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A GENERALIZED SCHWARZ SPLITTING METHOD BASED ON HERMITE COLLOCATION FOR ELLIPTIC BOUNDARY VALUE PROBLEMS *

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Abstract. The Schwarz Alternating Method (SAM) coupled with various numerical discretization schemes has been already established as an efficient alternative for solving differential equations on various parallel machines. In this paper we consider an extension of SAM (Generalized Schwarz Splitting-GSS) for solving elliptic boundary value problems with generalized interface conditions that depend on a parameter that might differ in each overlapping region [13]. The GSS utilized in this work is coupled with the cubic Hermite collocation discretization approach [8] to solve the corresponding boundary value problem in each subdomain. The main focus of this study is the iterative solution of the corresponding enhanced GSS collocation discrete matrix equation for a model elliptic boundary value problem. For this we carry out the spectral analysis of the associated enhanced block Jacobi iteration matrix. In the case of one-dimensional problems and assuming the same parameter in all overlapping regions, we determine the domain of convergence and find a subinterval of it in which the optimal parameter lies; moreover, we obtain sets of optimal parameters for the multi-parameter GSS case. In addition, we analyze the convergence properties of the one-parameter GSS case for a two-dimensional model problem. Finally, we apply GSS to a number of model and general one- and two-dimensional differential equation problems, present a number of numerical examples that verify the theoretical results obtained and compare the convergence rates of the SAM and GSS methods with minimum and maximum overlap.

1. Introduction. The Schwarz Alternating Method (SAM) was originally introduced in [11] over a hundred years ago to solve the Dirichlet problem for Laplace's equation on a plane domain by iterating over a sequence of Dirichlet subproblems defined on two overlapping subregions of the original domain. The coupling of these subproblems is enforced through the so called *interface conditions* defined on the subdomain boundaries in the interior of the whole domain (interfaces). The original formulation of SAM assumed Dirichlet interface conditions that depended on the solution of the neighbor subproblem(s). Its convergence properties are studied in detail in [3] and [4]. One of the early numerical formulations of SAM for elliptic boundary value problems can be found in [7]. The numerical SAM approach has recently become very popular in connection with the parallel solution of elliptic partial differential equations (PDEs). This is primarily due to its inherited coarse grain parallel structure. In this paper, we consider the SAM method with generalized interface conditions which are the linear combination of the solution and its normal derivative on the subdomain interfaces. Each of these conditions depends on a parameter associated with each overlapping region. This extension of SAM is called Generalized Schwarz Splitting (GSS) [13].

The Schwarz Alternating Method has been coupled with either finite difference or finite element discretization schemes to solve elliptic boundary value problems in complex geometries by many researchers. In some special cases, the convergence properties of SAM have been investigated at a functional level. Since its introduction, the convergence properties of the GSS with finite difference discretization have appeared in many studies including [13] and [5]. To our knowledge, there are a few researchers who

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have considered either SAM or GSS coupled with collocation discretization schemes. In [2] the authors apply SAM based on Legendre collocation discretization and spectral methods to solve elliptic problems and demonstrate its convergence for model problems. In [15] the formulation of SAM was considered for the Poisson equations with Dirichelet boundary conditions on an L-shaped region. Only experimental results are reported in [15]. The work in [2, 15] and our recent work in [5] and [6] has motivated us to study the convergence properties of GSS associated with the cubic Hermite collocation discretization technique [8].

The SAM approach can be formulated either on the continuous geometric and functional components of the PDE problem (referred as the functional level formulation) or on the corresponding discrete geometric and algebraic data structures associated with the numerical method selected (referred to as the matrix equation level formulation). In this paper, we consider the matrix formulation of SAM and GSS for elliptic PDE problems based on the Hermite collocation discretization procedure. Specifically, we derive the associated enhanced Hermite collocation matrix equation problem [13] for GSS and study its iterative solution.

This work is organized as follows. In Section 2, we give a brief description of the GSS on a rectangle at functional and matrix levels. In Section 3, first we define a matrix with a specific structure and then we investigate some basic properties associated with it. Using the results obtained, we derive the block Jacobi iteration matrix corresponding to applying the GSS with bicubic Hermite collocation discretization for the solution of the Poisson equation under Dirichlet boundary conditions on a rectangular domain split into overlapping stripes. In Section 4, we carry out a spectral analysis of the enhanced block Jacobi iteration matrix corresponding to the one-dimensional problem. Furthermore, we determine the domain of convergence and find a subinterval of it in which the optimal parameter for the one-parameter GSS case lies; moreover, we obtain sets of optimal parameters for the multi-parameter GSS case. In Section 5, we analyze the convergence properties of the one-parameter GSS case for the two-dimensional problem. Finally, in Section 6, we present a number of numerical examples in the one- and two-dimensional spaces that verify the theoretical results obtained in this paper. In addition, we compare the convergence rates of the SAM and GSS methods with minimum and maximum overlap and draw several conclusions.

2. A Generalized Schwarz Alternating Method. We consider the Dirichlet problem

$$\left\{ \begin{array}{rcl} Lu & = & f \ \ in \ \ \Omega, \\ u & = & g \ \ on \ \ \partial\Omega \end{array} \right. \tag{1}$$

where L is a second order linear elliptic partial differential operator, Ω is a rectangle $(a,b) \times (c,d) \in R^2$ and $\partial\Omega$ is its boundary.

In order to formulate the GSS for PDE problem (1), we decompose Ω into k overlapping rectangles (stripes) $\Omega_1, \ldots, \Omega_k$, defined as $\Omega_i = (t_{il}, t_{ir}) \times (c,d)$ with $a = t_{1l} < t_{2l} < \ldots < t_{kl} < b$ and $a < t_{1r} < t_{2r} < \ldots < t_{kr} = b$. Furthermore, for $k \geq 3$ we assume that $t_{2l} < t_{1r}$ and $t_{(i-2)r} < t_{il} < t_{(i-1)r}$ for $i = 3, \ldots, k$. This assumption guarantees that no three consecutive stripes can have a common overlapping area and that any two consecutive stripes do overlap. We set $\Gamma_{il} = \{t_{il}\} \times (c,d)$ and $\Gamma_{ir} = \{t_{ir}\} \times (c,d)$, plus assume that both sets Γ_{1l} and Γ_{kr} are empty. We also define

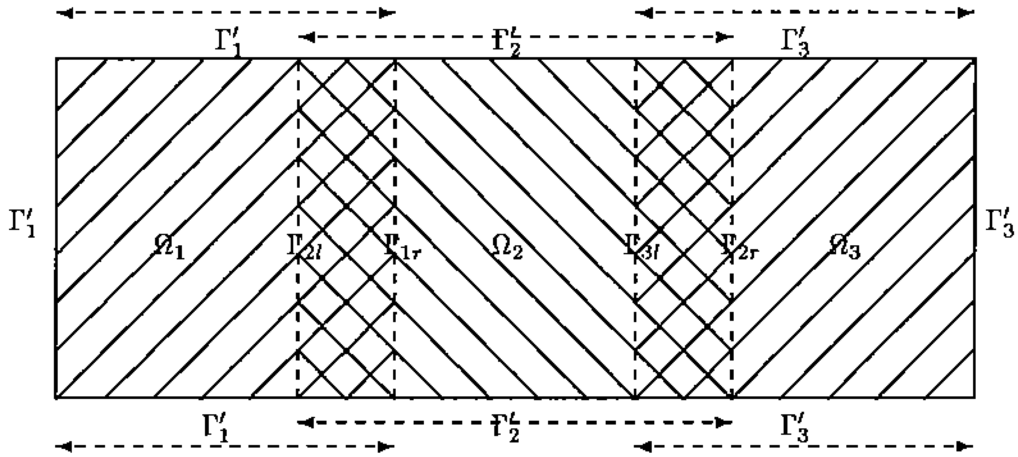

FIG. 1. *A decomposition of Ω for $k = 3$*

$\Gamma_i' = \partial\Omega_i - (\Gamma_{il} \cup \Gamma_{ir})$. An example of such a decomposition for $k = 3$ is depicted in Figure 1.

Then, the Generalized Schwarz Splitting method applied to problem (1), with a domain splitting as above, consists of solving the k coupled subproblems

$$(2) \qquad \begin{cases} L(u_i(x)) = f(x), & x \in \Omega_i, \\ u_i(x) = g(x), & x \in \Gamma_i', \\ \omega_l u_i(x) + (1-\omega_l)\frac{\partial u_i(x)}{\partial n} = \omega_l u_{i-1}(x) + (1-\omega_l)\frac{\partial u_{i-1}(x)}{\partial n}, & x \in \Gamma_{il}, \\ \omega_r u_i(x) + (1-\omega_r)\frac{\partial u_i(x)}{\partial n} = \omega_r u_{i+1}(x) + (1-\omega_r)\frac{\partial u_{i+1}(x)}{\partial n}, & x \in \Gamma_{ir}, \end{cases}$$

for $i = 1, \ldots, k$, where the ω's are user defined parameters.

Problem (2) can be solved iteratively for a given initial guess $(u_1^{(0)}, \ldots, u_k^{(0)})$. Following, we illustrate the application of Gauss-Seidel type iteration for the GSS PDE subproblems:

$$(3) \qquad \begin{cases} L(u_i^{(j)}(x)) = f(x), & x \in \Omega_i, \\ u_i^{(j)}(x) = g(x), & x \in \Gamma_i', \\ \omega_l u_i^{(j)}(x) + (1-\omega_l)\frac{\partial u_i^{(j)}(x)}{\partial n} = \omega_l u_{i-1}^{(j)}(x) + (1-\omega_l)\frac{\partial u_{i-1}^{(j)}(x)}{\partial n}, & x \in \Gamma_{il}, \\ \omega_r u_i^{(j)}(x) + (1-\omega_r)\frac{\partial u_i^{(j)}(x)}{\partial n} = \omega_r u_{i+1}^{(j-1)}(x) + (1-\omega_r)\frac{\partial u_{i+1}^{(j-1)}(x)}{\partial n}, & x \in \Gamma_{ir}, \end{cases}$$

where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots$. There are many ways of implementing a discrete analog of the algorithm (3). This is due to the many choices for the parameter ω and to the many alternatives of the discretization technique to be selected for each subproblem.

If the discretization scheme used to solve the subproblems in (2) is the same as the one used for the solution of the original problem (1) , then it is easy to see that problem (1) is reduced to the solution of a linear system, say $Ay = f$,

$$\begin{bmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & A_{32} & A_{33} & A_{34} & \\ & & A_{43} & A_{44} & A_{45} \\ & & & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

while problem (2) is reduced to solving the larger system (enhanced) $\bar{A}\bar{y} = \bar{f}$

$$
(4) \qquad
\begin{bmatrix}
A_{11} & A_{12} & & & & & & \\
A'_{21} & B'_{22} & C'_{22} & A'_{23} & & & & \\
A''_{21} & B''_{22} & C''_{22} & A''_{23} & & & & \\
 & & A_{32} & A_{33} & A_{34} & & & \\
 & & & & A'_{43} & B'_{44} & C'_{44} & A'_{45} \\
 & & & & A''_{43} & B''_{44} & C''_{44} & A_{45} \\
 & & & & & & A_{54} & A_{55}
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y'_2 \\ y_3 \\ y_4 \\ y'_4 \\ y_5 \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f'_2 \\ f''_2 \\ f_3 \\ f'_4 \\ f''_4 \\ f_5 \end{bmatrix}
$$

where $\bar{A}$ is a $k \times k$ block tridiagonal matrix. In both cases, we assume that the unknowns have been decomposed according to the splitting in Figure 1. Notice that, corresponding to the overlapping region $\Omega_1 \cap \Omega_2$, we have $f'_2 = \begin{bmatrix} f_2 \\ 0 \end{bmatrix}$, $f''_2 = \begin{bmatrix} 0 \\ f_2 \end{bmatrix}$, $A'_{21} = \begin{bmatrix} A_{21} \\ 0 \end{bmatrix}$, $A''_{23} = \begin{bmatrix} 0 \\ A_{23} \end{bmatrix}$, $A''_{21} = \begin{bmatrix} 0 \\ A_{21} \end{bmatrix}$, $A''_{23} = \begin{bmatrix} 0 \\ A_{23} \end{bmatrix}$, $B'_{22} = \begin{bmatrix} A_{22} \\ E_1 \end{bmatrix}$, $C'_{22} = \begin{bmatrix} 0 \\ -E_1 \end{bmatrix}$, $B''_{22} = \begin{bmatrix} -E_2 \\ 0 \end{bmatrix}$ and $C''_{22} = \begin{bmatrix} E_2 \\ A_{22} \end{bmatrix}$ with $E_1 = [0, 0, \ldots, 0, h_1(\omega_r), h_2(\omega_r)]$, $E_2 = [h_1(\omega_l), h_2(\omega_l), 0, \ldots, 0]$ plus 0 being zero vectors or submatrices, where h_i's are vectors derived from the interface boundary conditions. Similar relations hold for the equations associated with the overlapping region $\Omega_2 \cap \Omega_3$.

In view of the way algorithm (3) is derived, it is apparent that in order to study the convergence properties for a given discrete implementation of it, that is of (4), it suffices to study the corresponding properties of the block Jacobi iteration matrix associated with the enhanced linear system (4). On the other hand, one should bear in mind that for different implementations of the algorithm (3) the convergence properties of the corresponding iterative methods based on the linear system (4) may be different for the same problem. So, one may not have a single block Jacobi matrix to study for the different implementations of the algorithm (3). To simplify the subsequent discussion, we shall confine ourselves to selecting the cubic Hermite collocation discretization technique to discretize all the subproblems. For this specific implementation, we shall derive the corresponding block Jacobi iteration matrix for a model problem and shall study the impact of the various choices of the parameter ω, subject among others to the restriction $\omega_l = \omega_r$, on the spectral radius of the Jacobi matrix. For this study we will exploit some basic properties of a specific matrix structure in the section that follows.

3. Spectral Analysis of the Block Jacobi Iteration Matrices. In this section, we define a set of matrices which share a particular structure, study their properties, and develop the preliminaries needed for the rest of the analysis. Then, we use these results to derive the block Jacobi iteration matrix corresponding to a GSS scheme with bicubic Hermite collocation discretization technique for a model problem in the two-dimensional space. It is worth noticing that the analysis and the results of this section can also be used to handle the one-dimensional problem.

3.1. Preliminaries. First we define a matrix $T(m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ of order $4mn$ such that

$$
(5) \qquad \begin{bmatrix}
\alpha_1 A_1 + \beta_1 A_2 & A_3 & -A_4 & & & & & & & \\
\alpha_1 A_3 + \beta_1 A_4 & A_1 & -A_2 & & & & & & & \\
 & A_1 & A_2 & A_3 & -A_4 & & & & & \\
 & A_3 & A_4 & A_1 & -A_2 & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & A_1 & A_2 & A_3 & -A_4 & \\
 & & & & & A_3 & A_4 & A_1 & -A_2 & \\
 & & & & & & & A_1 & A_2 & \alpha_2 A_3 - \beta_2 A_4 \\
 & & & & & & & A_3 & A_4 & \alpha_2 A_1 - \beta_2 A_2
\end{bmatrix},
$$

where each A_i, $i = 1, \ldots, 4$, is a matrix of order $2m$ and $\alpha_1, \beta_1, \alpha_2, \beta_2$ are scalars. For simplicity, we denote it by T in the remaining of this section.

Next, we introduce the two matrices

$$
(6) \qquad N = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \text{ and } M = \begin{bmatrix} A_3 & -A_4 \\ A_1 & -A_2 \end{bmatrix}.
$$

We assume that N is nonsingular and its inverse is written in the same block form as N, namely $N^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. Then, it follows from the obvious relation $N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, where I denotes the identity matrix of order $2m$, that $M^{-1} = \begin{bmatrix} B_2 & B_1 \\ -B_4 & -B_3 \end{bmatrix}$. Based on the material introduced so far we can state and prove the following statement.

LEMMA 3.1. *If the matrices $-\beta_1 B_1 + \alpha_1 B_3$ and $\beta_2 B_1 + \alpha_2 B_3$ are invertible, then the following relations hold*

$$
(7) \quad T \begin{bmatrix} C_1 \\ (-N^{-1}M)^{n-1} \begin{bmatrix} \alpha_2 I \\ \beta_2 I \end{bmatrix} \\ \vdots \\ (-N^{-1}M) \begin{bmatrix} \alpha_2 I \\ \beta_2 I \end{bmatrix} \\ I \end{bmatrix} = \begin{bmatrix} S_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad T \begin{bmatrix} I \\ (-M^{-1}N) \begin{bmatrix} \alpha_1 I \\ \beta_1 I \end{bmatrix} \\ \vdots \\ (-M^{-1}N)^{n-1} \begin{bmatrix} \alpha_1 I \\ \beta_1 I \end{bmatrix} \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ S_2 \end{bmatrix}
$$

where

$$
(8) \qquad \begin{cases} S_1 = (-\beta_1 B_1 + \alpha_1 B_3)^{-1}([\beta_1 I, -\alpha_1 I](-N^{-1}M)^n \begin{bmatrix} \alpha_2 I \\ \beta_2 I \end{bmatrix}), \\ S_2 = (-\beta_2 B_1 - \alpha_2 B_3)^{-1}([\beta_2 I, -\alpha_2 I](-M^{-1}N)^n \begin{bmatrix} \alpha_1 I \\ \beta_1 I \end{bmatrix}) \end{cases}
$$

with C_1 and C_2 being matrices of order $2m$ that can be uniquely determined.

Proof. It is sufficient to show the first part of (7), since the second part follows by a similar argument. It is trivial to show, using (5) and (6), that the last $2n - 2$ block elements of both sides of (7) are equal. To determine S_1, we use the first two blocks of both sides to get

$$(9) \qquad N\begin{bmatrix}\alpha_1 I\\ \beta_1 I\end{bmatrix}C_1 + M(-N^{-1}M)^{n-1}\begin{bmatrix}\alpha_2 I\\ \beta_2 I\end{bmatrix} = \begin{bmatrix}S_1\\ 0\end{bmatrix}.$$

Then, premultiplying both members of (9) by $[\beta_1 I, -\alpha_1 I]N^{-1}$ we obtain

$$[\beta_1 I, -\alpha_1 I](-N^{-1}M)^n\begin{bmatrix}\alpha_2 I\\ \beta_2 I\end{bmatrix} = (-\beta_1 B_1 + \alpha_1 B_3)S_1$$

which determines S_1. C_1 can be determined uniquely from (9) by premultiplying it first by N^{-1} and then solving for C_1 from either the first or the second block component of the resulting equation. Since C_1 is not used later, its explicit expression is not needed. $\square$

Now, let us assume that T and $\alpha_1 A_3 + \beta_1 A_4$ are nonsingular and V_1 and V_2 denote the submatrices of the last $2n-1$ block components of the matrix products $T^{-1}[I, 0, \ldots, 0, 0]^T$ and $T^{-1}[0, I, 0, \ldots, 0]^T$ respectively. Then, since $T^{-1}T$ is the identity matrix, we obtain $V_1(\alpha_1 A_1 + \beta_1 A_2) + V_2(\alpha_1 A_3 + \beta_1 A_4) = 0$. This implies $V_2 = -V_1(\alpha_1 A_1 + \beta_1 A_2)(\alpha_1 A_3 + \beta_1 A_4)^{-1}$. Hence, the matrix of the last $2n-1$ block components of the matrix product $T^{-1}[A_1^T, A_3^T, 0 \ldots, 0]^T$ is

$$(10) \qquad V_1(A_1 - (\alpha_1 A_1 + \beta_1 A_2)(\alpha_1 A_3 + \beta_1 A_4)^{-1}A_3).$$

To simplify the above expression, we state and show the following lemma.

LEMMA 3.2. *If both A_1 and A_3 are nonsingular, then*

$$(\beta_1 B_1 - \alpha_1 B_3)(A_1 - (\alpha_1 A_1 + \beta_1 A_2)(\alpha_1 A_3 + \beta_1 A_4)^{-1}A_3) = \beta_1 I$$

Proof. We have

$$\begin{aligned}
&A_1 - (\alpha_1 A_1 + \beta_1 A_2)(\alpha_1 A_3 + \beta_1 A_4)^{-1}A_3\\
= \;& A_1(I - (\alpha_1 I + \beta_1 A_1^{-1}A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1})\\
= \;& A_1(\alpha_1 I + \beta_1 A_3^{-1}A_4 - \alpha_1 I - \beta_1 A_1^{-1}A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}\\
= \;& \beta_1 A_1(A_3^{-1}A_4 - A_1^{-1}A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}
\end{aligned}$$

and the fact that $N^{-1}N = I_{4m}$ implies $\begin{bmatrix}B_1 & B_2\\ B_3 & B_4\end{bmatrix}\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$. Thus, the following relations hold

$$\begin{aligned}
&(\beta_1 B_1 - \alpha_1 B_3)(A_1 - (\alpha_1 A_1 + \beta_1 A_2)(\alpha_1 A_3 + \beta_1 A_4)^{-1}A_3)\\
= \;& (\beta_1 B_1 - \alpha_1 B_3)\beta_1 A_1(A_3^{-1}A_4 - A_1^{-1}A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}\\
= \;& \beta_1(\beta_1 B_1 A_1 A_3^{-1}A_4 - \beta_1 B_1 A_2 - \alpha_1 B_3 A_1 A_3^{-1}A_4 + \alpha_1 B_3 A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}\\
= \;& \beta_1(\beta_1(I - B_2 A_3)A_3^{-1}A_4 - \beta_1 B_1 A_2 + \alpha_1 B_4 A_3 A_3^{-1}A_4 + \alpha_1 B_3 A_2)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}\\
= \;& \beta_1(\beta_1 A_3^{-1}A_4 - \beta_1(B_2 A_4 + B_1 A_2) + \alpha_1(B_4 A_4 + B_3 A_2))(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1}\\
= \;& \beta_1(\alpha_1 I + \beta_1 A_3^{-1}A_4)(\alpha_1 I + \beta_1 A_3^{-1}A_4)^{-1} = \beta_1 I. \;\square
\end{aligned}$$

Now, combining Lemma 3.2 with the expression in (10), we can easily show the first relation of the Lemma 3.3. Similarly, using the second equality of (7) we can derive the second relation of the same lemma.

LEMMA 3.3. *Let the assumptions of Lemma 3.1 hold and the matrices T, A_1, A_3 S_1, and S_2 be invertible. Then, we have*

$$T^{-1}\begin{bmatrix} A_1 \\ A_3 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1' \\ (-N^{-1}M)^{n-1}\begin{bmatrix}\alpha_2 I \\ \beta_2 I\end{bmatrix} \\ \vdots \\ (-N^{-1}M)\begin{bmatrix}\alpha_2 I \\ \beta_2 I\end{bmatrix} \\ I \end{bmatrix}(-\beta_1)([\beta_1 I, -\alpha_1 I](-N^{-1}M)^n\begin{bmatrix}\alpha_2 I \\ \beta_2 I\end{bmatrix})^{-1}$$

and

$$T^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ A_3 \\ A_1 \end{bmatrix} = \begin{bmatrix} I \\ (-M^{-1}N)\begin{bmatrix}\alpha_1 I \\ \beta_1 I\end{bmatrix} \\ \vdots \\ (-M^{-1}N)^{n-1}\begin{bmatrix}\alpha_1 I \\ \beta_1 I\end{bmatrix} \\ C_2' \end{bmatrix}(-\beta_2)([\beta_2 I, -\alpha_2 I](-M^{-1}N)^n\begin{bmatrix}\alpha_1 I \\ \beta_1 I\end{bmatrix})^{-1}$$

where C_1' and C_2' are matrices of order $2m$ that can be uniquely determined.

3.2. Derivation of the block Jacobi Iteration Matrix. In this section we consider the Dirichlet problem for Poisson equation on the rectangular domain Ω and the Ω splitting defined in Section 2. We use the bicubic Hermite collocation technique to discretize the corresponding continuous GSS PDE subproblems. To simplify the discussion, in the sequel, we use a uniform mesh with $m+1$ y-grid points and $l+1$ x-grid points for each subdomain. Moreover, it is assumed that the overlaps $\Omega_i \cap \Omega_{i+1}$, $i=1,\ldots,k-1$ are of equal size with (l_0+1) x-grid points in each of them, $h = h_x = \frac{d-c}{m} = \frac{b-a}{n} = h_y$ and $n = lk-(k-1)l_0$. In order to make the entries of the collocation coefficient matrix independent of the mesh size h, the basis functions for the standard bicubic Hermite collocation are modified as in [9], and instead of imposing the artificial boundary conditions

$$\begin{cases} \omega_l u_i(x) + (1-\omega_l)\frac{\partial u_i(x)}{\partial n} = \omega_l u_{i-1}(x) + (1-\omega_l)\frac{\partial u_{i-1}(x)}{\partial n}, & x \in \Gamma_{il}, \\ \omega_r u_i(x) + (1-\omega_r)\frac{\partial u_i(x)}{\partial n} = \omega_r u_{i+1}(x) + (1-\omega_r)\frac{\partial u_{i+1}(x)}{\partial n}, & x \in \Gamma_{ir}, \end{cases}$$

we impose

$$\begin{cases} \omega_l' u_i(x) + (1-\omega_l')h\frac{\partial u_i(x)}{\partial n} = \omega_l' u_{i-1}(x) + (1-\omega_l')h\frac{\partial u_{i-1}(x)}{\partial n}, & x \in \Gamma_{il}, \\ \omega_r' u_i(x) + (1-\omega_r')h\frac{\partial u_i(x)}{\partial n} = \omega_r' u_{i+1}(x) + (1-\omega_r')h\frac{\partial u_{i+1}(x)}{\partial n}, & x \in \Gamma_{ir}. \end{cases}$$

It is worth noticing that

$$\omega_l = \frac{\omega_l'}{h-\omega_l' h + \omega_l'} \quad \text{and} \quad \omega_r = \frac{\omega_r'}{h - \omega_r' h + \omega_r'}.$$

To form the corresponding linear system we use Papatheodorou's ordering (see also [9]) to order the unknowns and the equations. Therefore, the original problem (without applying the GSS scheme) leads to the solution of the linear system $Ay = f$ with the unknown and the right hand side vector being $[y_1^T, y_2^T, \ldots, y_{2n}^T]^T$ and $[f_1^T, f_2^T, \ldots, f_{2n}^T]^T$ respectively, where y_i and f_i are vectors of length $2m$. More specifically, the components of y_{2n} and y_{2i-1}, $i = 1, \ldots, n$, are the approximate values of u and $h\frac{\partial u}{\partial y}$ at the nodes on the corresponding x-grid line while y_1 and y_{2i}, $i = 1, \ldots, n-1$, are the approximate values of $h\frac{\partial u}{\partial x}$ and $h^2\frac{\partial^2 u}{\partial x \partial y}$ at the nodes on the corresponding x-grid line. The enhanced linear system (4), $\bar{A}\bar{y} = \bar{f}$, after eliminating the unknowns associated with the values of u_i and $\frac{\partial u_i}{\partial y}$ on the artificial boundaries by using the artificial boundary conditions from each subproblem, is expressed in a block form as follows

$$\begin{bmatrix} D_1 & U & & & \\ L & D_2 & U & & \\ & \ddots & \ddots & \ddots & \\ & & L & D_2 & U \\ & & & L & D_k \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \vdots \\ \bar{y}_k \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \vdots \\ \bar{f}_k \end{bmatrix}. \tag{11}$$

In (11),
$D_1 = T(m, l, 0, 1, \frac{\omega_r'-1}{\omega_r'}, 1)$, $D_2 = T(m, l, \frac{1-\omega_l'}{\omega_l'}, 1, \frac{\omega_r'-1}{\omega_r'}, 1)$ and $D_k = T(m, l, \frac{1-\omega_l'}{\omega_l'}, 1, 0, 1)$, while U is a matrix of block order $2l$ with $A_3, A_1, \frac{1-\omega_r'}{\omega_r'}A_3, \frac{1-\omega_r'}{\omega_r'}A_1$ as its $(2l-1, 2l_0)$, $(2l, 2l_0)$, $(2l-1, 2l_0+1)$, $(2l, 2l_0+1)$ block elements and 0's elsewhere and L is a matrix of block order $2l$ with $A_1, A_3, \frac{\omega_l'-1}{\omega_l'}A_1, \frac{\omega_l'-1}{\omega_l'}A_3$ as its $(1, 2l-2l_0)$, $(2, 2l-2l_0)$, $(1, 2l-2l_0+1)$, $(2, 2l-2l_0+1)$ block elements and 0's elsewhere. $\bar{y}_i$ and $\bar{f}_i$ are vectors consisting of $4ml$ elements each and their relations to those of the original linear system are the following $\bar{y}_i = [\bar{y}^T_{2(i-1)(l-l_0)+1}, \ldots, \bar{y}^T_{2(i-1)(l-l_0)+2l_0-1}, y^T_{2(i-1)(l-l_0)+2l_0}, \ldots, y^T_{2(i-1)(l-l_0)+2l-1}, \bar{y}^T_{2(i-1)(l-l_0)+2l+1}]^T$ $\bar{f}_i = [f^T_{2(i-1)(l-l_0)+1}, \ldots, f^T_{2(i-1)(l-l_0)+2l}]^T$. It is worth noticing that it can be shown that $\bar{y}_i = y_i$ if the matrix $\bar{A}$ on the left hand side of equation (11) is invertible (see, e.g., [12]).

Let J be the block Jacobi iteration matrix associated with the matrix coefficient $\bar{A}$ of (11). To simplify the notation, we assume that

$$D_1^{-1}[0, \ldots, 0, A_3^T, A_1^T]^T = [X_1^T, X_2^T, \ldots, X_{2l}^T]^T, \; D_2^{-1}[A_1^T, A_3^T, 0, \ldots, 0]^T = [Y_1^T, Y_2^T, \ldots, Y_{2l}^T]^T,$$
$$D_2^{-1}[0, \ldots, 0, A_3^T, A_1^T]^T = [Z_1^T, Z_2^T, \ldots, Z_{2l}^T]^T, \quad D_k^{-1}[A_1^T, A_3^T, 0, \ldots, 0]^T = [W_1^T, W_2^T, \ldots, W_{2l}^T]^T,$$

and introduce the new quantities

$$c_l = \frac{1-\omega_l'}{\omega_l'}, \quad c_r = \frac{1-\omega_r'}{\omega_r'}.$$

Then, it is easy to show that the spectrum $\sigma(J)$ of J satisfies $\sigma(J) = \sigma(J') \cup \{0\}$, where

$$J' = \begin{bmatrix} 0 & X & & & & & & \\ Y & 0 & 0 & Z & & & & \\ \bar{Y} & 0 & 0 & \bar{Z} & & & & \\ & & & \ddots & & & & \\ & & & & Y & 0 & 0 & Z \\ & & & & \bar{Y} & 0 & 0 & \bar{Z} \\ & & & & & & W & 0 \end{bmatrix}, \quad \begin{aligned} X &= \begin{bmatrix} X_{2l-2l_0} & c_r X_{2l-2l_0} \\ X_{2l-2l_0+1} & c_r X_{2l-2l_0+1} \end{bmatrix}, \\ \bar{Y} &= \begin{bmatrix} Y_{2l-2l_0} & -c_l Y_{2l-2l_0} \\ Y_{2l-2l_0+1} & -c_l Y_{2l-2l_0+1} \end{bmatrix}, \\ \bar{Z} &= \begin{bmatrix} Z_{2l-2l_0} & c_r Z_{2l-2l_0} \\ Z_{2l-2l_0+1} & c_r Z_{2l-2l_0+1} \end{bmatrix} \end{aligned}$$

$$Y = \begin{bmatrix} Y_{2l_0} & -c_l Y_{2l_0} \\ Y_{2l_0+1} & -c_l Y_{2l_0+1} \end{bmatrix}, Z = \begin{bmatrix} Z_{2l_0} & c_r Z_{2l_0} \\ Z_{2l_0+1} & c_r Z_{2l_0+1} \end{bmatrix}, W = \begin{bmatrix} W_{2l_0} & -c_l W_{2l_0} \\ W_{2l_0+1} & -c_l W_{2l_0+1} \end{bmatrix}.$$

In view of the structure of J', it is not difficult to see, through a similarity transformation using the matrix diag($\begin{bmatrix} I & -c_l I \\ 0 & I \end{bmatrix}$, $\begin{bmatrix} I & c_r I \\ 0 & I \end{bmatrix}$, ...), that $\sigma(J') = \sigma(J'') \cup \{0\}$, where J'' is of the same structure as J' with its entries being

$$\begin{aligned} &X = X_{2l-2l_0} - c_l X_{2l-2l_0+1}, \; Y = Y_{2l_0} + c_r Y_{2l_0+1}, \qquad Z = Z_{2l_0} + c_r Z_{2l_0+1}, \\ &\bar{Y} = Y_{2l-2l_0} - c_l Y_{2l-2l_0+1}, \quad \bar{Z} = Z_{2l-2l_0} - c_l Z_{2l-2l_0+1}, \; W = W_{2l_0} + c_r W_{2l_0+1}. \end{aligned}$$

Applying now Lemma 3.3 we can obtain that

$$\begin{cases} X &= -[I, -c_l I](-M^{-1}N)^{l-l_0} \begin{bmatrix} 0 \\ I \end{bmatrix} ([I, c_r I](-M^{-1}N)^l \begin{bmatrix} 0 \\ I \end{bmatrix})^{-1}, \\ Y &= -[I, c_r I](-N^{-1}M)^{l-l_0} \begin{bmatrix} -c_r I \\ I \end{bmatrix} ([I, -c_l I](-N^{-1}M)^l \begin{bmatrix} -c_r I \\ I \end{bmatrix})^{-1}, \\ Z &= -[I, c_r I](-M^{-1}N)^{l_0} \begin{bmatrix} c_l I \\ I \end{bmatrix} ([I, c_r I](-M^{-1}N)^l \begin{bmatrix} c_l I \\ I \end{bmatrix})^{-1}, \\ \tilde{Y} &= -[I, -c_l I](-N^{-1}M)^{l_0} \begin{bmatrix} -c_r I \\ 1 \end{bmatrix} ([I, -c_l I](-N^{-1}M)^l \begin{bmatrix} -c_r I \\ I \end{bmatrix})^{-1}, \\ \tilde{Z} &= -[I, -c_l I](-M^{-1}N)^{l-l_0} \begin{bmatrix} c_l I \\ 1 \end{bmatrix} ([I, c_r I](-M^{-1}N)^l \begin{bmatrix} c_l I \\ I \end{bmatrix})^{-1}, \\ W &= -[I, c_r I](-N^{-1}M)^{l-l_0} \begin{bmatrix} 0 \\ I \end{bmatrix} ([I, -c_l I](-N^{-1}M)^l \begin{bmatrix} 0 \\ I \end{bmatrix})^{-1}. \end{cases}$$

To simplify the notation further, we restrict ourselves to considering the case $c_r = c_l$. That is, we assume that the artificial boundary conditions are of the same type. Then, using the fact that $(M^{-1}N) = \mathrm{diag}(I, -I)(N^{-1}M)\mathrm{diag}(I, -I)$, it is shown that $X = W$, $Y = \bar{Z}$ and $Z = \bar{Y}$. Consequently, we take that $\sigma(J) = \sigma(-G_k) \cup \{0\}$, where

$$G_k = \begin{bmatrix} 0 & X & & & & & & \\ Y & 0 & 0 & Z & & & & \\ Z & 0 & 0 & Y & & & & \\ & & & \ddots & & & & \\ & & & Y & 0 & 0 & Z \\ & & & Z & 0 & 0 & Y \\ & & & & & X & 0 \end{bmatrix}, \tag{12}$$

$$\text{with } \begin{cases} X &= [I, c_r I](-N^{-1}M)^{l-l_0} \begin{bmatrix} 0 \\ I \end{bmatrix} ([I, -c_r I](-N^{-1}M)^l \begin{bmatrix} 0 \\ I \end{bmatrix})^{-1}, \\ Y &= [I, c_r I](-N^{-1}M)^{l-l_0} \begin{bmatrix} -c_r I \\ I \end{bmatrix} ([I, -c_r I](-N^{-1}M)^l \begin{bmatrix} -c_r I \\ I \end{bmatrix})^{-1}, \\ Z &= [I, -c_r I](-N^{-1}M)^{l_0} \begin{bmatrix} -c_r I \\ 1 \end{bmatrix} ([I, -c_r I](-N^{-1}M)^l \begin{bmatrix} -c_r I \\ I \end{bmatrix})^{-1}. \end{cases}$$

Note that G_k is a $2(k-1) \times 2(k-1)$ block matrix.

4. One-Dimensional Case.

4.1. The One-Parameter GSS. First, we consider the case where ω_l and ω_r are the same in each overlapping region. In this section, we consider the GSS algorithm (3) together with the cubic Hermite collocation discretization scheme for the boundary value problem

$$\begin{cases} u_{xx} = f, \ a < x < b, \\ u(a) = g_a, \ u(b) = g_b. \end{cases}$$

For this problem, we have $A_1 = -2\sqrt{3}$, $A_2 = -1-\sqrt{3}$, $A_3 = 2\sqrt{3}$ and $A_4 = -1+\sqrt{3}$, where the A_i's are defined in Section 3.1. Since these entities are scalars and not matrices of order $2m$, we can now write $(-N^{-1}M)$ explicitly. Simple computations show that $(-N^{-1}M) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. In turn, this implies that $(-N^{-1}M)^j = \begin{bmatrix} 1 & -j \\ 0 & 1 \end{bmatrix}$. Therefore, after some simplification of the previously found expressions takes place we can obtain that for the case $c_r = c_l$

$$\sigma(J) = \sigma(-G_k) \cup \{0\},$$

where G_k is the following matrix of order $2(k-1)$

$$G_k = \begin{bmatrix} 0 & \frac{(l-l_0)-c_r}{l+c_r} & & & & & \\ \frac{l-l_0}{l+2c_r} & 0 & 0 & \frac{l_0+2c_r}{l+2c_r} & & & \\ \frac{l_0+2c_r}{l+2c_r} & 0 & 0 & \frac{l-l_0}{l+2c_r} & & & \\ & & & \ddots & & & \\ & & & \frac{l-l_0}{l+2c_r} & 0 & 0 & \frac{l_0+2c_r}{l+2c_r} \\ & & & \frac{l_0+2c_r}{l+2c_r} & 0 & 0 & \frac{l-l_0}{l+2c_r} \\ & & & & & \frac{(l-l_0)-c_r}{l+c_r} & 0 \end{bmatrix}.$$

From the above expression, it is readily observed that G_k is block 2-cyclic consistently ordered or weakly cyclic of index 2 [14] (see also [16] or [1]), therefore $\sigma(G_k) = \sigma(-G_k)$. It is worth mentioning that in [5] a matrix of precisely the same structure is considered and recurrence relationships to minimize the spectral radius of G_k are obtained. However, for the cases $k = 4$ and 5 the expressions that can be obtained are very difficult to handle, while for $k > 5$ the equations that can be obtained can not be solved analytically. We have exactly the same situation. In the present work as in [5], it is shown that $\rho(G_k)$ can be made zero for $c_r = l - l_0$ and for the case $k = 2$ or $k = 3$. Thus, we have the theorem below.

THEOREM 4.1. *For $k = 2, 3$ and $c_r = l - l_0$, we have $\rho(J) = 0$.*

For the case $k > 3$, the analysis in [5] holds except that the expressions for the corresponding entries of the G_k matrix are different. However, in order to go a step further in the direction of determining the optimal value of c_r we shall focus on two issues: i) determine the interval of c_r for which the block Jacobi method converges and ii) determine a genuine subinterval of the interval in (i) in which the optimal c_r lies. For this we state and prove the following theorem.

THEOREM 4.2. *Under the assumptions made and the notation used so far the following relation holds*

$$\rho(J) (= \rho(G_k)) < 1$$

if and only if $c_r \in (-\frac{l_0}{2}, \infty)$. *Moreover, the minimum (optimal) value of* $\rho(J)$ *is attained at some* $c_r \in (l - l_0, \infty)$.

Proof. First, we consider the case $c_r < -l$. Since G_k is similar to

$$G_k' = \begin{bmatrix} 0 & \frac{(l-l_0)-c_r}{l+c_r} & & & & & \\ \frac{l-l_0}{l+2c_r} & 0 & 0 & -\frac{l_0+2c_r}{l+2c_r} & & & \\ -\frac{l_0+2c_r}{l+2c_r} & 0 & 0 & \frac{l-l_0}{l+2c_r} & & & \\ & & & \ddots & & & \\ & & & \frac{l-l_0}{l+2c_r} & 0 & 0 & -\frac{l_0+2c_r}{l+2c_r} \\ & & & -\frac{l_0+2c_r}{l+2c_r} & 0 & 0 & \frac{l-l_0}{l+2c_r} \\ & & & & & \frac{(l-l_0)-c_r}{l+c_r} & 0 \end{bmatrix}$$

and $-G_k'$ is an irreducible nonnegative matrix with all nonzero entries strictly increasing with c_r in $(-\infty, -l)$, it follows that $\rho(G_k)$ is strictly increasing. Moreover, it is easy to show that $lim_{c_r \to -\infty} \rho(G_k) = 1$. Consequently, we obtain $\rho(J) > 1$. In case $-l < c_r < -\frac{l}{2}$, we have

$$\begin{aligned} |\det(G_k)| &= |\det(G_k[e_2, e_1, e_4, e_3, \ldots, e_{2k-2}, e_{2k-3}])| \\ &= (\tfrac{l-l_0-c_r}{l+c_r})^2 |2\tfrac{l-l_0}{l+2c_r} - 1|^{k-2} > 1. \end{aligned}$$

Note that the inequality above is satisfied because $\frac{l-l_0-c_r}{l+c_r} > 1$ and $\frac{l-l_0}{l+2c_r} < 0$. Therefore, at least one of the eigenvalues of G_k must have modulus greater than 1, implying that $\rho(J) > 1$. In the case $-\frac{l}{2} < c_r \leq -\frac{l_0}{2}$, G_k' is an irreducible nonnegative matrix with all nonzero entries strictly decreasing with c_r increasing. Specifically, we have $\lim_{c_r \to (-l_0/2)^-} \rho(G_k) = 1$. This implies $\rho(J) \geq 1$. For the case $c_r > -\frac{l_0}{2}$ and $c_r \neq l - l_0$ we have $\rho(J) < 1$ because G_k is irreducible and the absolute sum of the first row is less than $||G_k||_\infty = 1$. As for the specific case $c_r = l - l_0$, G_k is reducible, since its first and last rows are null vectors. However, after deleting the first and last two rows as well as columns, the reduced matrix is irreducible and its spectral radius is the same as that of G_k. Then, following the same arguments as previously, we obtain again $\rho(J) < 1$. Coming to the second assertion of the present theorem, it is apparent that the minimum value of $\rho(J)$ is attained for some $c_r \in (-\frac{l_0}{2}, \infty)$. However, to obtain the genuine subinterval mentioned in the statement of the theorem a much deeper theoretical analysis, based on a number of other statements, is required. This analysis is presented in the Appendix. □

Note 1 We have $\omega_r = \frac{1}{1+c_r h}$, since $c_r = \frac{1-\omega_r'}{\omega_r'}$ and $\omega_r = \frac{\omega_r'}{h-\omega_r' h+\omega_r'}$. Thus the convergence interval in terms of ω $(= \omega_r = \omega_l)$ is $(0, \frac{2}{2-l_0 h}) \supset (0, 1]$ and the optimum occurs for some ω in the interval $(0, \frac{1}{1+(l-l_0)h}) \subset (0, 1)$. In addition, as $h \to 0^+$ the convergence interval tends to $(0, 1]$ while the interval in which the optimum occurs tends to $(0, 1)$.

Note 2 The problem of determining a "better" interval in which the optimum c_r lies than the one already obtained, i.e., $(l - l_0, \infty)$, is an open problem that is being investigated. However, a number of numerical experiments have shown that the value $c_r = l - l_0$ (i.e., $\omega = \frac{1}{1+(l-l_0)h}$) is a good approximation to the optimal value of c_r.

4.2. The Multi-Parameter GSS. As we have observed, there are many choices for the parameters ω in algorithm (3) and therefore in the linear system (11). Here, we shall consider the most general case, that is the one where there are two pairs of parameters $\omega_l^{(i)}$ and $\omega_r^{(i)}$ introduced for each subdomain Ω_i. Let $c_l^{(i)}$ and $c_r^{(i)}$ be defined in the same way as c_l and c_r were defined from ω_l and ω_r before. Let J be the block Jacobi iteration matrix associated with (11). Then, following similar analysis to that in Section 4.1, we get

$$\sigma(J) = \sigma(-G_k) \cup \{0\}$$

where

$$G_k = \begin{bmatrix} 0 & X & & & & & & \\ Y_2 & 0 & 0 & \tilde{Y}_2 & & & & \\ Z_2 & 0 & 0 & \bar{Z}_2 & & & & \\ & & & \ddots & & & & \\ & & & Y_{k-1} & 0 & 0 & \tilde{Y}_{k-1} \\ & & & Z_{k-1} & 0 & 0 & \bar{Z}_{k-1} \\ & & & & & \tilde{X} & 0 \end{bmatrix}$$

and

$$X = \frac{l-l_0-c_l^{(2)}}{l+c_r^{(1)}}, \quad Y_i = \frac{l-l_0-c_r^{(i-1)}+c_r^{(i)}}{l+c_r^{(i)}+c_l^{(i)}}, \ \tilde{Y}_i = \frac{l_0+c_r^{(i-1)}+c_l^{(i)}}{l+c_r^{(i)}+c_l^{(i)}},$$
$$\tilde{X} = \frac{l-l_0-c_r^{(k-1)}}{l+c_l^{(k)}}, \ Z_i = \frac{l_0+c_r^{(i)}+c_l^{(i+1)}}{l+c_r^{(i)}+c_l^{(i)}}, \quad \bar{Z}_i = \frac{l-l_0+c_l^{(i)}-c_l^{(i+1)}}{l+c_r^{(i)}+c_l^{(i)}}.$$

Following the same approach as in the proof of [5, Theorem 3.1], it can be shown that the next theorem holds.

THEOREM 4.3. *Let $c_l^{(i)} = (i-1)(l-l_0)$, $c_r^{(i)} = (k-i)(l-l_0)$, $i = j, \ldots, k$, where j is any integer in $\{1, \ldots, k\}$, and the remaining parameters $c_r^{(i)}$ and $c_l^{(i)}$ be any numbers such that $l + c_r^{(1)} \neq 0$ and $l + c_r^{(i)} + c_l^{(i)} \neq 0$. Then, we have $\rho(J) = 0$.*

Note In view of the structure of the corresponding G_k matrix, it is observed that among all the sets of parameters the set $c_l^{(i)} = (i-1)(l-l_0)$, $c_r^{(i)} = (k-i)(l-l_0)$, $i = 1, \ldots, k$, minimizes the maximum order of the Jordan blocks of G_k which is $k-1$. We have also observed from a number of experiments carried out that the maximum order of Jordan blocks affects very slightly the number of iterations required to achieve a specified accuracy.

5. Two-Dimensional Case. We consider the Poisson equation

$$\Delta u = f \ in \ \Omega$$

with boundary conditions

$$u \ = g \ on \ \partial\Omega$$

where Ω is a rectangle. For this problem, we have that $A_1 = -rT_1 + tT_2$, $A_2 = sT_1 + wT_2$, $A_3 = rT_1 + \bar{t}T_2$ and $A_4 = \bar{s}T_1 + \bar{w}T_2$, where

$$T_i = \left[\begin{array}{cc|cc} a_1 & a_2 & a_3 & -a_4 \\ a_3 & a_4 & a_1 & -a_2 \end{array}\right]_{(2m)}, i = 1, 2,$$

with

	a_1	a_2	a_3	a_4
T_1	t	w	$\bar{t}$	$\bar{w}$
T_2	$-r$	s	r	$\bar{s}$

where $r = 2\sqrt{3}, \quad s = -1-\sqrt{3}, \quad t = \frac{1}{2} + \frac{2}{3\sqrt{3}}, \; w = \frac{3+\sqrt{3}}{36}.$

Note that $\bar{t}$ denotes the "conjugate" of $t = t_1 + t_2\sqrt{3}$, i.e. $\bar{t} = t_1 - t_2\sqrt{3}$ and T_i is defined as in [9].

Following the proof of Lemma 5.1 in [6], it can be shown that there exists a nonsingular matrix V such that $VT_2 = DVT_1$ with

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2m}) = \mathrm{diag}\left(-12,\ -36,\ x_1^+, x_1^-, \ldots, x_{m-1}^+, x_{m-1}^-\right),$$

where

$$\begin{aligned} x_j^{\pm} &= \frac{12[(8+\cos\theta_j) \pm \sqrt{43+40\cos\theta_j - 2(\cos\theta_j)^2})]}{-7+\cos\theta_j}, \\ \theta_j &= \frac{j\pi}{m}. \end{aligned}$$

Thus, it follows that there exist nonsingular matrices P and Q of order $2m$ such that $PA_iQ = D_i$, $i = 1, \ldots, 4$, and each D_i is a diagonal matrix. Then, it can be shown that

$$\begin{aligned} &\begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} (-N^{-1}M) \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \\ = &\begin{bmatrix} (D_1D_4 - D_2D_3)^{-1} & 0 \\ 0 & (D_1D_4 - D_2D_3)^{-1} \end{bmatrix} \begin{bmatrix} D_1D_2 - D_3D_4 & D_4^2 - D_2^2 \\ D_3^2 - D_1^2 & D_1D_2 - D_3D_4 \end{bmatrix}. \end{aligned}$$

Let $\bar{P} = [e_1, e_{2m+1}, e_2, e_{2m+2}, \ldots, e_{2m}, e_{4m}]$, where e_i has 1 as its ith component and 0's elsewhere. It is clear that

$$\bar{P} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} (-N^{-1}M) \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \bar{P} = \mathrm{diag}(\bar{D}_1, \ldots, \bar{D}_{2m}),$$

where $\bar{D}_j = \begin{bmatrix} \bar{d}_{1j} & \bar{d}_{2j} \\ \bar{d}_{3j} & \bar{d}_{1j} \end{bmatrix}$ with the property $\det(\bar{D}_j) = 1$, $j = 1, \cdots, 2m$. On the other hand, it is easily found out that

$$\bar{d}_{1j} = \frac{(-r + t\lambda_j)(s + w\lambda_j) - (r + \bar{t}\lambda_j)(\bar{s} + \bar{w}\lambda_j)}{(-r + t\lambda_j)(\bar{s} + \bar{w}\lambda_j) - (s + w\lambda_j)(r + \bar{t}\lambda_j)} = \frac{432 - 192\lambda_j + 7\lambda_j^2}{432 + 24\lambda_j + \lambda_j^2}.$$

Also, it is observed that $\bar{d}_{1j} > 1$ because all λ_j's are less than 0. Hence, we may set $\bar{d}_{1j} = cosh\theta_j$ for some $\theta_j > 0$. Using the fact that $\det(\bar{D}_j) = 1$, it is proved that $\bar{d}_{2j}\bar{d}_{3j} = \sinh\theta_j$. Therefore there exist nonsingular matrices $Q_j = \begin{bmatrix} 1 & 1 \\ \sqrt{\frac{\bar{d}_{3j}}{\bar{d}_{2j}}} & -\sqrt{\frac{\bar{d}_{3j}}{\bar{d}_{2j}}} \end{bmatrix}$ such that $Q_j^{-1}\bar{D}_jQ_j = \mathrm{diag}(\cosh\theta_j - \sinh\theta_j, \cosh\theta_j + \sinh\theta_j)$. Let $\bar{Q} = \mathrm{diag}(Q_1, \ldots, Q_{2m})$, then we obtain that

$$\begin{aligned} &\bar{Q}^{-1}\bar{P} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} (-N^{-1}M)^p \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \bar{P}\bar{Q} \\ &= \mathrm{diag}(\cosh p\theta_1 - \sinh p\theta_1, \cosh p\theta_1 + \sinh p\theta_1, \ldots, \cosh p\theta_{2m} - \sinh p\theta_{2m}, \cosh p\theta_{2m} + \sinh p\theta_{2m}). \end{aligned}$$

Thus, using the equation

$$Q_j \begin{bmatrix} \cosh p\theta_j - \sinh p\theta_j & 0 \\ 0 & \cosh p\theta_j + \sinh p\theta_j \end{bmatrix} Q_j^{-1} = \begin{bmatrix} \cosh p\theta_j & -\sqrt{\frac{\bar{d}_{2j}}{\bar{d}_{3j}}} \sinh p\theta_j \\ -\sqrt{\frac{\bar{d}_{3j}}{\bar{d}_{2j}}} \sinh p\theta_j & \cosh p\theta_j \end{bmatrix},$$

we can summarize the above discussion and conclude that

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} (-N^{-1}M)^p \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \tilde{D}_{1p} & \tilde{D}_{2p} \\ \tilde{D}_{3p} & \tilde{D}_{1p} \end{bmatrix}, \tag{13}$$

where

$$\left\{ \begin{array}{rcl} \tilde{D}_{1p} & = & \operatorname{diag}(\cosh p\theta_1, \cosh p\theta_2, \ldots, \cosh p\theta_{2m}) \\ -\tilde{D}_{2p} & = & \operatorname{diag}(\sqrt{\frac{\bar{d}_{21}}{\bar{d}_{31}}} \sinh p\theta_1, \sqrt{\frac{\bar{d}_{22}}{\bar{d}_{32}}} \sinh p\theta_2, \ldots, \sqrt{\frac{\bar{d}_{2m}}{\bar{d}_{3m}}} \sinh p\theta_{2m}) \\ -\tilde{D}_{3p} & = & \operatorname{diag}(\sqrt{\frac{\bar{d}_{31}}{\bar{d}_{21}}} \sinh p\theta_1, \sqrt{\frac{\bar{d}_{32}}{\bar{d}_{22}}} \sinh p\theta_2, \ldots, \sqrt{\frac{\bar{d}_{3m}}{\bar{d}_{2m}}} \sinh p\theta_{2m}) \end{array} \right. . \tag{14}$$

Applying now (13) to express X, Y and Z of the matrix G_k defined in (12) we can come to the following conclusion.

THEOREM 5.1. *The spectrum of the block Jacobi iteration matrix associated with the enhanced linear system (11) for the two-dimensional Poisson model problem is given by*

$$\sigma(J) = \sigma(-G'_k) \cup \{0\}.$$

The matrix G'_k is of the same structure as G_k defined in (12) with

$$\begin{array}{c} X = (\tilde{D}_{2(l-l_0)} + c_\tau \tilde{D}_{1(l-l_0)})(\tilde{D}_{2l} - c_\tau \tilde{D}_{1l})^{-1}, \\ Y = (\tilde{D}_{2(l-l_0)} - c_\tau^2 \tilde{D}_{3(l-l_0)})(\tilde{D}_{2l} - 2c_\tau \tilde{D}_{1l} + c_\tau^2 \tilde{D}_{3l})^{-1}, \\ Z = (\tilde{D}_{2l_0} - 2c_\tau \tilde{D}_{1l_0} + c_\tau^2 \tilde{D}_{3l_0})(\tilde{D}_{2l} - 2c_\tau \tilde{D}_{1l} + c_\tau^2 \tilde{D}_{3l})^{-1} \end{array}$$

and $\tilde{D}_{ip}, i = 1, 2, 3$, being defined in (14).

COROLLARY 5.2. *The SAM algorithm converges for all possible combinations of l, l_0 and k.*

Proof. In the traditional approach to SAM (c_τ is chosen to be zero) X, Y and Z can be simplified to

$$X = Y = \operatorname{diag}(\frac{\sinh(2l - 2l_0)\theta_1}{\sinh 2l\theta_1}, \ldots, \frac{\sinh(2l - 2l_0)\theta_{2m}}{\sinh 2l\theta_{2m}})$$

and

$$Z = \operatorname{diag}(\frac{\sinh 2l_0\theta_1}{\sinh 2l\theta_1}, \ldots, \frac{\sinh 2l_0\theta_{2m}}{\sinh 2l\theta_{2m}}).$$

Then it follows from

$$\frac{\sinh(2l - 2l_0)\theta + \sinh 2l_0\theta}{\sinh 2l\theta} = \frac{2\cosh(l - 2l_0)\theta \sinh l\theta}{2\sinh l\theta \cosh l\theta} = \frac{\cosh(l - 2l_0)\theta}{\cosh l\theta}$$

that

$$\rho(G'_k) \leq ||G'_k||_\infty = \max_{\theta_i, i=1,\ldots,2m} \frac{\cosh(l - 2l_0)\theta_i}{\cosh l\theta_i} < 1. \quad \square$$

Note It is well understood from the proof of the corollary above that the amount of l_0/l is a key factor that affects the convergence rate of SAM.

TABLE 1

The convergence of the SAM, 1-GSS and m-GSS methods for one-dimensional model boundary value problem with exact solution $u(x) = e^{-100(x-0.1)^2}(x^2 - x)$. The number of subdomains (k), grid size, number of iterations taken for the splitting scheme to converge and the discretization error are displayed for two different domain splittings.

	1-GSS($c_r = l - l_0$)				SAM				m-GSS			
	$l_0 = l/2$		$l_0 = 1$		$l_0 = l/2$		$l_0 = 1$		$l_0 = l/2$		$l_0 = 1$	
$(k, grid)$	iter	error	iter	error	iter	error	iter	error	iter	error	iter	error
(2,10)	2	6.64e-3	2	6.64e-3	6	6.64e-3	13	6.64e-3	2	6.64e-3	2	6.64e-3
(2,22)	2	8.42e-5	2	8.42e-5	4	8.42e-5	10	8.36e-5	2	8.42e-5	2	8.42e-5
(3,17)	3	3.27e-4	3	3.27e-4	7	3.27e-4	23	3.27e-4	3	3.27e-4	3	3.27e-4
(3,29)	3	2.67e-5	3	2.67e-5	5	2.67e-5	38	2.57e-5	2	2.67e-5	3	2.67e-5
(4,26)	4	4.84e-5	4	4.84e-5	8	4.82e-5	73	4.72e-5	3	4.84e-5	3	4.84e-5
(4,46)	3	4.28e-6	3	4.25e-6	7	4.05e-6	125	6.12e-6	3	4.31e-6	3	4.30e-6

6. Numerical Examples. In this section, we present a number of numerical examples to verify the theoretical results obtained in the previous sections. We use the zero vector as the initial guess of the solution of the enhanced linear system (11). We display the maximum error $||u - u_{\frac{1}{n}}||_\infty$ based on an $n \times n$ grid of points, where u is the theoretical solution of the continuous problem and $u_{\frac{1}{n}}$ is the computed one. The iteration step ($iter$) denotes the number of the block Gauss-Seidel iterations required to satisfy the stopping criterion $\frac{||y^{(j)} - y^{(j-1)}||_\infty}{||y^{(j)}||_\infty} < \epsilon$, where $y^{(j)}$ is the jth iteration approximation to the solution of the linear system (11) and $\epsilon = 1.0e-6$ and $\epsilon = 5.0e-6$ for 1-D and 2-D problems, respectively. Throughout, we denote by 1-GSS the one parameter GSS and m-GSS the multi-parameter GSS.

For the one-dimensional case, we are using the boundary value problem

$$\begin{aligned} u''(x) &= f(x), \ x \in (0,1), \\ u(0) &= g_0, \quad u(1) = g_1, \end{aligned}$$

where $f(x)$, g_0 and g_1 are selected such that the exact solution is $u(x) = e^{-100(x-0.1)^2}(x^2 - x)$. We apply both the traditional SAM and the one-parameter GSS with $c_r = l - l_0$. This is the optimal value for the case $k = 2$ or 3 both for minimum and maximum overlaps. For the multi-parameter GSS and the domain split with minimum overlap, among the many choices of the parameters $c_r^{(i)}$ and $c_l^{(i)}$, we choose $c_l^{(i)} = (i-1)(l - l_0)$, and $c_r^{(i)} = (k - i)(l - l_0)$, $i = 1, \ldots, k$. The numerical results obtained are summarized in Table 1. The data in Table 1 verify the theoretical observation that the convergence of m-GSS and 1-GSS with optimal c_r is independent of the overlap for all k and $k = 2$, 3 respectively. The numerical results indicate that this behavior of 1-GSS holds for $k > 3$. All data indicate that GSS outperforms the traditional SAM.

Figure 2 displays the relation between the number of iterations and the parameters c_r for the 1-GSS for four pairs of (k, l), where k denotes the number of subdomains and l denotes the number of subintervals in each subdomain. Our experiments are carried out for maximum (half) overlap. From these plots, we can conclude that $c_r = l - l_0$ is indeed the optimal value for the case $k = 3$ while the optimal value of c_r for $k > 3$ is on the right of $l - l_0$ as this was shown in Section 4.1. Moreover, it appears that the optimal value of c_r can be expressed as $\alpha(l - l_0)$ for some number α, which seems to

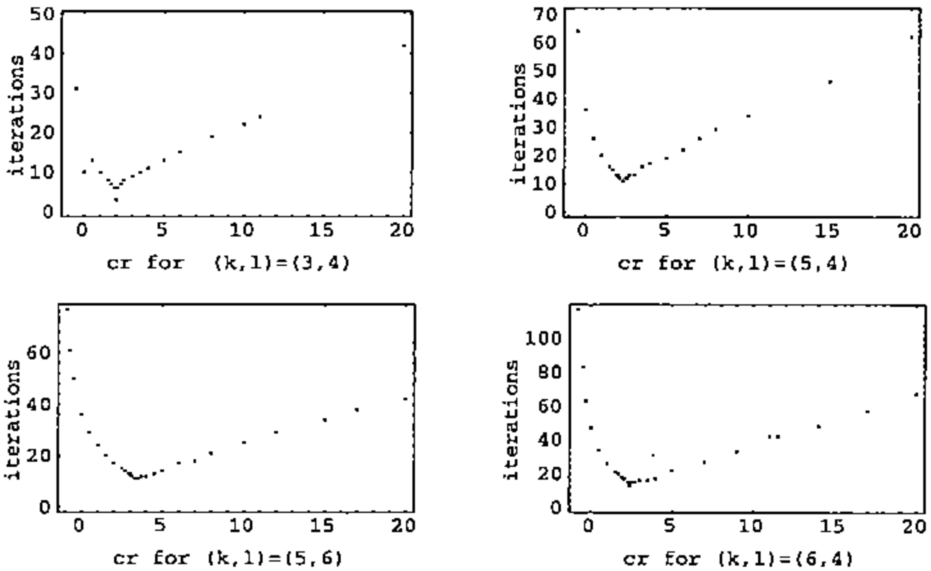

FIG. 2. *Plots of the number of iterations required by 1-GSS to achieve convergence. The number of iterations versus the parameter c_r for the one-dimensional problem with maximum overlap and different pairs (k,l), where k denotes the number of subdomains and l the number of subintervals in each subdomains.*

increase with k. Also, from the same plots, we can observe that the traditional SAM (case $c_r = 0$) has a very poor convergence rate compared to that of the one-parameter GSS with $c_r = l - l_0$.

For the two-dimensional case, let the domain Ω be the unit square. First we consider the Poisson equation

$$
\begin{aligned}
u_{xx} + u_{yy} &= f(x,y), \quad (x,y) \in \Omega \\
u(x,y) &= g(x,y), \quad (x,y) \in \partial\Omega,
\end{aligned} \tag{15}
$$

where $f(x,y)$ and $g(x,y)$ are selected so that

$$
u(x,y) = 10e^{-100(x-0.1)^2}(x^2 - x)e^{-100(y-0.1)^2}(y^2 - y).
$$

Then, using the same exact solution we consider the more general PDE, taken from [10],

$$
\begin{gathered}
[2 + (y-1)e^{-y^4}]u_{xx} + [1 + \tfrac{1}{(1+4x^2)}]u_{yy} + 5[x(x-1) + (y-0.3)(y-0.7)]u = f, \text{ in } \Omega, \\
u(x,y) = g \text{ on } \partial\Omega.
\end{gathered} \tag{16}
$$

For the Poisson problem using a 2-way splitting (k=2), we can numerically derive all the eigenvalues of the corresponding Jacobi matrix by Theorem 5.1 for any l and l_0. In Figure 3, the relations between the spectral radii and the parameters ω, c_r are depicted for maximum overlap. In these figures, we can see that for a fixed parameter ω the spectral radius decreases with the value of l increasing. In addition, we observe that for a given l the minimum of the spectral radius always occurs near $\omega = 0.8$. Note that $c_r = \frac{3l}{8}$, which is close to the value $l - l_0 = l/2$ for maximum overlap and for small values of l.

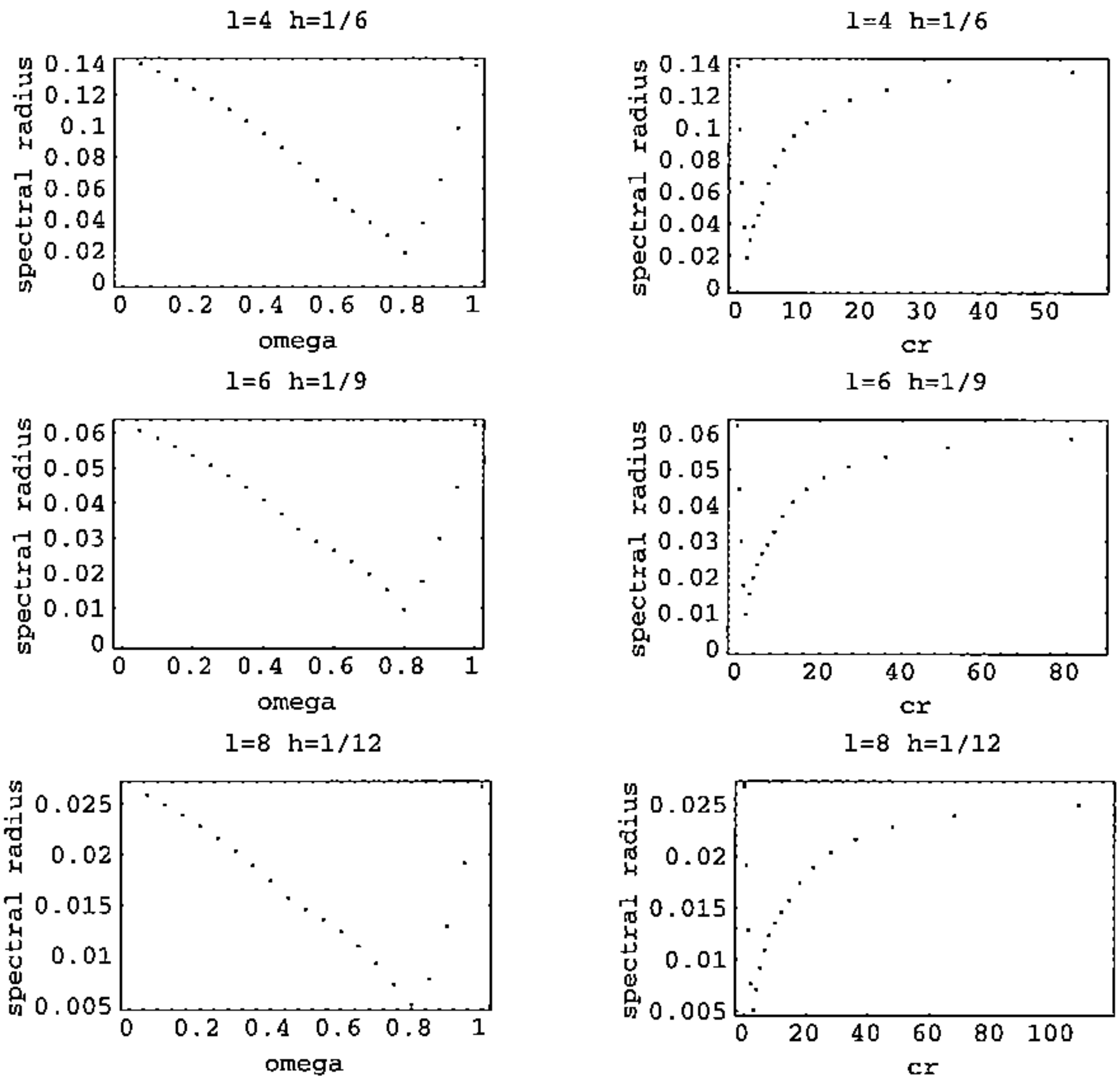

FIG. 3. *The Jacobi matrix spectral radius versus ω and c_r parameters for the two-dimensional Poisson model problem with 2-way domain splitting ($k = 2$) and maximum overlap.*

optimal value of c_r for $k > 3$ should lie. The determination of the optimal parameter c_r in question is still an open problem but our analysis suggests that this optimal value is a number greater than $l - l_0$. For the two-dimensional case, our analysis consists of Theorem 5.1. This theorem improves our understanding of the relation between the parameter c_r and the convergence properties of the corresponding block Jacobi iteration matrix. In addition, it provides a simpler matrix G'_k to determine this relation. In particular, for $k = 2$ we have experimented with several combinations of l and ω or c_r with $l_0 = l/2$ to obtain the corresponding spectral radius as shown in Figure 3. From the experiments, we can see that $\omega = 0.8$ is independent of l and may give an almost optimal convergence rate among cases with l being fixed.

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TABLE 2

The convergence of the 1-GSS ($c_r = l - l_0$) and SAM ($c_r = 0$) for the Dirichlet model problem (15) with minimum and maximum overlap splittings of the PDE domains. The exact solution is $u(x,y) = 10\phi(x)\phi(y)$, where $\phi(x) = e^{-100(x-0.1)^2}(x^2 - x)$. The number of subdomains (k), the number of subintervals in each subdomain (l), grid size, number of iterations and discretization error are displayed for both splittings.

	$l_0 = l/2$					$l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
$(k, grid)$	l	iter	error	iter	error	l	iter	error	iter	error
(2,10x10)	6	3	7.74e-3	3	7.74-3	5	4	7.74e-3	5	7.74e-3
(2,22x22)	14	2	1.53e-4	2	1.53e-4	11	2	1.53e-4	2	1.53e-4
(3,17x17)	8	2	6.08e-4	3	6.08e-4	6	5	6.08e-4	7	6.08e-4
(3,29x29)	14	2	4.69e-5	2	4.69e-5	10	6	4.69e-5	10	4.61e-5
(4,26x26)	10	3	9.09e-5	7	9.09e-5	7	6	9.08e-5	20	8.94e-5

	$l = 4, l_0 = l/2$					$l = 4, l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
k	grid	iter	error	iter	error	grid	iter	error	iter	error
2	7x7	3	1.74e-2	5	1.74e-2	8x8	5	2.29e-2	10	2.29e-2
3	9x9	5	1.56e-2	7	1.56e-2	11x11	4	3.19e-3	6	3.19e-3
4	11x11	4	3.19e-3	5	3.19e-3	14x14	6	7.34e-4	10	7.33e-4
5	13x13	5	3.24e-4	6	3.24e-4	17x17	7	6.08e-4	16	6.06e-4
6	15x15	4	8.51e-4	8	8.50e-4	20x20	7	2.21e-4	24	2.18e-4

	$l = 6, l_0 = l/2$					$l = 6, l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
k	grid	iter	error	iter	error	grid	iter	error	iter	error
2	10x10	3	7.74e-3	3	7.74e-3	12x12	3	1.13e-3	4	1.13e-3
3	13x13	2	3.24e-4	3	3.24e-4	17x17	5	6.08e-4	7	6.08e-4
4	16x16	3	7.64e-4	4	7.64e-4	22x22	6	1.53e-4	16	1.51e-4
5	19x19	5	3.21e-4	12	3.21e-4	27x27	7	7.37e-5	30	7.18e-5

Tables 2 and 3 display the convergence behavior of SAM and 1-GSS for the above two problems for different splittings and grids with maximum and minimum overlap. Since the theoretical values of the optimal parameter c_r for these problems are not known, we experimented with the value $c_r = l - l_0$ which corresponds to the case $k = 2$ as this can be seen from Figure 3. The entries in Tables 2 and 3 indicate that, regarding the accuracy of the solutions as well as the number of subdomains and the size of the overlapping region, the one-parameter GSS with $c_r = l - l_0$ requires less number of iterations than the traditional SAM.

7. Concluding Remarks and Discussion. In this paper we have studied the parameterized GSS at a discrete equation level (matrix formulation), coupled with the cubic Hermite collocation discretization scheme for both the one- and the two-dimensional model problems. For the one-dimensional problem, we have found the optimal parameter values which correspond to the smallest possible spectral radius of the block Jacobi iteration matrix associated with (11) for $k = 2, 3$ in the one-parameter case and for all k in the multi-parameter case. We also determined the interval in which the parameter c_r must lie so that the convergence of the Jacobi method would be guaranteed. Moreover, a subinterval of the previous one was found in which the

TABLE 3

The convergence of the 1-GSS ($c_r = l - l_0$) and SAM ($c_r = 0$) for the general PDE problem (16) with minimum and maximum overlap splittings of the unit square. The exact solution is $u(x,y) = 10\phi(x)\phi(y)$, where $\phi(x) = e^{-100(x-0.1)^2}(x^2 - x)$. The number of subdomains (k), the number of subintervals in each subdomain (l), grid size, number of iteration and discretization error are displayed for both splittings.

	$l_0 = l/2$					$l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
$(k, grid)$	l	iter	error	iter	error	l	iter	error	iter	error
(2,10x10)	6	3	7.72e-3	3	7.72e-3	5	4	7.72e-3	5	7.72e-3
(2,22x22)	14	2	1.54e-4	2	1.54e-4	11	2	1.54e-4	2	1.54e-4
(3,17x17)	8	3	6.08e-4	3	6.08e-4	6	5	6.08e-4	6	6.08e-4
(3,29x29)	14	2	4.70e-5	2	4.70e-5	10	6	4.70e-5	10	4.66e-5
(4,26x26)	10	3	9.12e-5	5	9.12e-5	7	6	4.70e-5	17	8.98e-5

	$l = 4, l_0 = l/2$					$l = 4, l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
k	grid	iter	error	iter	error	grid	iter	error	iter	error
2	7x7	3	1.75e-2	5	1.75e-2	8x8	5	2.31e-2	10	2.31e-2
3	9x9	5	1.57e-2	7	1.57e-2	11x11	4	3.17e-3	6	3.17e-3
4	11x11	3	3.17e-3	5	3.17e-3	14x14	6	7.34e-4	9	7.33e-4
5	13x13	5	3.24e-4	6	3.23e-4	17x17	6	6.08e-4	15	6.06e-4
6	15x15	6	8.50e-4	8	8.49e-4	20x20	6	2.21e-4	18	2.18e-4

	$l = 6, l_0 = l/2$					$l = 6, l_0 = 1$				
		$c_r = l - l_0$		$c_r = 0.$			$c_r = l - l_0$		$c_r = 0.$	
k	grid	iter	error	iter	error	grid	iter	error	iter	error
2	10x10	3	7.72e-3	3	7.72e-3	12x12	3	1.13e-3	3	1.13e-3
3	13x13	3	3.24e-4	3	3.24e-4	17x17	5	6.08e-4	6	6.08e-4
4	16x16	3	7.63e-4	3	7.63e-4	22x22	6	1.54e-4	14	1.54e-4
5	19x19	6	3.21e-4	11	3.21e-4	27x27	7	7.40e-5	38	7.37e-5

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Appendix. In this appendix we prove the second part of Theorem 4.2. This is accomplished after a number of statements presented as lemmas are proved.

LEMMA A.1. *Let $\rho(B)$ be the spectral radius of any matrix B of even order. Then, we have* $\det(B-\lambda I) > 0$ *for all* $\lambda > \rho(B)$.

Proof. Let $p(\lambda) = \det(B-\lambda I)$. It is clear that $p(\lambda)$ is a monic polynomial. It then follows that $p(\lambda) \to \infty$ as $\lambda \to \infty$. Suppose that $det(B-\rho_1 I) \leq 0$ for some $\rho_1 > \rho(B)$, then there must exist a number $\rho_2 \geq \rho_1$ such that $p(\rho_2) = 0$. This, however, contradicts the fact that $\rho(B)$ is the spectral radius of B. □

For the following statements, we define the three matrices below

$$B_{2n} = \begin{bmatrix} 0 & t & & & & & & \\ t & 0 & 0 & 1-t & & & & \\ 1-t & 0 & 0 & t & & & & \\ & & & \ddots & & & & \\ & & & t & 0 & 0 & 1-t & \\ & & & 1-t & 0 & 0 & t & \\ & & & & & t & 0 & \end{bmatrix}, A_{2n} = \begin{bmatrix} 0 & t & & & & & & \\ t & 0 & 0 & 1-t & & & & \\ 1-t & 0 & 0 & t & & & & \\ & & & \ddots & & & & \\ & & & t & 0 & 0 & 1-t & \\ & & & 1-t & 0 & 0 & t & \\ & & & & & 1 & 0 & \end{bmatrix},$$

and

$$C_{2n-1} = \begin{bmatrix} t & 0 & 1-t & & & & & & \\ 1-t & -\rho(t) & t & & & & & & \\ 0 & t & -\rho(t) & 0 & 1-t & & & & \\ 0 & 1-t & 0 & -\rho(t) & t & & & & \\ & & & & \ddots & & & & \\ & & & & t & -\rho(t) & 0 & 1-t \\ & & & & 1-t & 0 & -\rho(t) & t \\ & & & & & & t & -\rho(t) \end{bmatrix},$$

where $0 < t < 1$. Notice that the indices denote the order of the corresponding matrices and n is any nonnegative integer.

LEMMA A.2. *For the spectral radii $\rho(A_{2n})$ and $\rho(B_{4n})$ of A_{2n} and B_{4n} we have $\rho(A_{2n}) \leq \rho(B_{4n})$.*

Proof. Let $[x_1, x_2, \ldots, x_{2n}]^T$ be the eigenvector of the irreducible nonnegative matrix A_{2n} corresponding to the spectral radius $\rho(A_{2n})$. Then, it is easy to show that the vector $[x_1, x_2, \ldots, x_{2n}, x_{2n}, \ldots, x_2, x_1]^T$ is an eigenvector of B_{4n} with corresponding eigenvalue $\lambda = \rho(A_{2n})$ from which it follows that $\rho(A_{2n}) \leq \rho(B_{4n})$. □

LEMMA A.3. *If $\rho(t)$ is the spectral radius of B_{2n}, then* $\det(C_{2k-1}) > 0$ *for all* $k = 1, \ldots, n$.

Proof. It is easy to see that for $k = 1$ our assertion holds. For $k > 1$, we expand $\det(C_{2k-1})$ with respect to its first row to get $\det(C_{2k-1}) = t\det(B_{2(k-1)} - \rho(t)I) + (1-$

$t)^2\det(C_{2(k-1)-1})$. Since $B_{2(k-1)}$ is a principal submatrix of the nonnegative matrix B_{2n}, it will be $\rho(B_{2(k-1)}) \leq \rho(B_{2n})$. On the other hand, by Lemma A.1 we obtain that $\det(B_{2(k-1)} - \rho(t)I) \geq 0$ for $k = 2, \ldots, n$. Thus, the proof of the present lemma is completed by induction on k. □

LEMMA A.4. *The spectral radius $\rho(t) := \rho(B_{2n}(t))$ of B_{2n} strictly increases with t for $0 < t < 1$.*

Proof. We first observe that B_{2n} is a nonnegative and irreducible matrix as $0 < t < 1$. Then, it follows that $\rho(t)$ is a simple eigenvalue of B_{2n} and $\det(B_{2n} - \rho(t)I) = 0$. Taking the derivative of $\det(B_{2n} - \rho(t)I) = 0$ with respect to t and using the following two basic properties

$$\begin{aligned}\frac{d}{dt}(\det([a_1, a_2, \ldots, a_{2n}]^T)) =\ & \det([\tfrac{d}{dt}a_1, a_2, \ldots, a_{2n}]^T) + \det([a_1, \tfrac{d}{dt}a_2, \ldots, a_{2n}]^T) + \ldots \\ &+ \det([a_1, a_2, \ldots, \tfrac{d}{dt}a_{2n}]^T)\end{aligned}$$

and

$$\begin{aligned}&\det([a_1, \ldots, a_{i-1}, a_i + b_i, a_{i+1}, \ldots, a_{2n}]^T) \\ =\ &\det([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{2n}]^T) + \det([a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{2n}]^T),\end{aligned}$$

with each a_i or b_i denoting a vector of length $2n$, we obtain

$$(17)\quad 2\{\sum_{k=0}^{n-1} \det(B_{2k} - \rho(t)I)\det(B_{2(n-1)-2k} - \rho(t)I)\}\rho'(t)\rho(t) + \sum_{k=1}^{2n} \det(\bar{B}_k) = 0.$$

In (17), $\det(B_0 - \rho(t)I)$ is defined to be 1 and $\bar{B}_k$ is a matrix with the same entries as $B_{2n} - \rho(t)I$ except that its entries in the positions (k, k), $(k, k + 2(-1)^k)$ and $(k, k-(-1)^k)$ are 0, -1 and 1, respectively. Since $\rho(t)$ is a simple root of $\det(B_{2n} - \lambda I) = 0$ and from Lemma A.1 we have $\det(B_{2n} - \lambda I) > 0$ for $\lambda > \rho(t)$, it is implied that $\frac{d}{d\lambda}\det(B_{2n} - \lambda I) > 0$ for $\lambda = \rho(t)$. Thus, the coefficient of $\rho'(t)$ in (17) is positive, because it is equal to the value of $\frac{d}{d\lambda}\det(B_{2n} - \lambda I)$ at $\lambda = \rho(t)$. So, to show that $\rho'(t) > 0$, it suffices to show that $\sum_{k=1}^{2n}\det(\bar{B}_k) < 0$. For the terms corresponding to $k = 1$ and $k = 2n$, it is easy to show that $\det(\bar{B}_1) = \det(\bar{B}_{2n}) = -\det(C_{2n-1})$. Thus, from Lemma A.3 we obtain $\det(\bar{B}_1) = \det(\bar{B}_{2n}) < 0$. For the remaining terms, we shall consider pairs such that $k = 2i$ and $k = 2i + 1$ simultaneously. First, we switch the $(2i)$th row of $\bar{B}_{2i+1}$ with its $(2i+1)$st one and multiply the new $(2i)$th row by -1. Note that the determinant of the resulting matrix is equal to $\det(\bar{B}_{2i+1})$ and only its $(2i+1)$st row differs from that of the matrix $\bar{B}_{2i}$. Then, we apply the second property above to get $\det(\bar{B}_{2i}) + \det(\bar{B}_{2i+1}) = \det(T_i)$, where T_i is the matrix of order $2n$ shown below

$$\begin{array}{r} \\ \\ \\ \\ (2i)th \rightarrow \\ (2i+1)st \rightarrow \\ \\ \\ \\ \\ \end{array}
\begin{bmatrix}
-\rho(t) & t & & & & & & & & \\
t & -\rho(t) & 0 & 1-t & & & & & & \\
1-t & 0 & -\rho(t) & t & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & 1 & 0 & 0 & -1 & & & \\
 & & & 1 & -\rho(t) & -\rho(t) & 1 & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & t & -\rho(t) & 0 & 1-t \\
 & & & & & & 1-t & 0 & -\rho(t) & t \\
 & & & & & & & & t & -\rho(t)
\end{bmatrix}.$$

Expanding the determinant of T_i with respect to its $(2i)$th row, we obtain

$$(18) \det(T_i) = -\{\det(A_{2n-2i} - \rho(t)I)\det(C_{2i-1}) + \det(A_{2i} - \rho(t)I)\det(C_{2n-(2i+1)})\}.$$

Now, we will derive another expression for $\det(T_i)$. For this, first we add the $(2i)$th row of $B_{2n} - \rho(t)I$ to its $(2i+1)$st one. The resulting matrix has all its rows the same as those of the matrix T_i except for its $2i$th row; its determinant is zero because $\det(B_{2n} - \rho(t)I) = 0$. Then, we multiply its $(2i)$th row by $1/(1-t)$ and add the new matrix to the matrix T_i. Note that the determinant of the resulting matrix is the same as $\det(T_i)$. In view of the structure of the new resulting matrix, we can easily get

$$(19) \qquad \det(T_i) = \frac{1}{1-t}\det(A_{2n-2i} - \rho(t)I)\det(A_{2i} - \rho(t)I).$$

From (19), we readily see that $\det(T_i) = \det(T_{n-i})$. In the discussion that follows, we assume that $1 < i \le j$, where $j = [\frac{n}{2}]$ is the largest integer not exceeding $n/2$. Since A_{2i} is a principal submatrix of A_{2j}, we get $\rho(A_{2i}) \le \rho(A_{2j})$. Furthermore, by Lemma A.2 we have $\rho(A_{2i}) \le \rho(A_{2j}) \le \rho(B_{4j}) \le \rho(B_{2n})$. It then follows that $\det(A_{2i} - \rho(t)I) \ge 0$ by Lemma A.1. If we assume that $\det(T_i) > 0$ then both $\det(A_{2n-2i} - \rho(t)I)$ and $\det(A_{2i} - \rho(t)I)$ are nonnegative by (19). On the other hand, from Lemma A.3 we know that both $\det(C_{2i-1})$ and $\det(C_{2n-(2i+1)})$ are positive, therefore, the right hand side of (18) is nonpositive, which contradicts the assumption $\det(T_i) > 0$. Consequently, we obtain $\det(\bar{B}_{2i}) + \det(\bar{B}_{2i+1}) \le 0$ for $1 \le i < n$. This together with the negativeness of the first and the last terms completes the proof. □

Let us now consider the case where $c_r \in (-\frac{l_0}{2}, l - l_0)$. Since G_k is a nonnegative matrix, $\rho(G_k) \ge \rho(B_{2(k-3)})$ for $t = \frac{l-l_0}{l+2c_r}$ because $B_{2(k-3)}$ is a principal submatrix of G_k. On the other hand, it is easy to show that $\rho(G_k) = \rho(B_{2(k-3)})$ for $t = \frac{l-l_0}{3l-2l_0}$ and $c_r = l - l_0$. Therefore, applying Lemma A.4 and the fact that $\frac{l-l_0}{l+2c_r} > \frac{l-l_0}{3l-2l_0}$ for $c_r \in (-\frac{l_0}{2}, l - l_0]$ lead us to the conclusion that the spectral radius of the matrix G_k in (12) with $c_r = l - l_0$ is less than any one corresponding to $c_r \in (-\frac{l_0}{2}, l - l_0)$. This result with the first part of Theorem 4.2 show that the optimal value of c_r is attained at some point in the interval $[l - l_0, \infty)$. This completes the proof of the second part of Theorem 4.2.