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**EDGE WEIGHT REDUCTION PROBLEMS  
IN DIRECTED, ACYCLIC GRAPHS**

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# Edge Weight Reduction Problems in Directed, Acyclic Graphs

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## Abstract

Let  $G$  be a weighted dag in which each edge weight is not a static quantity, but can be reduced for a certain cost. In this paper we consider the problem of determining which edges to reduce so that the length of the longest paths is minimized and the total cost associated with the reductions does not exceed a given cost. We consider two types of edge reductions, linear reductions and 0/1 reductions, which model different applications. We present efficient algorithms for different classes of graphs, including trees, series-parallel graphs, and directed acyclic graphs, and we show other edge reduction problems to be NP-hard.

**Keywords:** Analysis of algorithms; directed, acyclic graphs; longest path computations; series-parallel graphs; trees.

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# 1 Introduction

Determining the longest path in a directed graph  $G$  is a problem with applications in scheduling task graphs, circuit layout compaction, and performance optimization of circuits. The problem can be solved in linear time when  $G$  is a directed, acyclic graph and it is NP-hard for general graphs [3, 4]. In some applications the weight of an edge is not a static quantity, but can be reduced for a certain cost. The problem arising is that of determining reductions on edge weights so that the length of the longest paths is minimized and the total cost associated with the reductions does not exceed a given cost. In this paper we consider two types of edge reductions, linear reductions and 0/1 reductions, which model different applications. We present efficient algorithms for different classes of graphs, including trees, series-parallel graphs, and directed acyclic graphs, and we show other edge reduction problems to be NP-hard.

Let  $G = (V, E)$  be a weighted, directed, and acyclic graph (dag) with  $n + 1$  vertices,  $v_0, v_1, v_2, \dots, v_n$ , and  $m$  edges. Edge  $(v_i, v_j)$  has weight  $d(v_i, v_j)$  with  $d(v_i, v_j) \geq 0$ . If not stated otherwise, we assume that  $G$  contains only one source  $v_0$  and one sink  $v_n$ . An *edge reduction*  $R$  assigns to every edge  $(v_i, v_j)$  a non-negative quantity  $r(v_i, v_j)$ . The *reduced weight*  $d_r(v_i, v_j)$  of edge  $(v_i, v_j)$  is a function of the edge's weight and its reduction. An edge reduction  $R$  is called a *linear reduction* if for every edge  $(v_i, v_j)$ ,  $r(v_i, v_j)$  is a non-negative real and

$$d_r(v_i, v_j) = d(v_i, v_j) - r(v_i, v_j).$$

An edge reduction is called a *0/1 reduction* if for every edge  $(v_i, v_j)$ ,  $r(v_i, v_j)$  is either 0 or 1 and

$$d_r(v_i, v_j) = \begin{cases} d(v_i, v_j) & \text{if } r(v_i, v_j) = 0 \\ \epsilon \times d(v_i, v_j) & \text{if } r(v_i, v_j) = 1 \end{cases}$$

where  $\epsilon$  is a given real with  $0 \leq \epsilon < 1$ . For both reductions we require  $d_r(v_i, v_j) \geq 0$ .

We briefly comment on where such edge reductions arise. Linear reductions model, for example, physical performance optimizations of circuits through gate resizing and buffer insertions [1, 2, 6, 7]. Such optimizations do not change the topology of the circuit and result in circuits having a smaller delay. At the same time, circuit size and power consumption increase. 0/1 reductions with  $\epsilon = 0$  are a basic operation in clustering heuristics for mapping task graphs to multiprocessors [5, 8]. In a task graph the edge weights represent the communication cost

and vertices mapped to the same processor experience no communication cost. For  $\epsilon > 0$ , 0/1 reductions can model scenarios in which there exist fast and slow buses for communication. Reducing an edge is then equivalent to assigning the corresponding communication to a fast bus.

Given a reduction  $R$  for graph  $G$ , the *reduced graph*  $G_R$  is obtained from  $G$  by replacing each edge weight  $d(v_i, v_j)$  by its reduced weight  $d_r(v_i, v_j)$ . Throughout,  $L(G_R)$  denotes the length of the longest path in  $G_R$  and  $M(G_R)$  denotes the *total reduction*; i.e.,  $M(G_R) = \sum_{(v_i, v_j) \in E} r(v_i, v_j)$ . In this paper we investigate the following three edge reduction problems:

- *(G, L)-problem*

Given  $L$ , find an edge reduction  $R^*$  such that  $L(G_{R^*}) \leq L$  and  $M(G_{R^*})$  is a minimum; i.e., for any edge reduction  $R'$  with  $L(G_{R'}) \leq L$ , we have  $M(G_{R^*}) \leq M(G_{R'})$ .

- *(G, M)-problem*

Given  $M$ , find an edge reduction  $R^*$  such that  $M(G_{R^*}) \leq M$  and  $L(G_{R^*})$  is a minimum; i.e., for any edge reduction  $R'$  with  $M(G_{R'}) \leq M$  we have  $L(G_{R^*}) \leq L(G_{R'})$ .

- *Tradeoff problem*

Given a tradeoff function  $f(G_R) = L(G_R) + \gamma \cdot M(G_R)$  defined for every edge reduction  $R$ , with  $\gamma$  being a constant, find an edge reduction  $R^*$  minimizing the tradeoff function.

In Section 2 we consider linear reductions in in-trees. An in-tree is a tree in which the out-degree of every vertex is at most 1. We present  $O(n)$  time algorithms for solving the  $(G, L)$ -,  $(G, M)$ - and the tradeoff problem in in-trees. Section 3 presents  $O(m^2)$  algorithms for the linear reduction problems in series-parallel graphs. The linear edge reduction problems can be solved in polynomial time for general dags by formulating the problems as linear programs. Sections 4 and 5 consider 0/1 reductions. We show that for series-parallel graphs each one of the three 0/1 reductions problems can be solved in  $O(m^2h)$  time, where  $h$  is the height of a bounded-degree decomposition tree of the series-parallel graph. For in-trees in which the degree of every node is bounded by a constant, the time bounds reduce to  $O(nh)$ , where  $h$  is the height of the in-tree. In Section 5 we show that the 0/1 reduction problems are NP-hard for general dags.

## 2 Linear reduction for in-trees

A directed tree is an *in-tree* if the out-degree of every vertex is at most 1. In this section we present  $O(n)$  time algorithms for the three different versions of linear edge reduction in in-trees. Clearly, our results also hold for out-trees. We point out that the algorithms for series-parallel graphs given in the next section result in  $O(n^2)$  time algorithms for in-trees. However, the algorithms given for series-parallel graphs can handle multiple edges between two vertices. The algorithms given below cannot be generalized to handle multiple edges.

Let  $v_n$  be the root of the in-tree. For convenience, we add an artificial source  $v_0$  and edges  $(v_0, v_i)$  with  $d(v_0, v_i) = 0$  for every leaf  $v_i$ . Even though the resulting graph is no longer an in-tree, the structure crucial to the algorithm is preserved and we refer to it as an in-tree.

### 2.1 Finding an optimal reduction for a given $L$

In the  $(G, L)$ -problem we generate an optimal reduction  $R^*$  satisfying  $L(G_{R^*}) \leq L$  and minimizing  $M(G_{R^*})$ . Reduction  $R^*$  generated by our algorithm satisfies a property, which we call the canonical property, and which is defined next. Let  $R$  be any reduction.  $R$  is *canonical* if for any reduction  $R'$  with  $L(G_R) = L(G_{R'})$  and  $M(G_R) = M(G_{R'})$  the length of the path from  $v_i$  to root  $v_n$  in  $G_R$  is not longer than its length in  $G_{R'}$ . Stated in terms of reductions, in a canonical reduction the reductions occur as close to the root as possible. Canonical reductions for in-trees have the following equivalent characterization used by our algorithms. We refer to an edge  $e$  with  $\tau(e) = d(e)$  (resp.  $\tau(e) = 0$ ) as an edge with *full* (resp. *zero*) reduction. An edge  $e$  with  $0 < \tau(e) < d(e)$  is called an edge with *partial* reduction.

**Lemma 2.1** *Let  $R$  be a canonical reduction and let  $P$  be any path from  $v_0$  to  $v_n$  in  $G_R$ . Then,  $R$  is canonical if and only if path  $P$  contains an edge  $(v_i, v_j)$  with  $0 \leq \tau(v_i, v_j) \leq d(v_i, v_j)$  such that every edge on the path from  $v_j$  to  $v_n$  has full reduction and every edge on the path from  $v_0$  to  $v_i$  has zero reduction.*

**Proof:** The lemma implies that path  $P$  contains at most one edge with partial reduction. Assume every path  $P$  in  $G_R$  can be characterized as stated. Since edges closest to the root are reduced first, there cannot exist another reduction  $R'$  (with the same  $L$ - and  $M$ -values) such

that the length of the path from  $v_i$  to  $v_n$  in  $G_{R'}$  is smaller than its length in  $G_R$ . Hence,  $R$  is canonical.

Assume now that  $R$  is a canonical reduction and  $G_R$  contains a path  $P$  not satisfying the characterization. Let  $(v_i, v_j)$  and  $(v_a, v_b)$  be two distinct edges on  $P$  such that every edge on  $P$  from  $v_b$  to  $v_n$  has full reduction, edge  $(v_a, v_b)$  has either partial or zero reduction, every edge on  $P$  from  $v_0$  to  $v_i$  has zero reduction, and edge  $(v_i, v_j)$  has either partial or full reduction. This implies that  $P$  contains the vertices  $v_0, \dots, v_i, v_j, \dots, v_a, v_b, \dots, v_n$ , with possibly  $v_j = v_a$ . Let  $R'$  be the reduction obtained from  $R$  by setting:

$$\begin{aligned} \tau'(v_a, v_b) &= \min\{d(v_a, v_b), \tau(v_a, v_b) + \tau(v_i, v_j)\} \\ \tau'(v_i, v_j) &= \max\{\tau(v_i, v_j) - (d(v_a, v_b) - \tau(v_a, v_b)), 0\} \\ \tau'(e) &= \tau(e) \text{ for every other edge } e. \end{aligned}$$

Clearly,  $M(G_R) = M(G_{R'})$ . The length of path  $P$  in  $R'$  is as in  $R$ . Furthermore, every path from  $v_0$  to  $v_n$  via edge  $(v_i, v_j)$  goes through edge  $(v_a, v_b)$ , and thus the length of any other path from  $v_0$  to  $v_n$  could only have decreased. Hence,  $L(G_R) = L(G_{R'})$  and  $M(G_R) = M(G_{R'})$ . Let  $v_k$  be a vertex on path  $P$  between (and including)  $v_j$  and  $v_b$ . The length of the path from  $v_k$  to  $v_n$  is smaller in  $G_{R'}$  than in  $G_R$ . This implies that  $R$  is not a canonical reduction and the lemma follows.  $\square$

While there can exist many optimal reductions, there exists only one optimal reduction that is also canonical. We next describe how to find this optimal reduction. Let  $L(v_i)$  be the length of the longest path from  $v_0$  to  $v_i$  in  $G$ . When  $L(v_n) \leq L$ , no edges need to be reduced and we have  $\tau^*(e) = 0$  for every edge  $e$ . Assume that  $L(v_n) > L$ . We determine  $R^*$  by setting, for every edge  $(v_i, v_j)$ ,

$$\tau^*(v_i, v_j) = \begin{cases} d(v_i, v_j) & \text{if } L \leq L(v_i) \\ L(v_i) + d(v_i, v_j) - L & \text{if } L(v_i) < L < L(v_i) + d(v_i, v_j) \\ 0 & \text{otherwise.} \end{cases}$$

The  $O(n)$  running time of the algorithm follows trivially. Clearly,  $L(G_{R^*}) = L$ . Consider next any path from  $v_0$  to  $v_n$  in  $G_{R^*}$ . This path contains at most one edge, say  $(v_a, v_b)$ , with partial reduction. All edges from  $v_0$  to  $v_a$  have zero reduction and all edges from  $v_b$  to  $v_n$  have full reduction. Hence,  $R^*$  is a canonical reduction. Optimality of  $R^*$  is established in the following lemma.

**Lemma 2.2** *Let  $R^*$  be the canonical reduction generated by our algorithm. Then,  $R^*$  is an optimal reduction.*

**Proof:** Assume  $R^*$  is not an optimal reduction. Then there exists another canonical reduction  $R'$  with  $M(G_{R^*}) > M(G_{R'})$  and  $L(G_{R'}) \leq L$ . This implies that in graph  $G_{R^*}$  there exists a path from  $v_0$  to  $v_n$  containing an edge  $e$  with  $\tau^*(e) > \tau'(e)$ . Let  $e = (v_i, v_j)$  be such an edge. Consider first the case when edge  $(v_i, v_j)$  has full reduction in  $R^*$ . Observe that this implies  $L \leq L(v_i)$ .  $R'$  is a canonical reduction and edge  $(v_i, v_j)$  has either partial or zero reduction in  $R'$ . Hence, any edge on a path from  $v_0$  to  $v_i$  has zero reduction in  $R'$ . The length of the longest path from  $v_0$  to  $v_j$  in  $G_{R'}$  is  $L(v_i) + d_{r'}(v_i, v_j) > L(v_i) + d_{r^*}(v_i, v_j) \geq L$ , contradicting our assumption of  $L(G_{R'}) \leq L$ .

Assume now that edge  $(v_i, v_j)$  has partial reduction in  $R^*$ . This implies  $L(v_i) < L < L(v_i) + d(v_i, v_j)$ . Since  $R^*$  is a canonical reduction, any edge on a path from  $v_0$  to  $v_i$  in  $G_{R^*}$  has zero reduction. The length of the longest path from  $v_0$  to  $v_j$  in  $G_{R^*}$  is equal to  $L(v_i) + d_{r^*}(v_i, v_j) = L$ . Since  $r'(v_i, v_j) < r^*(v_i, v_j) < d(v_i, v_j)$  and  $R'$  is a canonical reduction, any edge on a path from  $v_0$  to  $v_i$  in  $G_{R'}$  has zero reduction. Thus, the longest path length from  $v_0$  to  $v_j$  in  $G_{R'}$  is equal to  $L(v_i) + d_{r'}(v_i, v_j) > L(v_i) + d_{r^*}(v_i, v_j) = L$ , giving a contradiction. It thus follows that  $R^*$  is an optimal reduction.  $\square$

## 2.2 Finding an optimal reduction for a given $M$

We now turn to the  $(G, M)$ -problem in which we are given  $M$  and are to determine a reduction  $R^*$  with  $M(G_{R^*}) \leq M$  minimizing the length of the longest path from  $v_0$  to  $v_n$ . We first describe an  $O(n \log n)$  time algorithm and then describe how to improve its running time to  $O(n)$ .

Let  $\text{OPT\_L}(G, L)$  be the  $O(n)$  time algorithm for the  $(G, L)$ -problem described in the previous section. In the  $(G, M)$ -problem we are searching for the smallest  $L^*$  such that  $\text{OPT\_L}(G, L^*)$  generates a reduction  $R^*$  with  $M(G_{R^*}) \leq M$ .  $M(G_R)$  is a piecewise-linear, non-increasing function of  $L(G_R)$ . This allows us to perform a binary search for  $L^*$ . Actually, the binary search we perform may not produce  $L^*$ , but a value close to it. Let  $L(v_i)$  be again the length of the longest path from  $v_0$  to  $v_i$  in  $G$ . Edge  $(v_i, v_j)$  creates the entry  $L(v_i) + d(v_i, v_j)$  and let

$\mathcal{L} = \langle L_1, L_2, \dots, L_n \rangle$  be the list containing these entries in increasing order, with duplicates being removed. Obviously, list  $\mathcal{L}$  may contain fewer than  $n$  elements. Since  $L_{i-1} < L_i$ , we have  $M(G_{R_{i-1}}) > M(G_{R_i})$ . Assume invoking algorithm  $\text{OPT\_L}(G, L_i)$  generates reduction  $R_i$ . Let  $k$  be the index such that

$$M(G_{R_{k-1}}) \geq M > M(G_{R_k}).$$

By using algorithm  $\text{OPT\_L}$ , index  $k$  can be determined in  $O(n \log n)$  time. If  $M(G_{R_{k-1}}) = M$ , then  $R^* = R_{k-1}$  (and  $L^* = L_{k-1}$ ). Assume thus that  $M(G_{R_{k-1}}) > M > M(G_{R_k})$ . We next describe how to generate reduction  $R^*$  from the canonical reductions  $R_{k-1}$  and  $R_k$ . Clearly, an edge having full reduction in  $R_k$  has full reduction in  $R_{k-1}$ . Such an edge will receive full reduction in  $R^*$ . An edge having zero reduction in  $R_{k-1}$  has zero reduction in  $R_k$ . It will receive zero reduction in  $R^*$ . Consider now all the edges of  $G$  whose reduction in  $R_{k-1}$  is larger than in  $R_k$  (no edge can have a smaller reduction in  $R_k$ ). Let  $E_p$  be the set containing these edges. Let  $L_{k-1} + \delta = L_k$ ,  $\delta > 0$ . The following characterization of the edges in  $E_p$  is used in determining their reduction in  $R^*$ .

**Lemma 2.3** *Let  $P$  be a path from  $v_0$  to  $v_n$  in  $G$ . Then,  $P$  contains at most one edge belonging to set  $E_p$ . For any edge  $e$  in  $E_p$ , we have  $\tau_{k-1}(e) - \tau_k(e) = \delta$ .*

**Proof:** Assume there exists a path  $P$  containing two or more edges in set  $E_p$ . Let  $(v, w)$  be the edge on  $P$  in set  $E_p$  closest to root  $v_n$ . In  $R_k$ , edge  $(v, w)$  has either partial or zero reduction. We only give the argument for the case when  $(v, w)$  has partial reduction (zero reduction is handled in a similar way). Since  $R_k$  and  $R_{k-1}$  are canonical reductions, the following holds. Edge  $(v, w)$  has full reduction in  $R_{k-1}$  (if it had partial reduction, path  $P$  could not contain two edges belonging to  $E_p$ ). In addition, edge  $(u, v)$  on path  $P$  has zero reduction in  $R_k$  and either full or partial reduction in  $R_{k-1}$ . This also implies that the two edges of  $P$  belonging to  $E_p$  are adjacent. Hence, we have

$$L_{k-1} < L(u) + d(u, v) < L_k.$$

The left side of the inequality holds since edge  $(u, v)$  is reduced in  $R_{k-1}$ . The right side holds since edge  $(v, w)$  is partially reduced in  $R_k$  (the relation  $L(u) + d(u, v) \leq L_k$  would allow full

reduction on edge  $(v, w)$  in  $R_k$ ). The above inequality implies that entry  $L(u) + d(u, v)$  is not in list  $\mathcal{L}$ . Hence, path  $P$  cannot contain two edges belonging to set  $E_p$ .

Let edge  $e = (v, w)$  be an edge in  $E_p$ . Assume  $e$  has partial reduction in both  $R_k$  and  $R_{k-1}$ . The other three cases of possible reductions on edge  $e$  in  $R_k$  and  $R_{k-1}$  are handled in a similar manner. From algorithm OPT L it follows that  $r_k(v, w) = L(v) + d(v, w) - L_k$  and  $r_{k-1}(v, w) = L(v) + d(v, w) - L_{k-1} = L(v) + d(v, w) - L_k + \delta$ . Hence,  $r_{k-1}(e) - r_k(e) = \delta$  follows.  $\square$

We can now state how  $R^*$  is generated from reductions  $R_k$  and  $R_{k-1}$ . We set

$$r^*(v_i, v_j) = \begin{cases} d(v_i, v_j) & \text{if } r_k(v_i, v_j) = d(v_i, v_j) & (1) \\ 0 & \text{if } r_{k-1}(v_i, v_j) = 0 & (2) \\ r_k(v_i, v_j) + \frac{M - M(G_{R_k})}{|E_p|} & (v_i, v_j) \in E_p & (3) \end{cases}$$

The justifications for (1) and (2) have already been given.  $M - M(G_{R_k})$  represents the amount of reduction that can be spent in addition to  $M(G_{R_k})$ . This amount is evenly distributed among the edges in  $E_p$ . Lemma 2.3 implies  $M(G_{k-1}) - M(G_{R_k}) = |E_p| \times \delta$ . Since  $M(G_{k-1}) \geq M > M(G_k)$ , we have  $\frac{M - M(G_{R_k})}{|E_p|} \leq \delta$  and thus  $r^*(v_i, v_j) \leq d(v_i, v_j)$ . From (1), (2), and (3) it follows that  $R^*$  is a canonical reduction. Hence, given index  $k$ , the optimal reduction  $R^*$  can be generated in  $O(n)$  time and thus the  $O(n \log n)$  overall time bound follows.

The remainder of this section describes how to reduce the running time to  $O(n)$  by using prune-and-search. Our improved algorithm also performs  $O(\log n)$  searches to determine index  $k$ , but it reduces the amount of relevant data by a constant fraction after each search. Let  $\mathcal{L}$  now be the unsorted list containing the entries  $L(v_i) + d(v_i, v_j)$ . Assume that at the beginning of each iteration we have identified in list  $\mathcal{L}$  two entries  $L_a$  and  $L_b$  with  $L_a < L_k < L_b$ . For the first iteration we set  $L_a = 0$  and  $L_b = L_n$ . The edges of  $G$  are partitioned into 4 sets,  $E_z$ ,  $E_u$ ,  $E_p$ , and  $E_f$ . Set  $E_z$  contains the edges having zero reduction in both  $R_a$  and  $R_b$ . Set  $E_f$  contains the edges having full reduction in both  $R_a$  and  $R_b$ . As already argued in the  $O(n \log n)$  algorithm, edges in  $E_z$  receive zero reduction in  $R^*$  and edges in  $E_f$  receive full reduction in  $R^*$ . Set  $E_p$  contains edges for which it has already been determined that they receive partial reduction in  $R^*$ . An edge  $e = (u, v)$  that has partial reduction in both  $R_a$  and  $R_b$  clearly belongs to  $E_p$ . In addition,  $(u, v)$  belongs to  $E_p$  if it has full reduction in  $R_a$ , partial reduction in  $R_b$ ,

and every edge  $(x, u)$  has zero reduction in  $R_a$ . The amount of reduction on edge  $e = (u, v)$  in  $R^*$  is not yet known. However, for any  $L_k$  with  $L_a = L_k + \delta < L_b$ , we have  $r_k(e) = r_a(e) - \delta$ . Set  $E_u$  contains all remaining edges. Their type and amount of reduction remains to be decided.

Let  $M_f$  be the total reduction on the edges in set  $E_f$ ; i.e.,  $M_f = \sum_{(u,v) \in E_f} d(u, v)$ . Let  $M_{p,a}$  be the total reduction made on the edges in set  $E_p$  in reduction  $R_a$ . Let  $\mathcal{L}_{a,b}$  be the sublist of  $\mathcal{L}$  containing the entries  $L_j$  with  $L_a < L_j < L_b$ , with  $n_{ab}$  being the number of elements in list  $\mathcal{L}_{a,b}$ . If  $n_{ab} \leq 4$ , we check each one of the entries in  $\mathcal{L}_{a,b}$  as to whether it is  $L_{k-1}$ . Otherwise, we determine the  $\frac{n_{ab}}{4}$ -th smallest element in list  $\mathcal{L}_{a,b}$ . Let  $L_q$  be this element and let  $L_a + \delta = L_q$ . The reduction on the edges in  $E_u$  in  $R_q$  is determined by using the method described in Section 2.1. Doing so partitions  $E_u$  into 3 sets,  $E_{u,z}$ ,  $E_{u,p}$ , and  $E_{u,f}$ , depending on whether an edge of  $E_u$  receives zero, partial, or full reduction in  $R_q$ , respectively. The selection of  $L_q$  implies that  $|E_{u,z}| = n_{ab}/4$  and  $|E_{u,p}| + |E_{u,f}| = 3n_{ab}/4$ . We then have

$$M(G_{R_q}) = M_f + (M_{p,a} - \delta * |E_p|) + M_{u,f} + M_{u,p},$$

with  $M_{u,p} = \sum_{(u,v) \in E_{u,p}} r_q(u, v)$  and  $M_{u,f} = \sum_{(u,v) \in E_{u,f}} d_q(u, v)$ .

If  $M(G_{R_q}) = M$ , we have  $R^* = R_q$  and the algorithm terminates. Consider the case when  $M(G_{R_q}) > M$ .  $L_q$  is a new lower bound (since  $L_q < L_k < L_b$  holds) and, after updating the edge sets, the next iteration continues with  $L_q$  and  $L_b$ . The sets and reduction quantities are updated as follows:

- The edges in  $E_{u,z}$  are added to  $E_z$  and are deleted from  $E_u$ .
- Edges from  $E_{u,p}$  and  $E_{u,f}$  that qualify for  $E_p$  are moved from set  $E_u$  to  $E_p$ . The total reduction made on the edges in the new set  $E_p$  in reduction  $R_q$  (i.e., the new  $M_{p,a}$ ) is computed.

Assume now that  $M(G_{R_q}) < M$ . In this case we have found a new upper bound and continue with  $L_b = L_q$  (after updating the edge sets). The updating involves:

- The edges in  $E_{u,f}$  are added to  $E_f$  and are deleted from  $E_u$ .  $M_f$  is updated.
- Edges from  $E_{u,p}$  and  $E_{u,f}$  that qualify for  $E_p$  are moved from set  $E_u$  to  $E_p$ . The total reduction made on the edges in the new set  $E_p$  in reduction  $R_q$  is computed.

Clearly, the work done in an iteration is  $O(|E_u|)$ . In order to establish the  $O(n)$  overall time, we show that the size of set  $E_u$  reduces by a constant fraction each iteration. If  $M(G_{R_q}) > M$ , the edges belonging to  $E_{u,z}$  are deleted from  $E_u$ . Since  $|E_{u,z}| = n_{ab}/4$ , the size of  $E_u$  is at most  $3n_{ab}/4$  in the next iteration.

Consider now the case when  $M(G_{R_q}) < M$ . The deletion of the edges belonging to set  $E_{u,f}$  from  $E_u$  does not imply that the size of set  $E_u$  decreases by a constant fraction. Indeed,  $E_{u,f}$  could be the empty set. However, the following argument shows that the size of  $E_u$  is reduced by at least one half. Let  $(u, v)$  be an edge that has full reduction in  $R_a$ , partial reduction in  $R_q$ , and which does not qualify for set  $E_p$ . It does not qualify since there exists at least one edge  $(x, u)$  that has full or partial reduction in  $R_a$  and zero reduction in  $R_q$ . Edge  $(x, u)$  belongs to set  $E_{u,z}$  and  $(u, v)$  belongs to set  $E_{u,p}$ . Any such edge  $(u, v)$  can thus be assigned a unique edge  $(x, u)$  belonging to  $E_{u,z}$ . Both  $(u, v)$  and  $(x, u)$  remain in set  $E_u$ . Since  $|E_{u,z}| = n_{ab}/4$ , there can be at most  $n_{ab}/4$  edges like  $(u, v)$ . This implies that at least  $n_{ab}/2$  edges of set  $E_u$  either belong to set  $E_{u,f}$  or qualify for inclusion into set  $E_p$  (after being in set  $E_{u,p}$ ). Thus, the size of  $E_u$  is reduced by at least one half.

The  $O(n)$  time bound for the  $(G, M)$ -problem now follows easily. We first determine index  $k$  such that  $M(G_{R_{k-1}}) \geq M > M(G_{R_k})$  using the algorithm described above. We then generate reduction  $R^*$  from  $R_k$  and  $R_{k-1}$  in  $O(n)$  time as described earlier.

### 2.3 Optimal reduction for the tradeoff problem

The approach used for the  $(G, M)$ -problem in the last section can also be used to solve the tradeoff problem in in-trees in  $O(n)$  time. Recall that in the tradeoff problem we are to determine a reduction  $R^*$  minimizing  $f(G_R) = L(G_R) + \gamma M(G_R)$ . Let  $\mathcal{M}(L)$  represent the minimum total reduction needed to reduce the longest path length to  $L$ . It can be shown that  $\mathcal{M}(L)$  is piecewise-linear, non-increasing and concave-up. We can thus represent  $\mathcal{M}(L)$  by a sequence of linear functions,  $a_1 \times L + b_1, a_2 \times L + b_2, \dots, a_{n-1} \times L + b_{n-1}$ , with all  $a_j$ 's being negative. Function  $a_i \times L + b_i$  is associated with interval,  $[L_i, L_{i+1}]$ ,  $1 \leq i \leq n-1$ , where the  $L_i$ -values are as defined in the previous section. In interval  $[L_i, L_{i+1}]$ ,  $\mathcal{M}(L)$  is described by  $a_i \times L + b_i$ . Since  $\mathcal{M}(L)$  is concave-up, we have  $a_1 \leq a_2 \leq \dots \leq a_{n-1} < 0$ .

Function  $f(G_R)$  can be re-written as a function of the longest path length  $L$ ; i.e.,  $\mathcal{F}(L) =$

$L + \gamma \cdot \mathcal{M}(L)$ . Minimizing  $f(G_R)$  is equivalent to minimizing  $\mathcal{F}(L)$ . We distinguish between the following four cases.

**Case 1.**  $1 + \gamma \cdot a_{n-1} < 0$ .

In this case the minimum of  $\mathcal{M}(L)$  occurs at  $L = L_n$ .

**Case 2.**  $1 + \gamma \cdot a_1 > 0$ .

In this case the minimum of  $\mathcal{M}(L)$  occurs at  $L = L_1$ .

**Case 3.** There exists an  $a_j$  such that  $1 + \gamma \cdot a_j = 0$ .

In this case the minimum of  $\mathcal{M}(L)$  occurs at  $L = L_j$ .

**Case 4.** There exists an  $a_j$  such that  $1 + \gamma \cdot a_j < 0$  and  $1 + \gamma \cdot a_{j+1} > 0$ .

In this case the minimum of  $\mathcal{M}(L)$  occurs at  $L = L_{j+1}$ .

The heart of the algorithm is the search for index  $j$  in Cases 3 and 4. Index  $j$  can be determined in  $O(n)$  time by using an approach similar to the one used for the  $(G, M)$ -problem described in the previous section. We determine  $j$  by using a binary search combined with prune and search. In each iteration we again have a lower bound  $L_a$ , an upper bound  $L_b$ , and a new value  $L_q$ . The value of  $a_q$  can be determined in  $O(|E_u|)$  time. We omit the details of the  $O(n)$  time search algorithm.

### 3 Linear reduction for series-parallel graphs

In this section we describe  $O(m^2)$  time algorithms for the linear edge reduction problems in series-parallel graphs. The graphs can have multiple edges between two vertices and thus  $m = \Omega(n^2)$  is possible. Our algorithms use a greedy strategy and they decide which edges to reduce by using information generated by a minimum cut and longest path computations. We start by giving the necessary definitions regarding series-parallel graphs. In Section 3.1 we then describe our algorithms. Section 3.2 addresses their correctness.

A *series-parallel graph* (sp-graph for short)  $G$  is a dag with exactly one source  $v_0$  and one sink  $v_n$ , recursively defined as follows:

1. A dag consisting of a single edge from  $v_0$  to  $v_n$  is a series-parallel graph.

2. Given a set of series-parallel graph  $G_1, G_2, \dots, G_k$ , the dag  $G$  obtained by identifying the  $k$  sources with each other and by identifying the  $k$  sinks with each other is a series-parallel graph. This type of operation is called a *parallel composition*.
3. Given a set of series-parallel graph  $G_1, G_2, \dots, G_k$ , the dag  $G$  obtained by identifying the source of  $G_i$  with the sink of  $G_{i+1}$ , for  $1 \leq i \leq k - 1$ , is a series-parallel graph. This type of operation is called a *series composition*.

An sp-graph  $G$  can be represented by its *decomposition tree*  $D$ . Each node  $N$  of decomposition tree  $D$  corresponds to a subgraph  $G_N$  of  $G$ . If  $N$  is a leaf,  $G_N$  corresponds to the edge of  $G$  represented by the leaf. If  $N$  is an internal node of  $D$ , then  $G_N$  corresponds to the subgraph of  $G$  obtained by either a parallel or a series composition of the subgraphs associated with the children of  $N$ . In case of a parallel composition, we refer to internal node  $N$  as *p-node*, otherwise as an *s-node*. W.l.o.g., we make the following assumptions about decomposition tree  $D$  (this simplifies our correctness argument). We assume that a node and a child of this node are of different type (i.e., they cannot be both *s-* or *p-nodes*). An internal node has degree 1 iff its only child is a leaf. Both of these assumption result in a decomposition tree having a minimum number of nodes. Finally, we assume that the children of an *s-node* are ordered such that if child  $N_1$  is immediately to the left of child  $N_2$ , then the sink of  $G_{N_1}$  is identified with the source of  $G_{N_2}$ .

Testing whether a given dag  $G$  on  $n$  vertices and  $m$  edges is a series-parallel graph can be done in  $O(m)$  time [9]. Furthermore, the decomposition tree  $D$  for a given sp-graph  $G$  can be constructed in  $O(m)$  time by using the recognition algorithm in [9].

### 3.1 The algorithms for sp-grpahs

We first describe an  $O(m^2)$  time algorithm for the  $(G, L)$ -problem. Minor modifications in the termination of the algorithm solve the  $(G, M)$ -problem and the tradeoff problem, respectively.

Our algorithm for solving the  $(G, L)$ -problem generates an optimal reduction  $R^*$  minimizing  $M(G_{R^*})$  and satisfying  $L(G_{R^*}) \leq L$  over a number of iterations, with each iteration generating a new reduction. Let  $R_{i+1}$  be the reduction generated by the  $i$ -th iteration. The length of the longest paths decreases during the iterations, while the total reduction increases; i.e.,

$L(G_{R_i}) > L(G_{R_{i+1}})$  and  $M(G_{R_i}) < M(G_{R_{i+1}})$ .

Let  $R_i$  be the reduction available at the beginning of the  $i$ -th iteration, where  $R_1$  corresponds to a reduction of 0 on every edge of  $G$ . We use  $r_i(e)$  to denote the reduction done on edge  $e$  in  $R_i$ . The reduced weight  $d_i(e)$  is defined by  $d(e) - r_i(e)$ . Clearly, in order to reduce the length of the longest paths in  $G_{R_i}$ , edges on the longest paths from  $v_0$  to  $v_n$  have to receive a reduction. Let  $S_i$  be the subgraph of  $G_{R_i}$  containing only edges on a longest path from  $v_0$  to  $v_n$ . Let a minimum cut in an sp-graph be a cut containing a minimum number of edges without containing an edge of weight 0. Let  $C(S_i)$  be the set of edges corresponding to such a minimum cut in graph  $S_i$ . To generate  $R_{i+1}$ , we increase the reductions for the edges in cut  $C(S_i)$  by the same amount. The amount is determined by two conditions:

- (i) the weight of an edge in  $C(S_i)$  cannot be negative, and
- (ii) reduction on the edges in  $C(S_i)$  has to stop as soon as a path *not* in  $S_i$  becomes a longest path.

More precisely, let  $\gamma_i = \min_{e \in C(S_i)} \{d_i(e)\}$ . Assume temporarily that in  $R_{i+1}$  every edge in  $C(S_i)$  receives an additional reduction of  $\gamma_i$ . This temporary reduction may reduce the edges on the cut by too much (i.e., it can violate condition (ii)). We use this  $R_{i+1}$  to compute  $L(G_{R_{i+1}})$ , the length of the longest paths in  $G_{R_{i+1}}$ . If  $L(G_{R_{i+1}}) > L$ , the  $i$ -th iteration is not the last one and we set (and possibly decrease)  $r_{i+1}(e) = r_i(e) + (L(G_{R_i}) - L(G_{R_{i+1}}))$  for every edge  $e \in C(S_i)$  and  $r_{i+1}(e) = r_i(e)$  otherwise. If  $L(G_{R_{i+1}}) \leq L$ , we set  $r_{i+1}(e) = r_i(e) + (L(G_{R_i}) - L)$  for every edge  $e \in C(S_i)$  and  $r_{i+1}(e) = r_i(e)$  otherwise, and the algorithm terminates.

To complete the description of one iteration, we describe how to generate a minimum cut in a series-parallel graph  $G$  in  $O(m)$  time, where  $m$  is the number of edges in  $G$ . Let  $D$  be the decomposition tree of  $G$ . For every node  $N$  of  $D$ , we compute a set  $cut(N)$  which contains the edges in a minimum cut in  $G_N$  (recall that  $G_N$  is the subgraph of  $G$  corresponding to node  $N$  of  $D$ ). Assume  $N$  is a leaf of  $D$  and  $e$  is the edge of  $G$  corresponding to this leaf. If the weight of  $e$  is 0,  $cut(N)$  corresponds to a set having cardinality  $+\infty$ . Otherwise, we set  $cut(N) = \{e\}$ . When  $N$  is an internal node, we set

$$cut(N) = \begin{cases} cut(N_c) \text{ such that } |cut(N_c)| \leq |cut(N_d)|, \\ \text{where } N_c \text{ and } N_d \text{ are children of } N & \text{if } N \text{ is an } s\text{-node} \\ \bigcup_{N_c \text{ is a child of } N} cut(N_c) & \text{if } N \text{ is a } p\text{-node.} \end{cases}$$

Clearly, by traversing tree  $D$  from the leaves towards the root, a minimum cut can be determined in  $O(m)$  time.

From the above discussion it follows that the time of one iteration is bounded by  $O(m)$ . To bound the number of iterations, we first show that if an edge is included in  $S_i$ , then it is also in  $S_{i+1}$ . Let  $P$  be a longest path in  $S_i$ . Since  $S_i$  is a series-parallel graph,  $P$  contains exactly one edge belonging to cut  $C(S_i)$ . So, if we increase the reduction for each edge on  $C(S_i)$  by  $L(G_{R_i}) - L(G_{R_{i+1}})$ , the length of path  $P$  decreases by  $L(G_{R_i}) - L(G_{R_{i+1}})$ . Thus,  $P$  is still a longest path in  $G_{R_{i+1}}$  and every edge on  $P$  is in  $S_{i+1}$ . After the  $i$ -th iteration either one edge in  $S_i$  receives a weight of 0 or graph  $S_{i+1}$  contains at least one edge not in  $S_i$ . Graph  $G$  contains a total of  $m$  edges and thus the algorithm terminates after at most  $2m$  iterations. The  $O(m^2)$  overall time follows.

We next describe the changes to be made to the above algorithm in order to solve the  $(G, M)$ -problem and the tradeoff problem, respectively. Consider first the  $(G, M)$ -problem. Assume we have determined the minimum cut  $C(S_i)$  and the temporary value of reduction  $R_{i+1}$ . If  $M(G_{R_{i+1}}) < M$ , the  $i$ -th iteration is not the last one and we set  $R_{i+1}$  as done in the above algorithm. If  $M(G_{R_{i+1}}) \geq M$ , we set  $r_{i+1}(e) = r_i(e) + \frac{M - M(G_{R_i})}{|C(S_i)|}$  for every edge  $e \in C(S_i)$  and  $r_{i+1}(e) = r_i(e)$  otherwise and terminate the algorithm. Consider now the tradeoff problem with  $f(G_R) = L(G_R) + \gamma \times M(G_R)$  as the tradeoff function. An increase in the total reduction results in a decrease in the longest path length. We now terminate the algorithm when  $f(G_{R_i}) \leq f(G_{R_{i+1}})$  and output reduction  $R_i$  as the reduction minimizing the tradeoff function.

Before addressing the correctness of the above algorithms, we point out that, not surprisingly, the approach of repeatedly finding a minimum cuts fails for general dags. However, the linear reduction problems can be solved in polynomial time for general dags. All three versions can be phrased as linear programs. For the  $(G, L)$ -problem the formulation is as follows. Let  $t_0, t_1, \dots, t_n$  and  $r(v_i, v_j)$  for every edge  $(v_i, v_j)$  in  $G$  be the variables. Then,

$$\begin{array}{ll}
\text{Minimize} & t_n - t_0 \\
\text{subject to} & t_i + d(v_i, v_j) - r(v_i, v_j) \leq t_j \quad \text{for every } (v_i, v_j) \in E \\
& d(v_i, v_j) - r(v_i, v_j) \geq 0 \quad \text{for every } (v_i, v_j) \in E \\
& \sum_{(v_i, v_j) \in E} r(v_i, v_j) \leq M \\
& t_0 = 0 \text{ and } t_i \geq 0 \quad \text{for } 1 \leq i \leq n
\end{array}$$

### 3.2 Correctness of the series-parallel algorithm

In this section we show the correctness of the algorithm for the  $(G, L)$ -problem. The correctness arguments for the other two algorithms are almost identical. Assume now that the algorithm terminates after  $k$  iterations generating reduction  $R_{k+1}$ . We prove that  $R_{k+1}$  is an optimal reduction by showing that there exists an optimal solution  $R^*$  with  $r_{k+1}(e) = r^*(e)$  for every edge  $e$ . In order for this to be true we need that at the beginning of the  $i$ -th iteration we have  $r_i(e) \leq r^*(e)$ ,  $1 \leq i \leq k$ , which is satisfied for the first iteration. For notation, if  $r_i(e) \leq r^*(e)$  for every edge  $e$ , we say  $R_i \leq R^*$ . If this does not hold for one edge, we say  $R_i \not\leq R^*$ .

Assume  $R_i \leq R^*$  holds at the beginning of the  $i$ -th iteration and that after the  $i$ -th iteration we have  $R_{i+1} \not\leq R^*$ . We call an edge  $e$  with  $r_{i+1}(e) < r^*(e)$  a *surplus edge*. We next describe how to generate, from  $R^*$ , another optimal reduction  $R'$  which satisfies  $R_{i+1} \leq R'$ . Reduction  $R'$  is obtained from  $R^*$  by moving reduction from surplus edges to the edges in  $C(S_i)$  which violate  $R_{i+1} \leq R^*$ . Recall that in the  $i$ -th iteration only the edges in cut  $C(S_i)$  encounter a change in their reduction. The optimal reduction  $R'$  with  $R_{i+1} \leq R'$  is possibly obtained over a number of reduction re-allocations, with each re-allocation generating another optimal reduction. The re-allocations are guided by a labeling process in the decomposition tree of graph  $S_i$ . We next describe this labeling process.

Let  $D_i$  be the decomposition tree of the series-parallel graph  $S_i$ . Initially, only the leaves of  $D_i$  corresponding to an edge in cut  $C(S_i)$  are labeled. The leaf corresponding to an edge  $e$  in the cut is labeled "g" if  $r_{i+1}(e) \leq r^*(e)$ , and it is labeled "b" if  $r_{i+1}(e) > r^*(e)$ . Only nodes on the path from a labeled leaf to the root of  $D_i$  are labeled, with each such node labeled either "g" or "b". A node can only be labeled once all its children that can receive a label have been labeled. Labeling a node  $N$  of  $D_i$  "g" means that every edge  $e$  of subgraph  $G_N$  belonging to cut  $C(S_i)$  satisfies  $r_{i+1}(e) \leq r^*(e)$ . Labeling a node  $N$  of  $D_i$  "b" means that

- (i) there exists an edge  $e$  of  $G_N$  belonging to cut  $C(S_i)$  which satisfies  $r_{i+1}(e) > r^*(e)$ , and
- (ii) there exists no other cut in  $G_N$  for which reductions done in  $R^*$  can be re-allocated to edges in  $G_N$  belonging to cut  $C(S_i)$ .

The labeling process stops when the root of  $D_i$  is labeled. We will show that the root is always labeled "g".

We next give the rules for labeling  $s$ - and  $p$ -nodes and describe how re-allocations are done. Let  $R^*$  be the current optimal reduction and let  $N$  be an unlabeled node in  $D_i$ . When  $N$  is a  $p$ -node, no re-allocations take place. If all children of  $p$ -node  $N$  are labeled "g", we label  $N$  "g". If all children of  $N$  are labeled "b", we label  $N$  "b". We will show that it is not possible for a  $p$ -node to have one child labeled "b" and another labeled "g".

Assume that  $N$  is an  $s$ -node. Since  $C(S_i)$  is a minimum cut, at most one child of  $N$  can be labeled. Assume that  $N$  has a labeled child. If this child is labeled "g", we label  $N$  "g". Assume now that  $N$  has a child, say node  $N_b$ , labeled "b". Before  $N$  is labeled, we check whether re-allocations can be done. Re-allocations could result in changing all "b" labels of nodes in the subtree rooted at  $N_b$  to "g" labels. A cut  $C$  is a *surplus cut* if  $C$  is a cut containing only surplus edges. If  $G_N$  contains a surplus cut, we perform a re-allocation of reductions. Let  $N_u$  be a child of  $N$  so that  $G_{N_u}$  contains a surplus cut  $C$ . We choose  $N_u$  such that no sibling between  $N_u$  and  $N_b$  is associated with a subgraph containing a surplus cut. W.l.o.g. assume that  $N_u$  is to the left of  $N_b$ . Let  $c_b$  be the number of edges in  $G_{N_b}$  belonging to cut  $C(S_i)$  and let  $|C| = c_u$ . Since the algorithm finds minimum cuts in  $S_i$  and  $R^*$  is an optimal reduction, it follows that  $c_b = c_u$ . Let  $e$  be *any* edge belonging to  $C(S_i)$  and  $G_{N_b}$ . Define

$$\delta = \min\{\min_{\hat{e} \in C} \{r^*(\hat{e}) - \tau_{i+1}(\hat{e})\}, \tau_{i+1}(e) - r^*(e)\}.$$

In Lemma 3.1 we show that  $\tau_{i+1}(e) - r^*(e) = \tau_{i+1}(e') - r^*(e')$  for any two edges  $e$  and  $e'$  belonging to cut  $C(S_i)$  and to subgraph  $G_{N_b}$ . Since  $\tau_{i+1}(e) - r^*(e) > 0$  for any edge in  $C(S_i)$  and  $G_{N_b}$  and  $C$  is a surplus cut, we have  $\delta > 0$ . Let  $R'$  be the reduction obtained as follows:

$$r'(e) \leftarrow \begin{cases} r^*(e) + \delta & \text{if } e \text{ is in } C(S_i) \text{ and in } G_{N_b} \\ r^*(e) - \delta & \text{if } e \text{ is in } C \\ r^*(e) & \text{otherwise} \end{cases}$$

In Lemma 3.3 we will show that  $R'$  is another optimal reduction. In our labeling process we now have a new optimal solution and set  $R^* = R'$ . If in the new  $R^*$  we have  $\tau_{i+1}(e) \leq r^*(e)$  for every edge in subgraph  $G_N$ , every labeled node in the subtree rooted at node  $N_b$  and node  $N$  are labeled "g". Otherwise, we continue looking for surplus cuts to perform re-allocations. If no further surplus cut exists in  $G_N$ , we label node  $N$  "b".

We next prove crucial properties regarding the labels and establish the optimality of reduction  $R'$ .

**Lemma 3.1** *Let  $N$  be an internal node of  $D_i$  labeled “b”. Then, any two labeled leaf nodes belonging to the subtree rooted at  $N$  and corresponding to edges  $e$  and  $e'$ , are labeled “b” and  $\tau_{i+1}(e) - \tau^*(e) = \tau_{i+1}(e') - \tau^*(e')$ .*

**Proof:** From the rules given for labeling internal nodes it follows that, if  $N$  is labeled “b”, no node in the subtree rooted at node  $N$  is labeled “g”. Hence, all leaves in this subtree are labeled “b”. We next show that the edges corresponding to these leaves require the same amount of reduction to be re-allocated to them. Assume  $e = (a, b)$  and  $e' = (a', b')$  are two such edges with  $\tau_{i+1}(e) - \tau^*(e) \neq \tau_{i+1}(e') - \tau^*(e')$ . Let  $req(e) = \tau_{i+1}(e) - \tau^*(e)$  for any edge  $e$ . W.l.o.g. assume  $req(e) > req(e')$ . Let node  $N'$  be the lowest common ancestor of the nodes corresponding to  $e$  and  $e'$  in decomposition tree  $D_i$ . Observe that  $N'$  is a  $p$ -node labeled “b”. Let  $x$  be the source and  $y$  be the sink of sp-graph  $G_{N'}$ . We use  $\|u \overset{H}{\rightsquigarrow} v\|_G$  to denote the length of a longest path from a vertex  $u$  to vertex  $v$  in graph  $G$  when requiring the path go through subgraph  $H$ . From the construction of  $S_i$  and  $S_{i+1}$  in the algorithm we have

$$\|x \overset{e}{\rightsquigarrow} y\|_{S_i} = \|x \overset{e'}{\rightsquigarrow} y\|_{S_i} \text{ and } \|x \overset{e}{\rightsquigarrow} y\|_{S_{i+1}} = \|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}}.$$

Since there exist no surplus cuts in  $G_{N'}$  that could decrease  $req(e)$  we have

$$\|x \rightsquigarrow a\|_{G_{R^*}} = \|x \rightsquigarrow a\|_{S_{i+1}} = \|x \rightsquigarrow a\|_{S_i}.$$

Similar equalities hold for the length of the longest paths from  $x$  to  $a'$ , from  $b$  to  $y$ , and from  $b'$  to  $y$ . Observe that  $d^*(e) = d_{i+1}(e) + req(e)$  and  $d^*(e') = d_{i+1}(e') + req(e')$ . We thus have

$$\|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}} < \|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}} + req(e') = \|x \overset{e'}{\rightsquigarrow} y\|_{G_{R^*}} < \|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}} + req(e) = \|x \overset{e}{\rightsquigarrow} y\|_{G_{R^*}}.$$

Let  $\hat{R}$  be the reduction obtained from  $R^*$  by decreasing the reduction on edge  $e'$  by  $req(e) - req(e')$ . This corresponds to increasing the weight on edge  $e'$  to  $\hat{d}(e') = d_{i+1}(e') + req(e)$ . The total reduction in  $\hat{R}$  is smaller than the one in  $R^*$ . The length of the longest path from  $x$  to  $y$  via edge  $e'$  increases in  $\hat{R}$  by  $req(e) - req(e')$ , resulting in a total length of  $\|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}} + req(e)$ . Since this equals the length of the longest path from  $x$  to  $y$  via edge  $e$ , the longest path length is not increased. Hence,  $R^*$  is not an optimal reduction and we have  $req(e) = req(e')$ .  $\square$

**Lemma 3.2** *No  $p$ -node  $N$  of  $D_i$  has children with different labels.*

**Proof:** Assume  $N$  has a child  $N_g$  labeled “g” and another child  $N_b$  labeled “b”. Then, for every edge  $e$  in  $G_{N_b}$  and in cut  $C(S_i)$  we have  $\tau_{i+1}(e) - \tau^*(e) > 0$ . For every edge  $e'$  in  $G_{N_g}$  and in cut  $C(S_i)$  we have  $\tau_{i+1}(e') - \tau^*(e') \leq 0$ . Observe that the length of any path in  $G_{R^*}$  does not exceed the length of the same path in  $S_i$ . In addition, we have  $\|x \overset{e'}{\rightsquigarrow} y\|_{G_{R^*}} \leq \|x \overset{e'}{\rightsquigarrow} y\|_{S_{i+1}}$ . Consider the reduction  $\hat{R}$  obtained from  $R^*$  by decreasing the reduction on every edge  $e'$  by  $\tau_{i+1}(e) - \tau^*(e)$ . Using an argument similar to the one used in Lemma 3.1 one can show that the existence of  $\hat{R}$  contradicts the optimality of  $R^*$ .  $\square$

**Lemma 3.3** *Reduction  $R'$  is an optimal reduction.*

**Proof:** Recall that  $R'$  is obtained from optimal reduction  $R^*$  by decreasing the reduction done on surplus cut  $C$  in graph  $G_{N_u}$  by  $\delta$  and by increasing the reduction on the edges in cut  $C(S_i)$  belonging to  $G_{N_b}$  by  $\delta$ . We already argued that the total reduction of  $R'$  and  $R^*$  is the same. We next show that  $R'$  does not contain a path whose length exceeds  $L$ .

Let  $x$  and  $y$  be source and sink in subgraph  $G_N$ , let  $x_b$  and  $y_b$  (resp.  $x_u$  and  $y_u$ ) be source and sink of  $G_{N_b}$  (resp.  $G_{N_u}$ ). Recall that  $G_{N_u}$  was chosen so that there exists no surplus cut between  $y_u$  and  $x_b$  in  $G_N$ . The length of any path from  $x$  to  $y$  going through  $G_{N_u}$  and  $G_{N_b}$  is the same in  $G_{R^*}$  and  $G_{R'}$ . However, the length of a path going through  $G_{N_u}$ , but not through  $G_{N_b}$  increases by  $\delta$ . Assume there exists such a path in  $G_{R^*}$ . Let  $P$  be the path “by-passing”  $G_{N_b}$ , with  $x_p$  being the first and  $y_p$  being the last vertex on the path. Figure 1 shows such a path  $P$  and the subgraphs employed in our argument. We assume there exists no surplus cut between  $y_b$  and  $y_p$ . If one would exist, we make the leftmost such surplus cut the cut involved in the re-allocation.

Let  $\|x_p \overset{G_{N_b}}{\rightsquigarrow} y_p\|_{S_{i+1}} = h_1$ . Then,  $\|x_p \overset{G_{N_b}}{\rightsquigarrow} y_p\|_{G_{R^*}} = h_1 + \delta + \gamma$ , where  $\tau_{i+1}(e) - \tau^*(e) = \gamma + \delta$ , for any edge  $e$  in  $C(S_i)$  and  $G_{N_b}$ ,  $\gamma \geq 0$ . (The quantity  $\gamma$  corresponds to the amount of reductions still to be re-allocated in the future.) The last equation holds since there are no surplus cuts between  $x_p$  and  $y_p$ . Path  $P$  is not in  $S_i$  and thus every edge on  $P$  has a reduction of 0 in  $R_{i+1}$ . Hence,

$$\|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R'}} = \|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R^*}} \leq \|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R_{i+1}}} = \|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R_i}} \leq h_1.$$

Observe that we are again using the fact that the length of any path in  $G_{R^*}$  does not exceed the

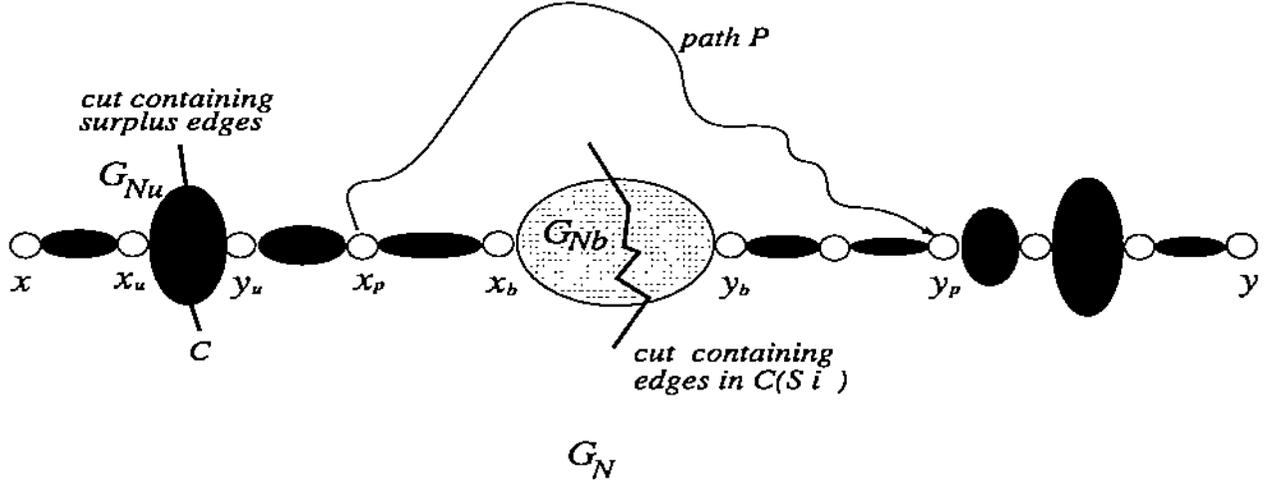


Figure 1: Graph  $G_N$  and its subgraphs  $G_{N_b}$ ,  $G_{N_u}$  and path  $P$ .

length of this path in  $G_{R_i}$ . Consider now, in  $R'$ , the longest path from  $v_0$  to  $v_n$  which contains path  $P$ . Its length is

$$\begin{aligned}
& \|v_0 \rightsquigarrow x_p\|_{G_{R'}} + \|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R'}} + \|y_p \rightsquigarrow v_n\|_{G_{R'}} = \\
& \|v_0 \rightsquigarrow x_p\|_{G_{R^*}} + \delta + \|x_p \overset{P}{\rightsquigarrow} y_p\|_{G_{R'}} + \|y_p \rightsquigarrow v_n\|_{G_{R^*}} \leq \\
& \|v_0 \rightsquigarrow x_p\|_{G_{R^*}} + \delta + h_1 + \|y_p \rightsquigarrow v_n\|_{G_{R^*}} \leq \\
& \|v_0 \rightsquigarrow x_p\|_{G_{R^*}} + \delta + h_1 + \gamma + \|y_p \rightsquigarrow v_n\|_{G_{R^*}} = \\
& \|v_0 \rightsquigarrow x_p\|_{G_{R^*}} + \|x_p \overset{G_{N_b}}{\rightsquigarrow} y_p\|_{G_{R^*}} + \|y_p \rightsquigarrow v_n\|_{G_{R^*}} \leq L.
\end{aligned}$$

Hence, the re-allocation of reduction cannot create a path whose length exceeds  $L$  and  $R'$  is another optimal solution.  $\square$

**Lemma 3.4** *The root of decomposition tree  $D_i$  is labeled "g" and  $R_{i+1} \leq R^*$ .*

**Proof:** Let  $N$  be the root of  $D_i$ . Assume first that  $N$  is an  $s$ -node with label "b" for which child  $N_b$  is also labeled "b". Since there exist no surplus cuts in the subgraphs associated with the siblings of  $N_b$ , as well as in  $G_{N_b}$ , we have  $\|v_0 \rightsquigarrow v_n\|_{G_{R^*}} > \|v_0 \rightsquigarrow v_n\|_{S_{i+1}} \geq L$ . This contradicts the assumption that  $R^*$  is an optimal reduction. When  $N$  is a  $p$ -node labeled "b", each child of  $N$  is labeled "b". Applying the above argument to each child given a contradiction to assuming that  $N$  is labeled "b". Hence, the root is labeled "g" and, by the definition of the label "g", it follows that  $R_{i+1} \leq R^*$ .  $\square$

From the above lemmas it follows that, if the algorithm terminates after  $k$  iterations, then reduction  $R_{k+1}$  generated by the algorithm is an optimal reduction.

## 4 0/1 reduction for series-parallel graphs and in-trees

We now turn to 0/1 edge reductions. Recall that in 0/1 reductions the weight of a reduced edge is  $\epsilon \times d(v_i, v_j)$ , where  $\epsilon$  is given,  $0 \leq \epsilon < 1$ . Let  $G$  be an sp-graph containing  $m$  edges, and let  $h$  be the height of a decomposition tree  $D$  of  $G$  in which every node has at most 2 children. We present an  $O(m^2h)$  time dynamic programming solution that can be used to solve each one of the three versions of the reduction problem. We again allow multiple edges between two vertices of  $G$ . We start by describing the approach used in this algorithm.

Let  $N_i$  be a node of  $D$ . Let  $G_i$  be the subgraph of  $G$  corresponding to the subtree of  $D$  rooted at vertex  $N_i$ . Assume that  $G_i$  has  $m_i$  edges. For vertex  $N_i$  we construct an array  $T_i$  of size  $m_i + 1$ . Entry  $T_i[j]$  represents the length of the longest path in  $G_i$  when at most  $j$  edges are reduced. We thus have  $T_i[0] \geq T_i[1] \geq T_i[2] \geq \dots \geq T_i[m_i - 1] \geq T_i[m_i]$ . The  $T_i$ -arrays are determined in a bottom-up fashion, with a node using the arrays generated for its children. The final answer is determined from the array generated for the root  $N_{root}$  of  $D$ .

If node  $N_i$  is a leaf of decomposition tree  $D$ ,  $G_i$  is an edge. Assume this edge is  $(v_a, v_b)$ . Array  $T_i$  has size two and we have  $T_i[0] = d(v_a, v_b)$  and  $T_i[1] = \epsilon \times d(v_a, v_b)$ . If  $N_i$  is not a leaf,  $T_i$  is constructed as follows.

Assume  $N_i$  is a  $p$ -node. Let  $N_l$  and  $N_r$  be the left child and right child of  $N_i$ , respectively. Assume  $T_l$  and  $T_r$  have already been determined. The entries in  $T_i$  can be defined by using  $T_r$  and  $T_l$  as follows:

$$T_i[j] = \min_{p+q=j} \{\max\{T_r[p], T_l[q]\}\}.$$

By making use of the fact that the entries in arrays  $T_r$  and  $T_l$  are sorted,  $T_i$  can be constructed in  $O(m_i)$  time. One possible solution is given below.

We determine  $T_i$  by scanning arrays  $T_l$  and  $T_r$  twice, each time from right to left. During the first scan of the arrays we determine the entries of  $T_i$  induced by entries in array  $T_r$ . Assume the scan in  $T_r$  is at position  $p$ . We determine the smallest  $q$  such that  $T_l[q - 1] > T_r[p] \geq T_l[q]$ . Let  $j = p + q$ . Then,  $T_r[p]$  is a possible solution for  $T_i[j]$ . If we already recorded a better solution for  $T_i[j]$ , we discard  $p$  and  $q$ . Otherwise, we record it as the currently best one. We then consider  $T_r[p - 1]$ . When we now search for an entry in array  $T_l$ , we search for an index  $q'$  with  $q' \leq q$ . Hence, all requests made to array  $T_l$  can be satisfied by executing one right to left

scan. We then scan both arrays again to determine the entries of  $T_i$  induced by entries in array  $T_l$ . Finally, a left to right scan of array  $T_i$  is performed. We may have recorded in  $T_i[j+l]$  a solution that is worse than the one recorded in  $T_i[j]$ . (Observe that a solution recorded for  $T_i[j]$  is also a solution for  $T_i[j+l]$  with  $l > 1$ .) Hence, we propagate the solution recorded in  $T_i[j]$  to the right until a better solution is encountered. In total, it takes  $O(m_i)$  times to generate  $T_i$  from lists  $T_l$  and  $T_r$ .

Assume now that  $N_i$  is an  $s$ -node. The entries in  $T_i$  can be defined by

$$T_i[j] = \min_{p+q=j} \{T_r[p] + T_l[q]\}.$$

We construct  $T_i$  by enumerating the values of  $T_r[p] + T_l[q]$  for all pairs of  $(p, q)$ ,  $0 \leq p, q \leq m_i$ . This takes  $O(m_i^2)$  time.

Determining array  $T_{root}$  associated with the root of decomposition tree  $D$  takes  $O(\sum_{i=1}^m m_i^2) = O(m^2h)$  time. Recall that  $D$  contains  $m-1$  interior vertices. The three reduction problems, the  $(G, L)$ -problem,  $(G, M)$ -problem and the tradeoff problem, can now be solved in  $O(m^2h)$  time as follows. For the  $(G, L)$ -problem we determine the smallest  $j$  such that  $T_{root}[j] \leq L$ . Quantity  $j$  represents the minimum number of edges that need to be reduced in order to achieve the path length of at most  $L$ . By traversing the tree from the root back to the leaves and using the list associated with each vertex, the edges receiving a reduction can be determined in an additional  $O(m)$  time. For  $(G, M)$ -problem, entry  $T_{root}[M]$  represents the minimum longest path length that can be obtained by reducing at most  $M$  edges. Clearly, the size of the array associated with a vertex does not have to exceed  $M$ . Again, determining which edges get reduced is done by traversing the tree once more. To find the optimal tradeoff between  $M$  and  $L$ , we evaluate  $T_{root}[j] + \gamma \cdot j$  for  $0 \leq j \leq m$ . The pair  $(T_{root}[j], j)$  resulting the minimum tradeoff value gives the solution to the tradeoff problem.

The approach describing above can be used as follows for in-trees. Assume  $G$  is an in-tree of height  $h$  in which the in-degree of every node is bounded by a constant. All three 0/1 reduction problems can then be solved in  $O(nh)$  time. We determine for every vertex  $v_i$  an array  $T_i$  defined as above. Array  $T_i$  is determined in  $O(n_i)$  time by using the arrays associated with  $v_i$ 's children, where  $n_i$  is the number of vertices of  $G$  in the subtree rooted at  $v_i$ . This is done by using the method described for handling  $p$ -nodes. When the vertices of  $G$  do not

have constant in-degree, the time bound of this approach is  $O(\min\{nhc, n^2\})$ , where  $c$  is the maximum in-degree of a vertex.

## 5 0/1 Reduction for general dags

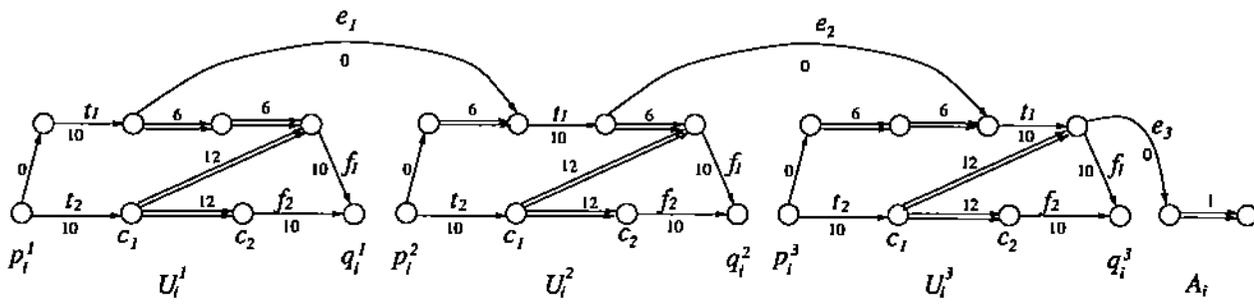
In this section we show that 0/1 reduction problems are NP-hard for general dags. The theorem below proves that the corresponding decision problem is NP-complete for  $\epsilon = 0$ . By changing the weights of the edges in the graph constructed, NP-completeness follows for other values of  $\epsilon$ . We discuss the weight changes for  $\epsilon = \frac{1}{2}$  at the end of this section.

**Theorem 5.1** *Given a weighted dag  $G$  and two positive reals  $M$  and  $L$ , it is NP-complete to decide whether there exists a 0/1 reduction  $R$  with  $\epsilon = 0$  such that  $M(G_R) \leq M$  and  $L(G_R) \leq L$ .*

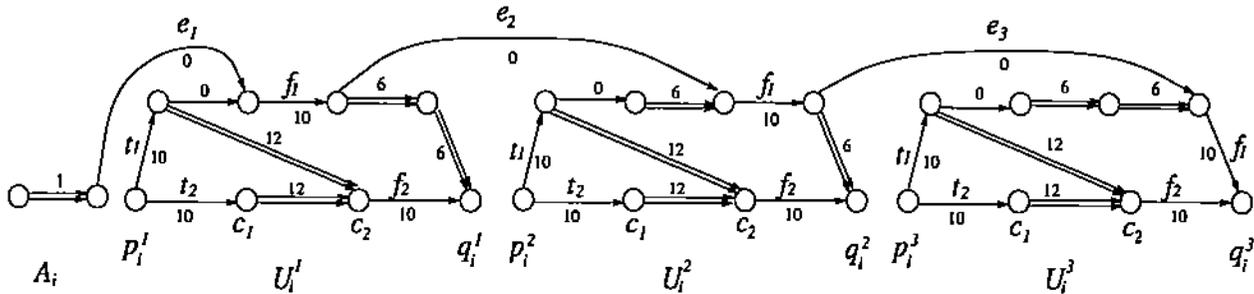
**Proof:** The problem is easily shown to be in NP. NP-completeness follows by a reduction from monotone 3-SAT [4]. Let  $X = \{x_1, x_2, \dots, x_n\}$  be  $n$  variables and  $C = C_1 \wedge C_2 \wedge \dots \wedge C_k$  be an instance of monotone 3-SAT. A clause containing only un-negated variables is called a *positive clause* and a clause containing only negated variables is called a *negative clause*. Let  $C_i = u_i^1 \vee u_i^2 \vee u_i^3$ , where  $u_i^j$  is referred to as a literal,  $1 \leq j \leq 3$ . We next describe how to construct a weighted dag  $G = (V, E)$  and determine  $M$  and  $L$  such that  $G$  has a 0/1 reduction  $R$  with  $M(G_R) \leq M$  and  $L(G_R) \leq L$  if and only if  $C$  is satisfiable.

Graph  $G$  contains  $k$  clause graph,  $G_1, G_2, \dots, G_k$ , which are connected by consistency edges. Clause graph  $G_i$  corresponds clause  $C_i$ , and we distinguish between positive and negative clause graphs (depending on the type of the corresponding clause). Each clause graph is made up of 3 components and one attachment. Each component is an 8-vertex graph and the attachment is a 2-vertex graph. Positive and negative clause graphs are constructed somewhat differently. Figure 2(a) shows a positive and Figure 2(b) shows a negative clause graph. A clause graph contains multiple edges between some of its vertices. Multiple edges between the same pair of vertices have the same weight and thus only one weight is shown.

Let  $U_i^1, U_i^2, U_i^3$ , and  $A_i$  be the three components and the attachment of clause graph  $G_i$ , respectively. In each component  $U_i^j$  we name the following vertices and edges as shown in Figure 2: edges  $t_1$  and  $t_2$  are called the *true-edges*, edges  $f_1$  and  $f_2$  are called the *false-edges*,



(a)



(b)

Figure 2: The clause graph  $G_i$  corresponding to clause  $C_i = u_i^1 \vee u_i^2 \vee u_i^3$ . (a) shows the clause graph corresponding to a positive clause and (b) shows the clause graph corresponding to a negative clause.

$p_i^j$  is the source and  $q_i^j$  is the sink of component  $U_i^j$ , and  $c_1$  and  $c_2$  are the vertices incident to the consistency edges. The path from  $p_i^j$  to  $q_i^j$  containing edges  $t_1$  and  $f_1$  is called the *upper path*, and the one containing  $t_2$  and  $f_2$  is called the *lower path*. The three components and the attachment are connected by edges of weight 0 as shown in Figure 2. Positive and negative clause graphs differ in the way the upper and lower path in a component interact, in the position of edges  $t_1$  and  $f_1$  on the upper path, and in how the components and the attachment are connected.

As already stated, the  $k$  clause graphs are connected by consistency edges. *Consistency edges* are edges of multiplicity 2 and each such edge has a weight of 12. Let  $u_i^a$  and  $u_j^b$ ,  $i < j$ , be two literals formed by the same variable, say  $x_i$ , and assume that  $x_i$  does not form a literal in clauses  $C_{i+1}, \dots, C_{j-1}$ . Graph  $G$  contains a consistency edge from vertex  $c_1$  in component  $U_i^a$  to vertex  $c_2$  in component  $U_j^b$ , and one from vertex  $c_1$  in component  $U_j^b$  to vertex  $c_2$  in component  $U_i^a$ . To complete the construction of  $G$ , we add a sink vertex  $p$  and a source vertex  $q$  and edges of weight 0 from  $p$  to every  $p_i^j$  and from every  $q_i^j$  to  $q$ . Figure 3 shows the graph  $G$  created for the formula  $C = \{(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_5 \vee \bar{x}_6)\}$ .

Clearly, given a monotone 3-SAT formula  $C$ , the corresponding graph  $G$  can be built in polynomial time.  $G$  has a total of  $26k + 2$  vertices. The length of the longest path from source  $p$  to sink  $q$  is 40.  $G$  contains  $k$  such longest paths, one for every clause. For a positive clause graph  $G_i$ , this path contains vertices  $p$  and  $p_i^1$ , edge  $t_1$  of component  $U_i^1$ , edge  $e_1$  of  $G_i$ , edge  $t_1$  of component  $U_i^2$ , edge  $e_2$ , edges  $t_1$  and  $f_1$  of component  $U_i^3$ , and vertex  $q_i^3$ . Figure 4(b) shows such a path. Finally, we set  $M = 6k$  and  $L = 30$ . We claim that  $G$  has a 0/1 reduction in which at most  $6k$  edges are reduced and the length of every path from  $p$  to  $q$  is at most 30 if and only if clause  $C$  is satisfiable.

Since there exist two edge-disjoint paths of length 32 (one is the upper path and the other is the lower path) in every one of the  $3k$  components, reducing the path length to 30 without reducing more than  $6k$  edges implies that we reduce exactly two edges per component. Furthermore, no multiple edges can be reduced. Assume that  $t : X \rightarrow \{T, F\}$  is a truth assignment satisfying  $C$ . We construct a 0/1 reduction  $R$  for  $G$  as follows. Let  $x_i$  be a variable with  $t(x_i) = T$ . Then, in every component  $U_j^b$  with  $u_j^b = x_i$  or  $u_j^b = \bar{x}_i$ , edges  $t_1$  and  $t_2$  are reduced.

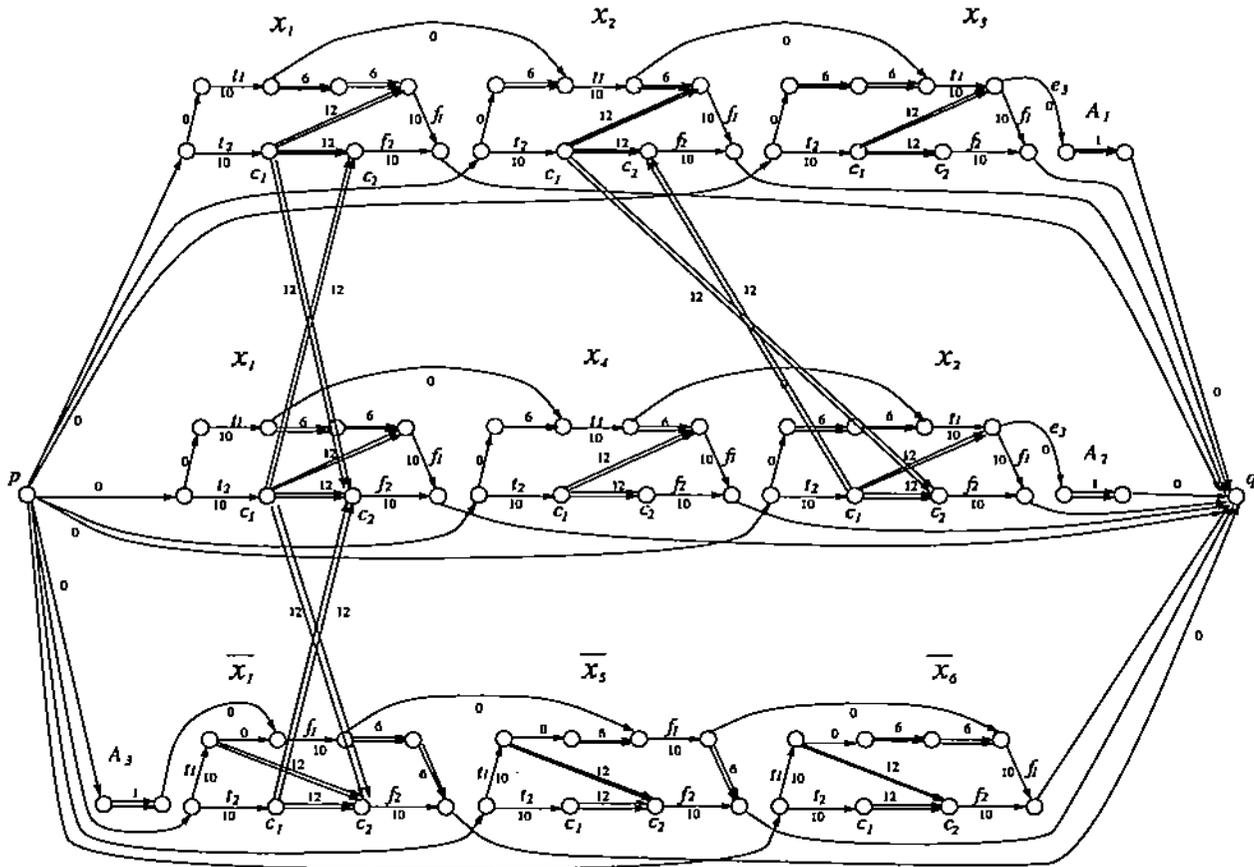


Figure 3: Graph  $G$  for formula  $C = \{(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_5} \vee \overline{x_6})\}$ .

On the other hand, if  $t(x_i) = F$ , then in every component  $U_j^b$  with  $u_j^b = x_i$  or  $u_j^b = \bar{x}_i$ , edges  $f_1$  and  $f_2$  are reduced. We are reducing exactly two edges per component and thus reduce a total of  $6k$  edges. It remains to be shown that the reduced graph  $G_R$  contains no path exceeding 30.

Let  $P$  be any path from  $p$  to  $q$ . The structure of  $P$  is one of the following:

- (i) Path  $P$  contains source  $p_i^j$  and sink  $q_i^j$  of some component  $U_i^j$ . Any such path has cost 32 in  $G$ . Since either  $t_1$  and  $t_2$  or  $f_1$  and  $f_2$  are reduced and  $P$  contains exactly one true and one false edge, the cost of  $P$  in  $G_R$  is 22.
- (ii) Assume  $P$  contains vertices of a single clause graph  $G_i$ , with the vertices belonging to different components or the attachment. The majority of the cases described below make use of the fact that any upper path in a component has either its true- or its false-edge reduced. Assume  $G_i$  is a positive clause graph. The situation for a negative clause graphs is symmetrical and is omitted.
  - (a)  $P$  goes through vertex  $p_i^1$ , edge  $t_1$  of  $U_i^1$ , edge  $e_1$ , edges  $t_1$  and  $f_1$  of  $U_i^2$  and vertex  $q_i^2$ , as shown in Figure 4(a). The length of  $P$  in  $G$  is 36 and it is at most 26 in  $G_R$ .
  - (b)  $P$  goes through vertex  $p_i^1$ , edge  $t_1$  of  $U_i^1$ , edge  $e_1$ , edge  $t_1$  of  $U_i^2$ , edge  $e_2$ , edges  $t_1$  and  $f_1$  of  $U_i^3$  and vertex  $q_i^3$ , as shown in Figure 4(b). The length of  $P$  in  $G$  is 40 and it is at most 30 in  $G_R$ .
  - (c)  $P$  goes through vertex  $p_i^1$ , edge  $t_1$  of  $U_i^1$ , edge  $e_1$ , edge  $t_1$  of  $U_i^2$ , edge  $e_2$ , edge  $t_1$  of  $U_i^3$ , edge  $e_3$ , and the attachment of clause graph  $G_i$ , as shown in Figure 4(c). The length of such a path in  $G$  is 31. Since at least one of the three literals of positive clause  $C_i$  is assigned "T", at least one of the three true-edges on the upper paths of the components of  $G_i$  is reduced. This implies that  $P$  is at most 21 in  $G_R$ .
  - (d)  $P$  goes through vertex  $p_i^2$ , edge  $t_1$  of  $U_i^2$ , edge  $e_2$ , edges  $t_1$  and  $f_1$  of  $U_i^3$  and vertex  $q_i^3$ , as shown in Figure 4(d). The length of  $P$  in  $G$  is 36 and its length in  $G_R$  is at most 26.
  - (e)  $P$  goes through vertex  $p_i^2$ , edge  $t_1$  of  $U_i^2$ , edge  $e_2$ , edge  $t_1$  of  $U_i^3$ , edge  $e_3$ , and the attachment of  $G_i$ , as shown in Figure 4(e). The length of  $P$  in  $G$  is 27 and does not need to be reduced.

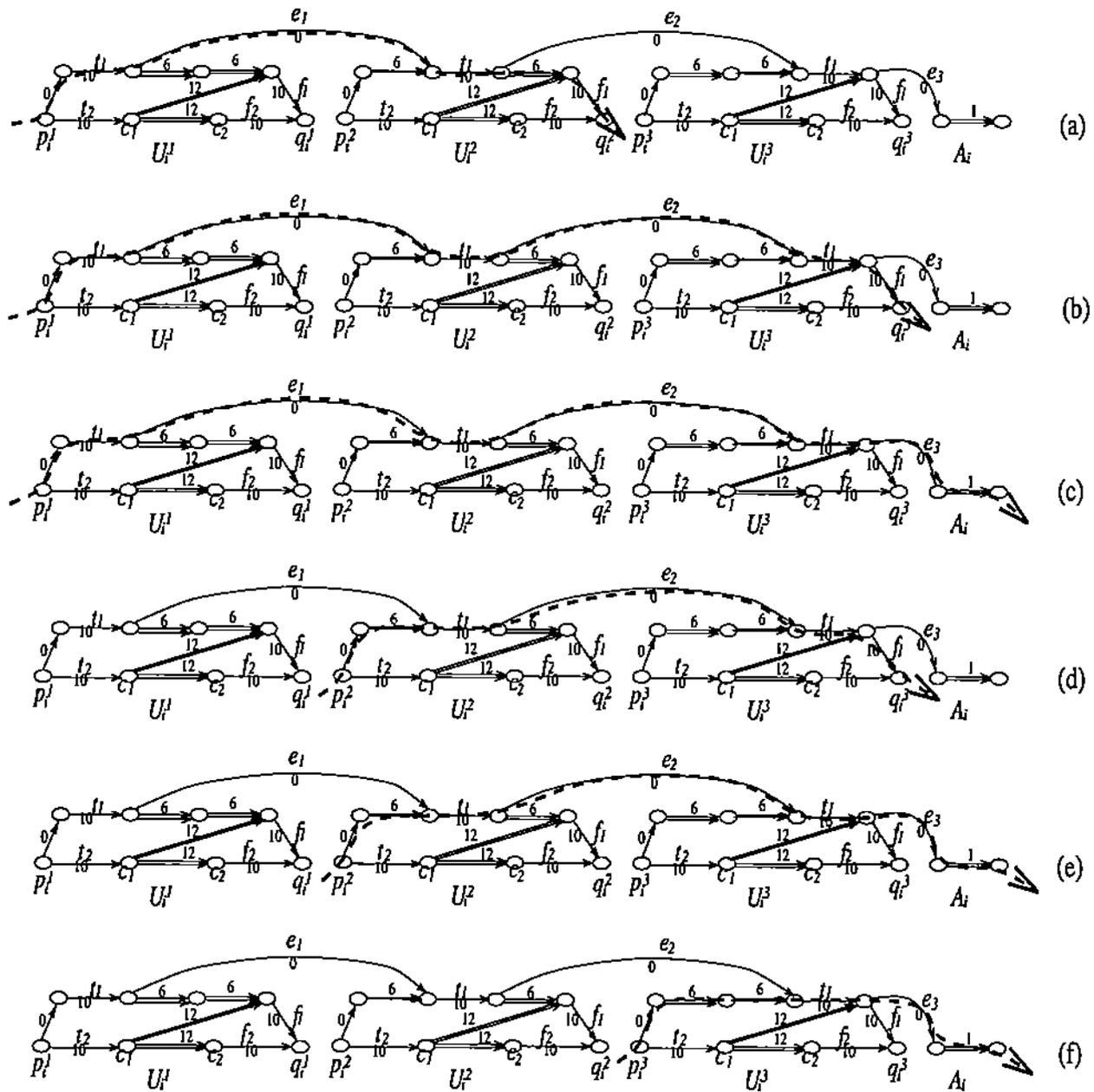


Figure 4: Paths in positive clause graph  $G_i$  going through different components and/or the attachment.

(f)  $P$  goes through vertex  $p_i^3$ , edge  $t_1$  of  $U_i^3$ , edge  $e_3$ , and the attachment of  $G_i$ , as shown in Figure 4(f). The length of  $P$  in  $G$  is 23 and does not need to get reduced.

(iii) Assume now that path  $P$  contains edges belonging to different clause graphs. Our construction of  $G$  allows such a path to contain edges of no more than two different clause graphs. Let  $P$  contain edges from components  $U_i^a$  and  $U_j^b$ ,  $i \neq j$ .  $P$  either contains vertices  $c_1$  of  $U_i^a$  and  $c_2$  of  $U_j^b$  or vertices  $c_1$  of  $U_j^b$  and  $c_2$  of  $U_i^a$ . Any such path has length 32 and it contains a  $t_2$  and an  $f_2$  edge belonging to different components. Components  $U_i^a$  and  $U_j^b$  correspond to literals formed by the same variable. We thus have in both components either all true or all false edges reduced. This implies that any such path has a length of exactly 22 in  $G_R$ .

Hence, reducing  $6k$  true- or false-edges according to the truth assignment satisfying  $C$  results in a reduced graph  $G_R$  containing no path exceeding 30. We now complete the proof by showing that if there exists a 0/1 reduction  $R$  with  $M(G_R) \leq 6k$  and  $L(G_R) \leq 30$ , then  $C$  can be satisfied. We start by giving properties that any such reduction  $R$  must satisfy.

**Property 5.1** *In a component  $U_i^a$  belonging to a positive clause graph the set of reduced edges is either  $\{t_1, t_2\}$ , or  $\{f_1, t_2\}$ , or  $\{f_1, f_2\}$ . In a component  $U_i^a$  belonging to a negative clause graph the set of reduced edges is either  $\{t_1, t_2\}$ , or  $\{t_1, f_2\}$ , or  $\{f_1, f_2\}$ .*

**Proof:** As already stated, in order to reduce the length of every path to 30 and reduce at most  $6k$  edges, two edges per component need to get reduced. Clearly, reduction  $R$  may reduce both true-edges or both false-edges. For components belonging to a positive clause graph it is also possible that edges  $f_1$  and  $t_2$  are reduced. Observe that reducing edges  $f_2$  and  $t_1$  preserves a path length of 32 within this component. In a symmetrical way, for components belonging to a negative clause graph, it is possible that edges  $f_2$  and  $t_1$  are reduced.  $\square$

**Property 5.2** *Let  $U_i^a$  and  $U_j^b$  be two components linked together by consistency edges. Then, either the  $t_2$  edges of  $U_i^a$  and  $U_j^b$  are reduced or the  $f_2$  edges of  $U_i^a$  and  $U_j^b$  are reduced.*

**Proof:** Assume the  $t_2$  edge of component  $U_i^a$  is reduced, but the  $t_2$  edge of component  $U_j^b$  is not. By Property 5.1, the  $f_2$  edge of component  $U_i^a$  is not reduced. This would imply that  $G_R$

contains a path of length 32 containing edge  $t_2$  and vertex  $c_1$  of  $U_j^b$  as well as vertex  $c_2$  and edge  $f_2$  of  $U_i^a$ . The other situations result in similar contradictions.  $\square$ .

**Property 5.3** *If  $G_i$  is a positive clause graph, at least one of the three  $t_1$  edges in  $G_i$  is reduced. If  $G_i$  is a negative clause graph, at least one of the three  $f_1$  edges in  $G_i$  is reduced.*

**Proof:** Let  $P$  be a path from source  $p$  to sink  $q$  going through clause graph  $G_i$  and containing edges  $e_1, e_2, e_3$  of  $G_i$ . Such path has length 31 in  $G$ . Since the edges in the attachment cannot be reduced, at least one of the three edges having weight 10 is reduced in  $R$ . These three edges correspond to true-edges in a positive clause graphs and correspond to false edges in a negative clause graph.  $\square$ .

Given a graph  $G$  and a reduction  $R$ , a truth assignment  $t : X \rightarrow \{T, F\}$  satisfying  $C$  is constructed as follows. For every variable  $x_i$ , find a component  $U_j^b$  corresponding to a literal  $u_j^b$  formed by  $x_i$ . If the  $t_2$  edge of component  $U_j^b$  is reduced, set  $t(x_i) = T$ . If the  $f_2$  edge of  $U_j^b$  is reduced, set  $t(x_i) = F$ . Property 5.2 guarantees that any literal formed by  $x_i$  induces the same truth assignment. By Property 5.3, at least one literal is true in each clause, and thus  $t : X \rightarrow \{T, F\}$  satisfies  $C$ . This concludes our NP-completeness proof.  $\square$

The assumption  $\epsilon = 0$  is not crucial to the argument used in the proof. For example, the following change in the edge weights of the multiple edges gives an NP-completeness proof for  $\epsilon = \frac{1}{2}$ . Multiple edges having a weight of 12 now have a weight of 16. The ones having a weight of 6 now have a weight of 8, and the edges in the attachment now have a weight of 6. The longest path length in  $G$  remains 40. An argument identical to the one already used shows that there exists a 0/1 reduction  $R$  with  $M(G_R) \leq 6k$  and  $L(G_R) \leq 35$  reducing at most  $6k$  edges if and only if  $C$  can be satisfied.

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