, with equality concerning limiting cases only. Therefore $\kappa_m(\omega)$ strictly decreases in I_{2k-1} . In case (c), where I_ℓ , $\ell = 2i + 2, 2i + 4, \dots$, is of the general type $[\frac{2}{\nu_{k-1} + \nu_k}, \frac{1}{\nu_k}]$, $k = \ell/2$, except the first and maybe the last interval, we have a similar situation to that in case (a). This time $\kappa_m(\omega)$ is given by the expression

$$\kappa_m(\omega) = \frac{1 - \lambda_k^m(\omega)}{1 - \lambda_1^m(\omega)}. \quad (3.20)$$

Since $\lambda_k(\omega) \geq 0 \geq \lambda_1(\omega)$ and both $\lambda_k(\omega)$ and $\lambda_1(\omega)$ decrease with ω increasing, $\kappa_m(\omega)$ strictly increases with ω . In case (d) we have a similar situation to that in case (b). The interval I_ℓ , $\ell = 2i + 3, 2i + 5, \dots$, is of the general type $[\frac{1}{\nu_k}, \frac{2}{\nu_k + \nu_{k+1}}]$, $k = (\ell - 1)/2$, except maybe the last one, and κ_m is given by (3.20), where this time $0 \geq \lambda_k(\omega) \geq \lambda_1(\omega)$, so both terms of the fraction in (3.20) decrease with ω increasing. On differentiation we have a series of relationships similar to those in (3.19) but this time

$$\frac{\partial \kappa_m}{\partial \omega} \sim \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_1(\omega)). \quad (3.21)$$

Based now on Lemma 3.1 we have again the desired result, namely that $\kappa_m(\omega)$ strictly increases on I_ℓ . Summarizing the conclusions of cases (a)–(d) leads to (3.16).

Case II: In case $\nu_{i+1} + \nu_i < 2$ we work in a similar way as in Case I. This time $1 \in [\frac{1}{\nu_i}, \frac{2}{\nu_i + \nu_{i+1}})$ and we have $2i$ subintervals to the left and at most $2(n - i) + 1$ ones to the right of 1. The function $\kappa_m(\omega)$ behaves in exactly the same way as before in the subintervals which are to the left and to the right of 1, as is readily checked. Consequently we arrive at exactly the same conclusion. \square

Suppose now that the eigenvalues of $I - G$ in (3.7) satisfy

$$0 < \nu_n \leq \dots \leq \nu_2 \leq \nu_1, \quad (3.22)$$

that is without the further assumption $\nu_1 = 2 - \nu_n$ of Thm 3.1. Suppose also that we extrapolate G using any parameter $\omega \in (0, \frac{2}{\nu_1})$. The answer to the question “What is the value of ω_{opt} in this case?” can be given immediately. This is because “The extrapolation with a parameter ω_2 of an extrapolation with parameter ω_1 is also an extrapolation with parameter $\omega = \omega_2\omega_1$ ”, which can be checked (see [9]), leads us to writing ω as $\omega = \omega_2\omega_1$, where $\omega_1 = \frac{2}{\nu_1 + \nu_n}$. But the eigenvalues $\nu'_i = \omega_1\nu_i$, $i = 1(1)n$, of $I - G_{\omega_1}$ satisfy all the assumptions of Thm 3.1. Specifically,

$$0 < \nu'_n \leq \dots \leq \nu'_2 \leq \nu'_1 = 2 - \nu'_n. \quad (3.23)$$

So, extrapolation of G_{ω_1} becomes optimal iff $\omega_2 = 1$. Thus we have just proved:

Theorem 3.2

Let the eigenvalues ν_i , $i = 1(1)n$, of $I - G$ in (3.7) satisfy (3.22). Then the condition number $\kappa_m = \kappa_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized with respect to $\omega \in (0, \frac{2}{\nu_1})$ for

$$\omega_{opt} = \frac{2}{\nu_1 + \nu_n}. \quad (3.24)$$

As an immediate corollary we have:

Corollary 3.1

Let A be real symmetric positive definite and point (or block) 2-cyclic consistently ordered and M , in the splitting $A = M - N$, be the diagonal (or the block diagonal part corresponding to the block partitioning) of A . Then the condition number $\kappa_m = \kappa_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized for $\omega_{opt} = 1$.

Note: If the only information available on the spectrum of G is its spectral radius $\rho(G) = \lambda_n < 1$, then ω_{opt} should be taken to be 1.

In Thm 3.2 it was proved that the optimal value of $\omega(\omega_{opt})$ is independent of m . This makes easy to examine how $\kappa_m(\omega_{opt})$, given by

$$\kappa_m(\omega_{opt}) = \frac{1 - (\min_i |1 - \frac{2\nu_i}{\nu_1 + \nu_n}|)^m}{1 - (1 - \frac{2\nu_n}{\nu_1 + \nu_n})^m}, \quad m(\geq 2) \text{ even}, \quad (3.25)$$

behaves when m increases. The behavior is a consequence of the statement below.

Lemma 3.2

Let $0 \leq x < y < 1$. Then the sequence

$$a_m = \frac{1 - x^m}{1 - y^m}, \quad m = 1, 2, 3, \dots, \quad (3.26)$$

strictly decreases, with

$$\lim_{m \rightarrow \infty} a_m = 1. \quad (3.27)$$

Proof: (3.27) trivially holds. Now, $\{a_m\}_{m=1}^{\infty}$ strictly decreases iff

$$\frac{1 - x^m}{1 - x^{m+1}} > \frac{1 - y^m}{1 - y^{m+1}} \quad (3.28)$$

holds. Since $(0 \leq) x < y (< 1)$, to prove (3.28) is equivalent to proving that the function $f(x) := \frac{1 - x^m}{1 - x^{m+1}}$ strictly decreases in $[0, 1)$. One readily finds out that

$$f'(x) \sim -x^{m+1} + (m+1)x - m =: g(x)$$

and $g'(x) \sim 1 - x^m > 0$. Since $g(0) = -m$ and $\lim_{x \rightarrow 1^-} g(x) = 0$, it is implied that $f'(x) < 0$ and the strictly decreasing nature of the sequence in (3.26) is proved. \square

Based on Lemma 3.2, we immediately obtain.

Theorem 3.3

Under the assumptions of Thm 3.2, $\kappa_m(\omega_{opt})$, given by (3.25), strictly decreases as a function of m , with $\lim_{m \rightarrow \infty} \kappa_m(\omega_{opt}) = 1$.

We turn now our attention to the determination of $\min_{\omega} \kappa_m$ in case $m(\geq 3)$ is odd, where $\kappa_m = \kappa_m(\omega)$ is given by the first expression in (3.11). After some simple manipulation we obtain

$$\begin{aligned} \frac{\partial \kappa_m}{\partial \omega} &\sim \nu_1 [1 - (1 - \omega \nu_n)^m] (1 - \omega \nu_1)^{m-1} \\ &- \nu_n [1 - (1 - \omega \nu_1)^m] (1 - \omega \nu_n)^{m-1} =: \chi(\omega). \end{aligned} \quad (3.29)$$

On differentiation of the function $\chi(\omega)$ in (3.29) and after simple operations, one obtains

$$\begin{aligned} \frac{\partial \chi(\omega)}{\partial \omega} &\sim - \nu_1^2 [1 - (1 - \omega \nu_n)^m] (1 - \omega \nu_1)^{m-2} \\ &+ \nu_n^2 [1 - (1 - \omega \nu_1)^m] (1 - \omega \nu_n)^{m-2} =: \psi(\omega). \end{aligned} \quad (3.30)$$

Based on the expressions of the functions $\chi(\omega)$ and $\psi(\omega)$ in (3.29) and (3.30), respectively, we prove the validity of the following statement.

Theorem 3.4

Let the eigenvalues ν_i , $i = 1(1)n$, of $I - G$ in (3.7) satisfy (3.22). Then the condition number $\kappa_m = \kappa_m(\omega)$ of $\tilde{M}_m^{-1}A$ given by (3.11) for odd $m \geq 3$ is minimized with respect to $\omega \in (0, \frac{2}{\nu_1})$ for $\omega = \omega_{opt}^{(m)}$. The optimal value $\omega_{opt}^{(m)}$ is the unique real root, in the interval $(\frac{1}{\nu_1}, \frac{2}{\nu_1 + \nu_n})$, of the $(2m - 4)^{th}$ degree polynomial equation:

$$\tau_m(\omega) := \frac{1}{\omega^2 \nu_1 \nu_n (\nu_n - \nu_1)} \chi(\omega) = 0, \quad (3.31)$$

where $\chi(\omega)$ was defined in (3.29). (**Note:** $2m - 4 = 2, 6, 10, \dots$, for $m = 3, 5, 7, \dots$)

Proof: For $\omega \in (0, \frac{1}{\nu_1}]$ we have that

$$0 \leq \lambda_1(\omega) = 1 - \omega \nu_1 < \lambda_n(\omega) = 1 - \omega \nu_n < 1. \quad (3.32)$$

A careful inspection of (3.29) and (3.32) reveals that the present situation is similar to that in Case Ib of Thm 3.1. Therefore by virtue of the second part of Lemma 3.1, κ_m strictly decreases. For $\omega \in (\frac{1}{\nu_1}, \min\{\frac{2}{\nu_1}, \frac{1}{\nu_n}\})$ it is

$$-1 < \lambda_1(\omega) = 1 - \omega \nu_1 < 0 < \lambda_n(\omega) = 1 - \omega \nu_n < 1. \quad (3.33)$$

In this case one has to appeal to the expression of $\psi(\omega)$ in (3.30). It is readily checked that in view of (3.33), $\psi(\omega) > 0$ for all ω in the interval of interest. This implies that $\chi(\omega)$ strictly increases.

On the other hand, it is found out that

$$\begin{aligned}\chi\left(\frac{1}{\nu_1}\right) &= -\nu_n \left(1 - \frac{\nu_n}{\nu_1}\right)^{m-1} < 0, \\ \chi\left(\frac{2}{\nu_1}\right) &= 2\nu_n \left[1 + \left(1 - \frac{2\nu_n}{\nu_1}\right) + \cdots + \left(1 - \frac{2\nu_n}{\nu_1}\right)^{m-2}\right] > 0, \text{ if } \frac{2}{\nu_1} \leq \frac{1}{\nu_n}, \\ \chi\left(\frac{1}{\nu_n}\right) &= \nu_1 \left(1 - \frac{\nu_1}{\nu_n}\right)^{m-1} > 0.\end{aligned}\tag{3.34}$$

Therefore κ_m strictly decreases for ω increasing from $\frac{1}{\nu_1}$ up to a certain value and then for ω increasing up to $\min\left\{\frac{2}{\nu_1}, \frac{1}{\nu_n}\right\}$, κ_m strictly increases. One more case remains to be examined. More specifically, the one where $\omega \in \left[\frac{1}{\nu_n}, \frac{2}{\nu_1}\right)$. This time it is

$$-1 < \lambda_1(\omega) = 1 - \omega\nu_1 < \lambda_n(\omega) = 1 - \omega\nu_n \leq 0.\tag{3.35}$$

Apparently, this case exists iff $\nu_1 < 2\nu_n$. Again we have a situation similar to that in Case Id of Thm 3.1. Consequently from the second part of Lemma 3.1, we have what κ_m strictly increases in the interval in question. Summarizing our partial conclusions so far regarding the monotonic behavior of κ_m we obtain the desired result by observing that $\frac{1}{\nu_1} < \frac{2}{\nu_1 + \nu_n} < \min\left\{\frac{2}{\nu_1}, \frac{1}{\nu_n}\right\}$ and that

$$\chi\left(\frac{2}{\nu_1 + \nu_n}\right) = \frac{(\nu_1 - \nu_n)^m}{(\nu_1 + \nu_n)^{2m-2}} [(\nu_1 + \nu_n)^{m-1} - (\nu_1 - \nu_n)^{m-1}] > 0\tag{3.36}$$

while simple manipulation on $\chi(\omega)$ results the expression for $\tau_m(\omega)$ in (3.31). \square

As is obvious from the degree $2m - 4$ of the polynomial equation $\tau_m(\omega) = 0$ in (3.29), the only odd value of $m \geq 3$ for which $\omega_{opt}^{(m)}$ can be found explicitly is $m = 3$. In all other cases $\omega_{opt}^{(m)}$ can only be found computationally from the values of ν_1 and ν_n . Specifically, for $m = 3$ we have:

Corollary 3.2

Under the assumptions of Thm 3.4 with $m = 3$, the optimal value $\omega_{opt}^{(3)}$ is given by

$$\omega_{opt}^{(3)} = \frac{3}{\nu_1 + \nu_n + \sqrt{\nu_1^2 + \nu_n^2 - \nu_1\nu_n}}.\tag{3.37}$$

Proof: Using (3.29) and (3.31) it can be found out that

$$\tau_3(\omega) := \nu_1\nu_n\omega^2 - 2(\nu_1 + \nu_n)\omega + 3.$$

Of the two zeros of $\tau_3(\omega) = 0$ the one in the interval $\left(\frac{1}{\nu_1}, \frac{2}{\nu_1 + \nu_n}\right)$ is that given in (3.37). \square

Remarks: (i) For $m = 1$, we note from (3.11) that $\kappa_1(\omega) = \frac{\nu_1}{\nu_n}$ that is independent of ω . So, if ω is kept fixed during the iterations no improvement over the original preconditioner should be expected. (ii) It must be noted that in [10] the m -step preconditioner given in (3.1) was used in conjunction with the block Jacobi iteration matrix (damped or underrelaxed Jacobi preconditioner) and some experimental results by using Parallel Computers and $m = 2$ were given (without giving the optimal value of the extrapolation parameter). (iii) Under the assumptions of Cor 3.1 or in case the only information available on the spectrum of G is its spectral radius $\rho(G) = \lambda_n < 1$, Thm 3.4 and Cor 3.2 should be applied with $\nu_1 = 1 - \lambda_1 = 1 + \rho(G)$ and $\nu_n = 1 - \lambda_n = 1 - \rho(G)$.

As in the previous case of even $m \geq 2$ it is possible to find out how $\kappa_m(\omega_{opt}^{(m)})$ for odd $m \geq 3$ behaves. This can be done despite the fact that $\omega_{opt}^{(m)}$ is a function of m and therefore not the same

for all odd m . More specifically, we have:

Theorem 3.5

Under the assumptions of Thm 3.4 $\kappa_m(\omega_{opt}^{(m)})$ strictly decreases as a function of the odd $m(\geq 3)$, with $\lim_{m \rightarrow \infty} \kappa_m(\omega_{opt}^{(m)}) = 1$.

Proof: Recall that $\omega_{opt}^{(m)} \in \left(\frac{1}{\nu_1}, \frac{2}{\nu_1 + \nu_n}\right)$. So, relationships (3.33) hold for any fixed ω in this interval. But then, it is easy to see, because of the signs of $\lambda_1(\omega)$ and $\lambda_n(\omega)$ and the fact that m is odd, that as m increases the numerator in $\kappa_m(\omega)$ strictly decreases while the denominator strictly increases making $\kappa_m(\omega)$ a strictly decreasing function of m for any fixed $\omega \in \left(\frac{1}{\nu_1}, \frac{2}{\nu_1 + \nu_n}\right)$. Consequently the following inequalities hold

$$\kappa_m(\omega_{opt}^{(m)}) > \kappa_{m+2}(\omega_{opt}^{(m)}) \geq \kappa_{m+2}(\omega_{opt}^{(m+2)}), \quad m = 3, 5, 7, \dots,$$

proving that our assertion holds true. Also, the limiting value of $\kappa_m(\omega_{opt}^{(m)})$ is trivially obtained, which concludes the proof. \square

So far we have found not only the optimal values of $\omega(\omega_{opt}^{(m)})$, and therefore $\kappa_m(\omega_{opt}^{(m)})$, for any integer $m \geq 2$, but also that as m increases taken on only even or only odd values, the corresponding $\kappa_m(\omega_{opt}^{(m)})$ strictly decrease. This theoretical result might not be of much practical value because of the additional number of matrix-vector multiplications introduced as m increases. On the other hand, a straightforward comparison between any two successive values $\kappa_m(\omega_{opt}^{(m)})$ and $\kappa_{m+1}(\omega_{opt}^{(m+1)})$, even under the simplified assumptions considered in [1], needs a numerical solution of algebraic equations of degree $2m - 4$ for various $\lambda_n = -\lambda_1 = \rho(G)$. In Table 1 we present for selected values of $\rho(G)$ the optimal condition numbers for $m = 2, 3$ and 4. It can be proved that for all $\rho(G) \in (0, 1)$, $\kappa_2(1) > \kappa_3(\omega_{opt}^{(3)}) > \kappa_4(1)$ hold. As can be seen from the table, κ_3 is slightly better than κ_2 , while κ_4 is much better than κ_3 tending to be half of it as $\rho(G)$ approaches 1.

Table 1
Optimal values of the condition numbers
for $m = 2, 3, 4$, $\lambda_n = -\lambda_1 = \rho(G)$ and
 $\lambda_j = 0$ for some j ($j = 1, 2, \dots, n$)

$\rho(G)$	$\kappa_2(1)$	$\kappa_3(\omega_{opt}^{(3)})$	$\kappa_4(1)$
0.1	1.01010	1.00199	1.00010
0.2	1.04167	1.01569	1.00160
0.3	1.09890	1.05239	1.00817
0.4	1.19048	1.12464	1.02627
0.5	1.33333	1.25098	1.06667
0.6	1.56250	1.46677	1.14890
0.7	1.96078	1.85455	1.31596
0.8	2.77778	2.66346	1.69377
0.9	5.26316	5.14272	2.90782
0.95	10.2564	10.1335	5.39102
0.99	50.2513	50.1265	25.3782
0.995	100.250	100.126	50.3766
0.999	500.230	500.109	250.365

The limiting ratio 0.5 of $\frac{\kappa_4(1)}{\kappa_2(1)}$ observed in the table is a particular case of a more general result as $\rho(G)$ approaches 1. In fact it can be checked that $\lim_{\rho(G) \rightarrow 1^-} (\frac{\kappa_{m+2}(1)}{\kappa_m(1)}) = \frac{m}{m+2}$ for even m , which becomes 0.5 when $m = 2$. Also, for $\rho(G)$ approaching 1 it can be proved that $\lim_{\rho(G) \rightarrow 1^-} \omega_{opt}^{(3)} = 0.75$. As one may have noticed, in the simplified case examined above the only information we can have about $\omega_{opt}^{(m)}$ for odd m as $\rho(G) \rightarrow 1^-$ is that it lies in the limiting interval $(0.5, 1)$. This lack of knowledge is the main reason we can not theoretically compare, in the general case, two optimal condition numbers corresponding to two consecutive values of the integer m for a given $\rho(G) < 1$.

We close this section by noting that the idea in [2] for defining m -step additive preconditioners of (1.1), where A is positive definite, can be extended. For this we consider the multisplitting

$$A = P_k - Q_k, \quad \det(P_k) \neq 0, \quad k = 1(1)p, \quad (3.38)$$

and the iteration matrix H of the corresponding multisplitting method (1.5) with $D_k = a_k I$, $k = 1(1)p$. Setting

$$G_k = P_k^{-1} Q_k, \quad M^{-1} = \sum_{i=1}^p a_i P_i^{-1}, \quad (3.39)$$

then

$$H = \sum_{i=1}^p a_i G_i \quad (3.40)$$

and the m -step additive preconditioner is defined by

$$M_m = M(I + H + H^2 + \dots + H^{m-1})^{-1}, \quad m \geq 1, \quad (3.41)$$

provided that M_m is positive definite (and $A \approx M_m$). We note that the m -step additive preconditioner is an m -step preconditioner (see (3.4)) related to the splitting defining a multisplitting method. Certainly, if M is positive definite and $\rho(H) < 1$, then M_m is also positive definite and $A \approx M_m$. In the following theorem we give sufficient conditions for M_m to be positive definite.

Theorem 3.6

Let A in (1.1) be positive definite and

$$A = P_k - Q_k, \quad k = 1(1)2q, \quad (3.42)$$

where

$$P_{q+i} = P_i^T, \quad i = 1(1)q. \quad (3.43)$$

Let the splittings (3.42), $k = 1(1)q$, be P -regular splittings of A . Then the m -step additive preconditioner (3.41), where

$$M = \left(\sum_{i=1}^{2q} a_i P_i^{-1} \right)^{-1}, \quad a_i = \frac{1}{2q}, \quad i = 1(1)2q, \quad H = \sum_{i=1}^{2q} a_i G_i, \quad G_i = P_i^{-1} Q_i, \quad (3.44)$$

is positive definite.

Proof: Since (3.42) for $k = 1(1)q$ are P -regular splittings and (3.43) hold, it follows that (3.42) for $k = q + 1(1)2q$ are also P -regular splittings of A . Thus $P_k + Q_k + (P_k + Q_k)^T = 2(P_k + P_k^T - A)$ is positive definite, $k = 1(1)2q$. Consequently $P_k + P_k^T$ is positive definite, $k = 1(1)2q$. Using (3.43), we find

$$M^{-1} = \sum_{i=1}^{2q} a_i P_i^{-1} = \frac{1}{2q} \sum_{i=1}^q (P_i^{-1} + P_{q+i}^{-1}) = \frac{1}{2q} \sum_{i=1}^q [(P_i^{-1})^T (P_i^T + P_i) P_i^{-1}]. \quad (3.45)$$

Since $P_i + P_i^T$ is positive definite, $i = 1(1)q$, and M^{-1} is a sum of positive definite matrices, M^{-1} and hence M is positive definite. Moreover, it is $\rho(H) < 1$ by Thm 2.2. Now, using Thm 3.1 of [6] we obtain the desired result. \square

4 Optimum SOR Additive Iterative Method

We again consider system (1.1), where

$$A = D - L - L^T \quad (4.1)$$

and A is positive definite. Given the splittings $A = P_k - Q_k$, $k = 1, 2$, with

$$P_1 = \frac{1}{\omega}(D - \omega L), \quad P_2 = P_1^T = \frac{1}{\omega}(D - \omega L^T) \quad (4.2)$$

and $\omega \in \mathbb{R} \setminus \{0\}$ a parameter, it can be shown that $A = P_1 - Q_1$ is a P -regular splitting of A , if $0 < \omega < 2$. Hence Thm 3.6 for $q = 1$ (see also Thm 2.2) implies that the SOR two-splitting or SOR-additive method [2]

$$x^{(m+1)} = Hx^{(m)} + c, \quad m = 0, 1, 2, \dots, \quad (4.3)$$

where

$$H = H(\omega) = \frac{1}{2}(G_1 + G_2), \quad c = \frac{1}{2}(P_1^{-1} + P_2^{-1})b, \quad G_i = P_i^{-1}Q_i, \quad i = 1, 2, \quad (4.4)$$

converges. Under the assumption that A has the 2-cyclic form

$$A = \begin{bmatrix} E_1 & -X \\ -X^T & E_2 \end{bmatrix} \quad (4.5)$$

(E_1, E_2 are diagonal matrices), it was proved in [2] that if λ is an eigenvalue of H , then

$$\lambda = \frac{1}{2}[\omega^2 \mu^2 + \omega(2 - \omega)\mu + 2(1 - \omega)], \quad (4.6)$$

where μ is an eigenvalue of the Jacobi iteration matrix $J = I - D^{-1}A$ for A . It is noted that J has real eigenvalues, which occur in \pm pairs and $\rho(J) < 1$. Moreover it was shown in [2] that $\min_{0 < \omega < 2} \rho(H(\omega)) = \rho(H(\omega_{opt}))$, where

$$\omega_{opt} = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}, \quad \mu_m = \rho(J). \quad (4.7)$$

Here it should be pointed out that the analysis in [2] was done to cover cases of practical importance where $\rho(J)$ is close to 1. Then the advantage of using the SOR method appears since it has much better convergence rates compared to those of the Jacobi method.

However, we can observe that $\lim_{\mu_m \rightarrow 0^+} \mu = 0$ for all the eigenvalues μ of J and from (4.6) we obtain $\lim_{\mu_m \rightarrow 0^+} \lambda = 1 - \omega$, which means that the optimum ω satisfies $\lim_{\mu_m \rightarrow 0^+} \omega_{opt} = 1$. On the other hand, (4.7) for $\mu_m = 0$ gives

$$\omega_{opt} \approx 0.9282 \neq 1. \quad (4.8)$$

This observation suggests that the determination of the optimum ω value must be completed to cover all possible theoretical cases too. In the following theorem we give the complete solution.

Theorem 4.1

Let A in (1.1) be positive definite, $A = D - L - L^T$ and have the form (4.5). Then the optimum value ω_{opt} for ω ($0 < \omega < 2$) of the SOR-additive method defined by (4.3) is given by

$$\omega_{opt} = \begin{cases} \frac{1 - \sqrt{1 - 2\mu_m^2}}{\mu_m^2}, & \text{if } 0 < \mu_m \leq \frac{1}{\sqrt{6}} \\ \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m + \mu_m^2}, & \text{if } \frac{1}{\sqrt{6}} \leq \mu_m < 1, \end{cases} \quad (4.9)$$

where $\mu_m = \rho(J)$ and $J = I - D^{-1}A$.

Proof: The problem we solve is: Find $\min_{\omega} \max_{\mu} |\lambda|$, where λ is given by (4.6), $0 < \omega < 2$, $\mu \in [-\mu_m, \mu_m]$ and $\mu_m < 1$. For this we have that $\frac{\partial \lambda}{\partial \mu} = 0$ iff $\mu = \frac{\omega - 2}{2\omega} \equiv \mu^*$. Moreover, $\mu^* \in [-\mu_m, \mu_m]$ iff $\omega^* \equiv \frac{2}{1 + 2\mu_m} \leq \omega < 2$.

With $\lambda = \lambda(\mu)$ we find

$$y = y(\omega) \equiv |\lambda(\mu_m)| = \frac{1}{2} |\omega^2 \mu_m^2 + \omega(2 - \omega)\mu_m + 2(1 - \omega)|,$$

$$z = z(\omega) \equiv |\lambda(-\mu_m)| = \frac{1}{2} |\omega^2 \mu_m^2 - \omega(2 - \omega)\mu_m + 2(1 - \omega)|,$$

$$v = v(\omega) \equiv |\lambda(\mu^*)| = \begin{cases} \frac{1}{8}(\omega^2 + 4\omega - 4), & \text{if } 2(\sqrt{2} - 1) \leq \omega < 2 \\ \frac{1}{8}(4 - 4\omega - \omega^2), & \text{if } 0 < \omega \leq 2(\sqrt{2} - 1) \end{cases}.$$

Hence $\max_{\mu} |\lambda| = \max\{y, z, v\}$. It can be proved that

(i) If $0 < \mu_m < \frac{\sqrt{2}}{2}$ and $0 < \omega \leq \omega_1 \equiv \frac{1 - \sqrt{1 - 2\mu_m^2}}{\mu_m^2}$ or $\frac{\sqrt{2}}{2} \leq \mu_m < 1$ and $0 < \omega < 2$, then

$$z \leq y = \frac{1}{2} [\omega^2 \mu_m^2 + \omega(2 - \omega)\mu_m + 2(1 - \omega)].$$

(ii) If $0 < \mu_m < \frac{\sqrt{2}}{2}$ and $\omega_1 \leq \omega < 2$, then

$$y \leq z = \frac{1}{2} [\omega(2 - \omega)\mu_m - \omega^2 \mu_m^2 - 2(1 - \omega)].$$

Thus, we distinguish the following cases:

Case I: $\frac{\sqrt{2}}{2} \leq \mu_m < 1$. Then it can be shown that $\omega^* \leq 2(\sqrt{2} - 1)$ and

$$\max\{y, z, v\} = \begin{cases} y & \text{if } 0 < \omega \leq \rho_2 \\ v & \text{if } \rho_2 \leq \omega < 2, \end{cases}$$

where

$$\rho_2 = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}.$$

Now, we find that $\frac{\partial y}{\partial \omega} < 0$ and $\frac{\partial v}{\partial \omega} > 0$, implying $\min_{\omega} y = y(\rho_2)$ and $\min_{\omega} v = v(\rho_2) = y(\rho_2)$. Hence we obtain $\omega_{opt} = \rho_2$ and $\min_{\omega} \max_{\mu} |\lambda| = y(\rho_2) = v(\rho_2) = \frac{1}{8}(\rho_2^2 + 4\rho_2 - 4)$.

Case II: $0 < \mu_m < \frac{\sqrt{2}}{2}$. Then it can be shown that:

(i) If $0 < \mu_m \leq \frac{1}{\sqrt{6}}$, then $2(\sqrt{2} - 1) < \omega_1 \leq \omega^*$.

(ii) If $\frac{1}{\sqrt{6}} \leq \mu_m < \frac{\sqrt{2}}{2}$, then $2(\sqrt{2} - 1) < \omega^* \leq \omega_1$.

Therefore we must distinguish the following subcases:

Case IIa: $0 < \mu_m \leq \frac{1}{\sqrt{6}}$. Then we find

$$\max\{y, z, v\} = \begin{cases} y, & \text{if } 0 < \omega \leq \omega_1 \\ z, & \text{if } \omega_1 \leq \omega \leq \omega^* \\ v, & \text{if } \omega^* \leq \omega \leq 2 \end{cases}$$

and

$$\min_{\omega} y(\omega) = y(\omega_1), \quad \min_{\omega} z(\omega) = z(\omega_1), \quad \min_{\omega} v(\omega) = v(\omega^*) = z(\omega^*) \geq z(\omega_1).$$

Hence we have $\omega_{opt} = \omega_1$ and $\min_{\omega} \max_{\mu} |\lambda| = y(\omega_1) = z(\omega_1)$.

Case IIb: $\frac{1}{\sqrt{6}} < \mu_m < \frac{\sqrt{2}}{2}$. Then it can be proved that

$$0 < 2(\sqrt{2} - 1) < \omega^* < \rho_2 < \omega_1 < 2$$

and

$$\max\{y, z, v\} = \begin{cases} y & \text{if } 0 < \omega \leq \rho_2 \\ v & \text{if } \rho_2 \leq \omega < 2. \end{cases}$$

As in Case I we find that $\omega_{opt} = \rho_2$ and $\min_{\omega} \max_{\mu} |\lambda| = y(\rho_2) = v(\rho_2) = \frac{1}{8}(\rho_2^2 + 4\rho_2 - 4)$. Combining the above results of Cases I, IIa, IIb we obtain (4.9). \square

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