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Extreme-Strike and Small-time Asymptotics for Gaussian Stochastic Volatility Models

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Extreme-Strike and Small-time Asymptotics for Gaussian Stochastic Volatility Models

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Date

EXTREME-STRIKE AND SMALL-TIME ASYMPTOTICS
FOR GAUSSIAN STOCHASTIC VOLATILITY MODELS

A Dissertation

Submitted to the Faculty

of

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Xin Zhang

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of

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This dissertation is dedicated to my parents, my husband, my daughter and my mother-in-law for their great support and unconditional love.

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ABSTRACT

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Asymptotic behavior of implied volatility is of our interest in this dissertation. For extreme strike, we consider a stochastic volatility asset price model in which the volatility is the absolute value of a continuous Gaussian process with arbitrary prescribed mean and covariance. By exhibiting a Karhunen-Loève expansion for the integrated variance, and using sharp estimates of the density of a general second-chaos variable, we derive asymptotics for the asset price density for large or small values of the variable, and study the wing behavior of the implied volatility in these models. Our main result provides explicit expressions for the first five terms in the expansion of the implied volatility, based on three basic spectral-type statistics of the Gaussian process: the top eigenvalue of its covariance operator, the multiplicity of this eigenvalue, and the L^2 norm of the projection of the mean function on the top eigenspace. Strategies for using this expansion for calibration purposes are discussed.

For small time, we consider the class of self-similar Gaussian stochastic volatility models, and compute the small-time (near-maturity) asymptotics for the corresponding asset price density, the call and put pricing functions, and the implied volatilities. Unlike the well-known model-free behavior for extreme-strike asymptotics, small-time behaviors of the above depend heavily on the model, and require a control of the asset price density which is uniform with respect to the asset price variable, in order to translate into results for call prices and implied volatilities. Away from the money, we express the asymptotics explicitly using the volatility process' self-similarity parameter H , its first Karhunen-Loève eigenvalue at time 1, and the latter's multiplicity. Several model-free estimators for H result is discussed. At the money, a separate

study is required: the asymptotics for small time depend instead on the integrated variance's moments of orders $\frac{1}{2}$ and $\frac{3}{2}$, and the estimator for H sees an affine adjustment.

1. Introduction

1.1 General Backgrounds

It has been known for decades that the Black-Scholes-Merton framework suffers from certain deficiencies, particularly the fact that volatility is not constant empirically. After the crash of 1987, practitioners began considering that extreme events were more likely than what a log-normal model will predict. Propositions to exploit this weakness in log-normal modeling systematically and quantitatively have grown ubiquitous to the point that implied volatility (IV), or the volatility level that market call option prices would imply if the Black-Scholes model were underlying, is now a vigorous topic of investigation, both at the theoretical and practical level. The initial evidence against constant volatility simply came from observing that IV as a function of strike prices for liquid call options exhibited non-constancy, typically illustrated as a convex curve, often with a minimum near the money as for index options.

In order to overcome these drawbacks, the researchers proposed vast class of stochastic volatility models, i.e., those continuous-time models where the relative noise intensity of return is itself a stochastic process which is at least partially driven by exogenous noise; the term ‘uncorrelated’ is added to refer to the submodel class in which the volatility process is independent of the noise term driving the stock price. A large number of articles and monographs on stochastic volatility (SV) can be consulted for empirical and economic justification of these models; we cite the classical text [1]. Of particular interest is SV models’ ability to reproduce some desirable market features of option prices, such as “smiles” and other non-flat shapes of the implied volatility.

One of the first mathematical treatments explaining empirically observed IV shapes was by Renault and Touzi in [2]. Note that the authors did not prove that

the IV is locally convex near the money, but their work still established stochastic volatility models as a main model class for studying IV; these models continued steadily to provide inspiration for IV studies. A current emphasis, which has become fertile mathematical ground, is on IV asymptotics, such as large/small-strike, large-maturity, or small-time-to-maturity behaviors.

In this dissertation, we present a study of the asymptotics for the asset price density, the call and put option, and the implied volatilities for the Gaussian stochastic volatility models. The structure of the dissertation is as follows: In chapter 2, we present the Karhunen-Loève expansion theorem and a brief discussion of the Stein-Stein model. As Karhunen-Loève expansion does not have a closed form in most of the models, we then introduce a numerical evaluation of Karhunen-Loève eigenvalues and eigenfunctions. We also discussed briefly some classical Gaussian models.

In chapter 3, we characterize the extreme-strike behavior of implied volatility curves for fixed maturity for uncorrelated Gaussian stochastic volatility models. We present our calibration strategy based on our explicit expression for expansion of the implied volatility.

In chapter 4, we discuss small-time asymptotics of implied volatilities for the class of continuous-time Black-Scholes models with Brownian noise and independent Gaussian self-similar volatility. The special case of the at the money is discussed separately, along with some numerical test on calibration of hurst parameter H .

1.2 Summary of main result

1.2.1 Extreme-strike asymptotics

In chapter 3, we consider the stock price model of the following form:

$$dS_t = rS_t dt + |X_t| S_t dW_t : t \in [0, T], \quad (1.1)$$

where the short rate r is constant, $X(t) = m(t) + \tilde{X}(t)$ with m an arbitrary continuous deterministic function on $[0, T]$ (the mean function), and \tilde{X} is a continuous

centered Gaussian process on $[0, T]$ independent of W , with arbitrary covariance Q . The initial condition for the stock price process will be denoted by s_0 . Note that it is not supposed in (1.1) that the process X is a solution to a stochastic differential equation as is often assumed in classical stochastic volatility models. A well-known special example of a Gaussian volatility model is the Stein-Stein model introduced in [3], in which the volatility process X is the so-called mean-reverting Ornstein-Uhlenbeck process satisfying

$$dX_t = \alpha (m - X_t) dt + \beta dZ_t \quad (1.2)$$

where m is the level of mean reversion, α is the mean-reversion rate, and β is level of uncertainty on the volatility; here Z is another Brownian motion, which may be correlated with W . In this dissertation, we adopt an analytic technique, encountered for instance in the analysis of the uncorrelated Stein-Stein model by this paper's first author and E.M. Stein in [4] (see also [5]).

Adopting the perspective that an asymptotic expansion for the IV can be helpful for model selection and calibration, our objective is to provide an expansion for the IV in a Gaussian volatility model relying on a minimal number of parameters, which can then be chosen to adjust to observed smiles. The restriction of non-correlated volatility means that the stock price distribution is a mixture model of geometric Brownian motions with time-dependent volatilities, whose mixing density at time T is that of the square root of a variable in the second-chaos of a Wiener process. That second-chaos variable is none other than the integrated variance

$$\Gamma_T := \int_0^T X_s^2 ds.$$

By relying on a general Hilbert-space structure theorem which applies to the second Wiener chaos, we prove that, in the most general case of a non-centered Gaussian stochastic volatility with a possible degeneracy in the eigenstructure of the covariance Q of X viewed as a linear operator on $L^2([0, T])$ (i.e. when the top eigenvalue λ_1 is allowed to have a multiplicity n_1 larger than 1), the large-strike IV asymptotics

can be expressed with three terms which depend explicitly on T and on the following three parameters: λ_1 , n_1 , and the ratio

$$\delta = \|P_{E_1} m\|^2 / \lambda_1$$

where $\|P_{E_1} m\|$ is the norm in $L^2([0, T])$ of the orthogonal projection of the mean function m on the first eigenspace of Q . We also push the expansion to five terms, and notice that the fifth term also only depends on λ_1 , n_1 , and δ , while the fourth term depends on all other eigenvalues and the action of m on all other eigenfunctions. Specifically, with $I(K)$, the IV as a function of strike K , letting $k := \log(K/s_0) - rT$ be the discounted log-moneyness, as $k \rightarrow +\infty$, we prove

$$\begin{aligned} I(K) = & M_1(T, \lambda_1)\sqrt{k} + M_2(T, \lambda_1, \delta) + M_3(T, \lambda_1, n_1)\frac{\log k}{\sqrt{k}} \\ & + M_4(T, \lambda_1, n_1, V)\frac{1}{\sqrt{k}} + M_5(T, \lambda_1, n_1, \delta)\frac{\log k}{k} + O\left(\frac{1}{\sqrt{k}}\right), \quad (1.3) \end{aligned}$$

where the constants M_1 , M_2 , M_3 , M_4 , and M_5 depend explicitly on T and λ_1 , M_2 also depends explicitly on δ , while M_3 also depends explicitly on n_1 , M_5 depends explicitly also on both n_1 and δ , and M_4 has an additional rather complex dependence on all the eigen-elements through a factor V ; this is all stated explicitly in Theorem 3.5 and formula (3.41). A similar asymptotic formula is obtained in the case where $k \rightarrow -\infty$, using symmetry properties of uncorrelated stochastic volatility models (see (3.49)).

The first-order constant M_1 is always strictly positive. The second-order term (the constant M_2) vanishes if and only if m is orthogonal to the first eigenspace of Q , which occurs for instance when $m \equiv 0$. The third-order and fifth-order terms vanish if and only if the top eigenvalue has multiplicity $n_1 = 1$, which is typical (the case $n_1 > 1$ can be considered degenerate, and does not occur in common examples). The behavior of M_1 and M_2 as functions of T is determined partly by how the top eigenvalue λ_1 depends on T , which can be non-trivial.

For fixed maturity T , assuming that Q has lead multiplicity $n_1 = 1$ for instance, a practitioner will have the possibility of determining a value λ_1 and a value δ to match the specific root-log-moneyness behavior of small- or large-strike IV; moreover in that

case, choosing a constant mean function m , one obtains $\delta = m^2 \lambda_1^{-1} \left| \int_0^T e_1(t) dt \right|^2$ where e_1 is the top eigenfunction of Q . Market prices may not be sufficiently liquid at extreme strikes to distinguish between more than two parameters; this is typical of calibration techniques for implied volatility curves for fixed maturity, such as the ‘stochastic volatility inspired’ (SVI) parametrization disseminated by J. Gatheral: see [6,7] (see also [8] and the references therein). Our result shows that Gaussian volatility models with non-zero mean are sufficient for this flexibility, and provide equivalent asymptotics irrespective of the precise mean function and covariance eigenstructure, since modulo the disappearance of the third-order term in the unit top multiplicity case $n_1 = 1$, only λ_1 and δ are relevant. The fourth-order term in our expansion can provide additional precision in calibration. Its use is illustrated in Section 4.9.

1.2.2 Small-time asymptotics

In chapter 4, we adopted the same framework established in chapter 3. The asset price process S satisfies equation (1.1). If $S_0 = s_0$, the call option on S with maturity T and strike price K has price $C(T, K)$; this price equals a price $C_{BS}(T, K; \sigma)$ in the Black-Scholes model with the volatility σ depending on T and K . That value of σ is called the *implied volatility* (IV) and is denoted by $I(T, K)$. Now we concentrate on the behavior of C and I for small T when K is fixed; consequently, we typically drop the dependence of C and I on K .

Of particular importance is the density p_T of the *integrated variance* $Y_T := \int_0^T X_t^2 dt$. In any case, the asymptotic behavior of p_T near $+\infty$, which was established in [9], depends on λ_1 , n_1 , and δ (see Theorem 4.1 below). When applied to the case of H -self-similar X , via the simple scaling formula $p_T(y) = T^{-2H-1} p_1(T^{-2H-1}y)$, the behavior of $p_T(\cdot)$ at $x \rightarrow +\infty$ translates into an expansion around $T \rightarrow 0^+$ of the density $\tilde{p}_T(x)$ of the rescaled square-rooted version of Y_T which is precise up to a factor $(1 + O_x(T^H))$ for any fixed $x > 0$: see asymptotic formula (4.13) in Theorem 4.2.

The independence of W and X imply that the density D_T of S_T is given by a mixing formula (4.1) involving \tilde{p}_1 via the self-similar scaling property $\tilde{p}_T(y) = T^{-H}\tilde{p}_1(T^{-H}y)$. A delicate use of Laplace's method then allows to translate Theorem 4.2 into small- T asymptotics for $D_T(x)$ for any x which is "out of the money" in the context of call pricing, in the sense that the big O term depends on a parameter $\varepsilon > 0$ to allow for $x > s_0 + \varepsilon$ (future stock price parameter x , which stands in for strike price K when one computes an IV, exceeds initial stock price s_0 by a margin ε). We find (Theorem 4.3) that for all for $x > s_0 + \varepsilon$

$$\begin{aligned}
D_T(x) &= \frac{\sqrt{s_0}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{4}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k}{2}} \\
&\times x^{-\frac{3}{2}} \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} T^{-\frac{(2H+1)n_1(1)}{4}} \left(\frac{x}{s_0} \right)^{-\frac{\sqrt{4+\lambda_1(1)T^{2H+1}}}{2\sqrt{\lambda_1(1)T^{H+\frac{1}{2}}}}} \\
&\times \left(1 + O(T^{2H+1}) \right) \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left(\log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right) \tag{1.4}
\end{aligned}$$

where the repeated notation (1) refers to KL elements for $T = 1$, and where n_k is the multiplicity of the k th largest KL eigenvalue ρ_k . The symbol O depends only on the covariance of X , but not on x or ε . The symbol O_ε depends on the covariance of X and on ε , but not x . We prove formula (1.4) under the assumptions that $r = 0$ and the volatility process $X^{(H)}$ is centered.

Being able to establish the precise x -behavior of the error terms above is crucial to transposing the behavior of $D_T(x)$ to the functions C and I . Specifically, we obtain the following for the out-of-the-money call as $T \rightarrow 0^+$ (Theorem 4.4) : for $K > s_0$,

$$C(T) = MT^{\frac{(2H+1)(4-n_1(1))}{4}} \left(\frac{s_0}{K} \right)^{\lambda_1(1) - \frac{1}{2}} T^{-H - \frac{1}{2}} \left(1 + O\left(T^{\frac{2H+1}{4}}\right) \right) \tag{1.5}$$

where the big O above does not depend on K if it is away from s_0 , and the constant M is explicit and proportional to the constant on the right-hand side of line (1.4). A nearly identical result is obtained for out-of-the-money put prices $P(T, K)$ (for $0 < K < s_0$) using symmetries of the problem (Theorem 4.6).

Ultimately, relying on a general result of Gao and Lee [10] for computing the small-time asymptotics of IV based on those of C , we obtain in Theorems 4.7 and 4.8 that for $0 < K \neq s_0$,

$$I(T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\left| \log \frac{K}{s_0} \right|}}{\sqrt{2}} T^{\frac{2H-1}{4}} + O\left(T^{\frac{6H+1}{4}} \log \frac{1}{T}\right) \quad (1.6)$$

where the big O is again uniform over K in any compact interval away from 0 and s_0 . The dominant factor in the expression (1.5) for C , and its analogue for P , is the exponential one. In the expression (1.6) for I , there is only one candidate for a dominant term. Consequently, one gets a way to estimate H using call or put prices or IVs away from the money as empirical statistics:

$$\begin{aligned} H &= \lim_{T \rightarrow 0} \frac{\log \log \frac{1}{C(T,K)}}{\log \frac{1}{T}} - \frac{1}{2} \\ &= \lim_{T \rightarrow 0} \frac{\log \log \frac{1}{P(T,K)}}{\log \frac{1}{T}} - \frac{1}{2} \\ &= 2 \lim_{T \rightarrow 0} \frac{\log \frac{1}{I(T,K)}}{\log \frac{1}{T}} + \frac{1}{2} \end{aligned}$$

where the first line holds for $K > s_0$, the second for $K < s_0$, and the third holds for all $K \neq s_0$ (Corollaries 4.2, 4.3, 4.4, and 4.5.) These expressions for H do not depend on any of the model parameters and statistics, and are in this sense model free within the class of self-similar models. However, in practice, since the regime $T \rightarrow 0$ is limited by the ability to trade options in a liquid way sufficiently close to maturity, the full asymptotics in (1.5) and (1.6) will typically be needed to help control the estimation error.

We notice that the above asymptotics for C and I formally lose information when $K = s_0$, since the expression $|\log(K/s_0)|$ is zero and thus kills the dominant terms. Hence the estimators for H above are not longer valid in that case. We investigate this at-the-money situation in some detail. The delicate calculations are largely performed “by hand”. The resulting asymptotics seem to rely on model statistics which cannot be related to the KL elements in any simple fashion, since they require computing the

moments $\mu_{1/2}$ and $\mu_{3/2}$ of order $1/2$ and $3/2$ of the non-explicit integrated variance's law. As $T \rightarrow 0$, we get in Corollary 4.6 that

$$C(T, s_0) = \frac{s_0 \mu_{1/2}}{\sqrt{2\pi}} T^{H+\frac{1}{2}} - \frac{s_0 \mu_{3/2}}{24\sqrt{2\pi}} T^{3H+\frac{3}{2}} + O\left(T^{5H+\frac{5}{2}}\right),$$

and in Theorem 4.10 that

$$I(T, s_0) = \mu_{1/2} T^H + \frac{(\mu_{1/2})^3 - \mu_{3/2}}{24} T^{3H+1} + O\left(T^{5H+2}\right). \quad (1.7)$$

Again, simple H -estimators can result, which do not rely on the moments $\mu_{1/2}$ and $\mu_{3/2}$, such as Theorem 4.11 :

$$H = \lim_{T \rightarrow 0} \frac{\log \frac{1}{I(T, s_0)}}{\log \frac{1}{T}}.$$

To illustrate the usage of our various asymptotic formulas numerically, we provide simulated stock prices, with corresponding call prices and IVs, from the self-similar volatility model, using a classical Monte-Carlo method. Using market-realistic parameter choices, we show how close prices and IVs are to our asymptotic formulas, noting that the fit is good in the call price case, and is excellent in the IV case, for time-to-maturity as large as 2 weeks. It is then not surprising when we show that our IV-based model-free calibration formulas for H are accurate to 2 decimal points up to 7 days in most cases, and 14 days in some cases. Being able to use the longest-possible time to maturity is important in practice because of liquidity considerations. This is all explained in Section 4.9.

2. PRELIMILARIES

2.1 Karhunen Loève expansion

In this dissertation, we find that the spectral structure of second-Wiener chaos variables allows us to express first several terms explicitly in the implied volatility asymptotics. The modelers may wish to know the ways to compute the said spectral parameters. We refer to the Karhunen-Loève expansion (see, e.g., [11], Section 26.1). Let $X : D \times \Omega \rightarrow R$ be a centered continuous stochastic process. There exist a non-increasing sequence of non-negative summable reals $\{\lambda_n : n = 1, 2, \dots\}$, an sequence of centered pairwise uncorrelated random variables $\{\varepsilon_n : n = 1, 2, \dots\}$, and a sequence of real-valued functions $\{e_n : n = 1, 2, \dots\}$ which form an orthonormal system in $L^2(D)$, such that

$$X(t, \omega) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varepsilon_i(\omega) e_i(t)$$

Moreover, if the process X is Gaussian, $\{\varepsilon_n : n = 1, 2, \dots\}$ are independent standard normal variates. The λ_n and e_n are the eigenvalues and eigenfunctions of the covariance Q of X acting on $L^2([0, T])$ as the following operator

$$\mathcal{K}(f)(t) = \int_0^T f(s) Q(t, s) ds, \quad f \in L^2([0, T]), \quad 0 \leq t \leq T,$$

For certain process driven by brownian motion, we can solve its K-L expansion analytically. For example,

- The Brownian motion, $(W_t)_{t \in [0, T]}$ with covariance function $Q(s, t) = \min(s, t)$,

$$\lambda_n = \left(\frac{T}{\pi \left(n - \frac{1}{2} \right)} \right)^2, \quad e_n(t) = \sqrt{\frac{2}{T}} \sin \left(\pi \left(n - \frac{1}{2} \right) \frac{t}{T} \right)$$

- The Brownian bridge can be derived from a brownian motion $W(t)$ by $X(t) = W(t) - tW(1)$, then

$$\lambda_n = \left(\frac{T}{\pi n}\right)^2, \quad e_n(t) = \sqrt{\frac{2}{T}} \sin\left(\pi n \frac{t}{T}\right)$$

Such formulas can be found in [12]. For OU bridges, one can consult [13, 14], and for the Gaussian process introduced in [15], the Karhunen-Loève decomposition can be found in the same paper.

2.1.1 Stein-Stein model and its KL expansion

Ornstein-Uhlenbeck process (O-U process) is another example that had closed form of KL expansion. The model with volatility being the absolute value of an O-U process is called Stein-Stein model (see [3]), which is an important special example of a Gaussian stochastic volatility model. In this section, we also consider a generalization of the Stein-Stein model, in which the initial condition for the volatility process is a random variable X_0 . Of our interest in the present section is a Gaussian stochastic volatility model with the process X satisfying the equation $dX_t = q(m - X_t)dt + \sigma dZ_t$. Here $q > 0$, $m \geq 0$, and $\sigma > 0$. It will be assumed that the initial condition X_0 is a Gaussian random variable with mean m_0 and variance σ_0^2 , independent of the process Z . It is known that

$$X_t = e^{-qt}X_0 + (1 - e^{-qt})m + \sigma e^{-qt} \int_0^t e^{qu} dZ_u, \quad t \geq 0. \quad (2.1)$$

If $\sigma_0 = 0$, then the initial condition is equal to the constant m_0 . The mean function of the process X is given by

$$m(t) = e^{-qt}m_0 + (1 - e^{-qt})m, \quad (2.2)$$

and its covariance function is as follows:

$$Q(t, s) = e^{-q(t+s)} \left\{ \sigma_0^2 + \frac{\sigma^2}{2q} (e^{2q \min(t,s)} - 1) \right\}.$$

Therefore, the following formula holds for the variance function:

$$\sigma_t^2 = \frac{\sigma^2}{2q} + e^{-2qt} \left(\sigma_0^2 - \frac{\sigma^2}{2q} \right),$$

and hence, if $\sigma_0^2 = \frac{\sigma^2}{2q}$, then the process $X_t - m(t)$, $t \in [0, T]$, is centered and stationary.

In this case, the covariance function is given by

$$Q(t, s) = \frac{\sigma^2}{2q} e^{-q|t-s|}.$$

The Karhunen-Loève expansion of the Ornstein-Uhlenbeck process is known explicitly (see [12]). Denote by w_n the increasingly sorted sequence of the positive solutions to the equation

$$\sigma^2 w \cos(wT) + (q\sigma^2 - w^2\sigma_0^2 - q^2\sigma_0^2) \sin(wT) = 0. \quad (2.3)$$

If $\sigma_0 = 0$, then the equation in (2.3) becomes

$$w \cos(wT) + q \sin(wT) = 0. \quad (2.4)$$

For the OU process in (2.1) with $\sigma_0 \neq 0$, we have $n_k = 1$ for all $k \geq 1$;

$$\lambda_n = \frac{\sigma^2}{w_n^2 + q^2} \quad (2.5)$$

for all $n \geq 1$; and

$$e_n(t) = K_n [\sigma_0^2 w_n \cos(w_n t) + (\sigma^2 - q\sigma_0^2) \sin(w_n t)] \quad (2.6)$$

for all $n \geq 1$ and $t \in [0, T]$. The constant K_n in (2.6) is determined from

$$\begin{aligned} \frac{1}{K_n^2} &= \frac{1}{2w_n} \sigma_0^2 (\sigma^2 - q\sigma_0^2) (1 - \cos(2w_n T)) + \frac{1}{2} \sigma_0^4 w_n^2 \left(T + \frac{1}{2w_n} \sin(2w_n T) \right) \\ &\quad + \frac{1}{2} (\sigma^2 - q\sigma_0^2)^2 \left(T - \frac{1}{2w_n} \sin(2w_n T) \right) \end{aligned} \quad (2.7)$$

for all $n \geq 1$. On the other hand, if $\sigma_0 = 0$, then λ_n is given by (2.5), while the functions e_n are defined by

$$e_n(t) = \frac{1}{\sqrt{\frac{T}{2} - \frac{\sin(2w_n T)}{4w_n}}} \sin(w_n t) \quad (2.8)$$

for all $n \geq 1$ and $t \in [0, T]$.

By the Karhunen-Loève theorem, the Ornstein-Uhlenbeck process X in (2.1) can be represented as follows:

$$X_t = e^{-qt}m_0 + (1 - e^{-qt})m + \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n$$

where $\{Z_n\}_{n \geq 1}$ is an i.i.d. sequence of standard normal variables. The eigenvalues λ_n , $n \geq 1$, and the eigenfunctions e_n , $n \geq 1$, are given by (2.5) and (2.6) if $\sigma_0 \neq 0$, and by (2.5) and (2.8) if $\sigma_0 = 0$. Recall that the numbers w_n , $n \geq 1$, in (2.5) are solutions to the equation in (2.3) if $\sigma_0 \neq 0$, and to the equation in (2.4) if $\sigma_0 = 0$. We refer the interested reader to [12] for more details.

Our next goal is to discuss the constants in the asymptotic formulas for the implied volatility at extreme strikes in the Stein-Stein model. Since $n_1 = 1$ for any OU process, the third and fifth terms in the expansion of Theorem 3.5 are zero, and with the exception of the term $V(1, 0)$ in M_4 , the only parameters needed to compute the above-mentioned constants are λ_1 and δ_1 . If $\sigma_0 \neq 0$, then we have

$$\lambda_1 = \frac{\sigma^2}{w_1^2 + q^2}, \quad (2.9)$$

where w_1 is the smallest strictly positive solution to the equation in (2.3).

The next assertion provides explicit formulas for the number $\delta_1 = \int_0^T m(t) e_1(t) dt$.

Lemma 2.1 (i) *For the generalized uncorrelated Stein-Stein model with $\sigma_0 \neq 0$,*

$$\begin{aligned} \delta_1 = & \frac{K_1 m (\sigma^2 - q \sigma_0^2) (1 - \cos(w_1 T))}{w_1} + K_1 \sigma_0^2 \sin(w_1 T) [(m_0 - m) e^{-qT} + m] \\ & + K_1 \sigma^2 (m_0 - m) \frac{w_1 [1 - e^{-qT} \cos(w_1 T)] - q e^{-qT} \sin(w_1 T)}{q^2 + w_1^2}, \end{aligned} \quad (2.10)$$

where the constant K_1 is determined from (2.7) with $n = 1$. The symbol w_1 in (2.10) stands for the smallest strictly positive solution to (2.3).

(ii) *For the uncorrelated Stein-Stein model with $X_0 = m_0$ \mathbb{P} -almost surely,*

$$\delta_1 = \frac{mq^2(1 - \cos(w_1 T)) + w_1^2(m_0 - m \cos(w_1 T))}{w_1(q^2 + w_1^2) \sqrt{\frac{T}{2} - \frac{\sin(2w_1 T)}{4w_1}}}. \quad (2.11)$$

Proof Taking into account (2.2) and (2.6), we see that

$$\begin{aligned}\delta_1 &= b_1 \int_0^T \cos(w_1 t) dt + b_2 \int_0^T e^{-qt} \cos(w_1 t) dt \\ &\quad + b_3 \int_0^T \sin(w_1 t) dt + b_4 \int_0^T e^{-qt} \sin(w_1 t) dt,\end{aligned}\tag{2.12}$$

where

$$\begin{aligned}b_1 &= mK_1\sigma_0^2 w_1, \quad b_2 = (m_0 - m)K_1\sigma_0^2 w_1, \\ b_3 &= mK_1(\sigma^2 - q\sigma_0^2), \quad \text{and} \quad b_4 = (m_0 - m)K_1(\sigma^2 - q\sigma_0^2).\end{aligned}\tag{2.13}$$

It remains to evaluate the integrals in (2.12). We have

$$\int_0^T \cos(w_1 t) dt = \frac{\sin(w_1 T)}{w_1},\tag{2.14}$$

$$\int_0^T e^{-qt} \cos(w_1 t) dt = \frac{q[1 - e^{-qT} \cos(w_1 T)] + w_1 e^{-qT} \sin(w_1 T)}{q^2 + w_1^2},\tag{2.15}$$

$$\int_0^T \sin(w_1 t) dt = \frac{1 - \cos(w_1 T)}{w_1},\tag{2.16}$$

and

$$\int_0^T e^{-qt} \sin(w_1 t) dt = \frac{w_1[1 - e^{-qT} \cos(w_1 T)] - qe^{-qT} \sin(w_1 T)}{q^2 + w_1^2}.\tag{2.17}$$

In the proof of (2.15) and (2.17), we use the integration by parts formula twice. Now, taking into account formulas (2.12-2.17) and making simplifications, we establish formula (2.10).

Next, suppose $\sigma_0 = 0$. Then (2.10) implies that

$$\begin{aligned}\delta_1 &= \frac{m}{\sqrt{\frac{T}{2} - \frac{\sin(2w_1 T)}{4w_1}}} \frac{1 - \cos(w_1 T)}{w_1} \\ &\quad + \frac{m_0 - m}{\sqrt{\frac{T}{2} - \frac{\sin(2w_1 T)}{4w_1}}} \frac{w_1[1 - e^{-qT} \cos(w_1 T)] - qe^{-qT} \sin(w_1 T)}{q^2 + w_1^2},\end{aligned}$$

where w_1 denotes the smallest strictly positive solution to (2.4). It is not hard to see, using the equality $w_1 \cos(w_1 T) + q \sin(w_1 T) = 0$, that

$$\delta_1 = \frac{m}{\sqrt{\frac{T}{2} - \frac{\sin(2w_1 T)}{4w_1}}} \frac{1 - \cos(w_1 T)}{w_1} + \frac{m_0 - m}{\sqrt{\frac{T}{2} - \frac{\sin(2w_1 T)}{4w_1}}} \frac{w_1}{q^2 + w_1^2},\tag{2.18}$$

and it is clear that (2.18) and (2.11) are equivalent.

This completes the proof of Lemma 2.1. ■

2.1.2 Numerical method to solve the K-L expansion

Unfortunately, even for classical fractional Gaussian processes, e.g., fBm or fOU, the Karhunen-Loève characteristics are not known. In [16] (see also [17]), Corlay developed a powerful numerical method to approximate Karhunen-Loève eigenvalues and eigenfunctions. Corlay uses the Nyström method associated with the trapezoidal integration rule combined with the Richardson-Romberg extrapolation in his work. In this section, we present a brief summary about this method.

We all know that, λ_n and e_n are the eigenvalues and eigenfunctions of Q acting on $L^2([0, T])$. It is often necessary to solve the equation

$$\int_0^T Q(t, s)e_n(s)ds = \lambda_n e_n(t) \quad (2.19)$$

First, we would like to choose some quadrature rule to approximate the integral, i.e.

$$\int_0^T f(t)dt \approx \sum_{i=1}^n \omega_i f(t_i) \quad (2.20)$$

Here, $0 = t_1 < t_2 < t_3 < \dots < t_n = T$ and $(\omega_j)_{1 \leq j \leq n}$ is a sequence of weights.

A simple example is uniform scheme, in which, $t_j = \frac{(j-1)T}{n-1}$ and

$$\int_0^T f(t)dt \approx \frac{1}{nT} \sum_{i=1}^n f(t_i)$$

Trapezium scheme is another integral numerical scheme frequently used. The t_i are defined as same as in uniform schme, and let $h = t_j - t_{j-1} = \frac{T}{n-1}$, then

$$\begin{aligned} \int_0^T f(t)dt &= \sum_{i=1}^{n-1} \frac{h}{2} (f(t_i) + f(t_{i+1})) \\ &= h \left(\frac{f(t_1)}{2} + f(t_2) + \dots + f(t_{n-1}) + \frac{f(t_n)}{2} \right) \end{aligned}$$

After the quadratrue rule is chosen, equation (2.19) can be approximated as:

$$\sum_{i=1}^n \omega_i Q(s, t_i) e_n(t_i) \approx \lambda_n e_n(s) \quad s \in [0, T] \quad (2.21)$$

We evaluate equation (2.21) at each point t_j that,

$$\sum_{i=1}^n \omega_i Q(t_j, t_i) e_n(t_i) \approx \lambda_n e_n(t_j) \quad j \in \{1, \dots, n\} \quad (2.22)$$

These n equations can be written in a matrix form as,

$$KWf = \lambda f \quad (2.23)$$

where

$$K = \begin{bmatrix} Q(t_1, t_1) & Q(t_1, t_2) & \cdots & Q(t_1, t_n) \\ Q(t_2, t_1) & Q(t_2, t_2) & \cdots & Q(t_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ Q(t_n, t_1) & Q(t_n, t_2) & \cdots & Q(t_n, t_n) \end{bmatrix}$$

$$W = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$

It is known that covariance matrix K is always symmetrical. However, W is not symmetric for most quadratic rule. We would like to restore the symmetricity for this matrix eigenvalue problem. Let $g = W^{\frac{1}{2}}f$, equation (2.23) becomes:

$$\left(W^{\frac{1}{2}}DW^{\frac{1}{2}}\right)g = \lambda g \quad (2.24)$$

Now, solving equation (2.24) give us a good estimation of first n KL eigenvalues $\lambda_1, \dots, \lambda_n$ and point value of corresponding eigenvectors $\{e_i(t_j)\}_{i,j \in [1,n]}$. Adopting an interpolation method, we obtain

$$e_k(t) = \frac{1}{\lambda_k} \sum_{j=1}^n \omega_j(t, t_j) e_k(t_j), \quad t \in [0, T], j = 1, \dots, n \quad (2.25)$$

Corlay starts with such estimates for Brownian motion, Brownian bridge, and OU process, for which explicit expressions for the eigenvalues are known. The resulting approximations are very close to the values obtained from the explicit formulas for the eigenvalues and eigenfunctions, which shows that the method used by Corlay is rather powerful. Corlay also estimates the five highest KL eigenvalues of fractional Brownian motion on $[0, 1]$ with the Hurst exponent $H = 0.7$. We used Corlay's method to compute first 500 highest eigenvalues and corresponding eigenfunctions for several OU and fOU processes, we are confident that the values we obtain have similar levels of accuracy to what is illustrated in [16].

2.2 Specific example of Gaussian process

2.2.1 Fractional brownian motion

Our model flexibility combined with known explicit spectral expansions and numerical tool allow practioners to compute the spectral parameters in a straightforward fashion based on smile features, while also allowing them to select their favorite Gaussian volatility model class. In this dissertation, we adopt OU process and fractional OU process, which were proposed early on for option pricing, and recently analyzed in citeCV,CR. We will illustrate how, in the case of the classical and fractional Stein-Stein models(OU and fOU process), the explicit, semi-explicit, or numerically accessible Karhunen-Loève expansion of X can be used in conjunction with our asymptotics for calibrating parameters. We start with simulated option prices data with predetermined parameters. Given a partition $0 = t_0 < t_1 < \dots < t_N = T$, in order to price the option, we use Euler scheme to simulate the SDEs in our model for $k = 0, \dots, N - 1$ recursively,

$$\begin{aligned} X_{k+1} &= X_k + \alpha (m - X_k) (t_{k+1} - t_k) + \beta (Z_{k+1} - Z_k) \\ S_{k+1} &= S_k \exp \left\{ \left(r - \frac{1}{2} Y_k^2 \right) (t_{k+1} - t_k) + |Y_k| \sqrt{t_{k+1} - t_k} W_k \right\} \end{aligned}$$

The process Z is brownian motion and fractional brownian motion in classical and fractional Stein-Stein models respectively. And W_0, \dots, W_{N-1} are iid standard normal variates. In this section, we will present how to generate fractional brwonian motion.

In general, the fBm is a centered continuous Gaussian process whose covariance function is

$$E(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|) \quad (2.26)$$

The increment of fBm is called fractional Gaussian noise as

$$N_k = B_H(k + 1) - B_H(k)$$

It can be proved that for each k , N_k has standard normal distribution with autocovariance

$$\gamma(k) = \frac{1}{2} [|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H}] \quad (2.27)$$

Let $N = 2^g$ be the sample size that we need to generate. We define the so-called circulant covariance matrix C by

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(N-1) & 0 & \gamma(N-1) & \gamma(N-2) & \cdots & \gamma(2) & \gamma(1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(N-2) & \gamma(N-1) & 0 & \gamma(N-1) & \cdots & \gamma(3) & \gamma(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma(N-1) & \gamma(N-2) & \cdots & \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(N-1) & 0 \\ 0 & \gamma(N-1) & \cdots & \gamma(1) & \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(N-2) & \gamma(N-1) \\ \gamma(N-1) & 0 & \cdots & \gamma(2) & \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(N-3) & \gamma(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma(1) & \gamma(2) & \cdots & 0 & \gamma(N-1) & \gamma(N-2) & \gamma(N-3) & \cdots & \gamma(1) & \gamma(0) \end{bmatrix}$$

Since C is positive definite and symmetric, there exist a diagonal matrix D of eigenvalues of C and an unitary matrix Q such that the matrix C can be written as

$$C = QDQ^* \quad (2.28)$$

Denote $S = QD^{\frac{1}{2}}Q^*$, then $SS^* = C$. If V is a vector of i.i.d standard normal random variates, it is clear that SV form a sample path of fGn. The steps are as follows:

- 1. Use Fast Fourier Transform to compute the eigenvalue of C by

$$\lambda_k = \sum_{j=0}^{2N-1} r_j \exp\left(2\pi i \frac{jk}{2N}\right) \quad k = 0, \dots, 2N-1$$

where r_j is the $(j+1)$ element of the first column of C .

- 2. Generate $2N$ i.i.d standard normal random variates V_0, \dots, V_{2N} and denote

$$W_j = \begin{cases} V_j & j = 0, N \\ \frac{1}{\sqrt{2}} (V_j + iV_{2N-j}) & j = 1, \dots, N-1 \\ \frac{1}{\sqrt{2}} (V_{2N-j} - iV_j) & j = N+1, \dots, 2N-1 \end{cases}$$

- 3. Use Fast Fourier Transform again to compute

$$Z_k = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} \sqrt{\lambda_j} W_j \exp\left\{-2i\pi \frac{jk}{2N}\right\}$$

The first N element of Z is the desired sample of fGn. Note that the other N terms also form a sample of fGn. But it is useless because it is not independent with the first sample.

The fractional Brownian motion is also used in chapter 4 since it is the only continuous self-similar centered Gaussian process with stationary increments.

Definition 2.1 *Let $0 < H < 1$. A stochastic process $X^{(H)}$ is called H -self-similar if for every $a > 0$, $X_{at}^{(H)} \stackrel{d}{=} a^H X_t^{(H)}$. Here $\stackrel{d}{=}$ means the equality of all finite-dimensional distributions.*

It is easy to see that if the process $X^{(H)}$ is H -self-similar, then $X_0^{(H)} = 0$. It will always be assumed in the sequel that the self-similar process $X^{(H)}$ is stochastically continuous. For a Gaussian process X , the H -self-similarity condition is expressed in terms of the covariance function C as follows:

$$C(at, as) = a^{2H} C(t, s), \quad (t, s) \in [0, T]^2.$$

We refer the interested reader to [18,19] for more information on self-similar stochastic processes. In this dissertation, we choose fractional brownian motion (fBm) as an example to illustrate our calibration strategy. The first step is to obtain a bunch of simulated data on IV by the model. In [20], the author present a list of method of generating fBm. We only discuss The Davies and Harte method since it is faster than others (of order $n \log n$).

2.2.2 Fractional OU process

The simulation of fractional Stein-Stein model is already presented. However, We still have a problem need to solve before calibrating. We know that the KL expansion can be computed numerically, but only if the covariance function is given. The fOU's covariance function does not have a closed form. Barboza's thesis [21] provide a method for us to compute it numerically.

The fOU satisfy the stochastic differential equation(Langevin equation)

$$dX_t = -qX_s ds + \sigma dB_t^H \quad (2.29)$$

where B^H is a fBm with hurst parameter H. It was shown that,

$$X_t = e^{-qt} X_0 + \sigma e^{-qt} \int_0^t e^{qu} dB_u^H, \quad t \geq 0. \quad (2.30)$$

is the unique almost surely continuous solution of equation (2.29). If we take $X_0 = \sigma \int_{-\infty}^0 e^{qu} dB_u^H$, this solution is stationay. Barboza proved that the autocovariance function of stationay X_t is given by

$$\rho_{H,q}(t) = 2\sigma^2 C_H \int_0^\infty \cos(tx) \frac{x^{1-2H}}{q^2 + x^2} dx \quad (2.31)$$

where $C_H = \frac{\Gamma(2H+1) \sin(\pi H)}{2\pi}$. He also proposed an alternative formula for $H \in (\frac{1}{2}, 1)$

$$\nu_{H,q}(t) = \sigma^2 H \lambda^{-2H} e^{-qt} \left[\Gamma(2H) + \frac{C_1(t, H, q) + C_2(t, H, q)}{2} \right] \quad (2.32)$$

where

$$C_1(t, H, q) = (e^{-2qt} - 1)(2H - 1)\Gamma(2H - 1, qt)$$

$$C_2(t, H, q) = (qt)^{2H-1}(e^{qt} - e^{-qt}) - \gamma(2H, qt) - \int_0^{qt} e^u u^{2H-1} du$$

The integral term $\int_0^{qt} e^u u^{2H-1} du$ can be obtained simply by applying Riemann approximation with grid size 1000. There is also a formula for short memeory which we do not cover in this dissertation.

3. EXTREME-STRIKE ASYMPTOTICS

3.1 Specific motivations and heuristics

Recent studies have looked into detail at the question of extreme strike asymptotics for implied volatility. Of note is the groundbreaking paper [22] of Lee. By exploiting a method of moments and the representation of power payoffs as mixtures of a continuum of calls with varying strikes, in a rather model-free context, he proved a celebrated formula that, for models with positive moment explosions, the squared IV's large strike behavior is of order the log-moneyness $\log\left(\frac{K}{s_0 e^{rT}}\right)$ times a constant which depends explicitly on supremum of the order of finite moments. A similar result holds for models with negative moment explosions, where the squared IV behaves like $K \mapsto \log\left(\frac{s_0 e^{rT}}{K}\right)$ for small values of K . More general formulas describing the asymptotic behavior of the IV in the 'wings' ($K \rightarrow 0$ or $+\infty$) were obtained in [10, 23–28] (see also the book [5]).

From the standpoint of modeling, one of the advantages of Lee's original result is the dependence of IV asymptotics merely on some simple statistics, namely as we mentioned, in the notation in [22], the maximal order \tilde{p} of finite moments for the underlying S_T , i.e.

$$\tilde{p}(T) := \sup \{p \in \mathbb{R} : \mathbb{E} [(S_T)^{p+1}] < \infty\}.$$

This allows the author to draw appropriately strong conclusions about model calibration. A special class of models in which \tilde{p} is positive and finite is that of Gaussian volatility models, which we introduce next.

For a Gaussian volatility model, value of \tilde{p} can sometimes be determined by simple calculations, which we illustrate here with an elementary example. Assume S is a geometric Brownian motion with random volatility, i.e. a model as in (1.1) where (abusing notation) $|X_t|$ is taken the non-time-dependent $\sigma |X|$ where σ is a constant

and X is an independent unit-variance normal variate (not dependent on t). Thus, at time T , with zero discount rate,

$$S_T = s_0 \exp(\sigma |X| W_T - \sigma^2 X^2 T/2)$$

To simplify this example to the maximum, also assume that X is centered; using the independence of X and W , we get that we may replace $|X|$ by X in this example, since this does not change the law of S_T (i.e. in the uncorrelated case, X 's non-positivity does not violate standard practice for volatility modeling). Then, using maturity $T = 1$, for any $p > 0$, the p th moment, via a simple change of variable, equals

$$\mathbb{E}[(S_1)^p] = \frac{s_0^p}{2\pi\sqrt{1+p\sigma^2}} \iint_{\mathbf{R}^2} dy dw \exp\left(-\frac{1}{2}\left(y^2 + w^2 - 2\frac{p\sigma}{\sqrt{1+p\sigma^2}}wy\right)\right)$$

which by an elementary computation is finite, and equal to $s_0^p/\sqrt{1+p\sigma^2-p^2\sigma^2}$, if and only if

$$p < \tilde{p} + 1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\sigma^2}}.$$

In the cases where the random volatility model X above is non-centered and is correlated with W , a similar calculation can be performed, at the essentially trivial expenses of invoking affine changes of variables, and the linear regression of one normal variate against another.

The above example illustrates heuristically that, by Lee's moment formula, the computation of \tilde{p} might be the quickest path to obtain the leading term in the large-strike expansion of the IV, for more complex Gaussian volatility models, namely ones where the volatility X is time-dependent. However, computing \tilde{p} is not necessarily an easy task, and appears, perhaps surprisingly, to have been performed rarely. For the Stein-Stein model, the value of \tilde{p} can be computed using the sharp asymptotic formulas for the stock price density near zero and infinity, established in [4] for the uncorrelated Stein-Stein model, and in [29] for the correlated one. These two papers also provide asymptotic formulas with error estimates for the IV at extreme strikes in the Stein-Stein model. Beyond the Stein-Stein model, little was known about the

extreme strike asymptotics of general Gaussian stochastic volatility models. In this chapter, we extend the above-mentioned results from [4] and [29] to such models.

3.2 General setup and second-chaos expansion of the integrated variance

Let X be an almost-surely continuous Gaussian process on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with mean and covariance functions denoted by $m(t) = \mathbb{E}[X_t]$ and

$$Q(t, s) = \text{cov}(X_t, X_s) = \mathbb{E}[(X_t - m(t))(X_s - m(s))],$$

respectively. While such processes used in a jump-free quantitative finance context for volatility modeling will require, in addition, that X be adapted to filtration of the Wiener process W driving the asset price (as in (1.1)), under our simplifying assumption that X and W be independent, this adaptability assumption can be considered as automatically satisfied, or equivalently, as unnecessary, since the filtration of W can be augmented by the natural filtration of X . Define the centered version of X : $\tilde{X}_t := X_t - m(t)$, $t \geq 0$.

Fix a time horizon $T > 0$. It is not hard to see that $Q(s, s) > 0$ for all $s > 0$. Since the Gaussian process X is almost surely continuous, the mean function $t \mapsto m(t)$ is a continuous function on $[0, T]$, and the covariance function $(t, s) \mapsto Q(t, s)$ is a continuous function of two variables on $[0, T]^2$. This is a consequence of the Dudley-Fernique theory of regularity, which also implies that m and Q boast moduli of continuity bounded above by the scale $h \mapsto \log^{-1/2}(h^{-1})$ (see [30]), but this can also be established by more elementary means.¹

¹The continuity of the process X implies its continuity in probability on Ω . Hence, the process X is continuous in the mean-square sense (see, e.g., [31], Lemma 1 on p. 5, or invoke the equivalence of L^p norms on Wiener chaos, see [32]). Mean-square continuity of X implies the continuity of the mean function on $[0, T]$. In addition, the autocorrelation function of the process X , that is, the function $R(t, s) = \mathbb{E}[X_t X_s]$, $(t, s) \in [0, T]^2$, is continuous (see, e.g., [33], Lemma 4.2). Finally, since $Q(t, s) = R(t, s) - m(t)m(s)$, the covariance function Q is continuous on $[0, T]^2$.

Applying the classical Karhunen-Loève theorem to \tilde{X} , we obtain the existence of a non-increasing sequence of non-negative summable reals $\{\lambda_n : n = 1, 2, \dots\}$, an i.i.d. sequence of standard normal variates $\{Z_n : n = 1, 2, \dots\}$, and a sequence of functions $\{e_n : n = 1, 2, \dots\}$ which form an orthonormal system in $L^2([0, T])$, such that

$$\tilde{X}_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n. \quad (3.1)$$

In (4.3), $\{e_n = e_{n,T}\}$ are the eigenfunctions of the covariance Q acting on $L^2([0, T])$ as the following operator

$$\mathcal{K}(f)(t) = \int_0^T f(s) Q(t, s) ds, \quad f \in L^2([0, T]), \quad 0 \leq t \leq T,$$

and $\{\lambda_n = \lambda_{n,T}\}$, $n \geq 1$, are the corresponding eigenvalues (counting the multiplicities). We always assume that the orthonormal system $\{e_n\}$ is rearranged so that

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n_1} > \lambda_{n_1+1} = \lambda_{n_1+2} = \dots = \lambda_{n_1+n_2} > \dots$$

In particular, λ_1 is the top eigenvalue, and n_1 is its multiplicity.

Using (4.3), we obtain

$$\int_0^T \tilde{X}_t^2 dt = \int_0^T \left(\sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n \right)^2 dt = \sum_{n=1}^{\infty} \lambda_n Z_n^2. \quad (3.2)$$

It is worth pointing out that this expression for the integrated variance of the centered volatility $\int_0^T \tilde{X}_t^2 dt$ is in fact the most general form of a random variable in the second Wiener chaos, with mean adjusted to ensure almost-sure positivity of the integrated variance. This is established using a classical structure theorem on separable Hilbert spaces, as explained in [32, Section 2.7.4]. In other words (also see [32, Section 2.7.3] for additional details), any prescribed mean-adjusted integrated variance in the second chaos is of the form

$$V(T) := \iint_{[0, T]^2} G(s, t) dZ(s) dZ(t) + 2 \|G\|_{L^2([0, T]^2)}^2$$

for some standard Wiener process Z and some function $G \in L^2([0, T]^2)$, and moreover one can find a centered Gaussian process \tilde{X} such that $V(T) = \int_0^T \tilde{X}_t^2 dt$ and one

can compute the coefficients λ_n in the Karhunen-Loève representation (3.2) as the eigenvalues of the covariance of \tilde{X} . When using the non-centered process X , this analysis immediately yields

$$\begin{aligned} \int_0^T X_t^2 dt &= \int_0^T \left(\tilde{X}_t + m(t) \right)^2 dt \\ &= \sum_{n=1}^{\infty} \lambda_n Z_n^2 + 2 \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left[\int_0^T m(t) e_n(t) dt \right] Z_n \\ &\quad + \int_0^T m(t)^2 dt. \end{aligned} \quad (3.3)$$

Set

$$\delta_n = \delta_{n,T} = \int_0^T m(t) e_n(t) dt, \quad n \geq 1, \quad (3.4)$$

and

$$s = s(T) = \int_0^T m(t)^2 dt. \quad (3.5)$$

Then Bessel's inequality implies that

$$\sum_{n=1}^{\infty} \delta_n^2 \leq s. \quad (3.6)$$

Denote

$$\tau = s - \sum_{n=1}^{\infty} \delta_n^2. \quad (3.7)$$

It is not hard to see that if the function $t \mapsto m(t)$ belongs to the image space $\mathcal{K}(L^2[0, T])$, then $\sum_{n=1}^{\infty} \delta_n^2 = s$, and hence $\tau = 0$. For instance, the previous equality holds for a centered Gaussian process X . Note that this equality also holds if $\lambda = 0$ is not an eigenvalue of the operator \mathcal{K} .

Equality (3.3) can be rewritten as follows:

$$\begin{aligned} \int_0^T X_t^2 dt &= \sum_{n=1}^{\infty} \lambda_n \left[Z_n^2 + 2 \frac{\delta_n}{\sqrt{\lambda_n}} Z_n \right] + s \\ &= \sum_{n=1}^{\infty} \lambda_n \left[Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2 + \left(s - \sum_{n=1}^{\infty} \delta_n^2 \right). \end{aligned} \quad (3.8)$$

Therefore, if the function $t \mapsto m(t)$ belongs to the image space $\mathcal{K}(L^2[0, T])$, then

$$\int_0^T X_t^2 dt = \sum_{n=1}^{\infty} \lambda_n \left[Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2.$$

Let us denote the noncentral chi-square distribution with the number of degrees of freedom k and the parameter of noncentrality λ by $\chi^2(k, \lambda)$ (more information on such distributions can be found in [5] or in any probability textbook). Define a random variable \tilde{Z}_n by

$$\tilde{Z}_n = \left[Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2.$$

It is clear that \tilde{Z}_n is distributed as $\chi^2(1, \frac{\delta_n^2}{\lambda_n})$. Set

$$\rho_1 = \lambda_1, \quad \rho_2 = \lambda_{n_1+1}, \quad \rho_3 = \lambda_{n_1+n_2+1}, \quad \dots,$$

and

$$\Lambda_T = \frac{1}{\lambda_1} \left(\int_0^T X_t^2 dt - \tau \right). \quad (3.9)$$

Then, using (3.8), we see that

$$\Lambda_T = \chi^2 \left(n_1, \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2 \right) + \frac{\rho_2}{\lambda_1} \chi^2 \left(n_2, \frac{1}{\rho_2} \sum_{n=n_1+1}^{n_1+n_2} \delta_n^2 \right) + \dots, \quad (3.10)$$

where the repeated chi-squared notation is used abusively to denote independent chi-squared random variables.

3.3 Asymptotics of the mixing density

The asymptotic behavior of the complementary distribution function of an infinite linear combination of independent central chi-square random variables was characterized by Zolotarev in [34]. Zolotarev's results were generalized to the case of noncentral chi-square variables by Beran (see [35]). Beran used some ideas from [36] in his work. We will employ Beran's results in the present paper. More precisely, Theorem 2 in [35] will be used below. This theorem provides an asymptotic formula for the complementary distribution function of an infinite linear combination of independent noncentral chi-square random variables. A sharper formula for the distribution density q_T of such a random variable can be extracted from the proof of Theorem 2 in [35] (see the very end of that proof). Adapting Beran's result to our case and taking into

account estimate (3.6) (this estimate is needed to check the validity of the conditions in Beran's theorem), we see that

$$\left| \frac{q_T(x)}{p_{\chi^2}\left(x; n_1, \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2\right)} - A \right| = O\left(x^{-\frac{1}{2}}\right) \quad (3.11)$$

as $x \rightarrow \infty$. In (3.11), the number A is given by

$$A = \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}} \times \exp \left\{ \frac{1}{2} \sum_{i=2}^{\infty} \frac{1}{\lambda_1 - \rho_i} \left(\sum_{n=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} \delta_n^2 \right) \right\}. \quad (3.12)$$

Set

$$\delta = \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2. \quad (3.13)$$

Then (3.11) gives

$$q_T(x) = A p_{\chi^2}(x; n_1, \delta) \left(1 + O\left(x^{-\frac{1}{2}}\right) \right). \quad (3.14)$$

For $\lambda > 0$, the following formula is known:

$$p_{\chi^2}(x; n, \lambda) = \frac{1}{2} \left(\frac{x}{\lambda} \right)^{\frac{n}{4} - \frac{1}{2}} e^{-\frac{x+\lambda}{2}} I_{\frac{n}{2}-1}(\sqrt{\lambda x}), \quad x > 0, \quad (3.15)$$

where I_ν is the modified Bessel function of the first kind (see, e.g., [5], Theorem 1.31).

It is easy to see that formula (3.15) and the formula

$$I_\nu(t) = \frac{e^t}{\sqrt{2\pi t}} \left(1 + O(t^{-1}) \right), \quad t \rightarrow \infty,$$

describing the asymptotic behavior of the I-Bessel function, imply that

$$p_{\chi^2}(x; n, \lambda) = \frac{1}{2\sqrt{2\pi}} \lambda^{-\frac{n-1}{4}} x^{\frac{n-3}{4}} e^{\sqrt{\lambda x}} e^{-\frac{x+\lambda}{2}} \left(1 + O\left(x^{-\frac{1}{2}}\right) \right) \quad (3.16)$$

as $x \rightarrow \infty$. On the other hand, if $\lambda = 0$, then

$$p_{\chi^2}(x; n, 0) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n-2}{2}} \exp\left\{-\frac{x}{2}\right\}, \quad x > 0, \quad (3.17)$$

(see, e.g., Lemma 1.27 in [5]).

Recall that we denoted by q_T the distribution density of the random variable Λ_T defined by (3.9). It follows from (3.14) and (3.16) that

$$q_T(x) = \frac{A}{2\sqrt{2\pi}} \delta^{-\frac{n_1-1}{4}} x^{\frac{n_1-3}{4}} e^{\sqrt{\delta}x} e^{-\frac{x+\delta}{2}} \left(1 + O\left(x^{-\frac{1}{2}}\right)\right) \quad (3.18)$$

as $x \rightarrow \infty$. The constants A and δ in (3.18) are defined by (3.12) and (3.13), respectively. For a centered Gaussian process X , (3.14) and (3.17) imply that

$$q_T(x) = \frac{A}{2^{\frac{n_1}{2}} \Gamma\left(\frac{n_1}{2}\right)} x^{\frac{n_1-2}{2}} \exp\left\{-\frac{x}{2}\right\} \left(1 + O\left(x^{-\frac{1}{2}}\right)\right), \quad x > 0, \quad (3.19)$$

as $x \rightarrow \infty$, where

$$A = \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k}\right)^{\frac{n_k}{2}}. \quad (3.20)$$

Indeed, in this case, we have $s = 0$, $\delta_n = 0$ for all $n \geq 1$, $\delta = 0$, and $\tau = 0$.

Our main goal is to characterize the asymptotic behavior of the distribution density p_T of the random variable

$$\Gamma_T = \int_0^T X_t^2 dt. \quad (3.21)$$

It follows from (3.7) and (3.9) that $\Gamma_T = \lambda_1 \Lambda_T + \tau$. Therefore,

$$p_T(x) = \frac{1}{\lambda_1} q_T\left(\frac{1}{\lambda_1}(x - \tau)\right). \quad (3.22)$$

Theorem 3.1 *Let p_T be the distribution density of the random variable Γ_T given by (3.21). If $\delta > 0$, then the following asymptotic formula holds:*

$$p_T(x) = C x^{\frac{n_1-3}{4}} \exp\left\{\sqrt{\frac{\delta}{\lambda_1}} \sqrt{x}\right\} \exp\left\{-\frac{x}{2\lambda_1}\right\} \times \left(1 + O\left(x^{-\frac{1}{2}}\right)\right) \quad (3.23)$$

as $x \rightarrow \infty$, where

$$C = \frac{A}{2\sqrt{2\pi}} \lambda_1^{-\frac{1}{2}} \left(\sum_{n=1}^{n_1} \delta_n^2\right)^{-\frac{n_1-1}{4}} \exp\left\{\frac{s - \sum_{n=1}^{\infty} \delta_n^2 - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1}\right\}. \quad (3.24)$$

In (3.23) and (3.24), the constants δ_n , s , A , and δ are defined by (3.4), (3.5), (3.12), and (3.13), respectively. On the other hand, for a centered Gaussian process X , we have

$$p_T(x) = Cx^{\frac{n_1-2}{2}} \exp\left\{-\frac{x}{2\lambda_1}\right\} \left(1 + O\left(x^{-\frac{1}{2}}\right)\right) \quad (3.25)$$

as $x \rightarrow \infty$, where

$$C = \frac{A}{2^{\frac{n_1}{2}} \Gamma\left(\frac{n_1}{2}\right)} \lambda_1^{-\frac{n_1}{2}}. \quad (3.26)$$

Proof Formulas (3.18) and (3.22) imply that

$$\begin{aligned} p_T(x) &= \frac{A}{2\sqrt{2\pi}} \frac{1}{\lambda_1} \lambda_1^{\frac{n_1-1}{4}} \delta^{-\frac{n_1-1}{4}} \\ &\times \left(\sum_{n=1}^{n_1} \delta_n^2\right)^{-\frac{n_1-1}{4}} \lambda_1^{-\frac{n_1-3}{4}} \exp\left\{\frac{\tau - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1}\right\} \\ &\times (x - \tau)^{\frac{n_1-3}{4}} \exp\left\{\sqrt{\frac{\delta(x - \tau)}{\lambda_1}}\right\} \exp\left\{-\frac{x}{2\lambda_1}\right\} \\ &\times \left(1 + O\left(x^{-\frac{1}{2}}\right)\right) \end{aligned} \quad (3.27)$$

as $x \rightarrow \infty$.

Next, taking into account the formulas

$$(x - \tau)^{\frac{n_1-3}{4}} = x^{\frac{n_1-3}{4}} (1 + O(x^{-1}))$$

and

$$\exp\left\{\sqrt{\frac{\delta(x - \tau)}{\lambda_1}}\right\} = \exp\left\{\sqrt{\frac{\delta}{\lambda_1}} \sqrt{x}\right\} (1 + O(x^{-\frac{1}{2}})),$$

and simplifying the expression on the right-hand side of (3.27), we obtain formula (3.23). The proof of formula (3.25) is similar. Here we use (3.19) and (3.22). \blacksquare

The next assertion follows from Theorem 3.1.

Corollary 3.1 *Let p_T be the distribution density of the random variable Γ_T given by (3.21). Then the following are true:*

1. If $n_1 = 1$, then

$$p_T(x) = Cx^{-\frac{1}{2}} \exp \left\{ \sqrt{\frac{\delta}{\lambda_1}} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \\ \times \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \quad (3.28)$$

as $x \rightarrow \infty$.

2. Suppose X is a centered Gaussian process with $n_1 = 1$. Then

$$p_T(x) = Cx^{-\frac{1}{2}} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \quad (3.29)$$

as $x \rightarrow \infty$.

3.4 Asset price asymptotics

In this section, we study stochastic volatility models, for which the volatility is described by the absolute value of a Gaussian process.

Recall that in the present paper we assume that the asset price process S satisfies the following linear stochastic differential equation:

$$dS_t = rS_t dt + |X_t|S_t dW_t, \quad (3.30)$$

where X is a continuous Gaussian process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. In (3.30), W is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}$, and the symbol r stands for the constant interest rate. It will be assumed that the processes X and W are independent. The initial price of the asset will be denoted by s_0 .

Since (3.30) is a linear stochastic differential equation, we have

$$S_t = s_0 \exp \left\{ rt - \frac{1}{2} \int_0^t X_s^2 ds + \int_0^t |X_s| dW_s \right\}.$$

The previous equality follows from the Doléans-Dade formula (see [37]). Therefore, the discounted asset price process is given by the following stochastic exponential:

$$\tilde{S}_t = e^{-rt} S_t = s_0 \exp \left\{ -\frac{1}{2} \int_0^t X_s^2 ds + \int_0^t |X_s| dW_s \right\}. \quad (3.31)$$

The next assertion states that under the restrictions on the volatility process used in the present paper, we are in a risk-neutral environment.

Lemma 3.1 *Under the restrictions on the volatility process X in (3.30), the discounted asset price process \tilde{S} is a $\{\mathcal{F}_t\}$ -martingale.*

Proof Lemma 3.1 is standard. Using Itô's formula, we first show that the process \tilde{S} in (3.31) is a positive local martingale. Hence, it is a supermartingale by Fatou's lemma. The conditional distribution of the stochastic integral $\int_0^t |X_s| dW_s$ given $|X|$ is normal with mean zero and variance $\int_0^t X_s^2 ds$. Hence by conditioning on $|X|$ and using the normal MGF, we can prove that $\mathbb{E}[\tilde{S}_t] = s_0$ for all t . However, a supermartingale with a constant expectation is a martingale. ■

Since the processes X and W are independent, the following formula holds for the distribution density D_t of the asset price S_t :

$$D_t(x) = \frac{\sqrt{s_0 e^{rt}}}{\sqrt{2\pi t}} x^{-\frac{3}{2}} \int_0^\infty y^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0 e^{rt}}}{2ty^2} + \frac{ty^2}{8} \right] \right\} \tilde{p}_t(y) dy. \quad (3.32)$$

In (3.32), \tilde{p}_t is the distribution density of the random variable

$$\tilde{Y}_t = \left\{ \frac{1}{t} \int_0^t X_s^2 ds \right\}^{\frac{1}{2}}.$$

The function \tilde{p}_t is called the mixing density. The proof of formula (3.32) can be found in [5] (see (3.5) in [5]). It is not hard to see that

$$\tilde{p}_t(y) = 2ty p_t(ty^2), \quad (3.33)$$

where the symbol p_t stands for the density of the realized volatility $Y_t = \int_0^t X_s^2 ds$.

Suppose first that the volatility process is a non-centered Gaussian process. It follows from formula (3.23) that

$$\begin{aligned} \tilde{p}_t(y) &= \tilde{A} y^{\frac{n_1-1}{2}} \exp \left\{ \tilde{B}y \right\} \exp \left\{ -\tilde{C}y^2 \right\} \\ &\times \left(1 + O(y^{-1}) \right) \end{aligned} \quad (3.34)$$

as $y \rightarrow \infty$, where

$$\tilde{A} = 2Ct^{\frac{n_1+1}{4}}, \quad \tilde{B} = \sqrt{\frac{\delta t}{\lambda_1}}, \quad \tilde{C} = \frac{t}{2\lambda_1}. \quad (3.35)$$

Our next goal is to estimate the function D_t . The asymptotic behavior as $x \rightarrow \infty$ of the integral appearing in (3.32) was studied in [4] (see also Section 5.3 in [5]). It is explained in [5] how to get an asymptotic formula for the integral in (4.1) in the case where an asymptotic formula for the mixing density is similar to formula (3.34). We refer the reader to the derivation of Theorem 6.1 in [5], which is based on formula (5.133) in Section 5.6 of [5] and Theorem 5.5 in [5]. The latter theorem concerns the asymptotic behavior of integrals with lognormal kernels. Having obtained an asymptotic formula for the distribution density of the asset price, we can find a similar asymptotic formula for the call pricing function C at large strikes, and then obtain an asymptotic formula for the implied volatility I (see Section 10.5 in [5]).

Theorem 5.5 in [5] provides an asymptotic formula as $w \rightarrow \infty$ for the integral

$$\int_0^\infty A(y) \exp \left\{ - \left(\frac{w^2}{y^2} + k^2 y^2 \right) \right\} dy,$$

where $k > 0$ is fixed, and it is assumed that

$$A(y) = e^{ly} \zeta(y) (1 + O(b(y)))$$

as $y \rightarrow \infty$. In the previous asymptotic formula, l is a real number, and ζ and b are functions satisfying certain conditions.

Let us fix $T > 0$. Our goal is to use Theorem 5.5 in [5] with

$$A(y) = y^{-1} \tilde{p}_T(y) \exp \left\{ \tilde{C} y^2 \right\},$$

$l = \tilde{B}$, $\zeta(y) = \tilde{A} y^{\frac{n_1-3}{2}}$, $b(y) = y^{-1}$, $w = (2T)^{-\frac{1}{2}} \log \frac{x}{s_0 e^{rT}}$, $k = \frac{\sqrt{8\tilde{C}+T}}{2\sqrt{2}}$, and $\gamma = 1$ (see the formulation of Theorem 5.5 in [5] for the meaning of the constant γ). This gives

$$\begin{aligned} & \int_0^\infty y^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0 e^{rT}}}{2ty^2} + \frac{ty^2}{8} \right] \right\} \tilde{p}_t(y) dy = \frac{\tilde{A} 2^{\frac{n_1-1}{4}} \sqrt{\pi}}{T^{\frac{n_1-3}{8}} (8\tilde{C}+T)^{\frac{n_1+1}{8}}} (s_0 e^{rT})^{\frac{\sqrt{8\tilde{C}+T}}{2\sqrt{T}}} \\ & \exp \left\{ \frac{\tilde{B}^2}{2(8\tilde{C}+T)} \right\} \left(\log \frac{x}{s_0 e^{rT}} \right)^{\frac{n_1-3}{4}} x^{-\frac{\sqrt{8\tilde{C}+T}}{2\sqrt{T}}} \exp \left\{ \frac{\tilde{B}\sqrt{2}}{T^{\frac{1}{4}}(8\tilde{C}+T)^{\frac{1}{4}}} \sqrt{\log \frac{x}{s_0 e^{rT}}} \right\} \\ & \left(1 + O \left(\left(\log \frac{x}{s_0 e^{rT}} \right)^{-\frac{1}{2}} \right) \right) \end{aligned} \quad (3.36)$$

as $x \rightarrow \infty$. Next, using (3.32) and (3.36), we obtain

$$\begin{aligned}
D_T(x) &= \frac{\tilde{A} 2^{\frac{n_1-3}{4}}}{T^{\frac{n_1+1}{8}} (8\tilde{C} + T)^{\frac{n_1+1}{8}}} (s_0 e^{rT})^{\frac{1}{2} + \frac{\sqrt{8\tilde{C}+T}}{2\sqrt{T}}} \exp \left\{ \frac{\tilde{B}^2}{2(8\tilde{C} + T)} \right\} \\
&\quad \left(\log \frac{x}{s_0 e^{rT}} \right)^{\frac{n_1-3}{4}} x^{-\left(\frac{3}{2} + \frac{\sqrt{8\tilde{C}+T}}{2\sqrt{T}}\right)} \exp \left\{ \frac{\tilde{B}\sqrt{2}}{T^{\frac{1}{4}}(8\tilde{C} + T)^{\frac{1}{4}}} \sqrt{\log \frac{x}{s_0 e^{rT}}} \right\} \\
&\quad \left(1 + O \left(\left(\log \frac{x}{s_0 e^{rT}} \right)^{-\frac{1}{2}} \right) \right)
\end{aligned} \tag{3.37}$$

as $x \rightarrow \infty$. Recall that

$$\begin{aligned}
\tilde{A} &= \frac{1}{\sqrt{2\pi}} \lambda_1^{-\frac{1}{2}} T^{\frac{n_1+1}{4}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}} \exp \left\{ \frac{1}{2} \sum_{i=2}^{\infty} \frac{1}{\lambda_1 - \rho_i} \left(\sum_{n=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} \delta_n^2 \right) \right\} \\
&\quad \left(\sum_{n=1}^{n_1} \delta_n^2 \right)^{-\frac{n_1-1}{4}} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \delta_n^2 - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1} \right\},
\end{aligned} \tag{3.38}$$

$$\tilde{B} = \sqrt{\frac{\delta T}{\lambda_1}}, \quad \tilde{C} = \frac{T}{2\lambda_1}. \tag{3.39}$$

(see (3.12), (3.24), and (3.35)).

The next assertion can be obtained by using (3.38) and (3.39) in (3.37) and simplifying the resulting expressions.

Theorem 3.2 *If the volatility is modeled by a noncentered Gaussian process, then*

$$\begin{aligned}
D_T(x) &= V \left(\log \frac{x}{s_0 e^{rT}} \right)^{\frac{n_1-3}{4}} x^{-\left(\frac{3}{2} + \frac{\sqrt{4+\lambda_1}}{2\sqrt{\lambda_1}}\right)} \exp \left\{ \frac{\sqrt{2\delta}}{\lambda_1^{\frac{1}{4}}(4 + \lambda_1)^{\frac{1}{4}}} \sqrt{\log \frac{x}{s_0 e^{rT}}} \right\} \\
&\quad \left(1 + O \left(\left(\log \frac{x}{s_0 e^{rT}} \right)^{-\frac{1}{2}} \right) \right)
\end{aligned} \tag{3.40}$$

as $x \rightarrow \infty$, where

$$\begin{aligned}
V &= \frac{2^{\frac{n_1-5}{4}} \lambda_1^{\frac{n_1-3}{8}}}{\sqrt{\pi}(4 + \lambda_1)^{\frac{n_1+1}{8}}} (s_0 e^{rT})^{\frac{1}{2} + \frac{\sqrt{4+\lambda_1}}{2\sqrt{\lambda_1}}} \exp \left\{ \frac{\delta}{2(4 + \lambda_1)} \right\} \\
&\quad \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}} \exp \left\{ \frac{1}{2} \sum_{i=2}^{\infty} \frac{1}{\lambda_1 - \rho_i} \left(\sum_{n=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} \delta_n^2 \right) \right\} \\
&\quad \left(\sum_{n=1}^{n_1} \delta_n^2 \right)^{-\frac{n_1-1}{4}} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \delta_n^2 - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1} \right\}.
\end{aligned} \tag{3.41}$$

Formula (3.40) describes the asymptotic behavior of the asset price density in a Gaussian stochastic volatility model in terms of the Karhunen-Loève parameters, the initial condition s_0 , the interest rate r , and the time horizon T . Note that the Karhunen-Loève parameters depend on T , while the constant V depends on s_0 and r . We will sometimes use the notation $V(s_0, r)$ to emphasize this dependence.

An asymptotic formula similar to that in (3.40) can be obtained in the case where the volatility is described by a centered Gaussian process. Here we reason exactly as in the proof of Theorem 3.2. However, we use formulas (3.25) and (3.26) instead of formula (3.23).

Theorem 3.3 *If the volatility is modeled by a centered Gaussian process, then*

$$D_T(x) = U \left(\log \frac{x}{s_0 e^{rT}} \right)^{\frac{n_1-2}{2}} x^{-\left(\frac{3}{2} + \frac{\sqrt{4+\lambda_1}}{2\sqrt{\lambda_1}}\right)} \left(1 + O \left(\left(\log \frac{x}{s_0 e^{rT}} \right)^{-\frac{1}{2}} \right) \right) \quad (3.42)$$

as $x \rightarrow \infty$, where

$$U = U(s_0, r) = \frac{1}{\Gamma\left(\frac{n_1}{2}\right) \lambda_1^{\frac{n_1}{4}} (4 + \lambda_1)^{\frac{n_1}{4}}} (s_0 e^{rT})^{\frac{1}{2} + \frac{\sqrt{4+\lambda_1}}{2\sqrt{\lambda_1}}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}}. \quad (3.43)$$

3.5 Asymptotics of the implied volatility

The call pricing function in the stochastic volatility model described by (3.30) will be denoted by C . We have

$$C(T, K) = e^{-rt} \mathbb{E} [(S_T - K)^+]$$

where T is the maturity and K is the strike price. We will fix T , and consider C as the function $K \mapsto C(K)$ of only the strike price K . The Black-Scholes implied volatility associated with the pricing function C will be denoted by I . More information on the implied volatility can be found in [5, 7].

The asymptotic behavior of the implied volatility for stochastic volatility models, in which the asset price density satisfies

$$D_T(x) = r_1 x^{-r_3} \exp\{r_2 \sqrt{\log x}\} (\log x)^{r_4} (1 + O((\log x)^{-\frac{1}{2}})), \quad x \rightarrow \infty, \quad (3.44)$$

where $r_1 > 0$, $r_2 \geq 0$, $r_3 > 2$, and $r_4 \in \mathbb{R}$, was characterized in [28]. However, there is an error in the expression for the fourth coefficient in formula (91) in [28]. The correct statement is as follows.

Theorem 3.4 *Suppose condition (3.44) holds. Then the following asymptotic formula is valid for the implied volatility:*

$$\begin{aligned}
I(K) &= \frac{\sqrt{2}}{\sqrt{T}}(\sqrt{r_3-1} - \sqrt{r_3-2})\sqrt{\log \frac{K}{s_0 e^{rT}}} + \frac{r_2}{\sqrt{2T}} \left(\frac{1}{\sqrt{r_3-2}} - \frac{1}{\sqrt{r_3-1}} \right) \\
&+ \frac{2r_4+1}{2\sqrt{2T}} \left(\frac{1}{\sqrt{r_3-2}} - \frac{1}{\sqrt{r_3-1}} \right) \frac{\log \log \frac{K}{s_0 e^{rT}}}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} \\
&+ \left[\frac{1}{\sqrt{2T}} \left(\frac{1}{\sqrt{r_3-1}} - \frac{1}{\sqrt{r_3-2}} \right) \log \frac{\sqrt{r_3-1} - \sqrt{r_3-2}}{2\sqrt{\pi}r_1} + \frac{r_2^2}{4\sqrt{2T}} \left(\frac{1}{(r_3-2)^{\frac{3}{2}}} - \frac{1}{(r_3-1)^{\frac{3}{2}}} \right) \right] \\
&\times \frac{1}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} + \frac{r_2(2r_4+1)}{4\sqrt{2T}} \left(\frac{1}{(r_3-2)^{\frac{3}{2}}} - \frac{1}{(r_3-1)^{\frac{3}{2}}} \right) \frac{\log \log \frac{K}{s_0 e^{rT}}}{\log \frac{K}{s_0 e^{rT}}} + O \left(\frac{1}{\log \frac{K}{s_0 e^{rT}}} \right)
\end{aligned} \tag{3.45}$$

as $K \rightarrow \infty$.

The proof of Theorem 3.4 is exactly the same as that of Theorem 17 in [28].

The next assertion provides an asymptotic formula for the implied volatility in the stochastic volatility model given by (3.30) in the case where the volatility process is noncentered.

Theorem 3.5 *Suppose the volatility is modeled by a noncentered Gaussian process. Then the following formula holds for the implied volatility $K \mapsto I(K)$:*

$$\begin{aligned}
I(K) &= M_1 \sqrt{\log \frac{K}{s_0 e^{rT}}} + M_2 + M_3 \frac{\log \log \frac{K}{s_0 e^{rT}}}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} \\
&+ M_4 \frac{1}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} + M_5 \frac{\log \log \frac{K}{s_0 e^{rT}}}{\log \frac{K}{s_0 e^{rT}}} + O \left(\frac{1}{\log \frac{K}{s_0 e^{rT}}} \right)
\end{aligned} \tag{3.46}$$

as $K \rightarrow \infty$, where

$$\begin{aligned}
M_1 &= \frac{\sqrt{2}}{\sqrt{T}} \left(\frac{\sqrt{\lambda_1}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}}, & M_2 &= \frac{\sqrt{\delta}}{\sqrt{T}} \left(\frac{\lambda_1}{\sqrt{4 + \lambda_1}(\sqrt{4 + \lambda_1} + 2)} \right)^{\frac{1}{2}}, & (3.47) \\
M_3 &= \frac{n_1 - 1}{4\sqrt{2T}} \left(\frac{\lambda_1^{\frac{3}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}}, \\
M_4 &= -\frac{1}{\sqrt{2T}} \left(\frac{\lambda_1^{\frac{3}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}} \log \left[\frac{1}{2\sqrt{\pi}V(1,0)} \left(\frac{\lambda_1^{\frac{1}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}} \right] \\
&\quad + \frac{\sqrt{2\delta}}{4\sqrt{T}} \left(\frac{\sqrt{\lambda_1}(\sqrt{4 + \lambda_1} - 2)}{4 + \lambda_1} \right)^{\frac{1}{2}} (\sqrt{4 + \lambda_1} + 1), \\
M_5 &= \frac{(n_1 - 1)\sqrt{\delta}}{8\sqrt{T}} \left(\frac{\lambda_1(\sqrt{4 + \lambda_1} - 2)}{\sqrt{4 + \lambda_1}} \right)^{\frac{1}{2}} (\sqrt{4 + \lambda_1} + 1),
\end{aligned}$$

where $V(1, 0)$ is the value of V in (3.41) with $s_0 = 1$ and $r = 0$.

Proof. Set $r_1 = V(1, 0)$, $r_2 = \frac{\sqrt{2\delta}}{\lambda_1^{\frac{1}{4}}(4 + \lambda_1)^{\frac{1}{4}}}$, $r_3 = \frac{3}{2} + \frac{\sqrt{4 + \lambda_1}}{2\sqrt{\lambda_1}}$, and $r_4 = \frac{n_1 - 3}{4}$. Next, using (3.40) and (3.45), and making straightforward simplifications, we get

$$\begin{aligned}
M_1 &= \frac{2\lambda_1^{\frac{1}{4}}}{\sqrt{T} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]}, \\
M_2 &= \frac{\sqrt{2\delta\lambda_1}}{(4 + \lambda_1)^{\frac{1}{4}}\sqrt{T} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]}, \\
M_3 &= \frac{(n_1 - 1)\lambda_1^{\frac{3}{4}}}{4\sqrt{T} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]}, \\
M_4 &= -\frac{\lambda_1^{\frac{3}{4}}}{\sqrt{T} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]} \\
&\quad \times \log \frac{\lambda_1^{\frac{1}{4}}}{\sqrt{2\pi}V(1,0) \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]} \\
&\quad + \frac{\delta\lambda_1^{\frac{1}{4}}}{8\sqrt{T}(4 + \lambda_1)} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{3}{2}} - (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{3}{2}} \right], \\
M_5 &= \frac{\sqrt{2\lambda_1\delta}(n_1 - 1)}{32\sqrt{T}(4 + \lambda_1)^{\frac{1}{4}}} \left[(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{3}{2}} - (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{3}{2}} \right].
\end{aligned}$$

Finally, by taking into account the equalities

$$\begin{aligned} (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} &= \sqrt{2}(\sqrt{4 + \lambda_1} + 2)^{\frac{1}{2}}, \\ (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} - (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} &= \sqrt{2}(\sqrt{4 + \lambda_1} - 2)^{\frac{1}{2}}, \\ (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{3}{2}} - (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{3}{2}} &= 2^{\frac{3}{2}}(\sqrt{4 + \lambda_1} - 2)^{\frac{1}{2}}(\sqrt{4 + \lambda_1} + 1), \end{aligned}$$

we obtain the formulas for the coefficients in Theorem 3.5.

The constant $V(1, 0)$, given by (3.41), depends on all the Karhunen-Loève parameters. However, this constant appears for the first time in the fourth term of the asymptotic expansion in (3.45). By keeping only three terms in (3.46), we obtain an asymptotic formula for the implied volatility, in which the coefficients do not depend on V . However, now we have the error term of the following form: $\left(\log \frac{K}{s_0 e^{rT}}\right)^{-\frac{1}{2}}$.

We will next suppose that the volatility is a centered Gaussian process, and study the wing behavior of the implied volatility in such a case. According to formula (3.42), we can take $r_1 = U(1, 0)$, $r_2 = 0$, $r_3 = \frac{3}{2} + \frac{\sqrt{4 + \lambda_1}}{2\sqrt{\lambda_1}}$, and $r_4 = \frac{n_1 - 2}{2}$. Here $U(1, 0)$ is defined by (3.43). Then, using Theorems 3.3 and 3.4, and reasoning as in the proof of Theorem 3.5, we obtain the following assertion.

Theorem 3.6 *Suppose the volatility is modeled by a centered Gaussian process. Then*

$$I(K) = L_1 \sqrt{\log \frac{K}{s_0 e^{rT}}} + L_2 \frac{\log \log \frac{K}{s_0 e^{rT}}}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} + L_3 \frac{1}{\sqrt{\log \frac{K}{s_0 e^{rT}}}} + O\left(\frac{1}{\log \frac{K}{s_0 e^{rT}}}\right) \quad (3.48)$$

as $K \rightarrow \infty$, where

$$\begin{aligned} L_1 &= \frac{\sqrt{2}}{\sqrt{T}} \left(\frac{\sqrt{\lambda_1}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}}, \quad L_2 = \frac{n_1 - 1}{2\sqrt{2T}} \left(\frac{\lambda_1^{\frac{3}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}}, \\ L_3 &= -\frac{1}{\sqrt{2T}} \left(\frac{\lambda_1^{\frac{3}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}} \log \left[\frac{1}{2\sqrt{\pi}U(1, 0)} \left(\frac{\lambda_1^{\frac{1}{2}}}{\sqrt{4 + \lambda_1} + 2} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Remark 3.1 *Since the processes X and W in (3.30) are independent, the model in (3.30) belongs to the class of the so-called symmetric models (see Section 9.8 in [5]).*

It is known that for a symmetric model,

$$I(K) = I\left(\frac{(s_0 e^{rT})^2}{K}\right) \quad \text{for all } K > 0. \quad (3.49)$$

It is clear that, using (3.49) and Theorem 3.5, we can characterize the left-wing asymptotic behavior of the implied volatility in the case of a noncentered Gaussian volatility. Similarly, (3.49) and Theorem 3.6 can be used in the case of a centered Gaussian volatility.

3.6 Numerical illustration

A basic calibration strategy when presented with asymptotic results such as those given in this chapter is to assume one can place oneself in the corresponding regime, and then determine model parameters by reading asymptotic coefficient off of market option prices. We now illustrate how this strategy can produce positive results, and discuss its limitations, when the top of the KL spectrum is simple ($n_1 = 1$). As noted in the introduction, in this case, the third and fifth terms in the expansion are null. The idea is to ignore the big O term in the asymptotic (3.47), and calibrate parameters to the remaining coefficients. Denoting the discounted log-moneyness $\log(S_0 e^{rT}/K)$ by k for compactness of notation, we thus have, for $|k|$ sufficiently large,

$$I(k) \simeq M_1 \sqrt{|k|} + M_2 + M_4 \frac{1}{\sqrt{|k|}}, \quad (3.50)$$

for three constants M_1 , M_2 , and M_4 , which can, in principle, be read off of market data. By the explicit expressions for the first two constants in (3.47) in terms of λ_1 and δ_1 , we then express the latter in terms of M_1 and M_2 as

$$\begin{aligned} \lambda_1 &= \frac{64T^2 M_1^4}{(4 - T^2 M_1^4)^2}, \\ \delta_1 &= \frac{4\sqrt{2T} M_2 \sqrt{4 + T^2 M_1^4}}{4 - T^2 M_1^4}. \end{aligned} \quad (3.51)$$

Here we use (3.50). One notices that, conveniently, λ_1 can be calibrated using only the coefficient M_1 only, while given M_1 , δ_1 is then proportional to M_2 .

At this stage, one may simply conclude that the extreme strike asymptotics given in the market are consistent with any Gaussian volatility model whose top of eigenstructure is represented by the values computed in the above expressions for λ_1 and

δ_1 . However, practitioners will prefer to determine a more specific model, perhaps by choosing a classical parametric one, and using other non-asymptotic-calibration techniques for estimating some of its parameters. The expressions in (3.51) can then be used to pin down other parameters by calibration, as long as one can relate the model's parameters to the pair (λ_1, δ_1) from the top of its KL spectrum, whether analytically or numerically. The expression for M_4 , given in (3.41) and (3.47), may be too complex to provide a reliable method for calibrating parameters beyond the pair (λ_1, δ_1) , but we will see below that the existence of the corresponding term in the expansion, combined with a truncation of the formula for M_4 , is very helpful for implementing the calibration based on (3.51).

We provide illustrations of this strategy in two cases: the stationary Stein-Stein model, where the KL expansion is known semi-explicitly, and the Stein-Stein model's long-memory version, where the volatility is also known as the fractional Ornstein-Uhlenbeck (fOU) model, and the KL expansion is computed numerically. The data we use is also generated numerically: for each model, we compute option prices and their corresponding implied volatilities, by classical Monte-Carlo, given that the underlying pair of stochastic processes is readily simulated. Specifically, in the Stein-Stein (standard OU) case, 10^6 paths are generated via Euler's method based on discretizing the stochastic differential equation satisfied by X started from a r.v. sampled from X 's stationary distribution, and the explicit expression for $\log S$ given X , also approximated via Euler with the same time steps; 10^3 time steps are used in $[0, T]$ for the various values of T we illustrate below (1, 2, 3 and 6 months, measured in years). Option prices are derived by computing average payoffs over the 10^6 paths. The details are well known, and are omitted. In the fOU case, the exact same methodology is used, except that one must specify the technique used to simulate increments of the fBm process which drives X : we used the circulant method, which is introduced in chapter 2.

Given this simulated data, before embarking on the task of calibrating parameters, to ensure that our methodology is relevant in practice, it is important to discuss

liquidity issues. It is known that the out-of-the-money call options market is poorly liquid, implying that the large strike asymptotics for call and IV prices are typically not visible in the data. We concentrate instead on small strike asymptotics. There, depending on the market segment, options with three-month maturity can be liquid with small bid-ask spread for log moneyness k as far down as -1 or even a bit further. Options with six-month maturity with very small bid-ask spread can be liquid as far down as -1.5 . Convincing visual evidence of this can be found in Figures 3 and 4 in [8] which report 2011 data for SPX options. We will also consider examples with one-month and two-month maturity, where liquidity will be assumed to exist down to $k = -0.8$, based on corresponding evidence in the same figures. We will illustrate calibration using intervals of relatively short length which start on the left side within these observed liquidity ranges. Beyond these lower bounds, liquidity is insufficient to measure IV. In these ranges of k , the constant term M_2 and the expressions $\sqrt{-k}$ and $1/\sqrt{-k}$ are of similar magnitude, which may call into question whether the expansion can be of any use in the range where liquidity exists. However, one may expect that the KL expansion converges fast enough that the three terms $M_1\sqrt{-k}$, M_2 , and $M_4/\sqrt{-k}$ are of different orders because the three constants are. This turns out to be the case in the two example classes we consider, so that our three-term expansion allows us to calibrate λ_1 and δ_1 to L and M as in (3.51). This works very well in practice, as our examples below now show.

We begin with the stationary uncorrelated Stein-Stein model with constant mean-reversion level m , rate of mean reversion q , and so called vol-vol parameter σ . Referring to the notation in Section 2.1.1, since now X is stationary, we have $m_0 = m$ and $\sigma_0^2 = \sigma^2/(2q)$, and the constant K_1 , which is determined from equation (2.7), will play an important role for us. The systems of equations needed to perform calibration here have a somewhat triangular structure. According to Section 2.1.1, if one were to calibrate q , access to δ_1 would be needed, if one were to rely on independent

knowledge of the level of mean reversion m . Specifically, one would solve the following system

$$\begin{aligned} q \sin(wT) + w \cos(wT) &= 0 \\ C_1 \left(\sin(wT) + \frac{q}{w}(1 - \cos(wT)) \right) &= \frac{\delta_1}{m} \end{aligned} \quad (3.52)$$

where $C_1 = K_1 \sigma_0^2$. As noted via (2.7), unfortunately the constant C_1 also depends on (q, w) in the following non-trivial way:

$$\frac{1}{C_1^2} = \frac{q}{2}(1 - \cos(2wT)) + \frac{w^2}{2} \left(T + \frac{\sin(2wT)}{2w} \right) + \frac{q^2}{2} \left(T - \frac{\sin(2wT)}{2w} \right). \quad (3.53)$$

When q is not fixed, the task of determining which value of w represents the minimal solution of the first equation above, given the large number of solutions to the above system, is difficult. We did not pursue this avenue further for this reason. Instead, we will assume that q , which determines the rate of mean reversion, is known, and we will calibrate the pair (m, σ) .

The equations for finding (σ, m) given prior knowledge of q , and given measurement of M_1 and M_2 which imply values of (λ_1, δ_1) via (3.51), are much simpler. Indeed, since q is assumed given, the base frequency w is computed easily as the smallest positive solution of (2.4). Then according to equation (2.9) and part (ii) of Lemma 2.1, with C_1 given by (3.53), we obtain immediately

$$\sigma^2 = \lambda_1 (w^2 + q^2); \quad (3.54)$$

$$m = \frac{\delta_1}{C_1 \left(\sin(wT) + \frac{q}{w}(1 - \cos(wT)) \right)}. \quad (3.55)$$

Any fitting method can in principle be used to estimate the coefficients M_1 , M_2 , and M_4 when working from a data-based IV curve. However, it turns out that, in the ranges of liquidity which we described above, any estimation will contain a certain amount of instability. We now give the details of an iterative technique which increases the stability of the method dramatically by exploiting the fact that M_4 is significantly smaller than M_1 and M_2 .

We use simulated IV data for the call option with $m = 0.2$ (signifying a typical mean level of volatility of 20%), $q = 7$ (fast mean reversion, every eight weeks or so),

and $\sigma = 1.2$ (high level of volatility uncertainty). How to estimate L from the data is not unambiguous. We adopt a least-squares method, on an interval of k -values of fixed length; after experimentation, as a rule of thumb, an interval of length 0.10 or 0.20 provides a good balance between providing a local estimate and drawing on enough datapoints. One should start the interval as far to the left as possible while avoiding any range with insufficient liquidity in practice. As a guide to assess this liquidity, we use the study reported in [8, Section 5.4], which depends heavily on the option maturity, as we mentioned in this section. The following are intervals employed.

Table 3.1.: Interval of data used for calibration with different maturities.

Maturity T	1/12 (1 mo.)	1/12 (1 mo.)	1/6 (2 mos.)	1/6 (2 mos.)
Interval used	$[-0.8, -0.6]$	$[-0.7, -0.6]$	$[-0.8, -0.6]$	$[-0.7, -0.6]$
Maturity T	0.25 (3 mos.)	0.25 (3 mos.)	0.5 (6 mos.)	0.5 (6 mos.)
Interval used	$[-1.1, -0.9]$	$[-1.0, -0.9]$	$[-1.4, -1.2]$	$[-1.3, -1.2]$

Graphs of the data versus the asymptotic curve in (3.50), showing excellent agreement, are given in Fig 3.1, though a case-by-case need for an analysis of the trade-off between this agreement and the liquidity-dictated calibration intervals, is apparent as one considers various possible maturities (note the difference in ranges for log-moneyness k on the horizontal axes).

Our stabilized calibration method starts with a least-squares measurement of M_1 and M_2 based on the asymptotic curve with only the first two terms. The value of M_1 is used to calibrate σ . A guess is expressed for m to initiate the procedure; in our examples we use $m = 0.22$, to signify an educated guess which misses the mark by 10%, as would be reasonable to expect when using a proxy such as the VIX to visually estimate this so-called vol-vol. The next step uses the values of σ and m previously determined, along with the known value q , to compute a large number of terms in

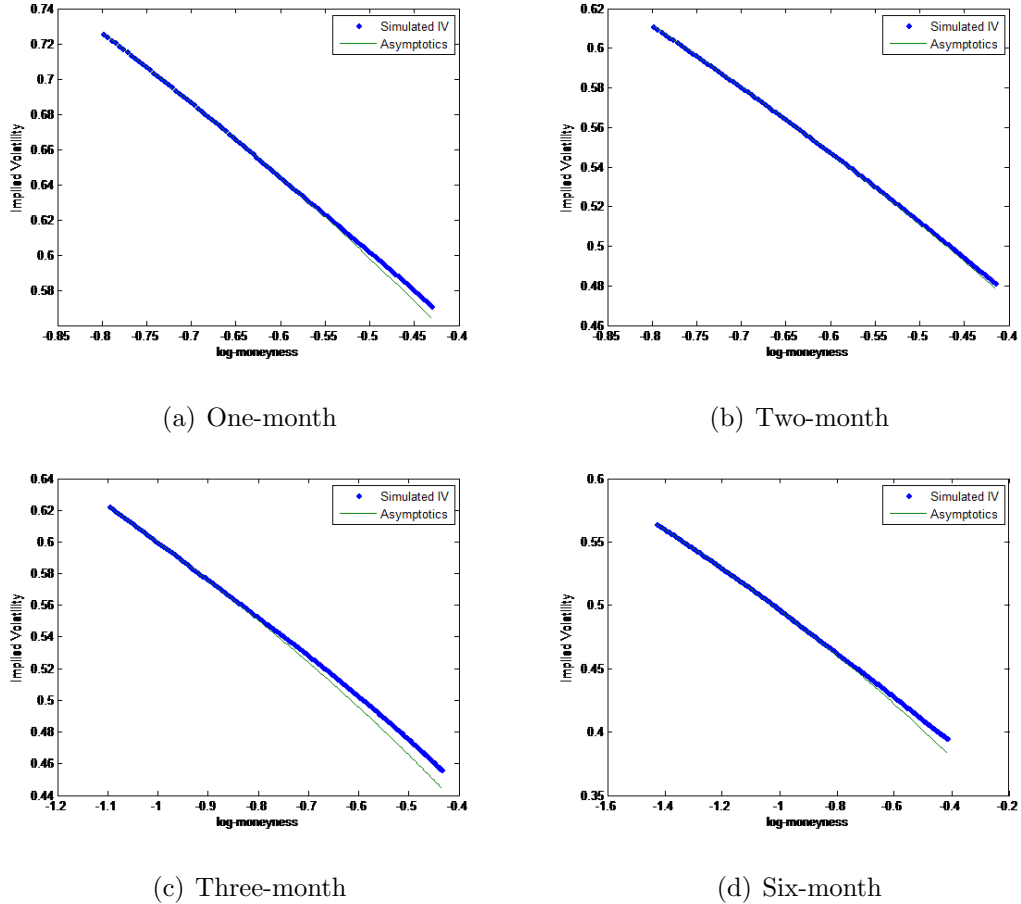


Figure 3.1.: IV for Stein-Stein model with parameters $m = 0.2$, $q = 7$, $\sigma = 1.2$

the KL expansion of the OU process (we use 500 terms), and uses those terms to compute M_4 via the expressions in (3.41) and (3.47). The value of M_4 just obtained is also used to refine the non-linear least-squares estimation of M_1 and M_2 based on the three-term asymptotic function in (3.50) where the term $M_4/\sqrt{|k|}$ is assumed known. The third step then calibrates σ and m based on the new values of M_1 and M_2 , and then recomputes M_4 using the same procedure as in the second step, which allows a new estimation of M_1 and M_2 using the full asymptotics including the just-updated term $M_4/\sqrt{|k|}$. One then repeats the third step iteratively, until one notices a stabilization. In the examples we report, the method either stabilizes on a single set of values for the pair (σ, m) , or loops between two very close sets of values; this

occurs after 6 or 7 steps. We think this needed number of repeats, and the precision obtained in the end, are typical, because they are functions of the small magnitude of M_4 compared to M_1 (order of 2% to 10% for our maturities from one month to six months), this M_4 being considered as a nuisance term whose rough estimation helps sharpen the estimation of the other two constants significantly. Summarizing the procedure, we have:

- (0) Assume q is known. Compute w as smallest frequency solving (3.52).
1. Use two-term asymptotics to estimate M_1 and M_2 , calibrate σ to M_1 via (3.51) and (3.55). Initialize m using a good guess for rate of mean reversion.
2. Use σ and m from step 1 (and q from step 0) to compute a large number (500) of terms in the KL expansion of X . Use truncated theoretical formula in (3.41) and (3.47) to compute K_4 from this expansion. Re-estimate M_1 and M_2 by using full three-term asymptotics (3.50) assuming $M_4/\sqrt{|k|}$ is known.
3. Calibrate σ from the new M_1 and m from the new pair (M_1, M_2) via (3.51), (3.55), and (3.54). Recompute the KL expansion of X based on the new (σ, m) , and recompute K_4 using the new KL expansion in the theoretical formula. Re-estimate M_1 and M_2 by using full three-term asymptotics (3.50) assuming $M_4/\sqrt{|k|}$ is known using the new M_4 .
4. Repeat step 3 iteratively until stabilization of (σ, m) occurs.

We report our findings for the calibration of (σ, m) in our 8 examples of interest in the following tables. The "true values" of M_1 , M_2 , and M_4 in these tables are those which are computed from the Stein-Stein model with $(\sigma, m, q) = (1.2, 0.2, 7)$ via its KL elements; as explained above, only the first-order KL eigen-elements are needed for M_1 , M_2 , while for M_4 , we use the full theoretical formula in which we ignore the eigen-elements after rank 500.

Table 3.2.: $T = 1/12$ (1 mo.) Calibration over the interval $[-0.8, -0.6]$

	M_1	M_2	M_4	σ	m
True values	0.7117	0.0706	0.0188	1.2	0.2
Step 1	0.6875	0.1113		1.1196	0.22
Step 2	0.6875	0.0777	0.0188		
Step 3	0.7145	0.0661	0.0183	1.2096	0.1873
Step 4	0.7138	0.0673	0.0184	1.2072	0.1907
Step 5	0.7140	0.0671	0.0184	1.2077	0.1900

Table 3.3.: $T = 1/12$ (1 mo.) Calibration over the interval $[-0.7, -0.6]$

	M_1	M_2	M_4	σ	m
True values	0.7117	0.0706	0.0188	1.2	0.2
Step 1	0.6859	0.1126		1.1142	0.22
Step 2	0.6859	0.0777	0.0187	1.2102	0.1872
Step 3	0.7147	0.0661	0.0183	1.2102	0.1872
Step 4	0.7141	0.0671	0.0184	1.2081	0.1901
Step 5	0.7143	0.0668	0.0184	1.2086	0.1894

We obtain excellent agreement of the calibration with the true values, with errors lower than 1% after 5 to 8 steps. Other calibrations, not reported here because of their similarity with these examples, show that calibration accuracy would increase with more liquid options since these allow being able to use intervals further to the left, ensuring a better match with the asymptotic regime (3.50). The examples reported above in full correspond to realistic liquidity assumptions.

We now propose a calibration method to estimate the memory parameter in the so-called continuous-time fOU volatility model. This model was introduced in [38] as a way to model long-range dependence in non-linear functionals of stock returns, while

Table 3.4.: $T = 1/6$ (2 mo.) Calibration over the interval $[-0.8, -0.6]$

	M_1	M_2	M_4	σ	m
True values	0.5743	0.0704	0.0245	1.2	0.2
Step 1	0.5370	0.1309		1.0490	0.22
Step 2	0.5370	0.0775	0.0232		
Step 3	0.5705	0.0752	0.0251	1.1839	0.2134
Step 4	0.5732	0.0706	0.0245	1.1953	0.2005
Step 5	0.5723	0.0720	0.0247	1.1917	0.2046
Step 6	0.5726	0.0716	0.0246	1.1929	0.2032
Step 7	0.5725	0.0718	0.0246	1.1923	0.2039

Table 3.5.: $T = 1/6$ (2 mo.) Calibration over the interval $[-0.7, -0.6]$

	M_1	M_2	M_4	σ	m
True values	0.5743	0.0704	0.0245	1.2	0.2
Step 1	0.5354	0.1322		1.0428	0.22
Step 2	0.5354	0.0775	0.0231		
Step 3	0.5711	0.0748	0.0251	1.1866	0.2123
Step 4	0.5742	0.0698	0.0244	1.1995	0.1982
Step 5	0.5731	0.0715	0.0246	1.2011	0.1997
Step 6	0.5734	0.0710	0.0246	1.1962	0.2017

preserving the uncorrelated semi-martingale structure at the level of returns themselves. This is the model for X in (1.2) where the process Z is a fractional Brownian motion, i.e. the continuous Gaussian process started at 0 with covariance determined by $\mathbf{E}[(Z_t - Z_s)^2] = |t - s|^{2H}$, with ‘‘Hurst’’ parameter $H \in (0.5, 1)$. In [39], it was shown empirically that standard statistical methods for long-memory data are inadequate for estimating H . This difficulty can be attributed to the fact that the

Table 3.6.: $T = 1/4$ (3 mo.) Calibration over the interval $[-1.1, -0.9]$

	M_1	M_2	M_4	σ	m
True values	0.5001	0.0702	0.0295	1.2	0.2
Step 1	0.4699	0.1299		1.0591	0.22
Step 2	0.4699	0.0773	0.0279		
Step 3	0.4980	0.0740	0.0300	1.1896	0.2107
Step 4	0.5001	0.0698	0.0294	1.1997	0.1987
Step 5	0.4995	0.0710	0.0295	1.1969	0.2021
Step 6	0.4996	0.0708	0.0295	1.1973	0.2016

Table 3.7.: $T = 1/4$ (3 mo.) Calibration over the interval $[-1.0, -0.9]$

	M_1	M_2	M_4	σ	m
True values	0.5001	0.0702	0.0295	1.2	0.2
Step 1	0.4655	0.1342		1.0396	0.22
Step 2	0.4655	0.0773	0.0275		
Step 3	0.4945	0.0777	0.0305	1.1732	0.2214
Step 4	0.4977	0.0716	0.0295	1.1883	0.2039
Step 5	0.4966	0.0736	0.0298	1.1832	0.2097
Step 6	0.4969	0.0730	0.0297	1.1847	0.2080
Step 7	0.4968	0.0732	0.0297	1.1842	0.2086

volatility process X can have non-stationary increments. In addition, some of the classical methods use path regularity or self-similarity as a proxy for long memory, which cannot be exploited in practice since there is a lower limit to how frequently observations can be made without running into microstructure noise. To make matters worse, the process X is not directly observed; in such a partial observation case, a general theoretical result was given in [40], by which the rate of convergence of any

Table 3.8.: $T = 1/2$ (6 mo.) Calibration over the interval $[-1.4, -1.2]$

	M_1	M_2	M_4	σ	m
True values	0.3838	0.0695	0.0428	1.2	0.2
Step 1	0.3521	0.1432		1.0094	0.22
Step 2	0.3521	0.0765	0.0385		
Step 3	0.3817	0.0757	0.0442	1.1869	0.2178
Step 4	0.3861	0.0657	0.0423	1.2144	0.1890
Step 5	0.3846	0.0690	0.0429	1.2052	0.1986
Step 6	0.3851	0.0679	0.0427	1.2081	0.1956
Step 7	0.3849	0.0683	0.0428	1.2071	0.1966
Step 8	0.3850	0.0681	0.0427	1.2076	0.1961

Table 3.9.: $T = 1/2$ (6 mo.) Calibration over the interval $[-1.3, -1.2]$

	M_1	M_2	M_4	σ	m
True values	0.3838	0.0695	0.0428	1.2	0.2
Step 1	0.3493	0.1464		0.9934	0.22
Step 2	0.3493	0.0765	0.0380		
Step 3	0.3797	0.0783	0.0446	1.1745	0.2255
Step 4	0.3850	0.0665	0.0423	1.2075	0.1915
Step 5	0.3832	0.0706	0.0430	1.1959	0.2034
Step 6	0.3837	0.0694	0.0428	1.1994	0.1998
Step 7	0.3836	0.0697	0.0429	1.1984	0.2008
Step 8	0.3836	0.0696	0.0428	1.1989	0.2003

estimator of H cannot exceed an optimal H -dependent rate which is always slower than $N^{-1/4}$, where N is the number of observations. Given the non-stationarity of the parameter H on a monthly scale, a realistic time series at the highest observation

frequency where microstructure noise can be ignored (e.g. one stock observation every 5 minutes) would not permit even the optimal estimators described in [40] from pinning down a value of H with any acceptable confidence level. The work in [39] proposes a calibration technique based on a straightforward comparison of simulated and market option prices to determine H . Our strategy herein is similar, but based on implied volatility.

Our goal is to calibrate the fOU model described above with the following parameters: $T = 1/4$, $m = 0.2$, $q = 7$, $\sigma = 1.2$, with different values of the Hurst parameter H , namely

$$H \in \{0.51, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85\} \quad (3.56)$$

As mentioned above, our simulated option prices use standard Monte Carlo, where the fOU process is produced by A.T. Dieker's circulant method. Since the values of λ_1 for each $H > 0.5$ are not known explicitly or semi-explicitly, we resorted to the method developed in by S. Corlay in [17] for optimal quantification: there, the infinite-dimensional eigenvalue problem is converted to a matrix eigenvalue problem which uses a low-order quadrature rule for approximating integrals (a trapezoidal rule is recommended), after which a Richardson-Romberg extrapolation is used to improve accuracy. We repeat this procedure for the fOU process with the above parameters, for each value of H from 0.50 to 0.99, with increments of 0.01. The corresponding values we obtain for λ_1 in each case are collected in the following table:

$H =$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
$\lambda_1 =$	0.0157	0.0155	0.0152	0.0150	0.0148	0.0146	0.0144	0.0142	0.0140	0.0138
$H =$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
$\lambda_1 =$	0.0136	0.0134	0.0132	0.0130	0.0128	0.0126	0.0124	0.0122	0.0120	0.0118
$H =$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
$\lambda_1 =$	0.0116	0.0115	0.0113	0.0111	0.0109	0.0108	0.0106	0.0104	0.0103	0.0101
$H =$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
$\lambda_1 =$	0.0100	0.0098	0.0097	0.0095	0.0094	0.0092	0.0091	0.0089	0.0088	0.0087

$H =$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
$\lambda_1 =$	0.0085	0.0084	0.0083	0.0082	0.0080	0.0079	0.0078	0.0077	0.0076	0.0075

Our illustration of the calibration method then consists of starting with simulated IV data for a fOU model with a fixed H from the set in (3.56), then, similarly to what we did for the Stein-Stein model, calibrate the value of λ_1 to the first term of the simulated IV curve over an interval of length 0.1. For our choice of $T = 1/4$ we use $k \in [-1.0, -0.9]$ to determine λ_1 , which is realistic in terms of liquidity constraints. We then match that value of λ_1 to the closest value in the above table, thereby concluding that the simulated data is consistent with the corresponding value of H in the table. Because of the instability in determining M_4 in (3.50) by curve fitting, as noted for the standard Stein-Stein model, rather than using the iterative technique described above, we fit our simulated data curve to the first two terms in this expansion only, resulting in a robust estimate for M_1 in all cases, from which our calibrated λ_1 results via (3.51). This is more efficient since we are only calibrating the single parameter H . The results of this method are summarized here.

Table 3.10.: $T = 1/4$ (3 mos.) Calibration of H via λ_1 over the interval $[-1.0, -0.9]$

True H	0.51	0.55	0.60	0.65	0.70	0.75	0.80	0.85
True λ_1	0.0155	0.0146	0.0136	0.0126	0.0116	0.0108	0.0100	0.00923
calibrated λ_1	0.0152	0.0147	0.0134	0.0127	0.0115	0.0109	0.0101	0.00937
calibrated H	0.52	0.55	0.61	0.64	0.71	0.74	0.79	0.84

Our method shows a good level of accuracy. One notes a bias between the curve $M_1\sqrt{-k} + M_2$ and the simulated IV data, as illustrated in Figures 2a to 2h, which appears to shift downward as H increases. Since we are only calibrating H via λ_1 which is inferred from M_1 , this bias has no influence on the calibration. At the cost of computing M_4 as we did for the Stein-Stein model, which would be more onerous in the fOU case because one would need to push Corlay's method much further

for estimating KL eigenelements, we could obtain the 3-term expansion in (3.50), resulting in curves which would have much less of a bias than in Figures 2a to 2h, but this would not improve the calibration of λ_1 and H .

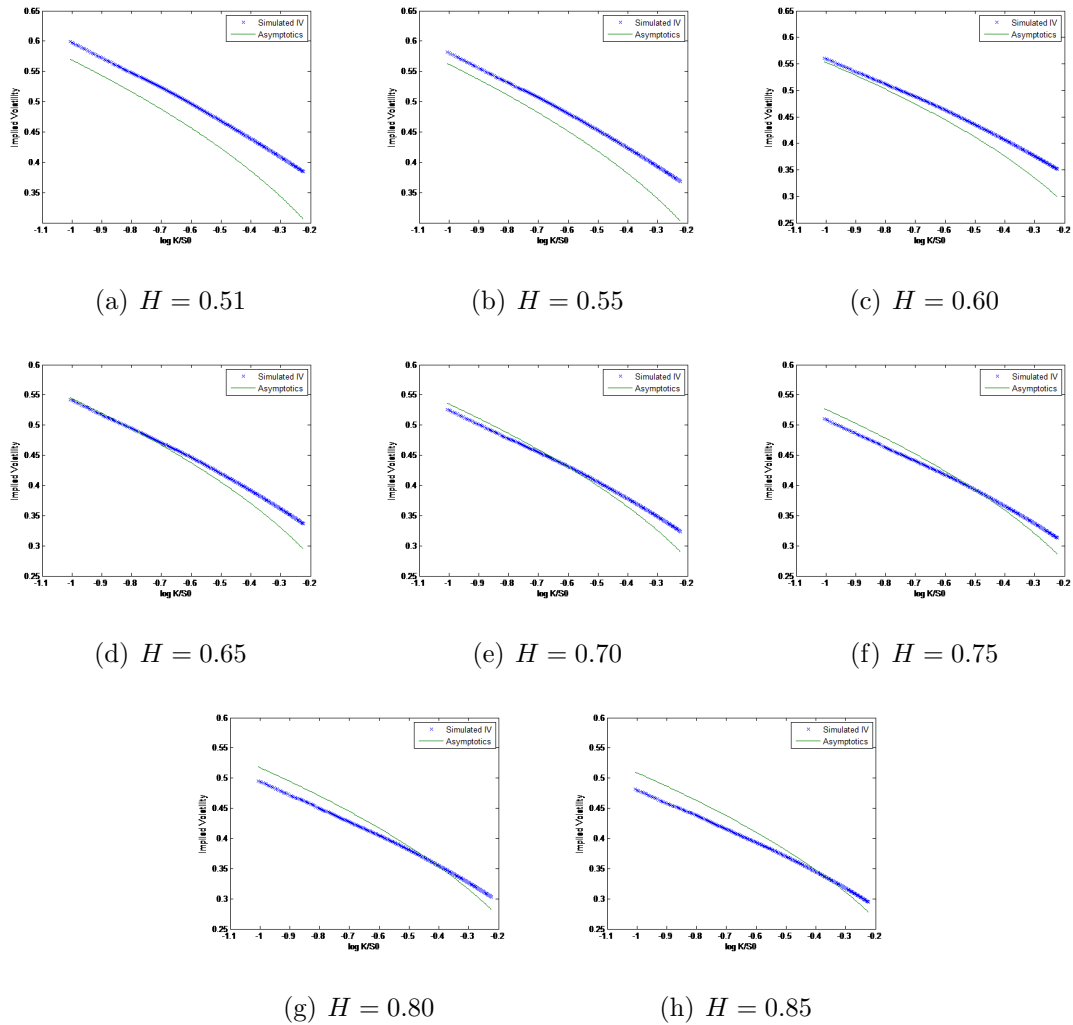


Figure 3.2.: Three-month IV for fOU model with parameters $m = 0.2$, $q = 7$, $\sigma = 1.2$

4. SMALL-TIME ASYMPTOTICS

4.1 Specific motivations and modeling choices

Small-time asymptotic behavior of densities, option pricing functions, and implied volatilities has been a popular topic of study. There are various model-independent results (see, e.g., [10, 41–43]), explaining how the asymptotics of the IV depend on those of option pricing functions. There are also papers discussing small-time asymptotics of the functions mentioned above in the case of stochastic volatility or local-stochastic volatility models (see [42, 44–50]), and for special models (see [51–56] (models with jumps), [57–60] (Heston model), [29, 61] (Stein-Stein model), [50, 62–64] (SABR model)).

The chapter follows up on our prior study in [9] by attempting to elucidate the small-time behavior of IV for a subclass of Gaussian volatility models, consisting of models with self-similar volatility processes. It turns out that establishing small-time asymptotics in a general Gaussian context is significantly more demanding than determining large-strike behavior. This can be understood as a manifestation of the fact that there is no model-free analogue of Lee’s moment formulas in the small or large time regimes. In this paper, we illustrate the challenge by specializing to the case of self-similar volatilities; we will see that the type of small-time behavior for both call price and IV is quite sensitive to the self-similarity parameter H . This is good news if one is to leverage these results to help determine H , as we will see.

Indeed, our study also allows us to investigate the question of long-memory SV calibration, since long-range dependence and self-similarity are proxies for each other in many known models, via their common Hurst parameter H . Based on a Gaussian long-memory model for log-volatility pioneered by Comte and Renault in [38], the work in [39] used an ad-hoc calibration method based on option prices to determine H

so as to best explain market prices. Fractional volatility models also appear in [40, 65–74]. In the current paper, we show that calibration of H near maturity can be given a stronger mathematical foundation under self-similarity assumptions for the volatility process. The parameter H can also be a proxy for local regularity measurements, in the sense of their paths' Hölder continuity parameter. Some recent papers and presentations, yet unpublished at the time of writing this article, appear to show that volatility is rough, in the sense that the log-volatility process is fractional and it is not Hölder continuous for $1/2 - \varepsilon < H < 1/2$, where ε is a positive number (see [67–69]). On the other hand, [39] and many studies before it (see references therein) indicate that $H > 1/2$ in terms of memory length. This is a demonstration that the use of H to measure self-similarity *and* long memory *and* path regularity/roughness, such as in the case of fractional Brownian motion (fBm), might be a misspecification in volatility modeling. The authors of [69] indicate that classical long-memory tests detect this property in their Gaussian rough volatility model, which is a geometric fBm or a geometric OU process with shorter memory ($H < 1/2$). The studies in [39] show on the other hand that no consistent memory estimation results in practice from any classical method when used on the non-self-similar stationary long-memory model of [38]. Our current work could help in elucidating the differences between these points of view; we do not comment on them further herein. An interesting discussion of long memory vs short memory problem can be found in Section 1.2 of [69]. In any case, the numerics which we include in this paper and will discuss at the end of this introduction show that our model class allows for a very sharp calibration tool.

Before providing a summary of our results, we discuss some classical Gaussian self-similar models. General details about this class are given in Section 4.3. These are the Gaussian processes X on $[0, T]$ such that for some $H \in (0, 1)$ and for any $a > 0$, the two processes $t \mapsto X_{at}$ and $t \mapsto a^H X_t$ have the same distribution (law). The best known among them is the fractional Brownian motion (fBm) B^H , the centered Gaussian process whose law is defined by $B^H(0) = 0$ and $\mathbf{E} \left[(B_t^H - B_s^H)^2 \right] = |t - s|^{2H}$. It is the only (continuous) self-similar centered Gaussian process with stationary in-

crements. Many texts can be consulted on B^H , including, e.g., [40, 75, 76]. Among the many other centered Gaussian self-similar models, which are all necessarily non-stationary, the easiest to construct is the Riemann-Liouville fBm, defined as $B_t^{H,RL} = \int_0^t (t-s)^{H-1/2} dW(s)$ where W is a standard Wiener process (see for instance [77]). This process, which is H -self-similar, has properties close to those of fBm, and can be more amenable to calculations. The so-called Bifractional Brownian motion depends on two similarity parameters H and K , has a more complex representation, as the sum of an fBm with parameter HK , and a process with C^∞ paths which is not adapted to a Brownian filtration: see [78], see also [79] and the references therein. This process, which is HK -self-similar, can model the effect of smoothly acquired exogenous information, and is an extension of the so-called sub-fractional Brownian motion (see [80]). Self-similar Gaussian processes can also be obtained as the solutions of stochastic partial differential equations: a class which includes solutions to fractional colored stochastic heat equations is studied in [81], which has the interesting property that its discrete quadratic variation has fluctuations which become non-Gaussian at a threshold of self-similarity which is lower than for fBm, and can be adjusted to be as low as desired. This can be helpful to model volatilities whose local behavior has heavier-tailed fluctuations than what standard fBm can allow, regardless of the volatility's self-similarity. It also allows the modeler to choose regularity and self-similarity properties independently of each other, which offers more flexibility than the models considered in [38, 39, 69]. More examples of Gaussian self-similar process can be found in [15, 80]. Interestingly, many of the Gaussian self-similar models share the same path regularity properties as fBm, because it can be shown that there are positive finite constants c, C for which $c|t-s|^{2H} < \mathbf{E}[|X_t - X_s|^2] < C|t-s|^{2H}$, where the symbol H stands for the self-similarity parameter of the model under consideration.

Finally, it bears noting that self-similarity implies that $X_0 = 0$ and that $Var[X_t]$ is proportional to t^{2H} . This is a strong assumption on X . An uncertainty level on volatility which increases with time is a reasonable conservative forecasting assump-

tion. That the volatility starts at 0 is more restrictive, since, in our IV context, it corresponds to saying that the underlying risky asset's movements tends towards certainty near the derivative's maturity. Such a behavior is characteristic of specific risky asset classes, such as fixed-income securities, e.g. treasury bonds, and the dividend streams in preferred stocks; it is atypical of common stocks. To soften the assumption that $X_0 = 0$, one can add a constant mean to each centered self-similar X . We have investigated this possibility; it appears that this will require additional non-trivial tools not contained herein. Given the length of the current article, we have opted to leave this improvement for another work. One may, however, include a non-zero mean for each X_t which is proportional to t^H ; this is the framework used herein throughout.

4.2 Mathematical background on Gaussian stochastic volatility models

In the present section, we consider the Gaussian stochastic volatility model defined by (3.30). Let us fix the time horizon $T > 0$, and denote by m and K the mean function and the covariance function of the process X given by $m(t) = \mathbb{E}[X_t]$, $t \in [0, T]$ and

$$K(t, s) = \mathbb{E}[(X_t - m(t))(X_s - m(s))], \quad t \in [0, T]^2,$$

respectively. It will be assumed that $K(s, s) > 0$ if $0 \leq s \leq T$.

The following formula is valid for the distribution density D_t of the asset price S_t in the Gaussian model described by (3.30):

$$D_t(x) = \frac{\sqrt{s_0 e^{rt}}}{\sqrt{2\pi t}} x^{-\frac{3}{2}} \int_0^\infty y^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0 e^{rt}}}{2ty^2} + \frac{ty^2}{8} \right] \right\} \tilde{p}_t(y) dy. \quad (4.1)$$

In (4.1), \tilde{p}_t is the distribution density of the random variable

$$\tilde{Y}_t = \left\{ \frac{1}{t} \int_0^t X_s^2 ds \right\}^{\frac{1}{2}}. \quad (4.2)$$

The function \tilde{p}_t is called the mixing density (see [5]). The proof of formula (4.1) can be found in [4, 5].

Applying the Karhunen-Loève theorem to the Gaussian process $\{X_t\}_{t \in [0, T]}$, we obtain

$$\tilde{X}_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n. \quad (4.3)$$

In (4.3), $\{e_n = e_{n,T}\}$ is an orthonormal system of eigenfunctions of the covariance operator

$$\mathcal{K}(f)(t) = \int_{0,T} f(s) K(t, s) ds, \quad f \in L^2[0, T], \quad 0 \leq t \leq T,$$

and $\{\lambda_n = \lambda_n(T)\}$, $n \geq 1$, are the corresponding eigenvalues (counting the multiplicities). The symbols $Z_n = Z_{n,T}$, $n \geq 1$, in (4.3) stand for a system of iid $\mathcal{N}(0, 1)$ random variables. We will always assume that the orthonormal system $\{e_n\}$ is rearranged so that

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n_1} > \lambda_{n_1+1} = \lambda_{n_1+2} = \dots = \lambda_{n_1+n_2} > \dots$$

For the sake of shortness, we introduce the following notation:

$$\begin{aligned} \rho_1 &= \lambda_1, \rho_2 = \lambda_{n_1+1}, \rho_3 = \lambda_{n_1+n_2+1}, \dots, \\ \delta_n &= \delta_n(T) = \int_0^T m(t) e_n(t) dt, \quad n \geq 1, \\ s &= s(T) = \int_0^T m(t)^2 dt, \quad \delta = \delta(T) = \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2. \end{aligned}$$

The mixing density \tilde{p}_T is related to the density p_T of the integrated variance

$$Y_T = \int_0^T X_t^2 dt$$

as follows:

$$\tilde{p}_T(y) = 2T y p_T(Ty^2). \quad (4.4)$$

The next theorem, characterizing the asymptotic behavior of the density p_T , was established in [9].

Theorem 4.1 *If $\delta > 0$, then the following asymptotic formula holds:*

$$\begin{aligned} p_T(x) &= C x^{\frac{n_1-3}{4}} \exp \left\{ \sqrt{\frac{\delta}{\lambda_1}} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \\ &\quad \times \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \end{aligned} \quad (4.5)$$

as $x \rightarrow \infty$, where

$$C = \frac{A}{2\sqrt{2\pi}} \lambda_1^{-\frac{1}{2}} \left(\sum_{n=1}^{n_1} \delta_n^2 \right)^{-\frac{n_1-1}{4}} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \delta_n^2 - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1} \right\}. \quad (4.6)$$

The constant A in (4.6) is given by

$$A = \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}} \exp \left\{ \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{\lambda_1 - \rho_k} \left(\sum_{n=n_1+\dots+n_{k-1}+1}^{n_1+\dots+n_k} \delta_n^2 \right) \right\}.$$

On the other hand, for a centered Gaussian process X , we have

$$p_T(x) = Cx^{\frac{n_1-2}{2}} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \quad (4.7)$$

as $x \rightarrow \infty$, where

$$C = \frac{1}{2^{\frac{n_1}{2}} \Gamma \left(\frac{n_1}{2} \right) \lambda_1^{\frac{n_1}{2}}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}}. \quad (4.8)$$

The next assertion follows from Theorem 4.1.

Corollary 4.1 *The following are true:*

1. If $n_1 = 1$, then

$$p_T(x) = Cx^{-\frac{1}{2}} \exp \left\{ \frac{\delta_1}{\lambda_1} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \times \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \quad (4.9)$$

as $x \rightarrow \infty$, where C is given by (4.6).

2. Suppose X is a centered Gaussian process with $n_1 = 1$. Then

$$p_T(x) = Cx^{-\frac{1}{2}} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left(1 + O \left(x^{-\frac{1}{2}} \right) \right) \quad (4.10)$$

as $x \rightarrow \infty$.

It was established in [9] that Gaussian stochastic volatility models are risk-neutral.

Lemma 4.1 *In the Gaussian stochastic volatility model, the discounted asset price process $t \mapsto e^{-rt} S_t$ is a $\{\mathcal{F}_t\}$ -martingale.*

4.3 Fractional Gaussian stochastic volatility models

Let us consider the following Gaussian stochastic volatility model:

$$dS_t = rS_t dt + |X_t^{(H)}| S_t dW_t, \quad S_0 = s_0, \quad (4.11)$$

where $s_0 > 0$ is the initial condition for the asset price process S , W is a standard Brownian motion, and $X^{(H)}$ is a continuous H -self-similar adapted Gaussian process. The process S characterizes the dynamics of the asset price in the stochastic volatility model, where the volatility is described by the absolute value of a self-similar Gaussian process. It will be assumed throughout the paper that the model in (4.11) is uncorrelated, which means that the processes $X^{(H)}$ and W are independent. We will often suppress the parameter H in various symbols used in the paper. A popular example of a self-similar Gaussian process is fractional Brownian motion $B^{(H)}$ (see, e.g., [75]). Note that fractional Brownian motion is the only process that is non-trivial, self-similar, Gaussian, and has stationary increments.

Exactly as in Section 4.2, we will denote by p_t the density of the integrated variance,

$$Y_t = \int_0^t (X_s^{(H)})^2 ds,$$

and by \tilde{p}_t the density of the random variable

$$\tilde{Y}_t = \left[\frac{1}{t} \int_0^t (X_s^{(H)})^2 ds \right]^{\frac{1}{2}}$$

(the mixing density). Since the process $X^{(H)}$ is self-similar, we have $Y_{at} = a^{2H+1} Y_t$. Moreover, the following equality holds: $\mathbb{P}(Y_t > y) = \mathbb{P}(Y_1 > t^{-2H-1} y)$, and hence,

$$p_t(y) = t^{-2H-1} p_1(t^{-2H-1} y). \quad (4.12)$$

The next assertion characterizes the small-time asymptotics of the mixing density.

Theorem 4.2 (i) For every $x > 0$, the following asymptotic formula holds for the mixing density \tilde{p}_T in the model described by (4.11):

$$\begin{aligned} \tilde{p}_T(x) &= 2CT^{-\frac{H(n_1(1)+1)}{2}} x^{\frac{n_1(1)-1}{2}} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} \frac{x}{T^H} \right\} \exp \left\{ -\frac{x^2}{2T^{2H}\lambda_1(1)} \right\} \\ &\quad \times (1 + O_x(T^H)) \end{aligned} \quad (4.13)$$

as $T \rightarrow 0$, where

$$\begin{aligned} C &= \frac{A}{2\sqrt{2\pi}} \lambda_1(1)^{-\frac{1}{2}} \left(\sum_{n=1}^{n_1(1)} \delta_n(1)^2 \right)^{-\frac{n_1(1)-1}{4}} \\ &\quad \times \exp \left\{ \frac{s(1) - \sum_{n=1}^{\infty} \delta_n(1)^2 - \sum_{n=1}^{n_1} \delta_n(1)^2}{2\lambda_1(1)} \right\}, \end{aligned} \quad (4.14)$$

and the constant A in (4.14) is given by

$$\begin{aligned} A &= \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k(1)}{2}} \\ &\quad \exp \left\{ \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{\lambda_1(1) - \rho_k(1)} \left(\sum_{n=n_1(1)+\dots+n_{k-1}(1)+1}^{n_1(1)+\dots+n_k(1)} \delta_n(1)^2 \right) \right\}. \end{aligned}$$

(ii) If the process $X^{(H)}$ is centered, then

$$\begin{aligned} \tilde{p}_T(x) &= 2CT^{-Hn_1(1)} x^{n_1(1)-1} \exp \left\{ -\frac{x^2}{2t^{2H}\lambda_1(1)} \right\} \\ &\quad (1 + O_x(T^H)) \end{aligned} \quad (4.15)$$

as $T \rightarrow 0$, where

$$C = \frac{1}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1^{-\frac{n_1(1)}{2}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k(1)}{2}}. \quad (4.16)$$

(iii) If the process $X^{(H)}$ is centered and $n_1(1) = 1$, then

$$\tilde{p}_T(x) = 2CT^{-H} \exp \left\{ -\frac{x^2}{2T^{2H}\lambda_1(1)} \right\} (1 + O_x(T^H)) \quad (4.17)$$

as $T \rightarrow 0$, where the constant C is given by (4.16) with $n_1(1) = 1$.

Proof. It follows from (4.4) and (4.12) that

$$\tilde{p}_T(x) = 2T^{-2H} x p_1(T^{-2H} x^2). \quad (4.18)$$

Since $X^{(H)}$ is a Gaussian process, we can use formula (4.5). This gives

$$\begin{aligned} p_1(x) &= C x^{\frac{n_1(1)-3}{4}} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1(1)} \right\} \\ &\quad \times \left(1 + O\left(x^{-\frac{1}{2}}\right) \right) \end{aligned} \quad (4.19)$$

as $x \rightarrow \infty$, where the constant C is given by (4.14). If the process $X^{(H)}$ is centered, then formulas (4.7) and (4.8) imply that

$$p_1(x) = C x^{\frac{n_1(1)-2}{2}} \exp \left\{ -\frac{x}{2\lambda_1(1)} \right\} \left(1 + O\left(x^{-\frac{1}{2}}\right) \right) \quad (4.20)$$

as $x \rightarrow \infty$, where the constant C is given by (4.16). Now, Theorem 4.2 can be derived from (4.18), (4.19), and (4.20).

4.4 Small-time asymptotics of the asset price density in self-similar Gaussian stochastic volatility models with centered volatility.

In this section, we restrict ourselves to the case where the process $X^{(H)}$ is an adapted continuous H -self-similar centered Gaussian process. Recall that we assume $r = 0$.

Of our interest in the present paper are asymptotic estimates of the density $D_T(x)$ as $T \rightarrow 0$, which are uniform with respect to the values of $x > 0$ separated from s_0 (away-from-the-money regime). Here we distinguish among two special cases. In the first case, we fix $\varepsilon > 0$, and consider asymptotic expansions as $t \rightarrow 0$, which are uniform with respect to $x > s_0 + \varepsilon$. The notation $O_\varepsilon(\phi(t, x))$ as $t \rightarrow 0$, where ϕ is a positive function of two variables, means that the O -large estimate holds as $t \rightarrow 0$ uniformly with respect to $x > s_0 + \varepsilon$. In the second case, we fix ε with $0 < \varepsilon < s_0$, and assume that $0 < x < s_0 - \varepsilon$. The same notation $O_\varepsilon(\phi(t, x))$ will be used in the second case.

Since $\tilde{p}_T(y) = T^{-H}\tilde{p}_1(T^{-H}y)$, formula (4.1) implies that

$$\begin{aligned}
D_T(x) &= \frac{\sqrt{s_0}}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} x^{-\frac{3}{2}} \\
&\quad \times \int_0^\infty y^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2Ty^2} + \frac{Ty^2}{8} \right] \right\} \tilde{p}_1(T^{-H}y) dy \\
&= \frac{\sqrt{s_0}}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} x^{-\frac{3}{2}} \\
&\quad \times \int_0^\infty u^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} \tilde{p}_1(u) du. \tag{4.21}
\end{aligned}$$

The next assertion is one of the main results of the present paper. It characterizes the small-time asymptotic behavior of the asset price density in a Gaussian model with a centered self-similar volatility process.

Theorem 4.3 *Fix $\varepsilon > 0$ and let $x > s_0 + \varepsilon$. Then as $T \rightarrow 0$, the following asymptotic formula holds for the asset price density D_T in the model described by (4.11):*

$$\begin{aligned}
D_T(x) &= \frac{\sqrt{s_0}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{4}} \prod_{k=2}^\infty \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k}{2}} x^{-\frac{3}{2}} \\
&\quad \times \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} T^{-\frac{(2H+1)n_1(1)}{4}} \left(\frac{x}{s_0} \right)^{-\frac{\sqrt{4+\lambda_1(1)T^{2H+1}}}{2\sqrt{\lambda_1(1)T^{H+\frac{1}{2}}}}} \\
&\quad \times \left(1 + O(T^{2H+1}) \right) \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left(\log \frac{x}{s_0} \right)^{-\frac{1}{2}} \right) \right). \tag{4.22}
\end{aligned}$$

Proof. Fix $x > 0$, and denote

$$J_x(T) = \int_0^\infty u^{-1} \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} \tilde{p}_1(u) du \tag{4.23}$$

It is clear from (4.21) that the small-time asymptotic behavior of the density $D_T(x)$ is determined by that of the integral $J_x(T)$.

The next lemma will allow us to use Theorem 4.2 to estimate the integral in (4.23).

Lemma 4.2 Fix $\alpha \in \mathbb{R}$, $b > 0$, and $\varepsilon > 0$. Let $x > s_0 + \varepsilon$, and suppose f is an integrable function on $[0, b]$. Then

$$\begin{aligned} & \int_0^b u^\alpha \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)| du \\ &= O_\varepsilon \left(\exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2b^2T^{2H+1}} \right\} \right) \end{aligned}$$

as $t \rightarrow 0$.

Proof. The lemma is trivial if $\alpha \geq 0$. For $\alpha < 0$, we have

$$\begin{aligned} & \int_0^b u^\alpha \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)| du \\ & \leq \int_0^b u^\alpha \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} \right\} |f(u)| du. \end{aligned} \tag{4.24}$$

The following equality holds for every $A > 0$:

$$\left(u^\alpha \exp \left\{ - \frac{A}{u^2} \right\} \right)' = [2Au^{\alpha-3} + \alpha u^{\alpha-1}] \exp \left\{ - \frac{A}{u^2} \right\}.$$

It follows that for $2A > -\alpha b^2$, the function

$$u \mapsto \frac{1}{u^\alpha} \exp \left\{ - \frac{A}{u^2} \right\}$$

is increasing on the interval $(0, b]$. Set

$$A = \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}}.$$

Using (4.24), we obtain

$$\begin{aligned} & \int_0^b u^\alpha \exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \frac{T^{2H+1}u^2}{8} \right] \right\} |f(u)| du \\ & \leq b^\alpha \exp \left\{ - \frac{\log^2 \frac{x}{s_0}}{2b^2T^{2H+1}} \right\} \int_0^b |f(u)| du, \end{aligned} \tag{4.25}$$

provided that $\log^2 \frac{x}{s_0} > b^2T^{2H+1}$. It is clear that the previous inequality holds for small enough values of T provided that $x > s_0 + \varepsilon$.

Finally, Lemma 4.2 follows from (4.25).

Using (4.18) and (4.20), we obtain

$$\tilde{p}_1(y) = \tilde{A} y^{n_1(1)-1} \exp \left\{ -\frac{y^2}{2\lambda_1(1)} \right\} (1 + O(y^{-1})) \quad (4.26)$$

as $y \rightarrow \infty$, where

$$\tilde{A} = \frac{2^{1-\frac{n_1(1)}{2}}}{\Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1^{-\frac{n_1(1)}{2}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k}{2}}. \quad (4.27)$$

It is not hard to see that Lemma 4.2 allows us to replace the function $\tilde{p}_1(u)$ in (4.23) by its approximation from (4.26). This gives the following:

$$\begin{aligned} J_x(T) &= \tilde{A} \int_0^{\infty} u^{n_1(1)-2} \\ &\exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \left(\frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)} \right) u^2 \right] \right\} (1 + O(u^{-1})) du \\ &+ O_{\varepsilon} \left(\exp \left\{ -\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}} \right\} \right) \end{aligned} \quad (4.28)$$

as $T \rightarrow 0$.

To study the asymptotics of the function $t \mapsto J_x(T)$ defined by (4.28), we consider the following two integrals:

$$\begin{aligned} \tilde{J}_x(T) &= \tilde{A} \int_0^{\infty} u^{n_1(1)-2} \\ &\exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \left(\frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)} \right) u^2 \right] \right\} du \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \hat{J}_x(T) &= \tilde{A} \int_0^{\infty} u^{n_1(1)-3} \\ &\exp \left\{ - \left[\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2} + \left(\frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)} \right) u^2 \right] \right\} du. \end{aligned} \quad (4.30)$$

Set

$$\beta_T = \frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}}, \quad \gamma_T = \frac{T^{2H+1}}{8} + \frac{1}{2\lambda_1(1)}.$$

Note that β_T depends on x , while γ_T does not. Then we have

$$\tilde{J}_x(T) = \tilde{A} \int_0^\infty u^{n_1(1)-2} \exp \left\{ - \left[\frac{\beta_T}{u^2} + \gamma_T u^2 \right] \right\} du$$

and

$$\hat{J}_x(T) = \tilde{A} \int_0^\infty u^{n_1(1)-3} \exp \left\{ - \left[\frac{\beta_T}{u^2} + \gamma_T u^2 \right] \right\} du.$$

Next, making a substitution

$$u = \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{1}{4}} v,$$

we transform the previous integrals as follows:

$$\tilde{J}_x(T) = \tilde{A} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-1}{4}} \int_0^\infty v^{n_1(1)-2} \exp \left\{ - \sqrt{\beta_T \gamma_T} \left[\frac{1}{v^2} + v^2 \right] \right\} dv$$

and

$$\hat{J}_x(T) = \tilde{A} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-2}{4}} \int_0^\infty v^{n_1(1)-3} \exp \left\{ - \sqrt{\beta_T \gamma_T} \left[\frac{1}{v^2} + v^2 \right] \right\} dv.$$

Let us denote

$$z(T) = \frac{1}{4} \sqrt{\frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}}}. \quad (4.31)$$

Then we have

$$\sqrt{\beta_T \gamma_T} = z(T) \left| \log \frac{x}{s_0} \right|. \quad (4.32)$$

Therefore,

$$\begin{aligned} \tilde{J}_x(T) &= \tilde{A} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-1}{4}} \\ &\quad \times \int_0^\infty v^{n_1(1)-2} \exp \left\{ -z(T) \left| \log \frac{x}{s_0} \right| \left[\frac{1}{v^2} + v^2 \right] \right\} dv \end{aligned} \quad (4.33)$$

and

$$\begin{aligned} \hat{J}_x(T) &= \tilde{A} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-2}{4}} \\ &\quad \times \int_0^\infty v^{n_1(1)-3} \exp \left\{ -z(T) \left| \log \frac{x}{s_0} \right| \left[\frac{1}{v^2} + v^2 \right] \right\} dv. \end{aligned} \quad (4.34)$$

It follows from (4.31) that $z(T) \rightarrow \infty$ as $T \rightarrow 0$. Our next goal is to apply Laplace's method to study the asymptotic behavior of the functions $T \mapsto \tilde{J}_x(T)$

and $T \mapsto \widehat{J}_x(T)$ as $T \rightarrow 0$. Note that the unique critical point of the function $\psi(v) = v^{-2} + v^2$ is at $v = 1$. Moreover, we have $\psi''(1) = 8 > 0$.

We will first reduce the integrals in (4.33) and (4.34) to the integrals over the interval $[0, 2]$ and give an error estimate. This next assertion will be helpful.

Lemma 4.3 *Suppose $a \in \mathbb{R}$ and $0 < \varepsilon < s_0$. Then*

$$\int_2^\infty v^a \exp \left\{ -\sqrt{\beta_T \gamma_T} \left[\frac{1}{v^2} + v^2 \right] \right\} dv = O_\varepsilon \left(\exp \left\{ -3\sqrt{\beta_T \gamma_T} \right\} \right)$$

as $t \rightarrow 0$.

Proof. Fix a small number $r > 0$. Then for $0 < T < T_0$, we have

$$\begin{aligned} & \int_2^\infty v^a \exp \left\{ -\sqrt{\beta_T \gamma_T} \left[\frac{1}{v^2} + v^2 \right] \right\} dv \leq \int_2^\infty v^a \exp \left\{ -\sqrt{\beta_T \gamma_T} v^2 \right\} dv \\ & \leq c_r \int_2^\infty \exp \left\{ -\left(\sqrt{\beta_T \gamma_T} - r \right) v^2 \right\} dv \\ & = c_r \left(\sqrt{\beta_T \gamma_T} - r \right)^{-\frac{1}{2}} \int_{2\sqrt{\beta_T \gamma_T - r}}^\infty e^{-u^2} du \leq \tilde{c}_r \exp \left\{ -4 \left(\sqrt{\beta_T \gamma_T} - r \right) \right\}. \end{aligned}$$

The proof of Lemma 4.3 is thus completed.

Now, we are ready to apply Laplace's method to the integrals in (4.33) and (4.34). The dependence of the parameter x in (4.33) and (4.34) is very simple. This allows us to obtain uniform error estimates. By taking into account Lemma 4.3, we see that for every $\varepsilon > 0$ and all $x > s_0 + \varepsilon$,

$$\begin{aligned} \tilde{J}_x(T) &= \frac{\tilde{A}\sqrt{\pi}}{2} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-1}{4}} \left(z(T) \left| \log \frac{x}{s_0} \right| \right)^{-\frac{1}{2}} \exp \left\{ -2z(T) \left| \log \frac{x}{s_0} \right| \right\} \\ & \left(1 + O_\varepsilon \left(\frac{1}{z(T) \left| \log \frac{x}{s_0} \right|} \right) \right) + O_\varepsilon \left(\exp \left\{ -3z(T) \left| \log \frac{x}{s_0} \right| \right\} \right) \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} \widehat{J}_x(T) &= \frac{\tilde{A}\sqrt{\pi}}{2} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-2}{4}} \left(z(T) \left| \log \frac{x}{s_0} \right| \right)^{-\frac{1}{2}} \exp \left\{ -2z(T) \left| \log \frac{x}{s_0} \right| \right\} \\ & \left(1 + O_\varepsilon \left(\frac{1}{z(T) \left| \log \frac{x}{s_0} \right|} \right) \right) + O_\varepsilon \left(\exp \left\{ -3z(T) \left| \log \frac{x}{s_0} \right| \right\} \right) \end{aligned} \quad (4.36)$$

as $T \rightarrow 0$. Recall that the O_ε estimates in (4.35) and (4.36) are uniform with respect to $x > s_0 + \varepsilon$. Since

$$J_x(T) = \tilde{J}_x(T) + O_\varepsilon\left(\widehat{J}_x(T)\right) + O_\varepsilon\left(\exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}}\right\}\right),$$

as $T \rightarrow 0$, formulas (4.35) and (4.36) imply that

$$\begin{aligned} J_x(T) &= \frac{\tilde{A}\sqrt{\pi}}{2} \left(\frac{\beta_T}{\gamma_T}\right)^{\frac{n_1(1)-1}{4}} \left(z(T) \left|\log \frac{x}{s_0}\right|\right)^{-\frac{1}{2}} \exp\left\{-2z(T) \left|\log \frac{x}{s_0}\right|\right\} \\ &\quad \left(1 + \left(\frac{\beta_T}{\gamma_T}\right)^{-\frac{1}{4}}\right) \left(1 + O_\varepsilon\left(\frac{1}{z(T) \left|\log \frac{x}{s_0}\right|}\right)\right) \\ &\quad + O_\varepsilon\left(\exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}}\right\}\right) + O_\varepsilon\left(\exp\left\{-3z(T) \left|\log \frac{x}{s_0}\right|\right\}\right) \end{aligned}$$

as $T \rightarrow 0$. Since for $T < 1$,

$$\frac{1}{4} \sqrt{\frac{\lambda_1(1) + 4}{\lambda_1(1)}} T^{-H-\frac{1}{2}} > z(T) > \frac{1}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}}, \quad (4.37)$$

we have

$$\begin{aligned} &O_\varepsilon\left(\exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}}\right\}\right) + O_\varepsilon\left(\exp\left\{-3z(T) \left|\log \frac{x}{s_0}\right|\right\}\right) \\ &= O_\varepsilon\left(\exp\left\{-3z(T) \left|\log \frac{x}{s_0}\right|\right\}\right) \\ &= O_\varepsilon\left(\exp\left\{-\frac{3}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \left|\log \frac{x}{s_0}\right|\right\}\right) \end{aligned}$$

as $T \rightarrow 0$, and therefore,

$$\begin{aligned} J_x(T) &= \frac{\tilde{A}\sqrt{\pi}}{2} \left(\frac{\beta_T}{\gamma_T}\right)^{\frac{n_1(1)-1}{4}} \left(z(T) \left|\log \frac{x}{s_0}\right|\right)^{-\frac{1}{2}} \exp\left\{-2z(T) \left|\log \frac{x}{s_0}\right|\right\} \\ &\quad \left(1 + \left(\frac{\beta_T}{\gamma_T}\right)^{-\frac{1}{4}}\right) \left(1 + O_\varepsilon\left(\frac{1}{z(T) \left|\log \frac{x}{s_0}\right|}\right)\right) \\ &\quad + O_\varepsilon\left(\exp\left\{-\frac{3}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \left|\log \frac{x}{s_0}\right|\right\}\right) \end{aligned}$$

as $T \rightarrow 0$. Moreover, for all $T < 1$ and $x > s_0 + \varepsilon$,

$$\left(\frac{\beta_T}{\gamma_T}\right)^{-\frac{1}{4}} \geq c_1 \frac{T^{\frac{2H+1}{4}}}{\sqrt{\left|\log \frac{x}{s_0}\right|}} \geq c_2 \frac{T^{\frac{2H+1}{2}}}{\left|\log \frac{x}{s_0}\right|} \geq c_3 \frac{1}{z(T) \left|\log \frac{x}{s_0}\right|},$$

and hence

$$\begin{aligned} & \left(1 + \left(\frac{\beta_T}{\gamma_T} \right)^{-\frac{1}{4}} \right) \left(1 + O_\varepsilon \left(\frac{1}{z(T) \left| \log \frac{x}{s_0} \right|} \right) \right) \\ &= \left(1 + O_\varepsilon \left(\left(\frac{\beta_T}{\gamma_T} \right)^{-\frac{1}{4}} \right) \right) = \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left| \log \frac{x}{s_0} \right|^{-\frac{1}{2}} \right) \right) \end{aligned}$$

as $T \rightarrow 0$. Finally,

$$\begin{aligned} J_x(T) &= \frac{\tilde{A}\sqrt{\pi}}{2} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-1}{4}} \left(z(T) \left| \log \frac{x}{s_0} \right| \right)^{-\frac{1}{2}} \exp \left\{ -2z(T) \left| \log \frac{x}{s_0} \right| \right\} \\ & \quad \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left| \log \frac{x}{s_0} \right|^{-\frac{1}{2}} \right) \right) \\ & \quad + O_\varepsilon \left(\exp \left\{ -\frac{3}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \left| \log \frac{x}{s_0} \right| \right\} \right) \end{aligned}$$

as $T \rightarrow 0$.

Recall that we assumed $r = 0$. It follows from (4.21) and (4.23) that

$$\begin{aligned} D_T(x) &= \frac{\sqrt{s_0}\tilde{A}}{2\sqrt{2}} T^{-H-\frac{1}{2}} x^{-\frac{3}{2}} \left(\frac{\beta_T}{\gamma_T} \right)^{\frac{n_1(1)-1}{4}} \left(z(T) \left| \log \frac{x}{s_0} \right| \right)^{-\frac{1}{2}} \\ & \quad \times \exp \left\{ -2z(T) \left| \log \frac{x}{s_0} \right| \right\} \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left| \log \frac{x}{s_0} \right|^{-\frac{1}{2}} \right) \right) \\ & \quad + O_\varepsilon \left(\exp \left\{ -\frac{3}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \left| \log \frac{x}{s_0} \right| \right\} \right) \end{aligned} \quad (4.38)$$

as $T \rightarrow 0$.

Our next goal is to remove the last O_ε -term from formula (4.38). Analyzing the expressions in (4.38), we see that in order to prove the statement formulated above, it suffices to show that there exists a constant $c > 0$ independent of $T < T_0$ and $x > s_0 + \varepsilon$ and such that

$$\begin{aligned} \left(\frac{x}{s_0} \right)^{-\frac{3}{2} \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}}} &\leq c T^{-H-\frac{1}{2}} x^{-\frac{3}{2}} \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-1}{4}} T^{-\frac{(2H+1)(n_1(1)-1)}{8}} T^{\frac{2H+1}{4}} \\ &\quad \times \left(\log \frac{x}{s_0} \right)^{-\frac{1}{2}} \left(\frac{x}{s_0} \right)^{-2z(T)} T^{\frac{2H+1}{4}} \left(\log \frac{x}{s_0} \right)^{-\frac{1}{2}}. \end{aligned} \quad (4.39)$$

The previous inequality is equivalent to the following:

$$\begin{aligned} \left(\frac{x}{s_0}\right)^{-\frac{3}{2}\lambda_1(1)^{-\frac{1}{2}}T^{-H-\frac{1}{2}}} &\leq cT^{-\frac{(2H+1)(n_1(1)-1)}{8}}x^{-\frac{3}{2}}\left(\frac{x}{s_0}\right)^{-2z(T)} \\ &\quad \times \left(\log\frac{x}{s_0}\right)^{\frac{n_1(1)-1}{4}-1} \end{aligned} \quad (4.40)$$

Since (4.37) holds, the inequality in (4.40) follows from the inequality

$$\begin{aligned} \left(\frac{x}{s_0}\right)^{-\frac{3}{2}\lambda_1(1)^{-\frac{1}{2}}T^{-H-\frac{1}{2}}} &\leq cT^{-\frac{(2H+1)(n_1(1)-1)}{8}} \\ &\quad \times \left(\frac{x}{s_0}\right)^{-\frac{3}{2}-\frac{1}{2}\lambda_1(1)^{-\frac{1}{2}}T^{-H-\frac{1}{2}}\sqrt{\lambda_1(1)T^{2H+1}+4}}\left(\log\frac{x}{s_0}\right)^{\frac{n_1(1)-1}{4}-1}. \end{aligned} \quad (4.41)$$

To prove the inequality in (4.41), we observe that for every small enough $\tau > 0$ there exists a constant $c_{\tau,\varepsilon}$ such that

$$c_{\tau,\varepsilon}\left(\frac{x}{s_0}\right)^{-\tau} \leq \left(\log\frac{x}{s_0}\right)^{\frac{n_1(1)-1}{4}-1}$$

for all $x > s_0 + \varepsilon$. Moreover, there exists $T_{\tau,\varepsilon} > 0$ such that

$$\left(\frac{x_0}{s_0}\right)^{-\tau T^{-H-\frac{1}{2}}} \leq \left(\frac{s_0 + \varepsilon}{s_0}\right)^{-\tau T^{-H-\frac{1}{2}}} \leq T^{-\frac{(2H+1)(n_1(1)-1)}{8}}$$

for all $T < T_{\tau,\varepsilon}$. Now, it is clear that (4.41) follows from the estimate

$$\begin{aligned} &\left(\frac{3}{2}\lambda_1(1)^{-\frac{1}{2}} - \tau\right)T^{-H-\frac{1}{2}} \\ &\geq \frac{3}{2} + \frac{1}{2}\lambda_1(1)^{-\frac{1}{2}}T^{-H-\frac{1}{2}}\sqrt{\lambda_1(1)T^{2H+1}+4} + \tau, \end{aligned} \quad (4.42)$$

for all $T < T_\tau$. It is not hard to see that there exist numbers τ and T_τ , for which the inequality in (4.42) holds. This establishes (4.39), and it follows that

$$\begin{aligned} D_T(x) &= \frac{\sqrt{s_0}\tilde{A}}{2\sqrt{2}}T^{-H-\frac{1}{2}}x^{-\frac{3}{2}}\left(\frac{\beta_T}{\gamma_T}\right)^{\frac{n_1(1)-1}{4}}\left(z(T)\log\frac{x}{s_0}\right)^{-\frac{1}{2}} \\ &\quad \times \exp\left\{-2z(T)\log\frac{x}{s_0}\right\}\left(1 + O_\varepsilon\left(T^{\frac{2H+1}{4}}\left(\log\frac{x}{s_0}\right)^{-\frac{1}{2}}\right)\right) \end{aligned} \quad (4.43)$$

as $T \rightarrow 0$, where \tilde{A} is given by (4.27). Formula (4.43) will help us to characterize the asymptotic behavior of the function $T \mapsto D_T(x)$.

Let us assume that $x > s_0 + \varepsilon$. Then we have

$$\left(\frac{\beta_T}{\gamma_T}\right)^{\frac{n_1(1)-1}{4}} = \lambda_1(1)^{\frac{n_1(1)-1}{4}} \left(\log \frac{x}{s_0}\right)^{\frac{n_1(1)-1}{2}} T^{-\frac{(2H+1)(n_1(1)-1)}{4}} (1+h)^{-\frac{n_1(1)-1}{4}}$$

where $h = \frac{\lambda_1(1)T^{2H+1}}{4}$. Therefore,

$$\begin{aligned} \left(\frac{\beta_T}{\gamma_T}\right)^{\frac{n_1(1)-1}{4}} &= \lambda_1(1)^{\frac{n_1(1)-1}{4}} \left(\log \frac{x}{s_0}\right)^{\frac{n_1(1)-1}{2}} T^{-\frac{(2H+1)(n_1(1)-1)}{4}} \\ &\quad (1 + O(T^{2H+1})) \end{aligned} \quad (4.44)$$

as $T \rightarrow 0$. Moreover,

$$\begin{aligned} z(T)^{-\frac{1}{2}} &= 2 \left[\frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}} \right]^{-\frac{1}{4}} \\ &= \sqrt{2} \lambda_1(1)^{\frac{1}{4}} T^{\frac{2H+1}{4}} (1 + O(T^{2H+1})) \end{aligned} \quad (4.45)$$

and

$$\exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} = \left(\frac{x}{s_0} \right)^{-\frac{\sqrt{4+\lambda_1(1)T^{2H+1}}}{2\sqrt{\lambda_1(1)}T^{H+\frac{1}{2}}}} \quad (4.46)$$

as $T \rightarrow 0$. Next, combining (4.27), (4.43), (4.44), (4.45), and (4.46), and simplifying the resulting expressions, we obtain formula (4.22).

This completes the proof of Theorem 4.3.

4.5 Asymptotic behavior of out-of-the-money call and put pricing functions

Let S be the asset price process in the model considered in (4.11). Define the call and the put pricing functions by

$$C(T, K) = \mathbb{E}[S_T - K]^+ \quad \text{and} \quad P(T, K) = \mathbb{E}[K - S_T]^+$$

where T is the maturity and K is the strike price. Recall that for a Gaussian stochastic volatility model with $r = 0$, the asset price process S is a martingale (see Lemma 4.1). Therefore, the put/call parity formula $C(T, K) = P(T, K) + s_0 - K$ holds.

In the present section, we consider the functions C and P as functions of the maturity for a fixed strike price, and we suppress the strike price in the symbols. Our goal is to characterize the asymptotic behavior as $T \rightarrow 0$ of the function $T \mapsto C(T)$ for $K > s_0$ (out-of-the money call) and of the function $T \mapsto P(T)$ for $0 < K < s_0$ (out-of-the-money put).

We will first consider the call pricing function $T \mapsto C(T)$ with $K > s_0$. It is known that

$$C(T) = \int_K^\infty (x - K) D_T(x) dx. \quad (4.47)$$

Therefore, we can use the uniform estimate in formula (4.22) to characterize the small-time behavior of the call pricing function. Let us consider the following integrals:

$$\begin{aligned} I_1(T) &= \int_K^\infty (x - K) x^{-\frac{3}{2}} \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} dx \\ &= s_0^{-\frac{1}{2}} \int_K^\infty \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp \left\{ - \left(\frac{1}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx \\ &\quad - s_0^{-\frac{3}{2}} K \int_K^\infty \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-2}{2}} \exp \left\{ - \left(\frac{3}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx \end{aligned} \quad (4.48)$$

and

$$\begin{aligned} I_2(T) &= \int_K^\infty (x - K) x^{-\frac{3}{2}} \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp \left\{ -2z(T) \log \frac{x}{s_0} \right\} dx \\ &= s_0^{-\frac{1}{2}} \int_K^\infty \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp \left\{ - \left(\frac{1}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx \\ &\quad - s_0^{-\frac{3}{2}} K \int_K^\infty \left(\log \frac{x}{s_0} \right)^{\frac{n_1(1)-3}{2}} \exp \left\{ - \left(\frac{3}{2} + 2z(T) \right) \log \frac{x}{s_0} \right\} dx, \end{aligned} \quad (4.49)$$

where we use the notation in (4.31) for the sake of shortness.

We will next make a substitution $u = (2z(T) - \frac{1}{2}) \log \frac{x}{s_0}$ in the integral on the second line in (4.48). The resulting expression is as follows:

$$s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2} \right)^{-\frac{n_1(1)}{2}} \int_{(2z(T)-\frac{1}{2}) \log \frac{K}{s_0}}^\infty u^{\frac{n_1(1)-2}{2}} e^{-u} du,$$

which is equal to

$$s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2} \right)^{-\frac{n_1(1)}{2}} \Gamma \left(\frac{n_1(1)}{2}, \left(2z(T) - \frac{1}{2} \right) \log \frac{K}{s_0} \right),$$

where the symbol Γ stands for the upper incomplete gamma function defined by

$$\Gamma(s, x) = \int_x^\infty v^{s-1} e^{-v} dv.$$

Making similar transformations in the other integrals in (4.48) and (4.49), we finally obtain

$$\begin{aligned} I_1(T) &= s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}, \left(2z(T) - \frac{1}{2}\right) \log \frac{K}{s_0}\right) \\ &\quad - s_0^{-\frac{1}{2}} K \left(2z(T) + \frac{1}{2}\right)^{-\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}, \left(2z(T) + \frac{1}{2}\right) \log \frac{K}{s_0}\right) \end{aligned}$$

and

$$\begin{aligned} I_2(T) &= s_0^{\frac{1}{2}} \left(2z(T) - \frac{1}{2}\right)^{-\frac{n_1(1)-1}{2}} \Gamma\left(\frac{n_1(1)-1}{2}, \left(2z(T) - \frac{1}{2}\right) \log \frac{K}{s_0}\right) \\ &\quad - s_0^{-\frac{1}{2}} K \left(2z(T) + \frac{1}{2}\right)^{-\frac{n_1(1)-1}{2}} \Gamma\left(\frac{n_1(1)-1}{2}, \left(2z(T) + \frac{1}{2}\right) \log \frac{K}{s_0}\right). \end{aligned}$$

It is known that

$$\Gamma(s, x) = x^{s-1} e^{-x} \left(1 + (s-1)x^{-1} + O(x^{-2})\right) \quad (4.50)$$

as $x \rightarrow \infty$. Formula (4.50) can be easily derived from the recurrence relation

$$\Gamma(s, x) = (s-1)\Gamma(s-1, x) + x^{s-1} e^{-x}$$

for the upper incomplete gamma function. It follows that

$$\begin{aligned} I_1(T) &= s_0^{2z(T)} K^{-2z(T)+\frac{1}{2}} \left(\log \frac{K}{s_0}\right)^{\frac{n_1(1)-2}{2}} \\ &\quad \left[\frac{1}{2z(T) - \frac{1}{2}} \left(1 + \frac{n_1(1)-2}{2(2z(T) - \frac{1}{2}) \log \frac{K}{s_0}} + O(T^{2H+1})\right) \right. \\ &\quad \left. - \frac{1}{2z(T) + \frac{1}{2}} \left(1 + \frac{n_1(1)-2}{2(2z(T) + \frac{1}{2}) \log \frac{K}{s_0}} + O(T^{2H+1})\right) \right] \\ &= s_0^{2z(T)} K^{-2z(T)+\frac{1}{2}} \left(\log \frac{K}{s_0}\right)^{\frac{n_1(1)-2}{2}} \left(\frac{1}{4z(T)^2 - \frac{1}{4}} + O\left(T^{3H+\frac{3}{2}}\right)\right) \end{aligned}$$

as $T \rightarrow 0$. Therefore,

$$I_1(T) = s_0^{2z(T)} K^{-2z(T)+\frac{1}{2}} \left(\log \frac{K}{s_0} \right)^{\frac{n_1(1)-2}{2}} \left(4z(T)^2 - \frac{1}{4} \right)^{-1} \left(1 + O\left(T^{H+\frac{1}{2}}\right) \right) \quad (4.51)$$

as $T \rightarrow 0$. Similarly,

$$I_2(T) = s_0^{2z(T)} K^{-2z(T)+\frac{1}{2}} \left(\log \frac{K}{s_0} \right)^{\frac{n_1(1)-3}{2}} \left(4z(T)^2 - \frac{1}{4} \right)^{-1} \left(1 + O\left(T^{H+\frac{1}{2}}\right) \right) \quad (4.52)$$

as $T \rightarrow 0$. It is not hard to see that

$$\left(4z(T)^2 - \frac{1}{4} \right)^{-1} = \lambda_1(1) T^{2H+1}.$$

It follows from (4.51) and (4.52) that

$$I_1(T) = \lambda_1(1) K^{\frac{1}{2}} \left(\log \frac{K}{s_0} \right)^{\frac{n_1(1)-2}{2}} \left(\frac{s_0}{K} \right)^{2z(T)} T^{2H+1} \left(1 + O\left(T^{H+\frac{1}{2}}\right) \right) \quad (4.53)$$

as $T \rightarrow 0$. Similarly,

$$I_2(T) = \lambda_1(1) K^{\frac{1}{2}} \left(\log \frac{K}{s_0} \right)^{\frac{n_1(1)-3}{2}} \left(\frac{s_0}{K} \right)^{2z(T)} T^{2H+1} \left(1 + O\left(T^{H+\frac{1}{2}}\right) \right) \quad (4.54)$$

as $T \rightarrow 0$.

The next assertion characterizes the small-time asymptotic behavior of the call pricing function.

Theorem 4.4 *Let $K > s_0$. Then the following asymptotic formula holds for the call pricing function in the model described by (4.11):*

$$C(T) = MT^{\frac{(2H+1)(4-n_1(1))}{4}} \left(\frac{s_0}{K} \right)^{\lambda_1(1)-\frac{1}{2}} T^{-H-\frac{1}{2}} \left(1 + O\left(T^{\frac{2H+1}{4}}\right) \right) \quad (4.55)$$

as $T \rightarrow 0$, where

$$M = \frac{(s_0 K)^{\frac{1}{2}}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{\frac{4-n_1(1)}{4}} \left(\log \frac{K}{s_0}\right)^{\frac{n_1(1)-2}{2}} \\ \times \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)}\right)^{\frac{n_k}{2}}. \quad (4.56)$$

Proof. Using (4.22), (4.47), (4.48) and (4.49), we see that

$$C(T) = \frac{\sqrt{s_0}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{4}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)}\right)^{\frac{n_k}{2}} \\ T^{-\frac{(2H+1)n_1(1)}{4}} (1 + O(T^{2H+1})) \left[I_1(T) + O\left(T^{\frac{2H+1}{4}} I_2(T)\right) \right]$$

as $T \rightarrow 0$. Next, (4.53) and (4.54), imply

$$C(T) = \frac{(s_0 K)^{\frac{1}{2}}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)-4}{4}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)}\right)^{\frac{n_k}{2}} \\ \left(\log \frac{K}{s_0}\right)^{\frac{n_1(1)-2}{2}} T^{\frac{(2H+1)(4-n_1(1))}{4}} \left(\frac{s_0}{K}\right)^{2z(T)} \left(1 + O\left(T^{\frac{2H+1}{4}}\right)\right) \quad (4.57)$$

as $T \rightarrow 0$. We also have

$$\sqrt{\frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}}} - \sqrt{\frac{4}{\lambda_1(1)T^{2H+1}}} = O\left(T^{H+\frac{1}{2}}\right) \quad (4.58)$$

as $T \rightarrow 0$. Therefore,

$$\left(\frac{s_0}{K}\right)^{2z(T)} = \exp\left\{-2z(T) \log \frac{K}{s_0}\right\} \\ = \exp\left\{-\frac{1}{2} \sqrt{\frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}}} \log \frac{K}{s_0}\right\} = \exp\left\{-\frac{1}{2} \sqrt{\frac{4}{\lambda_1(1)T^{2H+1}}} \log \frac{K}{s_0}\right\} \\ \exp\left\{-\frac{1}{2} \left[\sqrt{\frac{\lambda_1(1)T^{2H+1} + 4}{\lambda_1(1)T^{2H+1}}} - \sqrt{\frac{4}{\lambda_1(1)T^{2H+1}}} \right] \log \frac{K}{s_0}\right\}$$

as $T \rightarrow 0$. Using (4.58), we obtain

$$\left(\frac{s_0}{K}\right)^{2z(T)} = \left(\frac{s_0}{K}\right)^{\lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}}} \left(1 + O\left(T^{H+\frac{1}{2}}\right)\right) \quad (4.59)$$

as $T \rightarrow 0$.

Now, it is clear that Theorem 4.4 follows from (4.57) and (4.59).

The next statement allows us to recover the self-similarity index H from the asymptotics of the call pricing function.

Corollary 4.2 *Under the conditions in Theorem 4.4, for every $K > s_0$,*

$$H = \lim_{T \rightarrow 0} \frac{\log \log \frac{1}{C(T,K)}}{\log \frac{1}{T}} - \frac{1}{2}. \quad (4.60)$$

Proof. It follows from (4.55) that

$$\begin{aligned} \log \frac{1}{C(T)} &= \log \frac{1}{M} + \frac{(2H+1)(4-n_1(1))}{4} \log \frac{1}{T} \\ &\quad + \lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \log \frac{K}{s_0} + O\left(T^{\frac{2H+1}{4}}\right) \end{aligned} \quad (4.61)$$

as $T \rightarrow 0$. Hence,

$$\begin{aligned} \log \log \frac{1}{C(T)} &= \log \left[\lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \log \frac{K}{s_0} \right] \\ &\quad + \log \left(1 + O\left(T^{H+\frac{1}{2}} + T^{H+\frac{1}{2}} \log \frac{1}{T} + T^{H+\frac{1}{2}} O\left(T^{\frac{2H+1}{4}} \right) \right) \right) \\ &= \left(H + \frac{1}{2} \right) \log \frac{1}{T} + \log \left[\lambda_1(1)^{-\frac{1}{2}} \log \frac{K}{s_0} \right] + O\left(T^{H+\frac{1}{2}} \log \frac{1}{T} \right) \end{aligned} \quad (4.62)$$

as $T \rightarrow 0$.

Now, it is clear that (4.60) follows from the previous formula.

Next, we turn our attention to the out-of-the-money put pricing function $T \mapsto P(T)$ with $0 < K < s_0$. The asymptotic behavior of the put pricing function with $0 < K < s_0$ will be characterized using the symmetry properties of the model in (4.11). In ([5], Lemma 9.25), several equivalent conditions are given for the symmetry of a stochastic volatility model. One of them is as follows (see (9.79) in [5]):

$$D_T(x) = \left(\frac{s_0}{x} \right)^3 D_T \left(\frac{s_0^2}{x} \right) \quad (4.63)$$

for all $x > 0$ and $T > 0$. It is clear that for the model described by (4.11), the previous equality can be derived from formula (4.21). Next, using Theorem 4.3 and (4.63), we establish the following proposition.

Theorem 4.5 *Let $0 < \varepsilon < s_0$ and $0 < x < s_0 - \varepsilon$. Then as $T \rightarrow 0$, the following asymptotic formula holds for the asset price density D_T in the model described by (3.30):*

$$\begin{aligned} D_T(x) &= \frac{\sqrt{s_0}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{-\frac{n_1(1)}{4}} \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k}{2}} x^{-\frac{3}{2}} \\ &\times \left(\log \frac{s_0}{x} \right)^{\frac{n_1(1)-2}{2}} T^{-\frac{(2H+1)n_1(1)}{4}} \left(\frac{s_0}{x} \right)^{-\frac{\sqrt{4+\lambda_1(1)T^{2H+1}}}{2\sqrt{\lambda_1(1)T^{H+\frac{1}{2}}}}} \\ &\times \left(1 + O(T^{2H+1}) \right) \left(1 + O_\varepsilon \left(T^{\frac{2H+1}{4}} \left(\log \frac{s_0}{x} \right)^{-\frac{1}{2}} \right) \right). \end{aligned} \quad (4.64)$$

Since the model that we are studying is symmetric,

$$P(T, K) = \frac{K}{s_0} C \left(T, \frac{s_0^2}{K} \right). \quad (4.65)$$

(see condition 3 in Lemma 9.25 in [5]).

The next assertion follows from Theorem 4.4 and (4.65).

Theorem 4.6 *Let $0 < K < s_0$. Then the following asymptotic formula holds for the put pricing function in the model described by (4.11):*

$$\begin{aligned} P(T) &= \tilde{M} T^{\frac{(2H+1)(4-n_1(1))}{4}} \left(\frac{K}{s_0} \right)^{\lambda_1(1) - \frac{1}{2} T^{-H - \frac{1}{2}}} \\ &\left(1 + O \left(T^{\frac{2H+1}{4}} \right) \right) \end{aligned} \quad (4.66)$$

as $T \rightarrow 0$, where the constant \tilde{M} is given by

$$\begin{aligned} \tilde{M} &= \frac{(s_0 K)^{\frac{1}{2}}}{2^{\frac{n_1(1)}{2}} \Gamma\left(\frac{n_1(1)}{2}\right)} \lambda_1(1)^{\frac{4-n_1(1)}{4}} \left(\log \frac{s_0}{K} \right)^{\frac{n_1(1)-2}{2}} \\ &\times \prod_{k=2}^{\infty} \left(\frac{\lambda_1(1)}{\lambda_1(1) - \rho_k(1)} \right)^{\frac{n_k}{2}}. \end{aligned} \quad (4.67)$$

Next, using the same reasoning as in the proof of Corollary 4.2, we obtain the following statement.

Corollary 4.3 *Under the conditions in Theorem 4.6, for every $0 < K < s_0$,*

$$H = \lim_{T \rightarrow 0} \frac{\log \log \frac{1}{P(T, K)}}{\log \frac{1}{T}} - \frac{1}{2}. \quad (4.68)$$

4.6 Asymptotic behavior of the implied volatility

Theorems 4.4 and 4.6 characterize the small-time behavior of the call and put pricing functions in a stochastic volatility model with centered Gaussian self-similar volatility. In the present section, we study the small-time behavior of the implied volatility in such a model. We will use some of the results obtained by Gao and Lee in [10]. Gao and Lee establish certain asymptotic relations between the implied volatility and the call pricing function under very general conditions. They consider various asymptotic regimes, e.g., the extreme strike, the small/large time, or mixed regimes. Of our interest is formula (7.11) in Corollary 7.3 in [10], providing an asymptotic formula characterizing the small-time asymptotic behavior of the implied volatility in terms of the call pricing function. It follows from this formula that if $K \neq s_0$, then

$$\sqrt{T}I(T, K) = \frac{\left| \log \frac{K}{s_0} \right|}{\sqrt{2 \left| \log \frac{1}{C(T, K)} \right|}} \left(1 + O \left(\frac{\log \left| \log \frac{1}{C(T, K)} \right|}{\left| \log \frac{1}{C(T, K)} \right|} \right) \right)$$

as $T \rightarrow 0$. Therefore,

$$I(T, K) = \frac{\left| \log \frac{K}{s_0} \right|}{\sqrt{2T \left| \log \frac{1}{C(T, K)} \right|}} + O \left(\frac{\log \left| \log \frac{1}{C(T, K)} \right|}{\sqrt{T} \left| \log \frac{1}{C(T, K)} \right|^{\frac{3}{2}}} \right) \quad (4.69)$$

as $T \rightarrow 0$.

The following assertion can be derived from (4.55) and (4.69).

Theorem 4.7 *Let $K > s_0$. Then the following asymptotic formula holds for the implied volatility in the model described by (4.11):*

$$I(T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\log \frac{K}{s_0}}}{\sqrt{2}} T^{\frac{2H-1}{4}} + O \left(T^{\frac{6H+1}{4}} \log \frac{1}{T} \right) \quad (4.70)$$

as $T \rightarrow 0$.

Proof. It follows from (4.61) and (4.62) that

$$\log \frac{1}{C(T)} \approx T^{-H-\frac{1}{2}}$$

and

$$\log \log \frac{1}{C(T)} \approx \log \frac{1}{T}$$

as $T \rightarrow 0$. Moreover, the mean value theorem implies that

$$\begin{aligned} \left(\log \frac{1}{C(T)} \right)^{-\frac{1}{2}} &= \left(\lambda_1(1)^{-\frac{1}{2}} T^{-H-\frac{1}{2}} \log \frac{K}{s_0} \right)^{-\frac{1}{2}} + O \left(T^{\frac{6H+3}{4}} \log \frac{1}{T} \right) \\ &= \lambda_1(1)^{\frac{1}{4}} \left(\log \frac{K}{s_0} \right)^{-\frac{1}{2}} T^{\frac{2H+1}{4}} + O \left(T^{\frac{6H+3}{4}} \log \frac{1}{T} \right) \end{aligned}$$

as $T \rightarrow 0$. Now it is not hard to see that (4.70) follows from (4.69) and the previous formulas.

Remark 4.1 *Assume $K > s_0$. It follows from Theorem 4.7 that if the Hurst index satisfies $0 < H < \frac{1}{2}$, then the implied volatility $T \mapsto I(K, T)$ is singular at $T = 0$, and it behaves near zero like the function $T \mapsto T^{\frac{2H-1}{4}}$. For standard Brownian motion, $H = \frac{1}{2}$, and we have*

$$\lim_{T \rightarrow 0} I(K, T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\log \frac{K}{s_0}}}{\sqrt{2}}.$$

Finally, for $\frac{1}{2} < H < 1$, the implied volatility $T \mapsto I(K, T)$ tends to zero like the function $T \mapsto T^{\frac{2H-1}{4}}$.

The next statement is a corollary to Theorem 4.7. It provides a representation of the self-similarity index in terms of the implied volatility.

Corollary 4.4 *Let $K > s_0$. Then the following equality holds:*

$$H = 2 \lim_{T \rightarrow 0} \frac{\log \frac{1}{I(T, K)}}{\log \frac{1}{T}} + \frac{1}{2}. \quad (4.71)$$

In the case where $0 < K < s_0$, Theorem 4.7, Corollary 4.4, and the symmetry condition

$$I(T, K) = I \left(T, \frac{s_0^2}{K} \right)$$

(see [5], Lemma 9.25) imply the following assertions.

Theorem 4.8 *Let $0 < K < s_0$. Then the following asymptotic formula holds for the implied volatility in the model described by (4.11):*

$$I(T) = \frac{\lambda_1(1)^{\frac{1}{4}} \sqrt{\log \frac{s_0}{K}}}{\sqrt{2}} T^{\frac{2H-1}{4}} + O\left(T^{\frac{6H+1}{4}} \log \frac{1}{T}\right) \quad (4.72)$$

as $T \rightarrow 0$.

Corollary 4.5 *Let $0 < K < s_0$. Then equality (4.71) holds for the self-similarity index H .*

4.7 At-the-money options

In this section, we consider a stochastic volatility model, in which the volatility process $X^{(H)}$ is an adapted H -self-similar Gaussian process. As before, we assume $r = 0$. Let us also suppose $K = s_0$ (at-the-money case). Note that here we do not assume that the volatility process is centered.

Using (4.21) and the formula

$$C(T, K) = \int_K^\infty (x - K) D_T(x) dx,$$

we obtain the following equalities for the at-the-money call:

$$\begin{aligned} C(T, s_0) &= \frac{\sqrt{s_0}}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} \int_0^\infty u^{-1} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \tilde{p}_1(u) du \\ &\times \int_{s_0}^\infty (x - s_0) x^{-\frac{3}{2}} \exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2}\right\} dx \\ &= \frac{\sqrt{s_0}}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} \int_0^\infty u^{-1} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \tilde{p}_1(u) du \\ &\times \left[\int_{s_0}^\infty x^{-\frac{1}{2}} \exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2}\right\} dx - s_0 \int_{s_0}^\infty x^{-\frac{3}{2}} \exp\left\{-\frac{\log^2 \frac{x}{s_0}}{2T^{2H+1}u^2}\right\} dx \right]. \end{aligned}$$

It follows from the previous formula that

$$\begin{aligned} C(T, s_0) &= \frac{s_0}{\sqrt{2\pi}} T^{-H-\frac{1}{2}} \int_0^\infty u^{-1} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \tilde{p}_1(u) \\ &\times [\Phi_1(T, u) - \Phi_2(T, u)] du, \end{aligned} \quad (4.73)$$

where

$$\Phi_1(T, u) = \int_1^\infty y^{-\frac{1}{2}} \exp \left\{ -\frac{\log^2 y}{2T^{2H+1}u^2} \right\} dy \quad (4.74)$$

and

$$\Phi_2(T, u) = \int_1^\infty y^{-\frac{3}{2}} \exp \left\{ -\frac{\log^2 y}{2T^{2H+1}u^2} \right\} dy. \quad (4.75)$$

Our next goal is to estimate the functions Φ_1 and Φ_2 defined in (4.74) and (4.75).

We have

$$\begin{aligned} \Phi_1(T, u) &= \int_0^\infty \exp \left\{ -\left[\frac{w^2}{2T^{2H+1}u^2} - \frac{w}{2} \right] \right\} dw \\ &= \exp \left\{ \frac{T^{2H+1}u^2}{8} \right\} \int_0^\infty \exp \left\{ -\frac{1}{2T^{2H+1}u^2} \left(w - \frac{T^{2H+1}u^2}{2} \right)^2 \right\} dw \\ &= \exp \left\{ \frac{T^{2H+1}u^2}{8} \right\} \int_{-\frac{1}{2}T^{2H+1}u^2}^\infty \exp \left\{ -\frac{1}{2T^{2H+1}u^2} z^2 \right\} dz \\ &= T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^{2H+1}u^2}{8} \right\} \int_{-\frac{1}{2}T^{H+\frac{1}{2}}u}^\infty \exp \left\{ -\frac{y^2}{2} \right\} dy. \end{aligned}$$

Similarly,

$$\Phi_2(T, u) = T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^{2H+1}u^2}{8} \right\} \int_{\frac{1}{2}T^{H+\frac{1}{2}}u}^\infty \exp \left\{ -\frac{y^2}{2} \right\} dy.$$

Therefore

$$\Phi_1(T, u) - \Phi_2(T, u) = 2T^{H+\frac{1}{2}}u \exp \left\{ \frac{T^{2H+1}u^2}{8} \right\} \int_0^{\frac{1}{2}T^{H+\frac{1}{2}}u} \exp \left\{ -\frac{y^2}{2} \right\} dy. \quad (4.76)$$

The next lemma will be useful in the sequel. It will allow us to estimate the integral in (4.76).

Lemma 4.4 *Let $0 < a < 1$. Then the following inequalities are valid:*

$$a - \frac{a^3}{6} \leq \int_0^a \exp \left\{ -\frac{y^2}{2} \right\} dy \leq a - \frac{a^3}{6} + \frac{a^5}{40}. \quad (4.77)$$

On the other hand, if $a \geq 1$, then

$$\begin{aligned} \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{1}{a} \exp \left\{ -\frac{a^2}{2} \right\} &\leq \int_0^a \exp \left\{ -\frac{y^2}{2} \right\} dy \\ &\leq \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{a}{a^2+1} \exp \left\{ -\frac{a^2}{2} \right\}. \end{aligned} \quad (4.78)$$

Proof. The inequalities in (4.77) can be established using the Taylor expansion with two and three terms.

To prove the estimates in (4.78), we use the following known inequalities:

$$\frac{x}{x^2+1} \exp\left\{-\frac{x^2}{2}\right\} \leq \int_x^\infty \exp\left\{-\frac{y^2}{2}\right\} dy \leq \frac{1}{x} \exp\left\{-\frac{x^2}{2}\right\}, \quad (4.79)$$

for all $x > 0$. The previous inequalities follow from stronger estimates formulated in [82], 7.1.13. Now, (4.78) can be derived from (4.79) and the equality

$$\int_0^a \exp\left\{-\frac{y^2}{2}\right\} dy = \frac{\sqrt{\pi}}{\sqrt{2}} - \int_a^\infty \exp\left\{-\frac{y^2}{2}\right\} dy.$$

This completes the proof of Lemma 4.4.

The next assertion provides estimates for the at-the-money call.

Theorem 4.9 *The following inequalities are true for every $T > 0$:*

$$U_1(T) \leq C(T, s_0) \leq U_2(T),$$

where

$$\begin{aligned} U_1(T) &= \frac{s_0}{\sqrt{2\pi}} T^{H+\frac{1}{2}} \int_0^\infty \tilde{p}_1(u) u du - \frac{s_0}{24\sqrt{2\pi}} T^{3H+\frac{3}{2}} \int_0^\infty \tilde{p}_1(u) u^3 du \\ &+ \frac{2s_0}{\sqrt{2\pi} T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1\left(\frac{v}{T^{H+\frac{1}{2}}}\right) \left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{v}{2} + \frac{v^3}{48} - \frac{2}{v} \exp\left\{-\frac{v^2}{8}\right\} \right] dv \end{aligned}$$

and

$$\begin{aligned} U_2(T) &= \frac{s_0}{\sqrt{2\pi}} T^{H+\frac{1}{2}} \int_0^\infty \tilde{p}_1(u) u du - \frac{s_0}{24\sqrt{2\pi}} T^{3H+\frac{3}{2}} \int_0^\infty \tilde{p}_1(u) u^3 du \\ &+ \frac{s_0}{640\sqrt{2\pi}} T^{5H+\frac{5}{2}} \int_0^\infty \tilde{p}_1(u) u^5 du + \frac{2s_0}{\sqrt{2\pi} T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1\left(\frac{v}{T^{H+\frac{1}{2}}}\right) \\ &\left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{v}{2} + \frac{v^3}{48} - \frac{v^5}{1280} - \frac{2v}{v^2+4} \exp\left\{-\frac{v^2}{8}\right\} \right] dv \end{aligned}$$

Proof. It follows from (4.73), (4.76) and Lemma 4.4 that

$$\begin{aligned}
C(T, s_0) &\leq \frac{s_0}{\sqrt{2\pi}} \int_0^{\frac{2}{T^{H+\frac{1}{2}}}} \tilde{p}_1(u) \\
&\quad \left[T^{H+\frac{1}{2}}u - \frac{1}{24}T^{3H+\frac{3}{2}}u^3 + \frac{1}{640}T^{5H+\frac{5}{2}}u^5 \right] du \\
&\quad + \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2T^{H+\frac{1}{2}}u}{T^{2H+1}u^2 + 4} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \right] du \\
&= \frac{s_0}{\sqrt{2\pi}} \int_0^{\infty} \tilde{p}_1(u) \left[T^{H+\frac{1}{2}}u - \frac{1}{24}T^{3H+\frac{3}{2}}u^3 + \frac{1}{640}T^{5H+\frac{5}{2}}u^5 \right] du \\
&\quad - \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{1}{2}T^{H+\frac{1}{2}}u - \frac{1}{48}T^{3H+\frac{3}{2}}u^3 + \frac{1}{1280}T^{5H+\frac{5}{2}}u^5 \right] du \\
&\quad + \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2T^{H+\frac{1}{2}}u}{T^{2H+1}u^2 + 4} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \right] du. \tag{4.80}
\end{aligned}$$

and

$$\begin{aligned}
C(T, s_0) &\geq \frac{s_0}{\sqrt{2\pi}} \int_0^{\frac{2}{T^{H+\frac{1}{2}}}} \tilde{p}_1(u) \left[T^{H+\frac{1}{2}}u - \frac{1}{24}T^{3H+\frac{3}{2}}u^3 \right] du \\
&\quad + \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2}{T^{H+\frac{1}{2}}u} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \right] du \\
&= \frac{s_0}{\sqrt{2\pi}} \int_0^{\infty} \tilde{p}_1(u) \left[T^{H+\frac{1}{2}}u - \frac{1}{24}T^{3H+\frac{3}{2}}u^3 \right] du \\
&\quad - \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{1}{2}T^{H+\frac{1}{2}}u - \frac{1}{48}T^{3H+\frac{3}{2}}u^3 \right] du \\
&\quad + \frac{2s_0}{\sqrt{2\pi}} \int_{\frac{2}{T^{H+\frac{1}{2}}}}^{\infty} \tilde{p}_1(u) \left[\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{2}{T^{H+\frac{1}{2}}u} \exp\left\{-\frac{T^{2H+1}u^2}{8}\right\} \right] du. \tag{4.81}
\end{aligned}$$

Now, it is not hard to see, making the substitution $v = T^{H+\frac{1}{2}}u$, that Theorem 4.9 follows from (4.80) and (4.81).

The next statement characterizes the small-time asymptotic behavior of the at-the-money call pricing function in a Gaussian self-similar stochastic volatility model.

Corollary 4.6 *The following formula holds as $T \rightarrow 0$:*

$$C(T, s_0) = c_1 T^{H+\frac{1}{2}} - c_2 T^{3H+\frac{3}{2}} + O\left(T^{5H+\frac{5}{2}}\right), \tag{4.82}$$

where

$$c_1 = \frac{s_0}{\sqrt{2\pi}} \int_0^\infty p_1(u) u^{\frac{1}{2}} du \quad (4.83)$$

and

$$c_2 = \frac{s_0}{24\sqrt{2\pi}} \int_0^\infty p_1(u) u^{\frac{3}{2}} du. \quad (4.84)$$

Proof. For a centered volatility process X , we will use formula (4.26)). In the case of a noncentered volatility process X , we need the following formula:

$$\begin{aligned} \tilde{p}_1(x) &= 2Cx^{\frac{n_1(1)-1}{2}} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} x \right\} \exp \left\{ -\frac{x^2}{2\lambda_1(1)} \right\} \\ &\quad \times (1 + O(x^{-1})) \end{aligned} \quad (4.85)$$

as $x \rightarrow \infty$, where the constant C is given by (4.14). Formula (4.85) now derives easily from (4.18) and (4.19).

It follows from Theorem 4.9 that

$$\begin{aligned} C(T, s_0) - U_1(T) &\leq U_2(T) - U_1(T) \\ &\leq \frac{s_0}{640\sqrt{2\pi}} T^{5H+\frac{5}{2}} \int_0^\infty \tilde{p}_1(u) u^5 du \\ &\quad + \frac{2s_0}{\sqrt{2\pi} T^{H+\frac{1}{2}}} \int_2^\infty \tilde{p}_1 \left(\frac{v}{T^{H+\frac{1}{2}}} \right) \\ &\quad \left[\frac{2}{v} \exp \left\{ -\frac{v^2}{8} \right\} + \frac{2v}{v^2+4} \exp \left\{ -\frac{v^2}{8} \right\} + \frac{v^5}{1280} \right] dv. \end{aligned} \quad (4.86)$$

Let us next suppose the process X is centered. Then, using (4.26), we see that for $v > 2$ and for sufficiently small values of T ,

$$\begin{aligned} \frac{1}{T^{H+\frac{1}{2}}} \tilde{p}_1 \left(\frac{v}{T^{H+\frac{1}{2}}} \right) &\leq \alpha \left(\frac{v}{T^{H+\frac{1}{2}}} \right)^{n_1(1)-1} \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{2\lambda_1(1)T^{2H+1}} \right\} \\ &\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{4\lambda_1(1)T^{2H+1}} \right\} \\ &\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{1}{2\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\} \\ &\leq \alpha \exp \left\{ -\frac{1}{4\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\}. \end{aligned} \quad (4.87)$$

Here $\alpha > 0$ is a constant that may change from line to line.

Now assume the process X is noncentered. Then for $v > 2$ and for sufficiently small T ,

$$\begin{aligned} \frac{1}{T^{H+\frac{1}{2}}} \tilde{p}_1 \left(\frac{v}{T^{H+\frac{1}{2}}} \right) &\leq \alpha \left(\frac{v}{T^{H+\frac{1}{2}}} \right)^{\frac{n_1(1)-1}{2}} \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ \sqrt{\frac{\delta(1)}{\lambda_1(1)}} \frac{v}{T^{H+\frac{1}{2}}} \right\} \\ &\quad \exp \left\{ -\frac{v^2}{2\lambda_1(1)T^{2H+1}} \right\} \\ &\leq \alpha \frac{1}{T^{H+\frac{1}{2}}} \exp \left\{ -\frac{v^2}{4\lambda_1(1)T^{2H+1}} \right\} \\ &\leq \alpha \exp \left\{ -\frac{1}{4\lambda_1(1)T^{2H+1}} \right\} \exp \left\{ -\frac{v^2}{8\lambda_1(1)} \right\}. \end{aligned} \quad (4.88)$$

Finally, taking into account (4.86), (4.87), and (4.88), we obtain

$$C(T, s_0) - U_1(T) = O \left(T^{5H+\frac{5}{2}} \right) \quad (4.89)$$

as $T \rightarrow 0$. Now, it is not hard to see, using the definition of U_1 , (4.87), and (4.89) that

$$C(T, s_0) = b_1 T^{H+\frac{1}{2}} - b_2 T^{3H+\frac{3}{2}} + O \left(T^{5H+\frac{5}{2}} \right),$$

where

$$b_1 = \frac{s_0}{\sqrt{2\pi}} \int_0^\infty \tilde{p}_1(u) u du$$

and

$$b_2 = \frac{s_0}{24\sqrt{2\pi}} \int_0^\infty \tilde{p}_1(u) u^3 du.$$

Finally, using the equality $\tilde{p}_1(u) = 2up_1(u^2)$, we obtain $b_i = c_i$ for $i = 1, 2$.

This completes the proof of Corollary 4.6.

4.8 Implied volatility in at-the-money regime

The Black-Scholes call pricing function for $r = 0$ and $K = s_0$ is given by

$$C_{BS}(T, s_0, \sigma) = \frac{s_0}{\sqrt{2\pi}} \int_{-\frac{\sigma\sqrt{T}}{2}}^{\frac{\sigma\sqrt{T}}{2}} e^{-\frac{y^2}{2}} dy = s_0 \frac{2}{\sqrt{\pi}} \int_0^{\frac{\sigma\sqrt{T}}{2\sqrt{2}}} e^{-x^2} dx.$$

Hence,

$$C_{BS}(T, s_0, \sigma) = s_0 \operatorname{erf} \left(\frac{\sigma\sqrt{T}}{2\sqrt{2}} \right), \quad (4.90)$$

where erf is the error function defined by $\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2} dx$. The error function is a strictly increasing continuous function from $[0, \infty)$ onto $[0, 1)$. Its inverse function is denoted by erf^{-1} . It is known that the inverse error function has the following Maclorin's expansion:

$$\operatorname{erf}^{-1}(z) = \frac{\sqrt{\pi}}{2} \left(z + \frac{\pi}{12} z^3 + \frac{7\pi^2}{480} z^5 + \dots \right), \quad 0 \leq z \leq 1 \quad (4.91)$$

(see []). It follows from the definition of the implied volatility that

$$C_{BS}(T, s_0, I(T, s_0)) = C(T, s_0).$$

Therefore, (4.90) implies

$$I(T, s_0) = \frac{2\sqrt{2}}{\sqrt{T}} \operatorname{erf}^{-1} \left(\frac{C(T, s_0)}{s_0} \right).$$

Next, using (4.91), we obtain

$$I(T, s_0) = \frac{\sqrt{2\pi}}{\sqrt{T}} \left[\frac{C(T, s_0)}{s_0} + \frac{\pi}{12} \frac{C(T, s_0)^3}{s_0^3} + O(C(T, s_0)^5) \right] \quad (4.92)$$

as $T \rightarrow 0$.

Now, we are ready to characterize the small-time asymptotic behavior of the implied volatility in at-the-money regime.

Theorem 4.10 *The following asymptotic formula holds as $T \rightarrow 0$:*

$$\begin{aligned} I(T, s_0) &= T^H \int_0^\infty p_1(u) u^{\frac{1}{2}} du \\ &\quad + T^{3H+1} \frac{1}{24} \left[\left(\int_0^\infty p_1(u) u^{\frac{1}{2}} du \right)^3 - \int_0^\infty p_1(u) u^{\frac{3}{2}} du \right] \\ &\quad + O(T^{5H+2}). \end{aligned} \quad (4.93)$$

Proof. Our first goal is to obtain an asymptotic formula for the implied volatility with error term of the order $O(T^{5H+2})$, by using formula (4.82) in (4.92). Following this plan, we obtain

$$\begin{aligned} I(T, s_0) &= \frac{\sqrt{2\pi}}{s_0\sqrt{T}} \left(c_1 T^{H+\frac{1}{2}} - c_2 T^{3H+\frac{3}{2}} + O\left(T^{5H+\frac{5}{2}}\right) \right) \\ &+ \frac{\pi\sqrt{2\pi}}{12s_0^3\sqrt{T}} \left(c_1 T^{H+\frac{1}{2}} - c_2 T^{3H+\frac{3}{2}} + O\left(T^{5H+\frac{5}{2}}\right) \right)^3 + O(T^{5H+2}) \\ &= \frac{\sqrt{2\pi}c_1}{s_0} T^H + \left(\frac{\pi\sqrt{2\pi}c_1^3}{12s_0^3} - \frac{\sqrt{2\pi}c_2}{s_0} \right) T^{3H+1} + O(T^{5H+2}) \end{aligned} \quad (4.94)$$

as $T \rightarrow 0$. Now, it is not difficult to see that formula (4.93) follows from (4.83), (4.84), and (4.94).

This completes the proof of Theorem 4.10.

Remark 4.2 *It is clear that the following formulas are valid for the integrals in (4.93):*

$$\mu_{1/2} := \int_0^\infty p_1(u) u^{\frac{1}{2}} du = \mathbb{E} \left[\left(\int_0^1 X_s^2 ds \right)^{\frac{1}{2}} \right]$$

and

$$\mu_{3/2} := \int_0^\infty p_1(u) u^{\frac{3}{2}} du = \mathbb{E} \left[\left(\int_0^1 X_s^2 ds \right)^{\frac{3}{2}} \right].$$

Theorem 4.10 allows us to recover the self-similarity index H knowing the small-time behavior of the at-the-money implied volatility.

Theorem 4.11 *The following formula holds:*

$$H = \lim_{T \rightarrow 0} \frac{\log \frac{1}{I(T, s_0)}}{\log \frac{1}{T}}.$$

4.9 Numerical illustration

To illustrate the numerical potential of our asymptotic formulas in practice, we finish this article with a brief section comparing exact (Monte-Carlo-simulated) option prices and IVs with the asymptotics we have derived. Formulas such as (1.6) can be

used to calibrate various parameters which might be linked explicitly or empirically to $\lambda_1(1)$, assuming H is known. We refer to the numerics in our prior work in [9] for details on what can be done, leaving to the interested reader any details of how to translate the ideas therein which are for extreme strike asymptotics to the small time case. [9] also contains a description of how to simulate the fBm-driven models of interest to us, for Monte-Carlo purposes, we do not repeat this information here.

Our results in the at-the-money case are presumably harder to exploit along these lines because they depend on moment statistics $\mu_{1/2}$ and $\mu_{3/2}$ (Remark 4.2), which are not explicitly related to model parameters. An exception to this observation is in the case of models with a volatility scale parameter σ , by which we mean that one replaces model (3.30) with

$$dS_t = rS_t dt + \sigma |X_t| S_t dW_t. \quad (4.95)$$

Here the parameter σ is rather innocuous since, by self-similarity of $|X|$, this σ can be absorbed as a linear time change, but it represents a convenient parameter for tuning a model to realistic time-scales and volatility levels. We will use this device in this section. In particular, at the money, it is easy to see from Theorem 4.10 that one has

$$I(T, s_0) = \sigma \mu_{1/2} T^H + \frac{\sigma^3}{24} T^{3H+1} \left[(\mu_{1/2})^3 - \mu_{3/2} \right] + O(T^{5H+2})$$

where $\mu_{1/2}$ and $\mu_{3/2}$ are given in Remark 4.2. Thus at-the-money IV asymptotics can be used to calibrate σ in model (4.95). We do not comment on this further herein.

Instead, we provide a numerical analysis of our results' use in H 's calibration. Indeed, the reference [9] contains an effort to calibrate H itself, when other parameters have been estimated by other means, but left some stones unturned. We found therein that H calibration can be relatively successful in some cases in practice, though this is not necessarily backed up by any asymptotic theory. The model-free results such as Corollary 4.4 and Theorem 4.11 can provide excellent calibration of H in many cases. But in reality, the liquidity is low for options close to maturity. We choose to show the H calibration in the at-the-money case using Theorem 4.10 for the reason

that liquidity is also low for options away from the money near maturity, which all but dictates the use of at-the-money IV. The values $\mu_{1/2}$ and $\mu_{3/2}$ can be computed by using Monte Carlo method on an estimation of the integrated variance

$$\int_0^1 X_t^2 dt \approx \sum_{n=1}^N \lambda_n(1) Z_n^2.$$

with a large number N of terms in the KL expansion of the fBm process for $T=1$.

The setup we use is that of model (4.95) with $X = \text{fBm}$, $r = 0$, and $\sigma = 3$. The choice of σ is tailored to provide a realistic volatility level after 1 or 2 weeks, with time measured in years. Specifically, a practitioner may simply select the desired magnitude of σ by matching it to the mean magnitude of volatility in (4.95) via the formula

$$\mathbf{E}[\sigma | X_t] = \sigma t^H \sqrt{2/\pi}.$$

For example, with $H = 0.6$ and $\sigma = 3$ we get $\mathbf{E}[\sigma | X_t] \approx 0.22$ after one week ($t = 7/365 \approx 0.019$), and $\mathbf{E}[\sigma | X_t] \approx 0.34$ after two weeks ($t = 14/365 \approx 0.038$), which could represent a realistic scenario for a volatile short-term bond market. Values of σ closer to unity result in much smaller volatility values near maturity; these allow for an extremely sharp fit between theoretical call and IV values and our asymptotics, but would typically be unrealistically small, hence our choice of $\sigma = 3$.

Before using Theorem 4.10, a first question might be whether it would not be sufficient to use an asymptotic theory for call prices to estimate parameters. The use of IV over option prices has been advocated in many articles, including many of the ones cited herein, but the question is still legitimate since one rarely sees evidence in the literature that this is indeed preferable in practice. The following two images compare the fit between our asymptotic formulas (Corollary 4.6 and Theorem 4.10) and exact (simulated) call and IV values for times from 1 day to 2 weeks.

We chose the extreme case $H = 0.51$ because, as it turns out, the asymptotics' accuracy increase as H increases. We see from the above that the IV asymptotics are accurate at a roughly 5%-error level for more than 10 days, and remains fairly accurate up to 2 weeks, while the call asymptotics are only accurate at a 5%-error

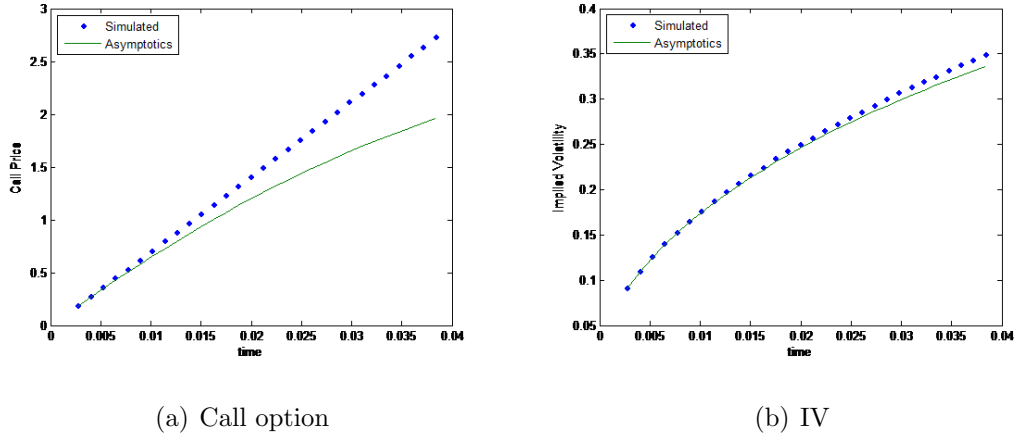


Figure 4.1.: Call options and IV with $\sigma = 3$, $H = 0.51$

level for 2 days, and deteriorate significantly thereafter. Other values of H show similar pictures. The choice to use IV over call prices for calibration purposes in small time is clear. This can of course be verified rigorously on our formulas since our coefficients can be computed numerically as well; this is omitted from our study. The next four pictures show the extremely sharp fit of IV asymptotics over two weeks as H increases, as we mentioned.

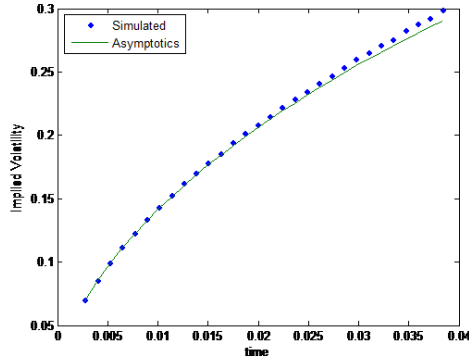
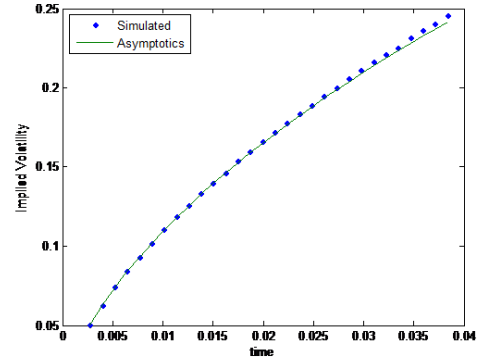
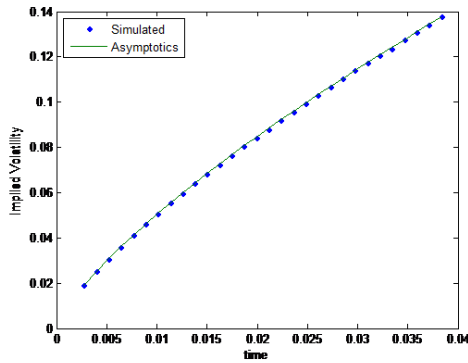
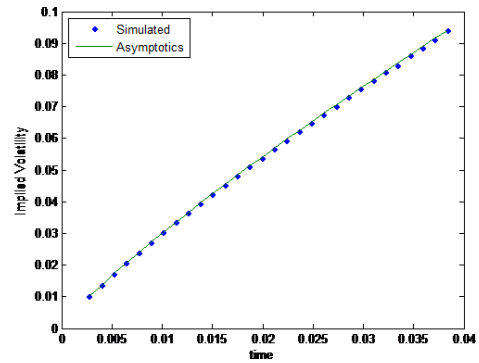
Since liquidity decreases as time to maturity decreases, it is desirable to use the largest possible time t_0 such that the relative error in IV approximation does not exceed a given error level, say 1% which would be a high level of accuracy. The table below give an idea of what this means in practice, by computing t_0 for a 1% level in the above realistic cases: with

$$t_0 = \max \left\{ t : \frac{\text{simulated IV}(t) - \text{asymptotic IV}(t)}{\text{simulated IV}(t)} < 0.01 \right\}$$

we find :

H	0.51	0.55	0.60	0.75	0.85
t_0 in days	2.3	4.6	10.4	14	14

These values of t_0 could be considered as rather conservative, due to the choice of 1% accuracy; practitioners may decide to choose a slightly more liberal level. This is

(a) $H = 0.55$ (b) $H = 0.60$ (c) $H = 0.75$ (d) $H = 0.85$ Figure 4.2.: IV with $\sigma = 3$, $t \in [1 \text{ day}; 2 \text{ weeks}]$ with different Hurst parameters

evident from the last tables below, in which we show the result of the calibration of H from exact (simulated) option prices, via Theorem 4.10.

In all cases, even with a 14-day time to maturity, the error in H -calibration is no greater than one hundredth (less than 2% relative error). The only difficulty we experience appears to be in differentiating between a model with Brownian scaling ($H = 0.50$, no memory in the volatility) and a model with $H > 0.50$, except for the very short times to maturity $t = 1, 2$ days. If liquidity at those levels is adequate, as it may be in heavily traded bond markets, then our calibration can be used with such short horizons. Otherwise a maturity of one week is preferable, particularly for self-similarity indices which are not too close to 0.50. A maturity of two weeks

Table 4.1.: Calibration of H in the at-the-money case near maturities $T = 1$ day

H used in simulation	0.50	0.51	0.55	0.60	0.75	0.85
H calibrated from IV via Theorem 4.11	0.50	0.51	0.55	0.60	0.75	0.85

 $T = 2$ days

H used in simulation	0.50	0.51	0.55	0.60	0.75	0.85
H calibrated from IV via Theorem 4.11	0.50	0.51	0.55	0.60	0.75	0.85

 $T = 7$ days

H used in simulation	0.50	0.51	0.55	0.60	0.75	0.85
H calibrated from IV via Theorem 4.11	0.50	0.50	0.55	0.60	0.75	0.85

 $T = 14$ days

H used in simulation	0.50	0.51	0.55	0.60	0.75	0.85
H calibrated from IV via Theorem 4.11	0.50	0.50	0.54	0.59	0.75	0.85

will work in all cases for scenarios where one is satisfied with a possible error of one hundredth on H calibration; this could be a realistic accuracy level for many users of stochastic volatility models who are currently not using any self-similarity or long-memory assumptions.

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