*DISTRIBUTION OF STRESSES
AND DISPLACEMENTS WITHIN
AND UNDER LONG ELASTIC
AND VISCOELASTIC
EMBANKMENTS*

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by
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Progress Report

DISTRIBUTION OF STRESSES AND DISPLACEMENTS WITHIN AND UNDER LONG ELASTIC AND VISCOELASTIC EMBANKMENTS

TO: Dr. G. A. Leonards, Director
Joint Highway Research Project

November 28, 1967

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FROM: Harold L. Michael, Associate Director
Joint Highway Research Project

Project No. C-36-5F

The attached Progress Report is submitted for the HPR-1(5) Part II research study entitled "Long-Time Deformation of Clay." The report is entitled "Distribution of Stresses and Displacements Within and Under Long Elastic and Viscoelastic Embankments." It has been authored by Mr. George Y. Baladi, research assistant on our staff, under the direction of Professor William H. Ferloff.

The purpose of this phase of the research was to develop a theory which would accurately predict stresses and displacements within and under long elastic and viscoelastic embankments. The embankments were assumed to be homogeneous, isotropic and continuous with the underlying material. A theory was developed which permits solution of a wide variety of plane strain problems with a variety of boundary conditions.

The report is presented to the Board for information and for the record. It will also be submitted to the ISHC and the BPR for their review, comments and approval.

Respectfully submitted,

Harold L. Michael/jgs

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Progress Report

DISTRIBUTION OF STRESSES AND DISPLACEMENTS WITHIN
AND UNDER LONG ELASTIC AND VISCOELASTIC EMBANKMENTS

by

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Project: C-36-5F

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Purdue University
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TABLE OF CONTENTS

	Page
LIST OF TABLES	vi
LIST OF FIGURES	vii
LIST OF SYMBOLS	xi
ABSTRACT	xiii
INTRODUCTION	1
REVIEW OF LITERATURE	3
Application of the Theory of Elasticity to Soil Media	3
Viscoelastic Material Representation	11
Review of Previous Contributions on the Specific Problem . .	12
Scope of the Problem Considered Herein	16
PART I - ELASTIC ANALYSIS OF THE EMBANKMENT	18
THE EQUATIONS FOR THE PLANE STRAIN THEORY OF ELASTICITY . . .	19
Introduction	19
Definition of the Stresses	21
Stress Function for Plane-Strain	22
COMPLEX REPRESENTATION OF THE GENERAL SOLUTION OF THE PLANE-STRAIN PROBLEM INCLUDING THE WEIGHT OF THE MATERIAL . .	25
Introduction	25
Complex Representation of the Stress Function	25
Complex Representation of Stresses	29
Complex Representation of the Displacements	32
APPLICATION OF CONFORMAL MAPPING	36
Introduction	36
The Transformation of the Geometric Boundary into a Half-Space	36
Operations Connected with Conformal Mapping into the Half-Space	39
The Complex Representation of Stresses in the t-plane	49

TABLE OF CONTENTS (Cont'd)

	Page
The Complex Representation of Displacements in the t-plane	50
Formula and Propositions for the t-plane	52
THE BOUNDARY CONDITIONS	57
The Boundary Condition in the z-plane	57
The Boundary Condition in the t-plane	58
THE SOLUTION OF THE PROBLEM	62
Introduction	62
Cauchy Integral	62
Determination of the Stresses by Using Cauchy Integral . .	63
Determination of the Displacements	
RESULTS AND DISCUSSION	75
Introduction	75
Analysis of the Vertical Normal Stress Distribution . . .	75
Analysis of the Horizontal Normal Stress Distribution . .	81
Analysis of the Horizontal and Vertical Shear Stress . . .	84
Analysis of the Maximum Shear Stress	89
Relationship of Results to In-Situ Stresses	96
Effect of Results on Stress Path Determination	97
PART II - ANALYSIS OF THE VISCOELASTIC EMBANKMENT	101
INTRODUCTION	102
TRIAXIAL LOAD AND CREEP TESTS	104
Material	104
Preparation of Soil Specimen	104
Experimental Apparatus and Procedure	107
RESULTS AND DISCUSSION ON CONFINED CREEP TESTS	109
Axial Strain	109
Creep Compliance	111
RESULTS AND DISCUSSION	114
Effect of Time on the Embankment Shape	114
Analysis of Vertical and Horizontal Displacement	116
PART III - APPLICATION OF METHOD TO ANALYSIS OF AN EXCAVATION	122

TABLE OF CONTENTS (cont'd)

	Page
INTRODUCTION	123
THE TRANSFORMATION OF THE GEOMETRIC BOUNDARY INTO A HALF-SPACE	125
RESULTS AND DISCUSSION	130
CONCLUSIONS AND RECOMMENDATIONS	139
Conclusions	139
Recommendations	141
BIBLIOGRAPHY	142
APPENDIX A	146
Typical Test Data	146
APPENDIX B	148
Computer Program for Calculation of the Transformation Integral Constants	148
APPENDIX C	151
Computer Program for the Determination of Stresses and Displacements Within and Under Linear Elastic and Viscoelastic Embankments	151
APPENDIX D	163
Design Charts for the Elastic Embankment	163
VITA	175

LIST OF TABLES

Table	Page
1. The Values of F and β for Various Shapes of Embankment. . .	45
2. Classification Properties of the Soil	104
Appendix	
Table	
A1. Typical Test Data	147

LIST OF FIGURES

Figure		Page
1.	Tensile Stress-Strain Curves from Beam Tests on Limestone Clay	5
2.	Stress-Strain Curves from Unconfined Compression Tests on Samples from Beams of Willard Dam Test Embankment	6
3.	Stress-Strain Curves from Unconfined Samples from Beams of Limestone Clay	7
4.	Comparison of Stress-Strain Curves from Unconfined Compression Tests on In-Situ and Recompacted Samples . . .	8
5.	Tensile Stress Versus Axial Tensile Strain for Kaolin-Sand Samples Under Direct Tension	9
6.	Problem Considered	14
7.	Two Dimensional Embankment	20
8.	An Element from the Boundary	20
9.	The Geometrical Transformation	37
10.	The Constant of the Transformation Integral for $L/H = 0.01$	40
11.	The Modulus of the Transformation Integral for $L/H = 0.01$	41
12.	The Modulus of the Transformation Integral as a Function of L/H	42
13.	The Modulus of the Transformation Integral as a Function of H/L	43
14.	The Value of the Constant, F , Which Precedes the Transformation Integral	44
15.	The Transformation of a Vector from the z -plane into the t -plane	48

Figure	Page
16. An Arbitrary Path in the Original and Transformed Regions	54
17. "Fictitious Boundary Stress" Due to the Weight of the Embankment	60
18. Contours for Vertical Stress	76
19. Distribution of Vertical Stress Along Vertical Sections for Varying L/H Ratios	78
20. Distribution of Vertical Normal Stress, $\sigma_v/\gamma H$ on the Base of the Embankment for Varying α , and L/H Ratios . . .	80
21. Contours for Horizontal Stress	82
22. Distribution of Horizontal Stress Along Vertical Sections for Varying L/H Ratios	83
23. Distribution of Horizontal Stress Along Vertical Sections for Varying L/H Ratios	85
24. Contours of Shear Stress, τ_{xy}	86
25. Distribution of Shear Stress, $\tau_{xy}/\gamma H$ on the Base of the Embankment for Varying α , and L/H Ratios	88
26. Contours for Max. Shear, $\tau_{max}/\gamma H$	90
27. Magnitude and Location of Maximum ($\tau_{max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 15^\circ$	91
28. Magnitude and Location of Maximum ($\tau_{max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 30^\circ$	92
29. Magnitude and Location of Maximum ($\tau_{max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 45^\circ$	93
30. Magnitude and Location of Maximum ($\tau_{max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 60^\circ$	94
31. Magnitude and Location of Maximum ($\tau_{max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 75^\circ$	95
32. Stress Path for Three Points Under the Center Line During Embankment Construction	99
33. Grain Size Distribution for Edgar Plastic Kaolin	105
34. Impact Compaction Curves for Edgar Plastic Kaolin	106

Figure	Page
35. Schematic Diagram of Radial Displacement and Specimen on Triaxial Cell Base	108
36. Creep Curve for Multiply Loaded Specimens of EPK	110
37. Creep Curve for Multiply Loaded Specimens of EPK	112
38. Effect of Time on Embankment Shape	115
39. Effect of Time on Vertical Displacements of Selected Points on and Within the Embankment	117
40. Effect of Time on the Relative Vertical Displacement Between the Points A and E	118
41. Effect of Time on the Relative Vertical Displacement Between the Points E and C	119
42. Effect of Time on the Relative Vertical Displacement Between the Points A and B	120
43. Effect of Time on Horizontal Displacements of Selected Points on the Embankment	121
44. Application of Stress on Base of Excavation	124
45. The Geometrical Transformation	125
46. The Modulus of the Transformation Integral as a Function of L/H	127
47. The Modulus of the Transformation Integral as a Function of H/L	128
48. The Value of the Constant, F , Which Precedes the Transformation Integral	129
49. Reduction in Vertical Stress Along Center Line Due to Excavation	131
50. Net Reduction in Computed Vertical Stress Along Center Line Due to Excavation, Compared to Boussinesq Value . . .	132
51. Reduction in Vertical Stress Under Edge of Excavation . . .	133
52. Net Reduction in Computed Vertical Stress Under Edge Due to Excavation, Compared to Boussinesq Value	135
53. Contours of Vertical Stress Due to Load on the Base of the Excavation, Weightless Material	136

Figure	Page
54. Net Increase in Vertical Stress Due to Combined Effects of Excavation and Application of Stress at Base of Excavation	137
Appendix	
Figure	
D1. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 15^\circ$, $\mu = 0.3$	164
D2. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 30^\circ$, $\mu = 0.3$	165
D3. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 45^\circ$, $\mu = 0.3$	166
D4. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 60^\circ$, $\mu = 0.3$	167
D5. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 75^\circ$, $\mu = 0.3$	168
D6. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 15^\circ$, $\mu = 0.3$	169
D7. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 30^\circ$, $\mu = 0.3$	170
D8. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 45^\circ$, $\mu = 0.3$	171
D9. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 60^\circ$, $\mu = 0.3$	172
D10. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 75^\circ$, $\mu = 0.3$	173
D11. Influence Diagrams for Shear Stress Along Selected Vertical Section for $\alpha = 45^\circ$, $\mu = 0.3$	174

LIST OF SYMBOLS

x, y	Cartesian coordinates.
z	Complex variable $x + iy$ (z-plane).
$\bar{z}$	Conjugate complex variable $x - iy$.
ξ, η	Orthogonal curvilinear coordinates, in z-plane; rectangular coordinates in t-plane.
t	Complex variable $\xi + i\eta$ (t-plane).
$\bar{t}$	Conjugate complex variable $\xi - i\eta$.
H	Height of the embankment.
$2L$	Width of the top of the embankment.
$\alpha = \frac{\pi}{N}$	Angle between the side of the embankment and the x-axis.
EI	Transformation formula, which transforms the z-plane into the t-plane.
C	Constant precedes the transformation integral EI.
β	Modulus of the transformation integral.
θ	Angle between the axes ξ and x , measured from the lat- ter anti-clockwise.
μ	Poisson's Ratio.
E	Modulus of elasticity.
G	Modulus of elasticity in shear. Modulus of rigidity.
E_c	Creep compliance.
g	Gravitational acceleration.
ρ	Density.
γ	Unit weight.

X, Y	Components of a body force per unit volume.
$\bar{X}, \bar{Y}$	Components of a distributed surface force per unit area.
N, T	Normal and tangential stresses.
σ_x, σ_y	Total normal stress components parallel to x- and y-axes.
σ'_x, σ'_y	Normal stress components due to geometric shape of the embankment.
σ_ξ, σ_η	Normal stress components parallel to ξ - and η -axes.
τ_{xy}	Shearing stress component in z-plane.
$\tau_{\xi\eta}$	Shearing stress component in t-plane.
ϵ_x, ϵ_y	Unit elongations in x- and y-directions.
u, v	Components of displacement in the x, y directions, respectively.
$\epsilon_\xi, \epsilon_\eta$	Unit elongations in ξ - and η -directions.
u, w	Components of displacement in the ξ, η directions, respectively.
$U(x, y)$	Stress function.
$\phi(z), \psi(z)$ $\phi(z), \psi(z)$ }	Complex potentials; functions of the complex variable, $Z = x + iy$.
$\phi(t), \psi(t)$ $\phi(t), \psi(t)$ }	Complex potentials; functions of the complex variable, $t = \xi + i\eta$.
$\bar{\phi}, \bar{\psi}$ $\bar{\phi}, \bar{\psi}$ }	Conjugate complex functions of $\phi, \psi, \bar{\phi}$, and $\bar{\psi}$.
T	Time.
∇^2	Harmonic operator.

ABSTRACT

Baladi, George Youssef, Ph.D., Purdue University, January 1968.
Distribution of Stresses and Displacements Within and Under Long Elastic
and Viscoelastic Embankments. Major Professor: Dr. William H. Perloff.

The research was initiated to predict stresses and displacements within and under long elastic and viscoelastic embankments. In both cases, the embankment was assumed to be homogeneous, isotropic and continuous with the underlying material.

As the first step in the development of a suitable theory, the total stress was defined to be the results of:

- 1 - The stress due to the geometric configuration of the boundary of the embankment.
- 2 - The stress due to the weight of the material.

This definition was used together with the Airy stress function to represent the stresses in terms of analytic functions.

The Schwarz-Christoffel transformation was used to transform both the geometric configuration and the boundary conditions for the purpose of matching the boundary of the embankment, with the stress functions. The Cauchy integral formula was applied to the transformed boundary conditions to yield the stresses and the displacements.

The material of the embankment and the foundation was assumed to behave as a linear viscoelastic body, the time dependent solution for the stresses and the displacements was obtained by applying the quasi-elastic approximation, proposed by Schapery [32]*, to the associated

* The numbers in brackets refer to references given at the end of the text.

elastic solution with creep functions in place of elastic constants.

Time-dependent material properties were selected from the results of triaxial creep tests on a compacted clay soil.

The numerical results of both the elastic and the viscoelastic solution were obtained by the aid of an IBM 7094 digital computer using a Fortran four source program.

The method developed permits solution of a wide variety of plane strain problems with variety of boundary conditions.

The distribution of vertical stress underneath the elastic embankment is more nearly uniform than is usually assumed.

The horizontal normal stresses was found to be significantly lower than those computed by the normal loading approximation.

The horizontal shear stress acted along the base of the embankment, was found to have a significant value, while it is assumed zero by the Boussinesq analysis.

The net increase in vertical normal stress within the compensated elastic foundation was found to be smaller than that usually assumed.

The viscoelastic analysis of the embankment indicates that, for the soil investigated, sixty per cent of the displacements which occur within the first ten years, is due to the elastic effect and ninety per cent of these displacements occur in the first month.

INTRODUCTION

The distribution of stresses and displacements within and under earth embankments, due to the embankment weight, is of great interest to civil engineers in a variety of applications. Consideration of deformations within embankments, consolidation of underlying compressible materials and other engineering applications require determination of the distribution of these stresses and/or displacements. Such a problem has been under active investigation by various investigators. These investigations are mostly concentrated on the use of the linear theory of elasticity. However, almost all these studies are primarily attempts of approaching a solution.

The stress distribution within a body depends on many parameters, such as:

- 1 - The physical properties of the body, which may be a function of load and deformation, and the stress history.
- 2 - The way in which the body is loaded.
- 3 - The shape of the loaded area.
- 4 - The geometric configuration of the body.

Any attempt of an exact solution to the problem must consider the above parameters properly.

The specific objective of this investigation is to develop a closed form solution to the embankment problem to predict the stress distributions as well as the elastic and viscoelastic displacements.

It is assumed herein that the embankment and the foundation material with which it is continuous, are composed of homogeneous, isotropic, linear viscoelastic material. Further, the embankment is assumed to be sufficiently long so that plane strain conditions apply.

The solution is obtained by transforming the region of the embankment where the solution is unknown, into a half-space where the solution can be found. Application of the Cauchy integral formulas to the boundary conditions permits determination of the stresses.

REVIEW OF LITERATURE

Application of the Theory of Elasticity to Soil Media

The determination of stresses and displacements within a loaded body requires assumptions about the nature of the loaded material and the magnitude of the resulting displacement. One theory available for the determination of such stresses and displacements assumes that the material is continuous, homogeneous and isotropic and that it recovers its original shape completely when the applied loads are removed. The theory based on these assumptions is called the theory of elasticity.

During the past, the theory of elasticity has found considerable application in the solution of engineering problems. But application of this theory in its general form, is frequently difficult, because the relation between stress and strain is generally a mathematical equation which may be described in the form of a power series. Since the behavior described by such a series is nonlinear, further formulation presents some difficulties, therefore further assumptions are introduced by neglecting all the terms of the series of order higher than unity. In this case, the strain is directly proportional to the applied stress and can be described by Hooke's law. The resulting theory is called the theory of linear elasticity. The manner in which the boundary forces are distributed within the soil mass depends on two sets of conditions: first, the way in which the load is placed on the soil, including the shape of the load over that area; second, the physical

properties of the elastic medium.

With a few exceptions there is no construction material whose real mechanical properties are simple enough to be acceptable as a basis for theoretical analysis. Therefore, simplifying assumptions have been made regarding these properties which make it possible to compute the stresses. These assumptions are always to a certain extent at variance with reality.

Taylor [33] said for the theory of elasticity to be applicable, the real requirement is not that the material be necessarily elastic, but that there must be constant ratios between the stresses and strains. Therefore, he suggested that the elastic theory may be applied to a non-elastic soil mass, to express the solution in mathematical form, provided such constant ratio exist.

Leonards and Narain [24] conducted bending and compressive tests on a variety of soils, both undisturbed and remolded. Figure 1 shows the results of bending stress-strain curves tests performed on beams (3 in. wide, 2 3/4 in. deep, and 22 1/8 in. long) of compacted limestone clay from Southern Indiana. Figure 2 and Figure 3 show the results of unconfined compressive stress-strain curves tests. These tests were performed on cylindrical specimens, (1.4 in. in diameter by 2.8 in. in height) on the soils mentioned before. Figure 4 shows stress-strain curves for compression tests on both undisturbed and recompacted clay samples. Dash [10] has conducted direct tension tests on consolidated clay samples (2.8 in. long, 1.4 in. in diameter). Figure 5 shows the results of this test. These results of bending compressive and tensile tests indicate that for small strains the resulting stress-strain curves

Figure 1. Tensile Stress-Strain Curves From Beam Tests on Limestone Clay

Figure 2. Stress - Strain Curves From Unconfined Compression Tests On Samples From Beams Of Willard Dam Test Embankment

Figure 3. Stress - Strain Curves From Unconfined Samples From Beams of Limestone Clay

Figure 4. Comparison Of Stress-Strain Curves From Unconfined Compression Tests On In-Situ And Recompacted Samples

Figure 5. Tensile Stress Versus Axial Tensile Strain for Kaolin - Sand Samples Under Direct Tension

are very nearly linear. On the other hand, when the shear failure is imminent, the strains are far from being proportional to the stresses.

An indication of whether theory may be applied in a given situation may be obtained by subjecting soil specimens in the laboratory to anticipated field stresses and plotting the elastic effect of stresses against strains. If straight line plots are obtained the theory of elasticity may be applied. The effect of time may also be important, viz. Figures 2 and 3. This is discussed below.

The application of the theory of elasticity to soils where the stress-strain ratios are functions of depth is awkward and difficult. Holl [18] solved the problem of a concentrated force acting on a surface of semi-infinite mass where the stress-strain ratio varies in the form:

$$E = E_1 z^\lambda$$

where E_1 is the modulus of elasticity at $z = 1$ and

$$\lambda + z = \frac{1}{\mu}$$

where μ is Poisson's ratio.

However, Taylor [33] suggests that in many soils this variation is not so large, and that the overall action is about the same as in the hypothetical case in which the stress-strain relationships are constant and equal to the average value.

The theories of behavior of soils under stress, are very complicated and the geometric configuration of the boundary of the soil mechanics problems are not simple, therefore, the use of the linear theory

of elasticity has been limited to problems which may be idealized to semi-infinite solids with loads on their boundary planes. Terzaghi [34] has pointed out that in applying the theory of elasticity, the investigator needs to know (a) how to compute the physical properties of the soils; (b) how the results thus obtained compare with experience. Nevertheless, the theoretical solution can give us insight into the problem, clarify both the limit of the solution and the boundary conditions; and the stress components at points in the region can be computed from known equations defining the stresses.

Viscoelastic Material Representation

A viscous material is one whose constitutive relations are time-dependent. A viscoelastic material is one which combines elastic and viscous response. As in elasticity, the materials are usually assumed to be linearly viscoelastic in order to simplify their representation and to be able to apply superposition.

Whether or not a real material can be described as linearly viscoelastic must be determined experimentally [21, 38].

For certain problems in linear viscoelasticity the correspondence principle can be used to calculate a time-dependent viscoelastic response from the solution of an "associated" elastic problem. This principle was deduced by Lee [22] for isotropic media and later by Biot [2] for anisotropic material. The basis of this principle is that, with zero initial conditions, the Laplace or Fourier time-transformed viscoelastic field equations and boundary conditions are formally identical with the equations for an elastic body of the same geometry. Thus transformed solutions can be obtained by standard elastic analysis,

and then inverted to obtain the time-dependent response. Lee and Roger [23] and Valanis and Lianis [37] have suggested some numerical methods which do not involve transforms, and are applicable when the relation between the stress and strain is expressed in the form of convolution-type integrals, involving experimentally measurable creep or relaxation functions. The combination of this stress-strain equation with the remaining equations of viscoelasticity, leads, in general, to integro-differential equations governing both space and time dependence of stresses and displacements. The solutions of these equations are approximated by using finite-difference integration with respect to time [37].

Schapery [32] has proposed a method which makes use of the hereditary form of the stress-strain equations. It consists first of calculating the response of a fictitious elastic body which is geometrically similar to the actual viscoelastic body, has the same mechanical loading, and has elastic properties which are numerically equal to the instantaneous relaxation moduli or creep compliances associated with the viscoelastic problem; these mechanical properties are expressed as functions of real time in isothermal problems. Two advantages of this method are that for many problems good accuracy is achieved with only the quasi-elastic approximations, the difference between the exact viscoelastic solution and this method can be neglected; and it is potentially applicable to nonlinear as well as linear problems [32].

Review of Previous Contributions to the Specific Problem Considered

Elastic theory has been used since 1920 to investigate stresses and deflection under long embankments. At that time Carothers [7]

determined the stresses within an homogeneous, isotropic elastic half-space resulting from a "long embankment" loading. It was assumed that the embankment can be represented by a series of independent line loads applied normal to the boundary with a magnitude which varies in such a manner that it graphically resembles an embankment cross-section, as shown in Figure 6(b). He applied superposition on equations developed by Boussinesq [4] and Flamant [15]. These results were presented in tabular form by Jurgenson [19]. In 1940, Newmark [26, 27] used the same assumption above to develop his solution which was superimposed later, by Osterberg [29], to obtain an influence chart for the determination of the magnitude of vertical stresses induced in an elastic half-space by a long embankment loading with a variety of cross-sections. These approaches to the problem do not consider the elastic characteristics of the embankment or the shearing stresses at the interface between the embankment and the foundation due to elastic body restraint. Thus, these solutions do not describe the real physical situation.

Trollope [36], and Davis and Taylor [11] computed the distribution of stresses underneath a granular embankment resting on a foundation which yielded an arbitrary amount. No attempt was made to compute the amount of foundation movement which would be created by the embankment. The impression given by these methods is that the nature of the problem is so complex that a large number of arbitrary assumptions must be introduced. In particular, the assumption of limiting equilibrium is a very discouraging feature of this type of approach to the problem.

Dingwall and Scrivner [12] used a finite difference method, to present an approach to an elastic embankment resting on a rock

(a) Long Symmetric Elastic Embankment
Continuous With Foundation

(b) Normal Loading Approximation

Figure 6. Problem Considered

foundation. This method was used by Carlton to study an elastic embankment continuous with an elastic foundation. Zienkiewicz [40, 43] applied a finite difference method to find the stress distribution within a gravity dam resting on an elastic foundation of different modulus.

Zienkiewicz and Cheung [41, 42] applied the finite element method to find the stresses within buttress dams resting on an elastic foundation. This method has been used by Clough and Chopra [9] and Finn [13] to the study of a triangular dam on a rigid foundation and a rock slope continuous with its elastic foundation, respectively. Brown, [6] and Goodman and Brown [17] investigated the case of a long elastic slope constructed incrementally; it is not clear to what degree their results are influenced by the fact that the compatibility equations are not satisfied by their solution method.

All the solutions obtained by applying numerical methods are approximate and were either restricted to a single embankment cross-section or a complete stress picture was not obtained.

In connection with stability analysis, Brahtz [5] developed a method for analyzing the stability of earth dams. Brahtz made use of equilibrium equations in addition to the assumption that, on the line of symmetry of the embankment, the vertical and horizontal normal stresses were proportional. Brahtz assumed that the boundary was free from external and internal stress. The stress distribution results from this method appear to be arbitrary, because: (a), the only basis of the suggested formula is static equilibrium, which is a necessary, but insufficient, requirement; (b), the boundary conditions are only partially considered; and (c), the notion of constant ratio between the

horizontal and vertical normal stresses acting at every point of the center of symmetry is not properly justified.

Glover [16] has considered the stability of an earth dam with triangular cross section, by dividing the foundation into a series of elastic and plastic regions. The alternating plastic and elastic zones are assumed bounded by straight lines. In each zone, the components of stress are supposed to vary linearly, and continuity of stress is assumed at the boundaries. This approach introduces a large number of arbitrary assumptions, which is a very discouraging feature of this type of approach to the problem of stability of slopes.

Terzaghi [34] described an effort by Rendulic [31] to evaluate the shear stresses transmitted to the foundation material by an embankment. The shearing stresses acting on the base of the embankment were computed by assuming that the embankment material was on the verge of failure and thus the shearing resistance of the embankment was fully mobilized in order to maintain equilibrium. This method applies the Mohr-Coulomb theory to the equations of equilibrium. Consequently, Terzaghi [34] suggested that the magnitude of the computed shear stress at the base was likely to be lower than the actual in-situ stresses.

Finn [14] suggested the use of the Schwarz-Christoffel transformation to map the embankment region, where the solution is very difficult to get, into a semi-infinite region, where the solution can be easily obtained. However, he did not carry out the suggested procedure.

Scope of the Problem Considered Herein

As stated above, the need for an exact solution to an embankment with its foundation is evident. In this investigation, a general exact

method of solving an elastic embankment continuous with the underlying material is developed. In fact, it is believed that this method can be applied to solve many plane-strain problems, under statics as well as dynamic loads. A quasi-elastic approximation is applied to obtain the time-dependent stresses and displacements for a viscoelastic material. The material properties for the viscoelastic response, are evaluated from laboratory test. Numerical results and influence charts for a variety of embankment cross-sections are presented. Some numerical results for a foundation are presented to show the potential of this method.

PART I
ELASTIC ANALYSIS OF THE EMBANKMENT

THE EQUATIONS FOR THE PLANE STRAIN THEORY OF ELASTICITY

Introduction

Let u , v and w be the components of the displacements in the x , y and z directions, respectively. A body is said to be in a state of plane strain, parallel to the plane oxy , Figure 7, if the component w is equal to zero and if the components u and v are independent of z . The problem of the determination of stresses and displacements of a long elastic embankment continuous with its foundation can be formulated analytically by using the equations of plane strain. The solution will be unique if it satisfies the equilibrium, the compatibility equations, and the boundary conditions [28].

The equations of equilibrium for plane strain are:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} - \rho g &= 0 \end{aligned} \tag{1}$$

in which ρg is the unit weight of the material, σ_x and σ_y are the total normal stress components parallel to the x - and y -axes, respectively, and τ_{xy} is the shear stress parallel to the x - and y -axes.

The compatibility equation for plane strain and for constant unit weight is:

Figure 7. Two Dimensional Embankment

Figure 8. An Element from the Boundary

$$\nabla^2(\sigma_x + \sigma_y) = 0 \quad (2)$$

where ∇^2 is the harmonic operator.

The boundary stress equations, Figure 8, are:

$$\begin{aligned} \bar{X} &= \sigma_x \cos(n,x) + \tau_{xy} \cos(n,y) \\ \bar{Y} &= \tau_{xy} \cos(n,x) + \sigma_y \cos(n,y) \end{aligned} \quad (3)$$

where $\bar{X}$ and $\bar{Y}$ are the x and y components, respectively, of a distributed surface force per unit area, $\cos(n,x) = \frac{dy}{ds}$ and $\cos(n,y) = -\frac{dx}{ds}$.

Definition of the Stresses

Define the total stress within the region under consideration as the following:

$$\begin{aligned} \sigma_x &= \sigma'_x + \frac{\mu}{1-\mu} \gamma y \\ \sigma_y &= \sigma'_y + \gamma y \\ \tau_{xy} &= \tau'_{xy} \end{aligned} \quad (4)$$

where σ'_x , σ'_y and τ'_{xy} are the stresses due to the shape of the region only and μ is Poisson's ratio. Therefore:

$$\sigma'_x = \sigma'_y = \tau'_{xy} = 0$$

for a semi-infinite region with no boundary loading. Substituting equations (4) into equations (1) and (2) gives:

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} = 0 \quad (5)$$

$$\frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau'_{xy}}{\partial x} = 0$$

and

$$\nabla^2(\sigma'_x + \sigma'_y) = 0 \quad (6)$$

which are the equilibrium and compatibility equations for a weightless material.

Stress Function for Plane Strain

The first of equations (1) represents the necessary and sufficient condition for the existence of some function $B(x, y)$ such that [35]:

$$\frac{\partial B(x, y)}{\partial x} = -\tau_{xy} \quad , \quad \frac{\partial B(x, y)}{\partial y} = \sigma_x - \frac{\mu}{1-\mu} \gamma y \quad (7)$$

The second of the equations (1) is the necessary and sufficient condition for the existence of some function $A(x, y)$ such that [35]:

$$\frac{\partial A(x, y)}{\partial x} = \sigma_y - \gamma y \quad , \quad \frac{\partial A(x, y)}{\partial y} = -\tau_{xy} \quad (8)$$

Comparison between equations (7) and (8) for τ_{xy} shows that:

$$\frac{\partial A(x, y)}{\partial y} = \frac{\partial B(x, y)}{\partial x} \quad (9)$$

whence follows the existence of some function $U(x, y)$ such that:

$$A(x, y) = \frac{\partial U(x, y)}{\partial x} \quad , \quad B(x, y) = \frac{\partial U(x, y)}{\partial y} \quad (10)$$

Substituting these values for $A(x, y)$ and $B(x, y)$ into equations (7) and (8), it can be seen that the stress can be expressed as:

$$\begin{aligned}\sigma_x &= \frac{\partial^2 U(x, y)}{\partial y^2} + \frac{\mu}{1-\mu} \gamma y \\ \sigma_y &= \frac{\partial^2 U(x, y)}{\partial x^2} + \gamma y \\ \tau_{xy} &= - \frac{\partial^2 U(x, y)}{\partial x \partial y}\end{aligned}\tag{11}$$

or, in terms of the "geometric" stress:

$$\begin{aligned}\sigma'_x &= \frac{\partial^2 U(x, y)}{\partial y^2} \\ \sigma'_y &= \frac{\partial^2 U(x, y)}{\partial x^2} \\ \tau'_{xy} &= \frac{\partial^2 U(x, y)}{\partial x \partial y}\end{aligned}\tag{12}$$

This fact was first noticed by G. B. Airy (1862). The function $U(x, y)$ is called a stress function, or the Airy stress function [35].

Assume that the stress components σ_x , σ_y and τ_{xy} are single-valued and continuous functions with continuous derivatives up to and including the second order throughout the region occupied by the body. Therefore, the function $U(x, y)$ must have continuous derivatives up to and including the fourth order and these derivatives, from the second order

onwards, must be single-valued functions throughout the region of the embankment.

If $U(x, y)$ has these properties, the function σ_x , σ_y and τ_{xy} will satisfy the equilibrium equations. However, for the uniqueness of the solution, the compatibility equation (2) must also be satisfied. Recognizing that:

$$\sigma_x + \sigma_y = \nabla^2 U(x, y) + \frac{\mu}{1-\mu} \gamma y$$

leads to:

$$\nabla^4 U(x, y) = 0 \quad (13)$$

or

$$\frac{\partial^4 U(x, y)}{\partial x^4} + 2 \frac{\partial^4 U(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x, y)}{\partial y^4} = 0$$

This equation is the biharmonic equation and its solution consists of biharmonic functions, which are the only functions that satisfy the biharmonic equation, the derivatives of which are continuous up to and including the fourth order and are single-valued throughout the region under consideration [35].

COMPLEX REPRESENTATION OF THE GENERAL SOLUTION
OF THE PLANE STRAIN PROBLEM
INCLUDING THE WEIGHT OF THE MATERIAL

Introduction

As mentioned above, the problem under consideration is formulated in terms of a biharmonic differential equation. Direct solution of such an equation is generally awkward and difficult. Therefore, the transformation of the biharmonic equation to a harmonic one by use of analytic complex function is expedient. This is done below by expressing stresses and displacements as complex harmonic functions in a manner similar to that developed by Muskhelishvili [25].

Complex Representation of the Stress Function

Introducing the following notation:

$$\nabla^2 U(x, y) = \sigma_x + \sigma_y - \frac{\mu}{1-\mu} \gamma y = \sigma'_x + \sigma'_y = P(x, y) \quad (14)$$

then

$$\nabla^4 U(x, y) = \nabla^2 P(x, y) = 0 \quad (15)$$

Thus the function $P(x, y)$ is a harmonic function.

Let $Q(x, y)$ be a harmonic function, conjugate to $P(x, y)$; i.e., a function satisfying the Cauchy-Riemann conditions

$$\frac{\partial P(x, y)}{\partial x} = \frac{\partial Q(x, y)}{\partial y}$$

(continued)

$$\frac{\partial P(x, y)}{\partial y} = - \frac{\partial Q(x, y)}{\partial x} \quad (16)$$

This function is determined for a given $P(x, y)$ apart from an arbitrary constant term. Then the following expression:

$$W(z) = P(x, y) + iQ(x, y) \quad (17)$$

will represent a function of the complex variable $z = x + iy$, harmonic in the region under consideration.

Furthermore, put

$$\phi(z) = p(x, y) + iq(x, y) = \frac{1}{4} \int W(z) dz \quad (18)$$

The derivative of equation (18) with respect to z is:

$$\phi'(z) = \frac{\partial p(x, y)}{\partial x} + i \frac{\partial q(x, y)}{\partial y} = \frac{1}{4} [P(x, y) + iQ(x, y)] \quad (19)$$

Whence, noting that by the Cauchy-Riemann conditions:

$$\frac{\partial p(x, y)}{\partial x} = \frac{\partial q(x, y)}{\partial y}, \quad \frac{\partial p(x, y)}{\partial y} = - \frac{\partial q(x, y)}{\partial x} \quad (20)$$

One obtains:

$$\frac{\partial p(x, y)}{\partial x} = \frac{\partial q(x, y)}{\partial y} = \frac{1}{4} P(x, y) \quad (21)$$

$$\frac{\partial p(x, y)}{\partial y} = - \frac{\partial q(x, y)}{\partial x} = - \frac{1}{4} Q(x, y)$$

Thus:

$$P(x, y) = 4 \frac{\partial p(x, y)}{\partial x} = 4 \frac{\partial q(x, y)}{\partial y} = \nabla^2 U(x, y) \quad (22)$$

Therefore:

$$\nabla^2 U(x, y) - 2 \frac{\partial p(x, y)}{\partial x} - 2 \frac{\partial q(x, y)}{\partial y} = 0 \quad (23)$$

This leads to:

$$\nabla^2 [U(x, y) - xp(x, y) - yq(x, y)] = 0 \quad (24)$$

The solution of this equation is a harmonic function in the region under consideration. Let $p_1(x, y)$ be such a harmonic function, then:

$$U(x, y) = xp(x, y) + yq(x, y) + p_1(x, y) \quad (25)$$

Now, let $X(x, y)$ denote the function of the complex variable $z = x + iy$, the real part of which is $p_1(x, y)$ and the imaginary part of which is a function $q_1(x, y)$, which is the harmonic conjugate to $p_1(x, y)$. Substituting this into equation (25) gives:

$$U(x, y) = xp(x, y) + yq(x, y) + \operatorname{Re} X(z) \quad (26)$$

Combining $xp(x, y)$ with $yq(x, y)$, gives:

$$U(x, y) = \operatorname{Re} [(x-iy)p(x, y) + i(x-iy)q(x, y)] + \operatorname{Re} X(z)$$

Substituting $x-iy = \bar{z}$ leads to:

$$U(x, y) = \operatorname{Re} \left\{ \bar{z} [p(x, y) + iq(x, y)] + X(z) \right\}$$

Recognizing that $p(x, y) + iq(x, y) = \phi(z)$; gives:

$$U(x, y) = \operatorname{Re}[\bar{z} \phi(z) + \chi(z)] \quad (27)$$

From the properties of the complex function (Churchill [8]), one can obtain:

$$2U(x, y) = \bar{z} \phi(z) + z \overline{\phi(z)} + \chi(z) + \overline{\chi(z)} \quad (28)$$

The partial derivatives of this equation with respect to x and y , respectively, are:

$$\frac{\partial U(x, y)}{\partial x} = \phi(z) + \bar{z} \phi'(z) + \overline{\phi(z)} + z \overline{\phi'(z)} + \chi'(z) + \overline{\chi'(z)} \quad (29)$$

$$\frac{\partial U(x, y)}{\partial y} = i[-\phi(z) + \bar{z} \phi(z) + \overline{\phi(z)} - z \overline{\phi'(z)} + \chi'(z) - \overline{\chi'(z)}] \quad (30)$$

Adding equations (29) and (30), after multiplying the second one by i , gives:

$$\frac{\partial U(x, y)}{\partial x} + i \frac{\partial U(x, y)}{\partial y} = \phi(z) + z \overline{\phi'(z)} + \overline{\chi'(z)} \quad (31)$$

Let

$$\chi'(z) = \frac{d\chi(z)}{dz} = \psi(z)$$

Then

$$\frac{\partial U(x, y)}{\partial x} + i \frac{\partial U(x, y)}{\partial y} = \phi(z) + z \overline{\phi'(z)} + \overline{\psi(z)} \quad (32)$$

Differentiating equation (29) with respect to x and equation (30) with respect to y and adding, leads to:

$$\nabla^2 U(x, y) = 2[\phi'(z) + \overline{\phi'(z)}] = 4 \operatorname{Re}[\phi'(z)] \quad (33)$$

Equation (33) shows that $\nabla^2 U(x, y)$ is completely determined by the real part of the function $\phi'(z)$. It is also convenient at this point to recognize that:

$$\nabla^2 U(x, y) = \sigma_x + \sigma_y - \frac{1}{1-\mu} \gamma y = \sigma'_x + \sigma'_y = 4 \operatorname{Re}[\phi'(z)] \quad (34)$$

Complex Representation of Stresses

It is shown in Figure 8 that the force $(\bar{X}ds, \bar{Y}ds)$ acting on an element ds , will be understood to be the force exerted on the side of the positive normal. Recalling equations (3)

$$\bar{X} = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) \quad (35)$$

$$\bar{Y} = \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y)$$

Combining equations (11) with equations (35), leads to:

$$\bar{X} = \frac{\partial^2 U(x, y)}{\partial y^2} \cos(n, x) - \frac{\partial^2 U(x, y)}{\partial x \partial y} \cos(n, y) + \frac{\mu}{1-\mu} \gamma y \cos(n, x) \quad (36)$$

$$\bar{Y} = -\frac{\partial^2 U(x, y)}{\partial x \partial y} \cos(n, x) + \frac{\partial^2 U(x, y)}{\partial x^2} \cos(n, y) + \gamma y(n, y)$$

Where:

$$\cos(n, x) = \cos(s, y) = \frac{dy}{ds}$$

$$\cos(n, y) = -\cos(s, x) = -\frac{dx}{ds}$$

s is the positive direction of the tangent.

Introducing these values into equations (36), gives:

$$\bar{X} = \frac{d}{ds} \left[\frac{\partial U(x, y)}{\partial y} \right] + \frac{\mu}{1-\mu} \gamma y \frac{dy}{ds} \quad (37)$$

$$\bar{Y} = - \frac{d}{ds} \left[\frac{\partial U(x, y)}{\partial x} \right] - \gamma y \frac{dx}{ds}$$

or in complex form:

$$\bar{X} + i\bar{Y} = \frac{d}{ds} \left[\frac{\partial U(x, y)}{\partial y} - i \frac{\partial U(x, y)}{\partial x} \right] + \gamma y \left[\frac{\mu}{1-\mu} \frac{dy}{ds} - i \frac{dx}{ds} \right] \quad (38)$$

$$(\bar{X} + i\bar{Y}) = - id \left[\frac{\partial U(x, y)}{\partial x} + i \frac{\partial U(x, y)}{\partial y} \right] - i\gamma y \left[dx + i \frac{\mu}{1-\mu} dy \right]$$

Substituting equation (32) into equation (38) gives:

$$(X + iY)ds = - id \left[\phi(z) + \overline{z\phi'(z)} + \overline{\psi(z)} \right] - i\gamma y \left[dx + i \frac{\mu}{1-\mu} dy \right] \quad (39)$$

First let the element ds have the direction of the axis oy . Then:

$$ds = dy$$

$$dz = idy$$

$$d\bar{z} = - idy$$

$$\bar{X} = \sigma_x$$

$$\bar{Y} = \tau_{xy}$$

Substituting these values into the above expression:

$$\sigma_x + i \tau_{xy} = \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\psi'(z)} + \frac{\mu}{1-\mu} y \quad (40)$$

Next, let ds have the direction of the axis ox . Then:

$$ds = dx$$

$$dz = d\bar{z} = dx$$

$$\bar{X} = -\tau_{xy}$$

$$\bar{Y} = -\sigma_y$$

From the same expression after multiplying both sides by i :

$$\sigma_y - i \tau_{xy} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)} + \gamma y \quad (41)$$

Equations (40) and (41) are the required expressions for the stress components. Adding and subtracting these formulas and replacing in the latter case i by $-i$, the following is obtained:

$$\begin{aligned} \sigma_x + \sigma_y &= 2[\phi'(z) + \overline{\phi'(z)}] + \frac{1}{1-\mu} \gamma y \\ &= 4 \operatorname{Re}[\phi(z)] + \frac{1}{1-\mu} \gamma y \\ &= 2[\phi(z) + \overline{\phi(z)}] + \frac{1}{1-\mu} \gamma y \end{aligned} \quad (42)$$

where:

$$\phi(z) = \phi'(z)$$

and:

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\phi(z) + \bar{\Psi}(z)] + \frac{1-2\mu}{1-\mu} \gamma y \quad (43)$$

where:

$$\bar{\Psi}(z) = \psi'(z)$$

Because $\phi(z)$ and $\bar{\Psi}(z)$ are analytic functions in the region under consideration and have continuous derivatives up to and including the

second order, the quantities

$$\sigma_x + \sigma_y = \frac{1}{1-\mu} \gamma y$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = \frac{1-2\mu}{1-\mu} \gamma y$$

are also analytic functions in the same region and have continuous derivatives up to and including the second order.

Complex Representation of the Displacements

The stress-strain relationships for plane strain are:

$$\frac{E}{1+\mu} \epsilon_x = \sigma'_x - \mu(\sigma'_x + \sigma'_y) \quad (44)$$

$$\frac{E}{1+\mu} \epsilon_y = \sigma'_y - \mu(\sigma'_x + \sigma'_y) + \frac{1-2\mu}{1-\mu} \gamma y$$

but

$$\sigma'_x + \sigma'_y = \nabla^2 U(x, y) = P(x, y)$$

then

$$\frac{E}{1+\mu} \epsilon_x = \frac{\partial^2 U(x, y)}{\partial y^2} - \mu \nabla^2 U(x, y) \quad (45)$$

$$\frac{E}{1+\mu} \epsilon_y = \frac{\partial^2 U(x, y)}{\partial x^2} - \mu \nabla^2 U(x, y) + \frac{1-2\mu}{1-\mu} \gamma y$$

where E is the elastic modulus.

Replacing in the first of equations (44) $\frac{\partial^2 U(x, y)}{\partial y^2}$ by $P(x, y) - \frac{\partial^2 U(x, y)}{\partial x^2}$

and in the second equation $\frac{\partial^2 U(x, y)}{\partial x^2}$ by $P(x, y) - \frac{\partial^2 U(x, y)}{\partial y^2}$, leads to:

$$2G \epsilon_x = - \frac{\partial^2 U(x, y)}{\partial x^2} + (1-\mu) P(x, y) \quad (46)$$

$$2G \epsilon_y = - \frac{\partial^2 U(x, y)}{\partial y^2} + (1-\mu) P(x, y) + \frac{1-2\mu}{1-\mu} \gamma y$$

Where:

$$G = \frac{E}{2(1+\mu)} \text{ is the shear modulus}$$

Substituting the value of $P(x, y)$ from equation (21) into the above equation:

$$2G \epsilon_x = - \frac{\partial^2 U(x, y)}{\partial x^2} + 4(1-\mu) \frac{\partial p(x, y)}{\partial x} \quad (47)$$

$$2G \epsilon_y = - \frac{\partial^2 U(x, y)}{\partial y^2} + 4(1-\mu) \frac{\partial q(x, y)}{\partial y} + \frac{1-2\mu}{1-\mu} \gamma y$$

Integrating the first equation with respect to x and the second with respect to y :

$$2Gu = - \frac{\partial U(x, y)}{\partial x} + 4(1-\mu) p(x, y) + f(y) \quad (48)$$

$$2Gv = - \frac{\partial U(x, y)}{\partial y} + 4(1-\mu) q(x, y) + \frac{1-2\mu}{1-\mu} \frac{\gamma y^2}{2} + g(x)$$

To determine the functions $f(y)$ and $g(x)$, differentiate the first

equation with respect to y and the second one with respect to x . Adding the results, gives:

$$2G \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] = -2 \frac{\partial^2 U(x, y)}{\partial x \partial y} + 4(1-\mu) \left[\frac{\partial p(x, y)}{\partial y} + \frac{\partial q(x, y)}{\partial x} \right] + f'(y) + g'(x) \quad (49)$$

Recognizing that

$$2G \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] = -2 \frac{\partial U^2(x, y)}{\partial x \partial y} \quad (50)$$

and

$$\frac{\partial p(x, y)}{\partial y} + \frac{\partial q(x, y)}{\partial x} = 0$$

results in

$$f'(y) + g'(x) = 0$$

Because $f'(y)$ is a function of y only and $g'(x)$ is a function of x only, this cannot be possible unless:

$$f(y) = cy + c_1 \quad (51)$$

$$g(x) = -cx + c_2$$

where c , c_1 and c_2 are arbitrary constants. Substituting these values into equations (48) leads to expressions for u and v which are definite apart from terms of the form:

$$u_1 = cy + c_1, \quad v_1 = -cx + c_2 \quad (52)$$

These terms express only rigid body displacement (in the oxy plane) and they do not influence stresses and strains [25]. The constants, c , c_1 and c_2 attain definite values, if one assumes as given the values of the components of initial rigid body displacements and rotation of the region under consideration.

Therefore, equations (48) can be written:

$$\begin{aligned} 2Gu &= - \frac{\partial U(x, y)}{\partial x} + 4(1-\mu)p(x, y) \\ 2Gv &= - \frac{\partial U(x, y)}{\partial y} + 4(1-\mu)q(x, y) + \frac{1-2\mu}{1-\mu} \frac{\gamma y^2}{2} \end{aligned} \quad (53)$$

also

$$\begin{aligned} 2G(u + iv) &= - \left[\frac{\partial U(x, y)}{\partial x} + i \frac{\partial U(x, y)}{\partial y} \right] + 4(1-\mu)\phi(z) \\ &+ i \frac{1-2\mu}{1-\mu} \frac{\gamma y^2}{2} \end{aligned} \quad (54)$$

Combining equations (32) and (54) gives:

$$2G(u + iv) = (3-4\mu)\phi(z) - z \overline{\phi'(z)} - \overline{\psi(z)} + i \frac{1-2\mu}{1-\mu} \gamma y^2 \quad (55)$$

which determine the displacement in terms of the complex functions $\phi(z)$ and $\overline{\psi(z)}$.

APPLICATION OF CONFORMAL MAPPING

Introduction

In order to find the value of the stress function derived above, the boundary of the problem has to be smooth. Since the boundary of the embankment is composed of several straight lines intersecting at angles between zero and π radian, i.e., the boundary is not smooth, then it must be transformed to a smooth curve. A conformal mapping is used to obtain a closed form solution.

The Transformation of the Geometric Boundary into a Half-Space

The boundary of the embankment is a polygon with one of its vertices at infinity. Thus the Schwarz-Christoffel transformation will map it conformally into the upper half of the t -plane [8], Figure 9.

$$z = M \int_0^t \frac{d\lambda}{(\lambda + 1)^{1-B/\pi} (\lambda + \beta)^{1-C/\pi} (\lambda - \beta)^{1-D/\pi} (\lambda - 1)^{1-E/\pi}} + M_1 \quad (56)$$

where M and M_1 are complex constants, B , C , D and E are the interior angles (in radians) of the embankment in the z -plane, and -1 , $-\beta$, β and 1 are points on the real axis of the t -plane corresponding to the respective vertices, B , C , D and E . The complex constant M_1 corresponds to the point on the perimeter of the embankment that has its image at $t = 0$. Substituting the values of B , C , D and E into equations (56) leads to:

(a) Z - Plane

(b) t - Plane

Figure 9. The Geometrical Transformation

$$z = M \int_0^t \left(\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right)^{\frac{1}{N}} d\lambda + M_1 \quad (57)$$

where $N = \frac{\pi}{\alpha}$, α is the positive angle of the slope of the embankment.

The constants M , β , and M_1 , are determined by considering the correspondence between the z -plane and the t -plane at points D, E and at the center of symmetry.

(a) - at point D, $t = \beta$, $z = L - iH$.

Letting $\int_0^\beta \left(\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right)^{\frac{1}{N}} d\lambda = F$ and substituting this into equation

(57), gives:

$$L - iH = MF + M_1 \quad (58)$$

(b) - at point E, $t = 1$, $z = L + H \cot \alpha$.

Letting $\int_0^1 \left(\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right)^{\frac{1}{N}} d\lambda = F_1 + iF_2$ and substituting into equation

(57) gives:

$$L + H \cot \alpha = M(F_1 + iF_2) + M_1 \quad (59)$$

(c) - at the center of symmetry, $t = 0$, $z = -iH$ and

$$\int \left(\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right)^{\frac{1}{N}} d\lambda = 0.$$

$$\text{Then } M_1 = -iH \quad (60)$$

From equations (58) and (60) one obtains:

$$M = \frac{L}{F} \quad (61)$$

$$M_1 = -iH \quad (62)$$

Substituting these values into equation (59) one obtains:

$$\frac{F}{F_2} = \frac{L}{H} \quad (63)$$

or

$$\frac{F_2}{F} = \frac{H}{L}$$

Equations (63) represent implicit expressions for F , F_2 and β in terms of $\frac{L}{H}$ or $\frac{H}{L}$.

Figures 10, 11, 12, 13 and 14 show the solution of these equations for various values of L/H and H/L . Table 1 gives the values of F and β which are used in the subsequent illustrations.

Substituting equations (61) and (62) into equations (57), gives:

$$\frac{z}{H} = \frac{L}{HF} \int_0^t \left(\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right)^{\frac{1}{N}} d\lambda - i \quad (64)$$

This equation is the transformation formula which maps the z -plane into the t -plane. The solution to equations (63) and (64) are evaluated numerically on the IBM 7094 digital computer by Simpson's rule. The steps of the numerical integration are obtained by trial, so that the numerical results are accurate to 0.01 per cent. The computer programs for the solutions of equations (63) and (64) are included in appendices B and C.

Operations Connected with Conformal Mapping into the Half-Space

Denote the region under consideration by S , its boundary by L , and the transformation formula, equation (64), by $f(t)$, then:

Figure 10. The Constant of the Transformation Integral For $L/H = 0.01$

Figure 11. The Modulus of the Transformation Integral For $L/H = 0.01$

Figure 12. The Modulus of the Transformation Integral as a Function of L/H

Figure 13. The Modulus of the Transformation Integral as a Function of H/L

The Modulus of the Transformation Integral

Figure 14. The Value of the Constant, F , Which Precedes the Transformation Integral

Table 1. The Values of F and β for Various Shapes of Embankment

$\alpha = \pi/N = 15^\circ$			$\alpha = \pi/N = 30^\circ$			$\alpha = \pi/N = 45^\circ$			$\alpha = \pi/N = 60^\circ$			$\alpha = \pi/N = 75^\circ$		
L/H	β	F	L/H	β	F	L/H	β	F	L/H	β	F	L/H	β	F
0.5	0.082	0.132	0.5	0.12	0.873	0.5	0.133	0.436	0.5	0.126	0.65	0.5	0.104	0.968
1.0	0.165	0.234	1.0	0.248	0.44	1.0	0.297	0.647	1.0	0.32	0.871	1.0	0.323	1.146
3.0	0.40	0.489	3.0	0.552	0.737	3.0	0.675	0.912	3.0	0.685	1.053	3.0	0.72	1.186
5.0	0.535	0.62	5.0	0.687	0.841	5.0	0.763	0.973	5.0	0.803	1.068	5.0	0.83	1.148

Note: β is the Modulus of the Transformation Integral

F is the Constant Which Precedes the Transformation Integral

$$\frac{z}{H} = f(t) = f(\xi + \eta) \quad (65)$$

As mentioned above, this function maps S into the lower half of the t -plane, i.e., into the half-plane $\eta > 0$, so that finite points correspond to finite points, Figure 9. Therefore, straight lines $\eta = \text{constant}$, lying in this half-plane correspond in the z -plane to some lines which go to infinity at both ends; these lines will be denoted by ξ .

Similarly, the semi-infinite straight lines $\xi = \text{constant}$ in the lower half of the t -plane correspond in the z -plane to lines η which begin on the boundary and go to infinity as shown in Figures 9 (a) and 9 (b).

Since in the region S a completely definite point $z = f(\xi + i\eta)$ of the z -plane corresponds to every point (ξ, η) for $\eta > 0$, the quantities ξ and η may be conceived as curvilinear coordinates in the z -plane. Due to the nature of the Schwarz-Christoffel transformation, the lines ξ and η form an orthogonal net of coordinate lines. If z is some point of the region S , Figure 15, the tangents to the lines ξ, η at z , in the directions of increasing ξ and η , will represent the axes of the curvilinear coordinates at the point z . These tangents are also denoted by ξ, η .

Let A be some vector starting from the point $z = f(\xi + i\eta)$ and let A_x, A_y be its projections on the x and y axes, respectively, and A_ξ, A_η , its projections on ξ and η axes, respectively. Then:

$$A_\xi + iA_\eta = (A_x + iA_y) e^{-i\theta} \quad (66)$$

where θ is the angle between the ox and $o\xi$ axes, measured from the latter anti-clockwise.

If the point z , Figure 15, be given a displacement dz in the tangential direction ξ , the corresponding point t will undergo a displacement $d\xi > 0$ in the direction ξ of the t -plane. Obviously,

$$dz = e^{i\theta} |dz| \quad (67)$$

whence

$$e^{i\theta} = \frac{dz}{|dz|} = \frac{f'(t) d\xi}{|f'(t)| |d\xi|}$$

or

$$e^{i\theta} = \frac{f'(t)}{|f'(t)|}, \quad e^{-i\theta} = \frac{\overline{f'(t)}}{|f'(t)|} \quad (68)$$

and

$$e^{2i\theta} = \frac{f'(t)}{\overline{f'(t)}} \quad (69)$$

Combining equations (66) and (69):

$$A_\xi + iA_\eta = (A_x + iA_y) \frac{\overline{f'(t)}}{|f'(t)|} \quad (70)$$

The relationships between the stress in the oxy and $o\xi\eta$ directions are [35]:

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y \quad (71)$$

$$\sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta} = (\sigma_y - \sigma_x + 2i\tau_{xy}) e^{2i\theta}$$

Figure 15. The Transformation of a Vector From The z -Plane into The t -Plane

Hence the following relationships hold:

$$\begin{aligned}\sigma_{\xi} &= \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2} \operatorname{Re} \left\{ \left[\sigma_y - \sigma_x + 2i\tau_{xy} \right] e^{2i\theta} \right\} \\ \sigma_{\eta} &= \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \operatorname{Re} \left\{ \left[\sigma_y - \sigma_x + 2i\tau_{xy} \right] e^{2i\theta} \right\} \\ \tau_{\xi\eta} &= \operatorname{Im} \left\{ \left[\frac{\sigma_y - \sigma_x}{2} + i\tau_{xy} \right] e^{2i\theta} \right\}\end{aligned}\quad (72)$$

and

$$\begin{aligned}\sigma_x &= \frac{\sigma_{\eta} + \sigma_{\xi}}{2} - \frac{1}{2} \operatorname{Re} \left\{ \left[\sigma_{\eta} - \sigma_{\xi} + 2i\tau_{\xi\eta} \right] e^{-2i\theta} \right\} \\ \sigma_y &= \frac{\sigma_{\eta} + \sigma_{\xi}}{2} + \frac{1}{2} \operatorname{Re} \left\{ \left[\sigma_{\eta} - \sigma_{\xi} + 2i\tau_{\xi\eta} \right] e^{-2i\theta} \right\} \\ \tau_{xy} &= \operatorname{Im} \left\{ \left[\frac{\sigma_{\eta} - \sigma_{\xi}}{2} + i\tau_{\xi\eta} \right] e^{-2i\theta} \right\}\end{aligned}\quad (73)$$

The Complex Representation of Stresses in the t-plane

The functions $\phi(z)$, $\Psi(z)$, $\phi'(z)$ and $\Psi'(z)$ can be written in terms of t , in the following forms:

$$\phi(z) = \phi[f(t)] = \phi(t)$$

$$\Psi(z) = \Psi[f(t)] = \Psi(t)$$

(continued)

$$\Phi'(z) = \frac{\Phi'(t)}{f'(t)}$$

$$\Psi'(z) = \frac{\Psi'(t)}{f'(t)} \quad (74)$$

Substituting these expressions into equations (42) and (43), gives:

$$\begin{aligned} \sigma_y + \sigma_x &= 2[\Phi(t) + \overline{\Phi(t)}] + \frac{1}{1-\mu} \gamma \cdot \text{Im}[f(t)] \\ &= 4 \text{Re}[\Phi(t)] + \frac{1}{1-\mu} \gamma \cdot \text{Im}[f(t)] \end{aligned} \quad (75)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\overline{f(t)} \frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] + \frac{1-2\mu}{1-\mu} \gamma \cdot \text{Im}[f(t)]$$

Combining equations (75) and equations (71) leads to:

$$\begin{aligned} \sigma_\eta + \sigma_\xi &= 4 \text{Re}[\Phi(t)] + \frac{1}{1-\mu} \gamma \text{Im}[f(t)] \\ \sigma_\eta - \sigma_\xi + 2i\tau_{xy} &= 2\left[\overline{f(t)} \frac{\Phi'(t)}{f'(t)} + \frac{f'(t)}{f'(t)} \Psi(t)\right] \\ &\quad + \frac{1-2\mu}{1-\mu} \gamma \text{Im}[f(t)] - \frac{f'(t)}{f'(t)} \end{aligned} \quad (76)$$

Equations (76) require determination of $\Phi(t)$ and $\Psi(t)$ in order to evaluate the stress.

The Complex Representation of Displacements in the t-plane

The displacement in the z-plane and in terms of the complex variable z, is given by equation (54). Replacing z by f(t) in this equation:

$$\begin{aligned}
2G(u + iv) = & (3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \\
& + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2
\end{aligned} \quad (77)$$

Applying equation (70) to equation (77), the components of the displacement in the t -plane can be written:

$$\begin{aligned}
2G(v + w) = & \left[(3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \right. \\
& \left. + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2 \right] \frac{\overline{f'(t)}}{|f'(t)|}
\end{aligned} \quad (78)$$

From equations (77) and (78), the following expressions hold:

$$\begin{aligned}
u = \frac{1}{2G} \operatorname{Re} \left[(3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \right. \\
\left. + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2 \right]
\end{aligned}$$

$$\begin{aligned}
v = \frac{1}{2G} \operatorname{Im} \left[(3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \right. \\
\left. + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2 \right]
\end{aligned}$$

$$\begin{aligned}
w = \frac{1}{2G} \operatorname{Re} \left\{ \left[(3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \right. \right. \\
\left. \left. + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2 \right] \frac{\overline{f'(t)}}{|f'(t)|} \right\} \quad (\text{continued})
\end{aligned}$$

$$\omega = \frac{1}{2G} \operatorname{Im} \left\{ \left[(3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \overline{\psi(t)} \right. \right. \\ \left. \left. + i \frac{1-2\mu}{1-\mu} \gamma \operatorname{Im}[f(t)]^2 \right] \frac{\overline{f'(t)}}{|f'(t)|} \right\} \quad (79)$$

Formulas and Propositions for the t-plane

As mentioned above, the expression $z = f(t)$ transforms the region of the embankment into the lower half of the t-plane, which is bounded by the $o\xi$ axis and consists of the points $\eta \geq 0$.

Assuming that the stress components, which satisfy the conditions of continuity, equilibrium and differentiability, tend to zero as $t \rightarrow \infty$, let the $\phi(t)$ and $\psi(t)$ functions be represented for large (t) by:

$$\phi(t) = \frac{c}{t} + \frac{v}{|t|^{1+\epsilon}} \quad (80)$$

$$\psi(t) = \frac{c_1}{t} + \frac{v}{|t|^{1+\epsilon}}$$

where c and c_1 are constants, v depends only on t and $v \rightarrow 0$ as $|t| \rightarrow \infty$, and ϵ is a positive constant which is arbitrarily small. Integrating the above equation:

$$\begin{aligned} \phi(t) &= c \log(t) + v + \text{constant} \\ \psi(t) &= c_1 \log(t) + v + \text{constant} \end{aligned} \quad (81)$$

in these expressions one definite branch of the multi-valued function $\log t$ must be selected, e.g., $\operatorname{Log} t + i\theta$, where θ (argument of t) varies

from $-\pi$ to π . In addition, the functions $\phi(t)$ and $\psi(t)$ will be harmonic in every finite region, contained in the lower half plane $\eta > 0$.

Finally, introduce the following condition: the resultant vector of the external forces, applied to a segment mn of the $o\xi$ axis, tends to a definite limit as m and n , Figure 16, move to infinity (m towards the left and n toward the right). This condition will always be satisfied if only a finite part of the boundary is loaded.

This last condition mathematically means, if X_ξ , X_η are the components of the resultant of the external forces, applied to mn , then one has by (32) and (38) the following:

$$\begin{aligned} X_\xi + iY_\eta &= -i \left[\frac{\partial U(x, y)}{\partial x} + i \frac{\partial U(x, y)}{\partial y} \right]_A^B \\ &= -i \left[\phi(t) + f(t) \frac{\overline{\phi'(t)}}{f'(t)} + \psi(t) \right]_A^B \end{aligned} \quad (82)$$

The negative sign has been chosen on the right-hand side of equation (82) because the region S lies on the right, and not on the left, as one moves from m to n .

When calculating the increase of the left-hand side of equation (82) for the transition from m to n , one may, instead of moving along the straight line mn , follow any curve which lies in S , such as the curve $m c n$, Figure 16.

If m and n lie sufficiently far away and on different sides of o , then equations (80), (81) and (82) give:

$$X_\xi + iY_\eta = -i \left[c \log \frac{R}{R_1} + c\pi i + c_1 \log \frac{R}{R_1} - \bar{c}_1 \pi i + v \right] \quad (83)$$

where R and R_1 are the distances of m and n from O and v is arbitrarily small and tends to zero as R and R_1 tend to infinity. In order that the preceding expression will remain finite for any arbitrarily large R and R_1 which are independent of each other, it is necessary and sufficient that:

$$c + \overline{c_1} = 0 \quad (84)$$

Under this condition the vector (X_ξ, X_η) of the external forces, applied to the whole of the $o\xi$ axis, will be given:

$$X_\xi + iY_\eta = -i \left[\phi(t) + f(t) - \frac{\phi'(t)}{f'(t)} - \overline{\psi(t)} \right]_{-\infty}^{+\infty} = \pi(c - \overline{c_1}) \quad (85)$$

It follows from (84) and (85) that:

$$c = \frac{X_\xi + iY_\eta}{2\pi}, \quad \overline{c_1} = -\frac{X_\xi - iY_\eta}{2\pi} \quad (86)$$

Hence, for large $|t|$:

$$\begin{aligned} \phi(t) &= -\frac{X_\xi + iY_\eta}{2\pi t} + \frac{v}{|t|^{1+\epsilon}} \\ \psi(t) &= \frac{X_\xi - iY_\eta}{2\pi t} + \frac{v}{|t|^{1+\epsilon}} \end{aligned} \quad (87)$$

$$\phi'(t) = \frac{X_\xi - iY_\eta}{2\pi t^2} + \frac{v|t|^2}{|t|^{2(1+\epsilon)}}|$$

$$\phi(t) = - \frac{X_{\xi} + iY_{\eta}}{2\pi} \log t + \nu + \text{constant}$$

$$\psi(t) = \frac{X_{\xi} - iY_{\eta}}{2\pi} \log t + \nu + \text{constant}$$

The quantities $Y_{\eta} = N(t)$ and $X_{\xi} = T(t)$ have to be found on the axis of ξ as a function of the abscissa ξ . On the basis of equation (87), for large t :

$$N = O\left(\frac{1}{\xi}\right), \quad T = O\left(\frac{1}{\xi}\right) \quad (88)$$

where:

$$\left|O\left(\frac{1}{\xi}\right)\right| < \frac{\nu}{|\xi|}$$

These conditions are necessary conditions in order to apply the Cauchy integral formula [25], which is introduced below.

BOUNDARY CONDITIONS

Boundary Conditions in the z-plane

From equation (75)

$$N + iT = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x + 2i\tau_{xy}}{2}$$

or

$$N + iT = \phi(t) + \overline{\phi(t)} + z\phi'(z) + \psi(z) + \gamma \operatorname{Im}[f(t)] \quad (89)$$

Denoting the boundary of the embankment by L, equation (89) becomes:

$$[N + iT]_L = [\phi(z) + \overline{\phi(z)} + z\phi'(z) + \psi(z) + \gamma \operatorname{Im}[f(t)]]_L \quad (90)$$

or

$$[N + iT]_L = [2 \operatorname{Re}[\phi(z)] + z\phi'(z) + \psi(z) + \gamma \operatorname{Im}[f(t)]]_L \quad (91)$$

where:

$z = x$	for $x \leq -(L + H \cot \alpha)$
$z = x - i \frac{L + H \cot \alpha + x}{\cot \alpha}$	for $-(L + H \cot \alpha) \leq x \leq -L$
$z = x - iH$	for $-L \leq x \leq L$
$z = x - i \frac{L + H \cot \alpha - x}{\cot \alpha}$	for $L \leq x \leq L + H \cot \alpha$
$z = x$	for $L + H \cot \alpha \leq x$

Boundary Conditions in the t-plane

From equations (76):

$$N + iT = \frac{\sigma_\eta + \sigma_\xi}{2} + \frac{\sigma_\eta + \sigma_\xi + 2i\tau_\xi \eta}{2}$$

or

$$N + iT = \phi(t) + \overline{\phi(t)} + \left[\overline{f(t)} \frac{\phi'(t)}{f'(t)} + \frac{f'(t)}{f'(t)} \Psi(t) \right] \\ + \frac{1}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(t)] + \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(t)] \frac{f'(t)}{f'(t)} \quad (92)$$

The boundary of the embankment in the z-plane corresponds to the axis $\eta = 0$ in the t-plane. Substituting this in equation (92) gives:

$$N + iT = \phi(\xi) + \overline{\phi(\xi)} + \left[\overline{f(\xi)} \frac{\phi'(\xi)}{f'(\xi)} + \frac{f'(\xi)}{f'(\xi)} \Psi(\xi) \right] \\ + \frac{1}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] + \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] \frac{f'(\xi)}{f'(\xi)} \quad (93)$$

Because the boundary is free of load, $N + iT = 0$, and equation (93) can be rearranged to yield:

$$\phi(\xi) + \overline{\phi(\xi)} + \left[\overline{f(\xi)} \frac{\phi'(\xi)}{f'(\xi)} + \frac{f'(\xi)}{f'(\xi)} \Psi(\xi) \right] = \\ - \frac{1}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] - \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] e^{2i\alpha} \quad (94)$$

where:

$$\begin{aligned}
 \alpha &= 0 & \text{for} & -\infty \leq \xi \leq -1 \\
 &= -\frac{\pi}{N} & \text{for} & -1 \leq \xi \leq -\beta \\
 &= 0 & \text{for} & -\beta \leq \xi \leq \beta \\
 &= \frac{\pi}{N} & \text{for} & \beta \leq \xi \leq 1 \\
 &= 0 & \text{for} & 1 \leq \xi \leq \infty
 \end{aligned}$$

and

$$\begin{aligned}
 \text{Im}[f(\xi)] &= 0 & \text{for} & -\infty \leq \xi \leq -1 \\
 &= \frac{-H(1+\xi)}{1-} & \text{for} & -1 \leq \xi \leq -\beta \\
 &= -H & \text{for} & -\beta \leq \xi \leq \beta \\
 &= \frac{-H(1-\beta)}{1-\beta} & \text{for} & \beta \leq \xi \leq 1 \\
 &= 0 & \text{for} & 1 \leq \xi \leq \infty
 \end{aligned}$$

Thus, the boundary conditions are as shown in Figure 17. These conditions will be called "Fictitious boundary stresses", because they arise from the weight of the embankment.

It is convenient to separate the real and imaginary parts of the "fictitious boundary stresses", i.e.,

$$N_{WE} = \text{Re} \left\{ -\frac{1}{1-\mu} \frac{\gamma}{2} \text{Im}[f(\xi)] - \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \text{Im} f(\xi) e^{2i\alpha} \right\} \quad (95)$$

and

$$T_{WE} = \text{Im} \left\{ -\frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \text{Im}[f(\xi)] - \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \text{Im} [f(\xi)] e^{2i\alpha} \right\} \quad (96)$$

where N_{WE} and T_{WE} are shown in Figure 17 (b) and Figure 17 (c),

(a) The Boundary of the Investigated Problem

(b) The Real Part of "Fictitious Stresses" on the Boundary in the t -Plane, N_{WE}

(c) The Imaginary Part of "Fictitious Stresses" on the Boundary in the t -Plane, T_{WE}

Figure 17. "Fictitious Boundary Stress" Due to the Weight of the Embankment

respectively.

Incorporating these notations into equation (94), leads to

$$N_{WE} + iT_{WE} = \Phi(\xi) + \overline{\Phi(\xi)} + \left[\overline{f(\xi)} \frac{\Phi'(\xi)}{f'(\xi)} + \frac{f'(\xi)}{f'(\xi)} \Psi(\xi) \right] \quad (97)$$

and by taking i to be $-i$:

$$N_{WE} - iT_{WE} = \overline{\Phi(\xi)} + \Phi(\xi) + \left[\overline{f(\xi)} \frac{\Phi'(\xi)}{f'(\xi)} + \frac{\overline{f'(\xi)}}{f'(\xi)} \Psi(\xi) \right]$$

These equations lead to the solution of the problem, as shown below.

THE SOLUTION OF THE PROBLEM

Introduction

In the following section the use will be made of so-called Cauchy integrals. A systematic study of the properties of these integrals may be found in reference [39], but for convenience, the essentials will be given in the next section.

Cauchy Integral

Let $h(\xi) = h_1(\xi) + h_2(\xi)$ may be some complex function given on the boundary L of the embankment. Assume that $h(\xi)$ is a finite and integrable function in an ordinary Riemann sense.

The integral of the form

$$\frac{1}{2\pi i} \int_L \frac{h(\xi) d\xi}{\xi - t}$$

taken over L with ξ some point on the boundary L , is called a Cauchy integral [39]. It is assumed that the point t does not lie on the boundary L .

The above integral represents a function of the complex variable t throughout the entire plane with the exception of the points L . Denote this function by $H(t)$, so that:

$$H(t) = \frac{1}{2\pi i} \int_L \frac{h(\xi) d\xi}{\xi - t} \quad (99)$$

The function $H(t)$ is harmonic inside the region, and if $h(t) \rightarrow \epsilon$ as $t \rightarrow \infty$, then $H(t) \rightarrow \epsilon$ [39], where ϵ is some finite value.

From the properties of Cauchy integrals, the following theorems can be stated [39].

1 - If the function $H(t)$ is harmonic and continuous inside the region and $H(t)|_{t=\infty} = 0$. Then

$$\frac{1}{2\pi i} \int_L \frac{H(\xi) d\xi}{\xi - t} = H(t) \quad \text{inside the region} \quad (100)$$

and

$$\frac{1}{2\pi i} \int_L \frac{H(\xi) d\xi}{\xi - t} = 0 \quad \text{outside the region} \quad (101)$$

2 - If the function $H(t)$ is harmonic, continuous outside the region and $H(t)|_{t=\infty} = 0$. Then

$$\frac{1}{2\pi i} \int_L \frac{H(\xi) d\xi}{\xi - t} = 0 \quad \text{inside the region} \quad (102)$$

and

$$\frac{1}{2\pi i} \int_L \frac{H(\xi) d\xi}{\xi - t} = -H(t) \quad \text{outside the region} \quad (103)$$

The expressions (100), (101), (102) and (103) are called the Cauchy formulas.

Determination of the Stresses by Using Cauchy Integral

It was shown above that the mapping formula is given by:

$$\frac{z}{H} = \frac{L}{HF} \int_0^t \left[\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right]^{\frac{1}{N}} d\lambda - i = f(t) \quad (104)$$

When the point t moves along the ξ axis from left to right, the corresponding point z/H moves along the boundary likewise from left to right.

It is readily verified that the equation (104) maps the region under consideration, into the half-plane $\eta > 0$.

Repeating equation (97) for reference,

$$\Phi(\xi) + \overline{\Phi(\xi)} + \left[\overline{f(\xi)} \frac{\Phi'(\xi)}{f'(\xi)} + \frac{f'(\xi)}{\overline{f'(\xi)}} \Psi(\xi) \right] = N_{WE} + T_{WE} \quad (105)$$

In this equation, the unknown functions $\Phi(t)$, $\Psi(t)$ which are harmonic in the lower half of the t -plane, satisfy, on the basis of (87), the conditions:

$$\begin{aligned} \Phi(t)|_{t \rightarrow \infty} &= \frac{\nu}{|t|} \\ \Phi'(t)|_{t \rightarrow \infty} &= \frac{\nu}{|t|^2} \\ \Psi(t)|_{t \rightarrow \infty} &= \frac{\nu}{|t|} \end{aligned} \quad (106)$$

Therefore, on the basis of equations (100), (101), (102) and (103) the Cauchy integral can be applied on equation (105).

The function $\overline{f(\xi)} \frac{\Phi'(\xi)}{f'(\xi)}$ determined by (105) is the boundary value

of the function $g(t) \frac{\Phi'(t)}{f'(t)}$ harmonic in the lower half of the

t -plane and vanishing at infinity, and the function $\frac{f(\xi)}{f'(\xi)} \Psi(\xi)$ is

the boundary value of the function $\frac{f'(t)}{f'(t)} \Psi(t)$, harmonic in the

lower half of the t -plane and vanishing at infinity. Applying the Cauchy integral formula to equation (105), after taking the complex conjugate form of it, gives:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\Phi(\xi)} d\xi}{\xi - t} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi(\xi) d\xi}{\xi - t} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f(\xi) \frac{\overline{f'(\xi)}}{f'(\xi)} \frac{d\xi}{\xi - t} \\ & \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{f'(\xi)}}{f'(\xi)} \frac{\overline{\Psi(\xi)}}{\xi - t} d\xi = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(N_{WE} - iT_{WE})}{\xi - t} \end{aligned} \quad (107)$$

where t is a point of the lower half of the t plane; noting that $\Phi(\xi)$ is the boundary value of $\Phi(t)$, harmonic in the lower half of the t -plane and vanishing at infinity, and that $\overline{\Phi(\xi)}$, $f(\xi) \frac{\overline{f'(\xi)}}{f'(\xi)}$ and $\frac{\overline{f'(\xi)}}{f'(\xi)} \overline{\Psi(\xi)}$, are the boundary values of $\overline{\Phi(t)}$, $f(t) \frac{\overline{f'(t)}}{f'(t)}$ and $\frac{\overline{f'(t)}}{f'(t)} \overline{\Psi(t)}$, harmonic in the upper half of the t -plane and vanishing at infinity. From the properties of the Cauchy integral, it is clear that the first, third and the fourth integrals vanish and the second integral is equal to $\Phi(t)$. Therefore:

$$\Phi(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(N_{WE} - T_{WE})}{\xi - t} d\xi \quad (108)$$

The function $\Psi(t)$ is now easily determined from its boundary value given by equation (105):

$$\Psi(t) = -g(t) \frac{\Phi'(t)}{f'(t)} - q(t) \frac{\overline{f'(t)}}{f'(t)} + \frac{\overline{f'(t)}}{f'(t)} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(N_{WE} + iT_{WE})}{\xi - t} d\xi \quad (109)$$

where

$$g(t) = \frac{L}{HF} \int_0^t \overline{\left[\frac{\lambda^2 - 1}{\lambda^2 - \beta^2} \right]^{\frac{1}{N}}} d\lambda + i \quad (110)$$

$$N_{WE} + iT_{WE} = -\frac{1}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] - \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] e^{2i\alpha} \quad (111)$$

$$N_{WE} - iT_{WE} = -\frac{1}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] - \frac{1-2\mu}{1-\mu} \frac{\gamma}{2} \operatorname{Im}[f(\xi)] e^{-2i\alpha} \quad (112)$$

Substituting these values into equations (108), leads to:

$$\begin{aligned} \Phi(t) = & \frac{1}{2\pi i} \int_{-1}^{-\beta} \frac{\gamma H}{2(1-\mu)(1-\beta)} \cdot \frac{1+\xi}{\xi-t} d\xi \\ & + \frac{1}{2\pi i} \int_{-\beta}^{+\beta} \frac{\gamma H}{2(1-\mu)} \cdot \frac{d\xi}{\xi-t} \\ & + \frac{1}{2\pi i} \int_{\beta}^1 \frac{\gamma H}{2(1-\mu)(1-\beta)} \cdot \frac{1-\xi}{\xi-t} d\xi \\ & + \frac{e^{-2i\alpha} 1}{2\pi i} \int_{-1}^{-\beta} \frac{\gamma H}{2(1-\mu)(1-\beta)} \cdot \frac{1+\xi}{\xi-t} d\xi \\ & + \frac{e^{-2i\alpha} 2}{2\pi i} \int_{-\beta}^{+\beta} \frac{\gamma H}{2(1-\mu)} \cdot \frac{d\xi}{\xi-t} \\ & + \frac{e^{-2i\alpha} 3}{2\pi i} \int_{\beta}^1 \frac{\gamma H}{2(1-\mu)(1-\beta)} \cdot \frac{1-\xi}{\xi-t} d\xi \end{aligned} \quad (113)$$

where:

$$\alpha_1 = -\frac{\pi}{N}$$

$$\alpha_2 = 0$$

$$\alpha_3 = \frac{\pi}{N}$$

Integrating and elaborating:

$$\begin{aligned} \frac{\Phi(t)}{\gamma_H} = & -\frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1-\beta \right] \\ & - \frac{i}{4\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\ & - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \\ & - \frac{i e^{-2i\alpha_1}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \\ & - \frac{i e^{-2i\alpha_2}}{4\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\ & - \frac{i e^{-2i\alpha_3}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \quad (114) \end{aligned}$$

Differentiating this equation gives:

$$\begin{aligned} \frac{\Phi'(t)}{\gamma_H} = & -\frac{i}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t+1)(t+\beta)} \log \frac{t+\beta}{t+1} - \frac{t(1-\beta)}{(t+1)(t+\beta)} \right] \\ & - \frac{i}{2\pi(1-\mu)} \left[\frac{\beta}{t^2 - \beta^2} \right] \end{aligned}$$

(continued)

$$\begin{aligned}
& - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right] \\
& - \frac{i e^{-2i\alpha_1}}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t+1)(t+\beta)} + \log \frac{t+\beta}{t+1} - \frac{t(1-\beta)}{(t+1)(t+\beta)} \right] \\
& - \frac{i e^{-2i\alpha_2}}{2\pi(1-\mu)} \left[\frac{\beta}{t^2 - \beta^2} \right] \\
& - \frac{i e^{-2i\alpha_3}}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right] \quad (115)
\end{aligned}$$

Combining equations (109), (110), (111), (114) and (115), gives:

$$\begin{aligned}
\frac{\Psi(t)}{\gamma H} = & - \frac{g(t)}{f'(t)} \left\{ - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t+1)(t+\beta)} + \log \frac{t+\beta}{t+1} \right. \right. \\
& + \left. \frac{t(1-\beta)}{(t+1)(t+\beta)} \right] - \frac{i}{2\pi(1-\mu)} \left[\frac{\beta}{t^2 - \beta^2} \right] \\
& - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right] \\
& - \frac{i e^{-2i\alpha_1}}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t+1)(t+\beta)} + \log \frac{t+\beta}{t+1} + \frac{t(1-\beta)}{(t+1)(t+\beta)} \right] \\
& - \frac{i e^{-2i\alpha_2}}{2\pi(1-\mu)} \left[\frac{\beta}{t^2 - \beta^2} \right] \\
& - \frac{i e^{-2i\alpha_3}}{4\pi(1-\mu)(1-\beta)} \left[\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right] \Big\} \\
& - \frac{f'(t)}{f'(t)} \left\{ - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \right.
\end{aligned}$$

(continued)

$$\begin{aligned}
& - \frac{i}{4\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \\
& - \frac{i e^{-2i\alpha_1}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \\
& - \frac{i e^{-2i\alpha_2}}{4\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& - \frac{i e^{-2i\alpha_3}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \Big\} \\
& + \overline{\frac{f'(t)}{f(t)}} \left\{ - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \right. \\
& - \frac{i}{4\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& - \frac{i}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \\
& - \frac{i e^{+2i\alpha_1}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \\
& - \frac{i e^{+2i\alpha_2}}{4\pi(1-\mu)} \log \frac{t-\beta}{t+1} \\
& \left. - \frac{i e^{+2i\alpha_3}}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \right\} \quad (116)
\end{aligned}$$

Combining equations (114), (115) and (116), gives:

$$\begin{aligned}
\frac{\sigma_Y}{\gamma_H} = & 2 \operatorname{Re} \left\{ - \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \right. \\
& - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \right\} \\
& + \operatorname{Re} \left\{ \left[\frac{\overline{f(t)} - g(t)}{f'(t)} \right] \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t+1)(t+\beta)} + \log \frac{t+\beta}{t+1} \right. \right. \right. \\
& \left. \left. + \frac{t(1-\beta)}{(t+1)(t+\beta)} \right) - \frac{i}{\pi(1-\mu)} \left(\frac{\beta}{t^2 - \beta^2} \right) \right. \\
& \left. \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right) \right] \right. \\
& - \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \\
& - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \right] \\
& + \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{+2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \\
& - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
& \left. \left. - \frac{i(1+e^{+2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \right] \right\}
\end{aligned}$$

(continued)

$$+ \operatorname{Im}[f(t)] \quad (117)$$

$$\begin{aligned} \frac{\sigma_x}{\gamma H} = & 2 \operatorname{Re} \left\{ - \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right] \right. \\ & - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\ & \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left[\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right] \right\} \\ & - \operatorname{Re} \left\{ \left[\frac{\overline{f(t)} - \overline{g(t)}}{f'(t)} \right] \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t+1)(t+\beta)} + \log \frac{t+\beta}{t+1} \right. \right. \right. \\ & \left. \left. + \frac{t(1-\beta)}{(t+1)(t+\beta)} \right) - \frac{i}{\pi(1-\mu)} \left(\frac{\beta}{t^2 - \beta^2} \right) \right. \\ & \left. \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right) \right] \right. \\ & - \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \\ & - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\ & \left. \left. - \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \right] \right. \\ & \left. + \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{+2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \right. \\ & \left. \left. - \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \right] \right\} \end{aligned}$$

(continued)

$$\begin{aligned}
& - \frac{i(1+e^{2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \Bigg\} \\
& + \frac{\mu}{1-\mu} \operatorname{Im}[f(t)] \quad (118)
\end{aligned}$$

$$\begin{aligned}
\frac{\tau_{xy}}{\gamma H} &= \operatorname{Im} \left\{ \left[\frac{\overline{f(t)} - g(t)}{f'(t)} \right] \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t+1)(t+\beta)} \right. \right. \right. \\
&+ \log \frac{t+\beta}{t+1} + \frac{t(1-\beta)}{(t+1)(t+\beta)} \Bigg) - \frac{i}{\pi(1-\mu)} \left(\frac{\beta}{t^2 - \beta^2} \right) \\
&- \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\frac{1-\beta}{(t-1)(t-\beta)} - \log \frac{t-1}{t-\beta} - \frac{t(1-\beta)}{(t-1)(t-\beta)} \right) \Bigg] \\
&- \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{-2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \\
&- \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
&- \frac{i(1+e^{-2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \Bigg] \\
&+ \frac{\overline{f'(t)}}{f'(t)} \left[- \frac{i(1+e^{2i\alpha_1})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t+\beta}{t+1} + t \log \frac{t+\beta}{t+1} + 1 - \beta \right) \right. \\
&- \frac{i}{2\pi(1-\mu)} \log \frac{t-\beta}{t+\beta} \\
&- \frac{i(1+e^{+2i\alpha_3})}{4\pi(1-\mu)(1-\beta)} \left(\log \frac{t-1}{t-\beta} - t \log \frac{t-1}{t-\beta} - 1 + \beta \right) \Bigg] \Bigg\} \quad (119)
\end{aligned}$$

Determination of the Displacements

The components of the displacements are given by equation (77), repeated here for reference:

$$\frac{2G}{\gamma H^2} [u + v] = (3-4\mu)\phi(t) - \frac{f(t)}{f'(t)} \overline{\phi'(t)} - \psi(t) + i \frac{1-2\mu}{1-\mu} \operatorname{Im}[f(t)]^2 \quad (120)$$

Substituting the integral of equations (114) and (116) and the complex conjugate of equation (114), into equation (120), leads to:

$$\begin{aligned} \frac{E}{\gamma H^2 (1+\mu)} [u + v] = & - \frac{i(3-4\mu)[1+(1-2\mu)e^{-2i\alpha_1}]}{4\pi(1-\beta)(1-\mu)} \left[(1-\beta)(t+\beta)\log(t+\beta) \right. \\ & + \frac{(t+\beta)^2}{2} \log(t+\beta) - \frac{(t+1)^2}{2} \log(t+1) \\ & + \left(\frac{1-\beta}{4} \right) (1+2t+\beta) - \beta \log \beta + \frac{\beta^2}{2} \log \beta \\ & \left. + \frac{\beta^2}{4} - \frac{1}{4} \right] \\ & - \frac{i(3-4\mu)[1+(1-2\mu)e^{-2i\alpha_3}]}{4\pi(1-\beta)(1-\mu)} \left[(\beta-1)(t-\beta)\log(t-\beta) \right. \\ & + \frac{(t-\beta)^2}{2} \log(t-\beta) - \frac{(t-1)^2}{2} \log(t-1) \\ & + \frac{(\beta-1)}{4} (2t-1-\beta) - \beta \left(\frac{\beta}{2} + 1 \right) \log \beta + \frac{7}{4} \beta^2 \\ & \left. + i\pi(1-\beta) \left(\frac{3+\beta}{2} \right) + \frac{1}{4} \right] \\ & - \frac{i(3-4\mu)}{2\pi} \left[(t-\beta) \log(t-\beta) - (t+\beta) \log(t+\beta) \right. \\ & \left. + 2\beta \log \beta + i\pi\beta \right] \end{aligned}$$

(continued)

$$\begin{aligned}
& - \frac{i[1+(1-2\mu)e^{-2i\alpha_1}]}{4\pi(1-\beta)(1-\mu)} \left[(1-\beta)(\bar{t}+\beta) \log (\bar{t}+\beta) \right. \\
& + \frac{(\bar{t}+\beta)^2}{2} \log (\bar{t}+\beta) - \frac{(\bar{t}+1)^2}{2} \log (\bar{t}+1) \\
& + \left(\frac{1-\beta}{4} \right) (1+2\bar{t}+\beta) - \beta \log \beta + \frac{\beta^2}{2} \log \beta + \frac{\beta^2}{4} - \frac{1}{4} \left. \right] \\
& - \frac{i[1+(1-2\mu)e^{-2i\alpha_3}]}{4\pi(1-\beta)(1-\mu)} \left[(\beta-1)(\bar{t}-\beta) \log (\bar{t}-\beta) \right. \\
& + \frac{(\bar{t}-\beta)^2}{2} \log (\bar{t}-\beta) - \frac{(\bar{t}-1)^2}{2} \log (\bar{t}-1) \\
& + \frac{(\beta-1)}{4} (2\bar{t}-1-\beta) - \beta \left(\frac{\beta}{2} + 1 \right) \log \beta + \frac{7}{4} \beta^2 \\
& + i\pi(1-\beta) \left(\frac{3+\beta}{2} \right) + \frac{1}{4} \left. \right] \\
& - \frac{i}{2\pi} \left[(\bar{t}-\beta) \log (\bar{t}-\beta) - (\bar{t}+\beta) \log (\bar{t}+\beta) \right. \\
& + 2\beta \log \beta + i\pi\beta \left. \right] \\
& - \left[\frac{f(t) - \overline{f(t)}}{f'(t)} \right] \overline{\phi(\bar{t})} + i \frac{1-2\mu}{1-\mu} \operatorname{Im}[f(t)]^2 \quad (121)
\end{aligned}$$

The horizontal displacement, u , is the real value of equation (121) and the vertical displacement, v , is the imaginary value of equation (121).

RESULTS AND DISCUSSION

Introduction

The stress distribution within and under long linear elastic, isotropic homogeneous embankments, continuous with the underlying material of similar properties, has been investigated for the cases shown in Table 1 for Poisson's ratio, $\mu = 0.3$ and 0.5 . These results were compared with those obtained by the normal loading approximation, viz. Figure 6 (b). All the results below show only the effect of the embankment weight. Thus, at depths below the base of the embankment ($y/H = 0$) the material is assumed weightless. The effect of the medium weight can be superimposed upon these values to give the total stress acting at a point.

Analysis of the Vertical Normal Stress Distribution

Figure 18 shows a typical result of the vertical normal stress contours in dimensionless form, $\sigma_y/\gamma H$, for an embankment with $\alpha = 45^\circ$, $L/H = 3$, and Poisson's ratio, $\mu = 0.3$.* The dashed lines in this figure show stress contours, in terms of $\sigma_y/\gamma H$, for the usual normal loading approximation corresponding to this embankment. These indicate that the vertical normal stresses produced in the foundation material below the elastic embankment are generally smaller than those computed for the normal loading approximation.

The stress distribution due to the normal loading approximation

* Experiments suggest $\mu = 0.3$ is a reasonable value for unsaturated soils.

Figure 18. Contours for Vertical Stress

is independent of Poisson's ratio; the stresses due to the elastic embankment are dependent upon μ . However, the vertical stresses are relatively insensitive to its magnitude; changing μ from 0.3 to 0.5 changes the vertical stress at a point by less than five per cent.

At the base of the embankment ($y/H = 0$) there is a sharp change in slope of the contour lines. This results from the difference in the way that a weightless body is stressed under an elastic body force applied to its surface. The expression for the vertical normal stress, equation (117) has two parts. The second part of this equation is a linear function of the unit body weight and position. It is expressed by $\gamma \text{Im}[f(t)]$. In the foundation γ is zero. Therefore, as the stress contour passes over the interface line a linear term is dropped from the equation for stress. This results in the observed sharp change in the contour lines.

The effect of L/H ratio on the vertical stress along vertical sections through the center line of the embankment and the toe of the slope, is illustrated in Figure 19, for $\alpha = 45^\circ$, $\mu = 0.3$. The figure is a composite diagram showing the embankment schematically, and the magnitude of the vertical stress at each section as a function of depth. Clearly, the L/H ratio has a pronounced effect on the distribution of vertical stress. As L/H decreases, the stress decreases. Furthermore, a smaller L/H ratio produces a more rapid dissipation of stress with depth, and elevates the horizontal section where the vertical normal stress becomes practically uniform.

The dashed lines show the vertical normal stress for the normal loading approximation equivalent in shape to the embankment for which

(a) At Centerline

(b) At Toe of Slope

Figure 19. Distribution of Vertical Stress Along Vertical Sections for Varying L/H Ratios

$L/H = 1$. It is clear that the vertical stress beneath the embankment for the corresponding elastic embankment is smaller.

Figure 20 shows the distribution of vertical normal stress along the base of the embankment for $\mu = 0.3$, four values of α , and several embankment shapes shown schematically in the figure. The curved solid lines represent the distribution of stress against the base; dashed lines show the distribution of stress assumed in the usual normal loading approximation. It is clear that the stress distribution is much more uniform under the elastic embankment than is ordinarily assumed. The difference becomes especially apparent as the L/H ratio of the steeper embankment decreases. Moreover, the magnitude of the stress under the central zone of the elastic embankment is less than that shown by the dashed curves. Again the effect is enhanced for narrow, steep embankments (for $\alpha = 45^\circ$, and $L/H = 0$, the vertical normal stress is only 65 per cent of that usually assumed.)

Because there are no shearing stresses on the axis of symmetry of the embankment, the conditions of equilibrium require that the sum of the vertical normal stresses acting on the interface must be equal to the weight of the embankment. Therefore, the area under corresponding dashed and solid curves must be the same. Hence the difference between these curves becomes less pronounced, at least near the central portion of the embankment, as the L/H ratio increase. However, near the outer edge of the embankment, the stresses are still significantly larger on a proportional basis, than indicated by the normal loading approximation. Thus, for embankments with moderate L/H ratios, the normal loading approximation leads to larger estimates of differential settlement,

Figure 20. Distribution of Vertical Normal Stress, $\sigma_z/\gamma H$ On The Base of The Embankment For Varying α , And L/H Ratios

assuming one-dimensional compression, than would be computed by the method presented herein.

Analysis of the Horizontal Normal Stress Distribution

The horizontal normal stress contours, $\sigma_x/\gamma H$, are shown in Figure 21, for $\alpha = 45^\circ$, $L/H = 3$ and $\mu = 0.3$. The dashed lines are contours determined from the usual normal loading approximation. The figure shows that the maximum horizontal stress occurs within the body of the embankment and decreases with increasing depth. In the foundation material in the vicinity of the elastic embankment, $\sigma_x/\gamma H$ is less than half of that usually assumed, because, the normal loading approximation neglects the shear stress, $\tau_{xy}/\gamma H$, transmitted from the embankment to the foundation.

The effect of the ratio, L/H , on the horizontal stress along vertical sections through the center line of the embankment and the toe of the slope, is illustrated in Figure 22 for $\alpha = 45^\circ$ and $\mu = 0.3$. As the embankment becomes narrower ($L/H \leq 1$), the stress is actually negative at some points below the center line. That is, the embankment causes a reduction in horizontal stress at these points. However, the horizontal normal stress within the embankment at the center line increases as L/H increases and becomes larger than $\frac{\mu}{1-\mu}$ for $L/H \geq 3$. This is due to the shear stress acting along the base of the embankment.

The dashed line shows the stresses determined from the normal loading approximation for $L/H = 1$. The stress is larger than that due to the elastic embankment at all depths. In fact, in the vicinity of the embankment it is more than five times as large under the center line and twice as large under the toe. In contrast to the elastic embankment, the normal loading approximation does not produce negative

Figure 21. Contours for Horizontal Stress

(a) At Centerline (b) At Toe of Slope

Figure 22. Distribution of Horizontal Stress Along Vertical Sections for Varying L/H Ratios

horizontal stress at any depth. The reason for this difference becomes apparent when the shear stresses transmitted by the embankment to the foundation material are considered. This is discussed below.

The effect of Poisson's ratio on the horizontal stress is illustrated in Figure 23. This figure shows the horizontal stress along vertical sections through the center line and toe of the embankment for $\alpha = 45^\circ$ and $\mu = 0.5$. The dashed line shows the stress due to the normal loading approximation for $L/H = 1$. Comparison with Figure 22 indicates that a change in Poisson's ratio from 0.3 to 0.5 changes the stress at shallow depths below the central portion of the embankment by as much as a factor of two. The difference decreases as the L/H ratio increases. The influence of μ is less pronounced below the toe than below the center line.

Analysis of Horizontal and Vertical Shear Stress

Contours of horizontal and vertical shear stress, $\tau_{xy}/\gamma H$, are shown in Figure 24, for $\alpha = 45^\circ$, and $L/H = 3$. The solid contours are for Poisson's ratio of 0.3. The long dashed contours are for $\mu = 0.5$, and the short dashed lines are for the normal loading approximation. The figure indicates the existence of horizontal shear stresses within the body of the embankment, increasing to a value at the base near the toe of the slope, in excess of $0.2\gamma H$. However, the maximum value of horizontal shear stress (approximately $0.3\gamma H$) occurs below the base of the embankment.

Like the horizontal normal stress, the shear stress, $\tau_{xy}/\gamma H$, is affected markedly by the magnitude of Poisson's ratio. However, the effect observed depends upon the position of the point considered,

(a) At Centerline (b) At Toe of Slope

Figure 23. Distribution of Horizontal Stress Along Vertical Sections for Varying L/H Ratios

Figure 24. Contours of Shear Stress, T_{xy}

relative to the base of the embankment. In the zone below the embankment to a depth of y/H equal approximately two to three, the shear stresses in the incompressible material ($\mu = 0.5$) are less than for the case in which $\mu = 0.3$. At greater depths, the reverse is true. The shear stress determined from the normal loading approximation is less than that for either μ above a depth factor of approximately three to five, and more at greater depths. The magnitude of this effect depends upon the horizontal location considered, as shown in the figure.

The normal loading approximation assumes that there is no shear stress at the base of the embankment. Figures 24 and 25 indicate that, for the elastic embankment, this assumption is not reasonable. Figure 25 shows the horizontal shear stress, $\tau_{xy}/\gamma H$, at the base of the embankment for $\mu = 0.3$, four values of α , and a variety of embankment shapes shown schematically in the figure. The horizontal shear stress is zero at the center line, as required by symmetry, and reaches a maximum near the toe of the slope. The magnitude of the maximum and its location depend upon α and the embankment shape. As L/H decreases for a given α , the maximum $\tau_{xy}/\gamma H$ increases, and moves closer to the toe of the slope. The magnitude of the increase is slight for $\alpha = 15$ degrees, but becomes more significant as α increases. Note that a maximum $\tau_{xy}/\gamma H$ in excess of 0.4 implies that the horizontal shear stress at the base of a forty foot high embankment may be greater than one ton per square foot (unless the shear strength of the material is such that failure is induced).

Figure 25. Distribution of Shear Stress, $T_{xy}/\gamma H$ On The Base of The Embankment For Varying α , And L/H Ratios

Analysis of the Maximum Shear Stress

It is often useful to consider whether the maximum (i.e., principal) shear stress, $\tau_{xy}/\gamma H$, at any depth beneath the embankment exceeds the available shear strength. Thus, it is desirable to know the magnitude and distribution of maximum shear stresses due to the embankment. Contours of $\tau_{\max}/\gamma H$ are shown in Figure 26 for the embankment section of Figures 18, 21 and 24. Note that the magnitude of $\tau_{\max}$ transmitted from the embankment to the foundation material is approximately $0.25\gamma H$ at the base of the embankment in the vicinity of the toe. However, the largest shear stress, $0.33\gamma H$, occurs beneath the center line at $y/H = 1.8$. It is also interesting to observe that within the embankment, the maximum shear stresses are larger near the top than in the mid-depth region, and that they increase again as depth increases. This is believed due to the relatively large horizontal stresses which are induced by the deformation mode of the embankment, (cf. Figure 21).

Two contours of $\tau_{\max}/\gamma H$ for the normal loading approximation corresponding to the embankment considered are shown in Figure 26 as dashed lines. They indicate a shear stress less than that produced by the elastic embankment in a shallow zone below the embankment, but larger shear stresses at depth.

Data for a variety of embankment shapes, with $\mu = 0.3$, and $\alpha = 15^\circ$, 30° , 45° , 60° , and 75° are shown in Figures 27 to 31, respectively. In these figures, the maximum value of $\tau_{\max}$ at a particular depth is plotted as a function of depth for various L/H ratios. The horizontal location of the point at which this maximum value occurs is also shown. It can be observed that in Figure 27, as the L/H ratio increases for

Figure 26. Contours for Max. Shear, $T_{\max}/YH$

Figure 27. Magnitude and Location of Maximum $(T_{\max}/\gamma H)$ as a Function of Depth for $\mu=0.3$, $\alpha=15^\circ$

Figure 28. Magnitude and Location of Maximum ($T_{\max}/\gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 30^\circ$

Figure 29. Magnitude and Location of Maximum ($T_{\max} / \gamma H$) as a Function of Depth for $\mu = 0.3$, $\alpha = 45^\circ$

Figure 30. Magnitude and Location of Maximum $(T_{\max}/\gamma H)$ as a Function of Depth for $\mu = 0.3$, and $\alpha = 60^\circ$

Figure 31. Magnitude and Location of Maximum $(T_{\max}/\gamma H)$ as a Function of Depth for $\mu = 0.3$ and $\alpha = 75^\circ$

$\alpha = 15^\circ$, the peak $\tau_{\max}$ increases in magnitude, and acts at an increasing depth below the embankment. The horizontal location of the maximum shear stress at a particular depth moves from a position near the toe of the slope immediately beneath the embankment to the center line of the embankment at a depth which depends upon the L/H ratio.

A similar trend is shown in Figure 28 for $\alpha = 30^\circ$. However, in Figure 29 ($\alpha = 45^\circ$), the largest shear stress occurs near the toe of the slope for L/H = 0.0. Although the smallest L/H ratio shown in Figures 30 and 31 is 0.5, the development of large shear stress near the toe of narrow steep embankments is clearly indicated.

The dashed line in Figure 29 shows the magnitude of the maximum $\tau_{\max}$ as a function of depth for the normal loading approximation corresponding to the 45° embankment for which L/H = 1. It is evident that the peak magnitudes are nearly the same for the two cases, but it occurs at approximately twice the depth in the case of the normal loading approximation. A similar effect is evident in Figure 26 for L/H = 3. Thus the influence of the elastic embankment is more pronounced nearer the surface where softer soils might be expected. As a result, it may be that current estimates of stability, potential creep and other shear stress related phenomena, for soils at shallow depths beneath embankments, are unconservative.

Relationship of Results to In-Situ Stresses

It is not immediately clear what relationship these results have to stresses which actually exist in the field. In the case of a built-up embankment, it is likely that the embankment material will exhibit significantly different mechanical properties from the foundation

material. For a cut-down slope, the assumption of homogeneity in the two zones may be more nearly justified. The non-linearity in the mechanical response of most natural materials will undoubtedly also influence the results. However, the feature which may be most significant, at least in the case of built-up embankments, is the fact that they are constructed in layers rather than instantaneously. Thus when the top-most lift is placed on an earth embankment, the upper material does not undergo strain due to elastic deformation of the embankment resulting from the stresses imposed by the entire mass. Rather, the strains are due only to the increment of stress imposed by this layer. The degree to which the results would be changed is not clear. However, it is believed that the results presented herein, provide a more realistic estimate of stress conditions than that computed from the normal loading approximation.

Effect of Results on Stress Path Determination

Lambe [21] has suggested that the "stress path" method for prediction of vertical settlements of cohesive soils is superior to conventional analyses in cases where compression is clearly not one-dimensional. This approach involves three basic steps (Lambe, 21):

1. Estimation of the effective stress path of an "average" element in the compressible layer, for the field loading.
2. Performance of a laboratory compression test which duplicates, insofar as practicable, the field effective stress path.
3. Computation of settlement by multiplying the thickness of the layer considered by the axial (vertical) strain from the laboratory test.

Because the strains in the laboratory sample depend upon the applied stresses, the method requires a means of correctly assessing the in-situ stresses.

A comparison of the total stress paths for several points under the center line of an elastic embankment ($\alpha = 30^\circ$, $L/H_{\text{final}} = 0.5$, $\mu = 0.3$), with those computed using the normal loading approximation, is shown in Figure 32. The dashed "initial stress" line shows the state of stress in an elastic half-space, for which $\mu = 0.3$, before construction of the embankment. The three points shown on the line correspond to the stress states depths of 0.5, 1.0 and 2.0 times the final height of the embankment, H_{final} . The open points show the stress paths during "construction" of an elastic embankment continuous with the foundation material. The solid points show the stresses for corresponding embankment heights, determined by the conventional method. Several features of this comparison are especially noteworthy:

1. At relatively shallow depths ($y/H = 0.5$), the conventional method leads to a total stress path which lies entirely below the k_0 line. That is, one would predict relatively small shear settlements. However, on the basis of the elastic embankment analysis, the estimated shear induced settlement would likely be larger, and compression settlement would be less.
2. At greater depth ($y/H_{\text{final}} = 2.0$) both methods lead to stress paths which lie above the k_0 line. The two paths are closer, and the shear stress under the elastic embankment is actually less than that due to the normal loading approximation.
3. At intermediate depths ($y/H_{\text{final}} = 1.0$) the normal loading approximation remains relatively close to the k_0 line. The stress path due to the elastic embankment is still considerably steeper.

Because of the influence of the applied stress path on the measured

Figure 32. Stress Path for Three Points Under the Center Line During Embankment Construction

laboratory settlements, and therefore on the computed field settlements, it would seem essential to estimate the predictive capability of the method to the field stresses accurately. In the case considered, the stresses produced by the elastic embankment are significantly different from those due to the normal loading approximation, at least at shallow depths. The effect of this difference on the results predicted by the stress path method is not obvious, however, this question would appear to deserve further attention.

PART II

ANALYSIS OF THE VISCOELASTIC EBANKMENT

INTRODUCTION

Many embankments are composed of cohesive soils. The deformation of these embankments are composed of three parts: (a) the immediate deformations which are due to the elastic characteristics of the embankment and the underlying foundation; (b) long-term deformation which is probably attributed to a creep mechanism in response to shear stress; (c) the deformation due to consolidation of the soils underlying the embankment.

In this part of this dissertation a viscoelastic embankment is analyzed. The material comprising the embankment and the foundation was assumed to behave in a linear viscoelastic manner.

The creep curve used was obtained from the results of triaxial creep tests performed on kaolin soil from Edgar, Florida compacted at optimum water content for the compaction method used [30].

The solution of the problem was obtained by applying the quasi-elastic approximation proposed by Schapery [32], to the associated elastic solution, with creep function in place of elastic constants. That is, if

$$\begin{aligned} u &= E_f(x, y, \mu) \\ v &= E_g(x, y, \mu) \end{aligned} \tag{122}$$

are the elastic displacements, then the viscoelastic displacements, for

a linear viscoelastic material, are:

$$u(T) = E_c(T) f[x, y, \mu(T)] \quad (123)$$

$$v(T) = E_c(T) g[x, y, \mu(T)]$$

where:

$E_c(T)$ is the creep compliance of both the embankment and the foundation.

$\mu(T)$ is Poisson's ratio in creep.

T is the time.

$u(T)$ is the horizontal time dependent displacement.

$v(T)$ is the vertical time dependent displacement.

If Poisson's ratio in creep is independent of time, then the quasi-elastic method is exact.

TRIAxIAL LOAD AND CREEP TESTS

Material

As mentioned above, the soil used in the determination of the creep compliance, E_c , is a moderately well crystallized kaolin soil from Edgar, Florida.

Classification properties of the soil are given in Table 2. The grain size distribution and compaction characteristics of the soil are shown in Figures 33 and 34.

Table 2

Classification Properties of the Soil

Liquid Limit %	58.5
Plasticity Index %	22.0
Specific Gravity of Solids	2.6
Clay Fraction %	78.0

Preparation of Soil Specimens

Test specimens were prepared at the desired moisture content by impact compaction in a 1.3 inch diameter mold with a special drop hammer weighing 3.62 pounds and height of fall of 10 inches [30]. Specimens were prepared in three layers. Impact compaction curves are shown in Figure 34. Note that the curves are essentially similar in appearance to those of many naturally occurring silty clays, with a moderately well defined peak point.

Figure 33. Grain Size Distribution for Edgar Plastic Kaolin

Figure 34. Impact Compaction Curves For Edgar Plastic Kaolin

Experimental Apparatus and Procedure

The triaxial apparatus used was of the "Norwegian" type accommodating of 1.3 inch diameter by 2.8 inch long specimen, described in detail by Andresen et al [1]. Lateral pressures were applied through a confining fluid to which pressure is applied by small "constant pressure cells". Axial load is applied through a piston by placing weights on the load hanger.

The creep test was conducted by permitting the specimen to come to equilibrium under a desired confining fluid pressure. After the specimen reached equilibrium, the creep stresses were applied axially.

The axial displacement of the specimen was determined by measuring the vertical displacement of the cross bar of the loading frame by a linear variable differential transformer (LVDT) which converts displacement into an electrical signal.

The radial displacement of the specimen is determined by a radial displacement sensor which is a modification of the lateral strain indicator suggested by Bishop and Henkel [3], and is shown schematically in Figure 35. The apparatus and experimental techniques are discussed in detail by Perloff [30].

Figure 35. Schematic Diagram of Radial Displacement Sensor and Specimen on Triaxial Cell Base (Perloff)

RESULTS AND DISCUSSION ON CONFINED CREEP TESTS

Axial Strain

A typical creep curve plotted to logarithmic scales is shown in Figure 36. This figure indicates the axial strain, ϵ_1 , as a function of time for an impact compacted kaolin specimen. This result is typical of the tests which have been run [30], including tests of 1000 minutes duration. It is clear that the creep curve appears as a straight line on the logarithmic plot. The equation of such a line is:

$$\log \epsilon_1 = b \log T + \log a \quad (124)$$

or

$$\epsilon_1 = aT^b \quad (125)$$

where T is time, and a and b are constants of the equation. The constant a is readily determined as the magnitude of the creep strain at a time of one minute. The constant b is the logarithmic slope of the straight line, i.e.,

$$b = \frac{\log \left[\frac{(\epsilon_1)_2}{(\epsilon_1)_1} \right]}{\log \left[\frac{T_2}{T_1} \right]} \quad (126)$$

Because the materials comprising the embankment and the foundation are assumed to behave in a linear viscoelastic fashion, a and b must be

Figure 36. Creep Curve for Multiply Loaded Specimens of EPK

assumed independent of the stress level. Perloff [30] has shown that the constant b is indeed independent of the stress level, but the constant a is not, at least for levels exceeding 30%.

The ratio of the axial to the radial strains is shown in Figure 37. This figure indicates that this ratio, which is equal to Poisson's ratio μ , is independent of time, i.e., the relationship between Poisson's ratio and time is a straight line parallel to the time axis. Therefore, the stresses, which are functions of Poisson's ratio are time-independent.

Creep Compliance

The creep compliance can be determined from:

$$E_c = \frac{\sigma_1 - \sigma_3}{\epsilon_1} \quad (127)$$

Combining equations (125) and (127):

$$E_c = \frac{\sigma_1 - \sigma_3}{aT^b} \quad (128)$$

To determine the constant a and b , consider the correspondence between ϵ_1 and T in equation (125).

$$1 - \text{at } T = 1 \text{ min.}, \quad \epsilon_1 = 9.24 \times 10^{-4}$$

Substituting into equation (125):

$$a = 9.24 \times 10^{-4}$$

$$2 - \text{at } T = 10 \text{ min.}, \quad \epsilon_1 = 9.81 \times 10^{-4}. \text{ From equation (125):}$$

$$b = 0.0259$$

Therefore:

Figure 37. Creep Curve for Multiply Loaded Specimens of EPK

$$E_c = \frac{\sigma_1 - \sigma_3}{9.24 \times 10^{-4} T^{0.0259}} \quad (129)$$

Let E_{c_0} be the creep compliance at $T = 0.001$ min. Substituting into equation (129):

$$E_{c_0} = \frac{\sigma_1 - \sigma_3}{9.24 \times 10^{-4} \times 0.8358} \quad (130)$$

Dividing equation (129) over equation (130), one obtains:

$$\frac{E_c}{E_{c_0}} = \frac{0.8358}{T^{0.0259}} \quad (131)$$

where T has to be in minutes.

or

$$\frac{E_c}{E_{c_0}} = \frac{8358}{11112 T^{0.0259}} \quad (132)$$

where T has to be in hours.

RESULTS AND DISCUSSION

Introduction

The displacement within and under a long viscoelastic, isotropic and homogeneous embankment continuous with the underlying material of similar properties, has been investigated for $\alpha = 30^\circ$, $L/H = 1$, $\mu = 0.3$ and creep characteristics shown in Figure 36.

The magnitude of Poisson's ratio was chosen to be 0.3, which is different from the test data given in Figure 37. This discrepancy occurs because the test was conducted on a compacted unsaturated specimen, while the material underneath the embankment is likely to be saturated. The value $\mu = 0.3$ was considered to be an appropriate average. Further, it is assumed that the embankment is constructed instantaneously.

Effect of Time on the Embankment Shape

A typical result of the analysis is illustrated in Figure 38. This figure shows the change of the shape of the embankment with respect to time, in dimensionless form, $VE_c/\gamma H^2$ (where $V = u + iv$). The dashed lines show the displacements of the base and under the edge of the embankment. These results indicate that, for the soil considered, the immediate displacement is almost sixty per cent of the displacement occurring within the first ten years. Ninety per cent of this displacement occurs within the first month. The maximum value of the displacement

Figure 38. Effect of Time on Embankment Shape

occurs along the center line and decreases as x/H increases and becomes very small at large x/H values, i.e., less than 0.012 for $x/H = 50$.

Analysis of Vertical and Horizontal Displacement

Figure 39 shows the effect of time on the vertical displacements of five points from the embankment, shown schematically in the figure. These results are typical of all the points in the region. The observed straight line logarithmic plot of displacement as a function of time is implied by equations (123) and the experimental data, Figure 36.

Figure 40 shows the effect of time on the compression of the embankment discussed above, between points A and E. This figure indicates that the vertical displacements induced within the embankment at any time are approximately fifteen per cent of the total displacement.

Figure 41 shows the relative vertical displacement between points E and C, that is, the difference between curves E and C in Figure 39. Thus the base of the embankment is settling at a faster rate in the center than at the toe.

The effect of time on the relative vertical displacement between the center top and the top edge of the embankment is shown in Figure 42. This indicates that very little differential settlement between points A and B, i.e., for $L = H = 30$ ft., $\alpha = 30^\circ$, $\gamma = 120$ pounds per cubic foot and $E_c = E_{c_0}$, the computed differential settlement between A and B during the first ten years is 0.059 inches.

The relative horizontal displacements are also small, as indicated in Figure 43. This figure shows the effect of time on the horizontal displacements of five points from the embankment. It is clear that the horizontal displacement at points A and E is zero, this is due to the symmetry of the problem.

Figure 39. Effect of Time on Vertical Displacements of Selected Points on and Within the Embankment

Figure 40. Effect of Time on the Relative Vertical Displacement Between the Points A and E

Figure 4I. Effect of Time on the Relative Vertical Displacement Between the Points E and C

Figure 42. Effect of Time on the Relative Vertical Displacement Between the Points A and B

Figure 43. Effect of Time on Horizontal Displacements of Selected Points on the Embankment

PART III

APPLICATION OF METHOD TO ANALYSIS OF AN EXCAVATION

INTRODUCTION

The method presented above can solve many plane strain problems of homogeneous, isotropic material and can be extended to include anisotropic material and layered systems. In fact, the method does not differentiate in any way between embankments and foundations, or between slopes and cuts. However, the transformation formula, which transforms the geometric boundary into a half-space differs in each case.

As an illustration of the application of the method, the stresses induced within an elastic, homogeneous and isotropic mass due to excavation of an open trench in a semi-infinite mass, and the application of a uniform strip load on the base of the excavation are analyzed.

The problem, illustrated in Figure 44 (a), is that of a long compensated foundation, in which a depth of soil is excavated to reduce the net load applied to compressible subsurface materials. The usual method for estimating the stress distribution is to assume that the net load applied at the base of the excavation is equal to the foundation load, less the unit weight of soil times the depth of excavation, and that this stress distributes in the same fashion as a strip load on the surface of a half-space placed at the elevation of the base of the excavation. This is referred to herein as the Boussinesq approximation, and is illustrated in Figure 44 (b).

Figure 44. Application of Stress on the Base of the Excavation

THE TRANSFORMATION OF THE GEOMETRIC
BOUNDARY INTO A HALF-SPACE

Applying the Schwarz-Christoffel transformation (56) to transform Figure 45 (a) into Figure 45 (b), leads to:

$$\frac{z}{H} = \frac{L}{HF} \int_0^t \left[\frac{\lambda^2 - \beta^2}{\lambda^2 - 1} \right]^{\frac{1}{N}} d\lambda + i \quad (133)$$

where:

β is the modulus of the integral.

F is the value of the integral at point β .

Figures 46 and 47 give the value of β as a function of L/H or H/L .

Knowing β , the value of F is determined from Figure 48.

Figure 45. The Geometrical Transformation

Figure 46. The Modulus of the Transformation Integral as a Function of L/H

Figure 47. The Modulus of the Transformation Integral as a Function of H/L

The Modulus of the Transformation Integral

Figure 48. The Value of the Constant, F , Which Precedes the Transformation Integral

RESULTS AND DISCUSSION

Figure 49 shows the reduction in vertical stress, $\sigma_y/\gamma H$, along the center line due to the excavation, calculated by this method and by the Boussinesq approximation for $L/H = 1$ and $\mu = 0.4$. Here again, the stress determined by the Boussinesq analysis is independent of Poisson's ratio; the stress determined herein depends upon μ . Figure 49 indicates that the reduction in vertical stress, at depths between $y/H = 0$ and $y/H = 1$, is less than that computed by the Boussinesq approximation and higher elsewhere. The difference between the computed vertical stress reduction and Boussinesq value along the center line, expressed as a percentage of the Boussinesq value, is shown in Figure 50, for $L/H = 0.5, 1$ and 2 . For all L/H ratios, the reduction in vertical stress, between the bottom of the foundation ($y/H = 0$) and approximately $y/H = 1$, is slightly smaller than computed by the normal loading approximation, but is significantly larger below the depth $y/H = 1$. Furthermore, this figure shows that L/H ratio has pronounced effect on the difference between both methods; a smaller L/H ratio leads to a greater disparity between the results of the two methods.

The reduction in vertical stress under the edge of the excavation, is shown in Figure 51. The vertical stress reduction is significantly higher than that computed by the Boussinesq approximation. The difference becomes especially apparent at the bottom of the foundation, where the vertical stress reduction is approximately 1.8 times that

Figure 49. Reduction in Vertical Stress Along Center Line Due to Excavation

Figure 50. Net Reduction in Computed Vertical Stress Along Center Line Due to Excavation, Compared to Boussinesq Value

Figure 51. Reduction in Vertical Stress Under Edge of Excavation

usually assumed.

Figure 52 shows the percentage differences between the excavation analysis and Boussinesq value, below the edge of the excavation, for $L/H = 0.5, 1$ and 2 , and $\alpha = 90^\circ$. It can be seen clearly from this figure that the difference is larger at the bottom of the foundation and decreases as y/H increases reaching a minimum value at approximately $y/H = 1$. At greater depth, this figure shows only a slight increase in the difference for high value of L/H and significant increases in the difference for small value of L/H , and the difference is almost constant below $y/H = 10$.

Figure 53 shows contours of vertical normal stress, σ_y/q , due to a uniform strip load, q , placed on the base of the foundation, for $L/H = 1$, $\mu = 0.4$ and $q = 1$. The dashed lines show stress contours in terms of σ_y/q , for the Boussinesq approximation. It is clear that the stress is smaller in the vicinity of the loaded area than that computed by Boussinesq value and larger elsewhere, this is a reverse picture of the stress reduction shown in Figures 49 to 52.

The net increase in vertical normal stress due to the combined effect of excavation and application of stress at the base of the compensated foundation is shown in Figure 54, for $L/H = 1$, $\mu = 0.4$ and $q = 1.47H$. The results depend, of course, upon the magnitude of q relative to the depth of the excavation. A value of $q = 1.47H$ is representative of that for a multistory structure. Figure 54 (a) shows that the net increase in vertical stress along the center line has the same value at $y/H = 0$, but it dissipates faster than that computed by the Boussinesq analysis. Figure 54 (b) shows that the net increase in

Figure 52. Net Reduction in Computed Vertical Stress Under Edge Due to Excavation, Compared to Boussinesq Value

Figure 53. Contours of Vertical Stress Due to Load on the Base of the Excavation, Weightless Material

Figure 54. Net Increase in Vertical Stress Due to Combined Effects of Excavation and Application of Stress at Base of Excavation

vertical stress under the edge of the excavation is smaller than that usually assumed especially at the bottom of the foundation where the Boussinesq value is 35 per cent greater than the calculated value. Thus, one-dimensional settlements calculated using the Boussinesq approximation will be larger than those determined by the excavation analysis.

CONCLUSION AND RECOMMENDATIONS

Conclusions

The conclusions to be drawn from the analysis presented herein fall into three major categories: those relating to the method, those relating to the elastic results and those relating to the viscoelastic results.

The conclusions drawn from the method are:

- 1 - The method developed permits solution of a wide variety of plane strain problems including:
 - a) Plane strain problems with free boundary loading.
 - b) Plane strain problems with external static stresses applied to the boundary.
 - c) Plane strain problems where the displacements of points of the boundary is given.
 - d) Mixed plane strain problems.
 - e) Plane strain problems where its boundary is under dynamic effect.
- 2 - In order to apply this method, the stresses have to be defined as a result of:
 - a) The stress due to the geometric shape of the boundary.
 - b) The stress due to the weight of the medium.

The conclusions drawn from the results of the elastic analysis are:

- 1 - The distribution of stresses within and beneath an elastic embankment.

- a) The horizontal distribution of vertical stress beneath the embankment is more nearly uniform than is usually assumed. Thus, the differential settlements computed using the normal loading approximation will be larger than those determined using the stress distribution presented herein.
 - b) The horizontal and vertical shear stresses created in the foundation material by the embankment, are found to significantly higher at shallow depths for the elastic embankment than for the normal loading approximation.
 - c) The horizontal location of the maximum shear stress at a particular depth moves from a position near the toe of the slope immediately beneath the embankment to the center line of the embankment at a depth which depends upon the L/H ratio.
- 2 - The distribution of stresses within an elastic foundation.
- a) The reduction in the vertical normal stress due to the excavation is, in general, larger than that computed by the Boussinesq analysis.
 - b) The net increase in vertical normal stress due to the combined effects of excavation and application of stress at the base of a compensated foundation is smaller than that usually assumed. Thus, the one-dimensional settlement calculated by this method will be smaller than those determined by the Boussinesq analysis.

The conclusions drawn from the viscoelastic results, for the soil tested, indicate that sixty per cent of the total displacement which occurs in the first ten years is due to the elastic effect. However,

ninety per cent of this displacement occurs in the first month and most of it is in the foundation.

Recommendations

- 1 - The method presented herein can be extended to include:
 - a) The effect of a difference in the stiffness of the embankment and the foundation on the distribution of stresses and displacements.
 - b) An anisotropic embankment resting on an anisotropic foundation material.
 - c) The layered system embankment resting on layered foundation system.
 - d) The effect of a dynamic load on the boundary of the embankment.
 - e) The non-linear viscoelastic embankment.
- 2 - The application of the method presented herein on different plane strain problems is highly recommended.
- 3 - The possibility of verifying the results included in this investigation, experimentally, should be studied.

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APPENDIX A

APPENDIX A

Typical Test Data

Soil - kaolin

Initial height (L_o) = $2 \frac{103}{128}$ in

Diameter (D_o) = $1 \frac{40}{128}$ in

Compaction - impact - 3 layers - 15 blows each layer

Confining pressure (σ_3) = 3 kg cm^2

Existing axial load = 0
 (Area)($\sigma_1 - \sigma_3$)₀

Additional axial load = 18.4
 (Area) ($\sigma_1 - \sigma_3$)

Initial net weight of the specimen = 119.32

Final net weight of the specimen = 119.46

Dry weight of the specimen = 94.06

Initial moisture content = 27%

Table A1 - Typical Test Data

Time in min.	ΔV in $\times 10^{-2}$	ϵ_1^d	radial rdg, Δd in $\times 10^{-2}$	ϵ_3^d	ϵ_3/ϵ_1	$(\epsilon_1 - \epsilon_3)^d$
1 60	0.144	0.0780	0.0410	0.01320	0.169	0.0650
1 30	0.154	0.0817	0.0425	0.01365	0.167	0.0681
1 15	0.164	0.0853	0.0440	0.01415	0.166	0.0712
1 6	0.166	0.0860	0.0455	0.01460	0.170	0.0714
1 3	0.168	0.0867	0.0470	0.01510	0.174	0.0716
1	0.184	0.0924	0.0494	0.01590	0.172	0.0765
2	0.186	0.0931	0.0500	0.01605	0.172	0.0771
4	0.188	0.0938	0.0506	0.01625	0.173	0.0776
6	0.190	0.09456	0.0515	0.01655	0.173	0.0780
10	0.200	0.0981	0.0530	0.01705	0.174	0.0811
15	0.204	0.0995	0.0533	0.01715	0.174	0.0824
30	0.214	0.1031	0.0560	0.01800	0.175	0.0851
45	0.220	0.1052	0.0569	0.01830	0.175	0.0869
60	0.224	0.1066	0.0590	0.01895	0.176	0.0877

APPENDIX B

APPENDIX B

COMPUTER PROGRAM FOR CALCULATION OF
THE TRANSFORMATION INTEGRAL CONSTANTS

```

$ID 3384*2*10**BALADI*GEORGE
$EXECUTE IBJOB
$IBJOB GO
$IBFTC TRIAL
C M MUST BE ODD
  COMPLEX LF,F,F1,FM,EI(20),D1,D2,EIF,EIG,RIA,EIB,FF,CIBC
  REAL N
200 READ(5,99) DBET,NN
  99 FORMAT(F10.5,I10)
  WRITE(6,11)
  11 FORMAT(15X,3HEIF,25X,3HEIG,25X,3HEIA,22X,3HEIB)
100 READ(5,101) N
  101 FORMAT(F10.5)
  WRITE(6,12) N
  12 FORMAT(5X,2HN=,F10.5)
  READ(5,41) BETT
  41 FORMAT(F10.5)
  DO 109 J=1,NN
    PP=J-1
    BET=BETT+PP*DEET
    WRITE(6,13) BET
  13 FORMAT(5X,5HBETA=,F10.5)
    K=1
    M=251
    T1=0.0
    DELT=0.9*BET
102 D1=(0.0,0.0)
    D2=(0.0,0.0)
    DO 104 I=1,M
      P=M-1
      Q=I-1
      T=T1+DELT*Q/P
      FF=(T**2-1.0)/(T**2-BET**2)
      IF(REAL(FF).EQ.0.0.AND.AIMAG(FF).EQ.0.0)F=(0.0,0.0)
      IF(REAL(FF).EQ.0.0.AND.AIMAG(FF).EQ.0.0)GO TO 190
      LF=CLOG(FF)/N
      F=CEXP(LF)

```

```

190 IF(I.EQ.1) F1=F
    IF(I.EQ.1) GO TO 104
    IF(I.EQ.M) FM=F
    IF(I.EQ.M) GO TO 104
    IF((I-1)/2.EQ.I/2) GO TO 103
    D2=D2+F
    GO TO 104
103 D1=D1+F
104 CONTINUE
    EI(K)=(DELT/(3.0*P))*(F1+2.0*D1+4.0*D2+FM)
    K=K+1
    GO TO (105,105,106,111,121,122,126,123,124,112,125,113,114,116,110
1),K
105 T1=0.9*BET
    DELT=0.09*BET
    GO TO 102
106 T1=0.99*BET
    DELT=0.009*BET
    GO TO 102
111 T1=0.999*BET
    DELT=0.0009*BET
    GO TO 102
121 T1=0.9999*BET
    DELT=0.00009*BET
    GO TO 102
122 T1=0.99999*BET
    DELT=0.000009*BET
    GO TO 102
126 T1=0.999999*BET
    DELT=0.0000009*BET
    GO TO 102
123 T1=1.0000001*BET
    DELT=0.0000009*BET
    GO TO 102
124 T1=1.000001*BET
    DELT=0.000009*BET
    GO TO 102
112 T1=1.00001*BET
    DELT=0.00009*BET
    GO TO 102
125 T1=1.0001*BET
    DELT=0.0009*BET
    GO TO 102
113 T1=1.001*BET
    DELT=0.009*BET
    GO TO 102
114 T1=1.01*BET
    DELT=0.09*BET
    GO TO 102
116 T1=1.10*BET
    DELT=1.0-1.1*BET

```

```
      GO TO 102
110 EIF=EI(1)+EI(2)+EI(3)+EI(4)+EI(5)+EI(6)+EI(7)
    EIG=EIF+EI(8)+EI(9)+EI(10)+EI(11)+EI(12)+EI(13)+EI(14)
    EIA=EIF/EIG
    EIB=EIG/EIF
    CIB=AIMAG(EIG)
    CIBC=EIF/CIB
109 WRITE(6,108) EIF,EIG,EIA,EIB,CIBC
108 FORMAT(5X,2F10.5,5X,2F10.5,5X,2F10.5,5X,2F10.5,5X,2F10.5)
      GO TO 200
      END
$DATA
```

APPENDIX C

APPENDIX C

COMPUTER PROGRAM FOR THE DETERMINATION OF
STRESSES AND DISPLACEMENTS WITHIN AND UNDER
LINEAR ELASTIC AND VISCOELASTIC EMBANKMENTS

```
$ID 3384*9*50**BALADI*GEORGE
$EXECUTE IBJOB
$IBJOB GO
$IBFTC BALADI
```

```
C LIST OF SYMBOLS
C *****
C * PO=SQRT(-1.0) *
C *-----*
C * ZZ(II)=MU IS THE POISSON'S RATIO *
C *-----*
C * CC=H/L, WHERE H IS THE HEIGHT OF THE *
C * EMBANKMENT AND 2L IS THE WIDTH OF THE *
C * TOP OF THE EMBANKMENT *
C *-----*
C * N1 IS THE ANGLE BETWEEN THE SIDE OF THE *
C * EMBANKMENT AND THE X-AXIS *
C *-----*
C * A(J,K)=(VERTICAL STRESS)/(GAMA*H) *
C *-----*
C * B(J,K)=(SHEAR STRESS)/(GAMA*H) *
C *-----*
C * S(J,K)=(HORIZONTAL STRESS)/(GAMA*H) *
C *-----*
C * BMAX(J,K)=(MAX.SHEAR)/(GAMA*H) *
C *-----*
C * AMAX(J,K)=(MAX.STRESS)/(GAMA*H) *
C *-----*
C * SMIN(J,K)=(MIN.STRESS)/(GAMA*H) *
C *-----*
C * ALFA1 IS THE ORIENTATION ANGLE OF MAJOR *
C * PRINCIPAL STRESS MEASURED COUNTER- *
C * CLOCKWISE FROM THE HORIZONTAL *
C *-----*
C * ALFA2 IS THE ORIENTATION ANGLE OF MINOR *
C * PRINCIPAL STRESS MEASURED COUNTER- *
C * CLOCKWISE FROM THE HORIZONTAL *
C *-----*
```

```

C      * V(J,K)=(VERT.DISPLACEMENT*E)/(GAMA*H**2) *
C      *-----*
C      * U(J,K)=(HORIZ.DISPLACEMENT*E)/(GAMA*H**2)*
C      *-----*
C      * E IS THE ELASTIC MODULUS *
C      *-----*
C      * EC IS THE CREEP COMPLIANCE *
C      *-----*
C      *****

```

```

COMPLEX PO,LAM(15,20),EI(15,20),EIC(15,20),DLJ,DLK,PHIE(15,20),DEL
1T(15,20),SPHI(15,20),RI(15,20),SHI(15,20),GI(15,20)
REAL N,NL,ZZ(4)

```

```

COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
COMMON/FUN/PHIE,DELT
COMMON/STR/A(15,20),B(15,20),S(15,20),BMAX(15,20),AMAX(15,20),SMIN
1(15,20),ALFA1(15,20),ALFA2(15,20)
COMMON/PH/SPHI,SHI
COMMON/GES/GI
COMMON/DIS/U(15,20),V(15,20)
COMMON/POIS/ZZ
COMMON/VET/NL,DT,DET
COMMON/CHIF/ADJ

```

```

      READ(5,11) LL
11  FORMAT(I10)
      READ(5,12) (ZZ(II),II=1,LL)
12  FORMAT(2F10.5)
      READ(5,14) NL,DT,DET
14  FORMAT(I10,2F10.5)
100 READ(5,101) NN,MM,DEJ,DEK
101 FORMAT(2I10,2F10.5)
      READ(5,102) N,C,BETA,CC
102 FORMAT(4F10.5)
      READ(5,103) PR
103 FORMAT(F10.5)
      READ(5,16) ADJ
16  FORMAT(F10.5)
      NL=180.0/N
      PO=1.0*(0.0,1.0)

```

```

      CALL TRANSF

```

```

C      STRESS CALCULATION
C      -----

```

```

      DO 13 II=1,LL

```

```

C      THE STRESS FUNCTION
C      -----

```

```

      CALL FUNCT(II)

```

```

      CALL STRESS(II)

C      THE INTEGRAL OF THE STRESS FUNCTIONS FOR THE DISPLACEMENTS
C      -----

      CALL PHI(II)

      CALL PSI(II)

C      THE ELASTIC DISPLACEMENT
C      -----

      CALL DISPLA(II)

C      THE VISCOELASTIC DISPLACEMENT
C      -----

      CALL VISCO(II)

      WRITE(6,21) NN,MM,DEJ,DEK
21  FORMAT(///1H0,3HNN=,I3,9X,3HMM=,I3,9X,4HDEJ=,F9.5,4X,4HDEK=,F9.5)
      WRITE(6,22)
22  FORMAT(1H1,5X,104HSTRESS DISTRIBUTION WITHIN AND UNDER LONG ELASTI
1C AND VISCOELASTIC EMBANKMENT WITH FREE BOUNDARY LOADING)
      WRITE(6,23) ZZ(II)
23  FORMAT(///10X,3HMMU=,F10.5)
      WRITE(6,24) N1,C,BETA,CC
24  FORMAT(5X,3HNL=,F10.5,5X,2HC=,F10.5,5X,5HBETA=,F10.5,5X,4HH/L=,F10
1.5)
      WRITE(6,25) ((EI(J,K),A(J,K),B(J,K),S(J,K),AMAX(J,K),BMAX(J,K),SM
1IN(J,K),ALFA1(J,K),ALFA2(J,K),U(J,K),V(J,K),K=1,MM),J=2,NN)
25  FORMAT(6X,3HX/H,7X,3HY/H,5X,8HSIGMA(Y),3X,5HSHEAR,3X,8HSIGMA(X),1X
1,10HSIGMA(MAX),1X,9HMAX.SHEAR,1X,10HSIGMA(MIN),2X,5HALFA1,5X,5HALF
2A2,5X,4HU(X),6X,4HV(Y)///(1X,10F10.5,2F9.4))
13  CONTINUE
      GO TO 100
      END

$IBFTC SUB1
      SUBROUTINE TRANSF

C      THE TRANSFORMATION FROM THE Z-PLANE ONTO THE T-PLANE
C      -----

      COMPLEX PO,LAM(15,20),EI(15,20),EIC(15,20),DLJ,DLK,D1,D2,ED1,ED2,T
1,FF,LF,F,EF,F1,Ef1,FM,EFM
      REAL N,N1

C      *****
C      * BETA IS THE MODULUS OF THE EMBANKMENT TRANSFORMATION INTEGRAL *
C      *-----*

```

```

C      * EI(J,K)=X/H+PO*Y/H THE TRANSFORMATION FORMULAE      *
C      *-----*
C      * EIC(J,K) IS THE HARMONIC CONJUGATE OF EI(J,K) AT THE BOUNDARY *
C      *-----*
C      * 1/C, IS A CONSTANT PRECEDE THE TRANSFORMATION INTEGRAL EI(J,K) *
C      *-----*

```

```

C      M MUST BE ODD
C      -----

```

COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK

```

LAM(1,1)=(0.0,0.0)
EI(1,1)=-PO
EIC(1,1)=PO
DLJ=DEJ*PO
DLK=-DEK*(PO**2)
DO 104 J=2,NN
PP=J-1
DO 104 K=1,MM
VV=K-1
LAM(J,K)=PR*DLJ+(PP-1.0)*DLJ+VV*DLK
IF(J.EQ.2.AND.K.EQ.1) GO TO 105
M=101
GO TO 106
105 G1=100.0*PR
M=2*IFX(G1)+1
106 P=M-1
D1=(0.0,0.0)
D2=(0.0,0.0)
ED1=(0.0,0.0)
ED2=(0.0,0.0)
DO 107 I=1,M
Q=I-1
IF(K.NE.1) GO TO 108
IF(J.EQ.2) GO TO 109
T=LAM(J-1,1)+DLJ*Q/P
GO TO 110
109 T=LAM(1,1)+PR*DLJ*Q/P
GO TO 110
108 T=LAM(J,K-1)+DLK*Q/P
110 FF=(T**2-1.0)/(T**2-BETA**2)
LF=CLOG(FF)/N
F=CEXP(LF)
EF=REAL(F)-AIMAG(F)*PO
IF(I.EQ.1) F1=F
IF(I.EQ.1) EF1=EF
IF(I.EQ.1) GO TO 107
IF(I.EQ.M) FM=F
IF(I.EQ.M) EFM=EF
IF(I.EQ.M) GO TO 107
IF((I-1)/2 .EQ. I/2) GO TO 111

```

```

      D2=D2+F
      ED2=ED2+EF
      GO TO 107
111 D1=D1+F
      ED1=ED1+EF
107 CONTINUE
      IF(K.EQ.1) GO TO 112
      EI(J,K)=(DLK/(3.0*P))*(F1+2.0*D1+4.0*D2+FM)
      EI(J,K)=(1.0/(CC*C))*EI(J,K)+EI(J,K-1)+PO
      EI(J,K)=EI(J,K)-PO
      EIC(J,K)=(DLK/(3.0*P))*(EF1+2.0*ED1+4.0*ED2+EFM)
      EIC(J,K)=(1.0/(CC*C))*EIC(J,K)+EIC(J,K-1)-PO
      EIC(J,K)=EIC(J,K)+PO
      GO TO 104
112 IF(J.EQ.2) GO TO 113
      EI(J,1)=(DLJ/(3.0*P))*(F1+2.0*D1+4.0*D2+FM)
      EI(J,1)=(1.0/(CC*C))*EI(J,1)+EI(J-1,1)+PO
      EI(J,1)=EI(J,1)-PO
      EIC(J,1)=(DLJ/(3.0*P))*(EF1+2.0*ED1+4.0*ED2+EFM)
      EIC(J,1)=(1.0/(CC*C))*EIC(J,1)+EIC(J-1,1)-PO
      EIC(J,1)=EIC(J,1)+PO
      GO TO 104
113 EI(2,K)=(PR*DLJ/(3.0*P))*(F1+2.0*D1+4.0*D2+FM)
      EI(2,K)=(1.0/(CC*C))*EI(2,K)+EI(1,1)+PO
      EI(2,K)=EI(2,K)-PO
      EIC(2,K)=(PR*DLJ/(3.0*P))*(EF1+2.0*ED1+4.0*ED2+EFM)
      EIC(2,K)=(1.0/(CC*C))*EIC(2,K)+EIC(1,1)-PO
      EIC(2,K)=EIC(2,K)+PO
104 CONTINUE
      RETURN
      END

```

```

$IBFTC SUB2
      SUBROUTINE FUNCT(II)

```

```

C      THE CALCULATION OF THE STRESS FUNCTIONS
C      -----

```

```

      COMPLEX PO,LAM(15,20),EI(15,20),EIC(15,20),DLJ,DLK,D1,D2,ED1,ED2,T
      1,FF,LF,F,EF,F1,EF1,FM,EFM,CD,DCD,CE,CCE,PEN,BCB,DD,EE,R,RR,PHI1,PH
      2I2,PHI3,PHI9,PHI4,PHI5,PHI6,PHI7,PHI8,PHIE(15,20),DELT1,DELT2,DELT
      33,DELT4,DELT5,DELT6,DELT7,DELT8,DELT9,DELT11,DELT12,DELT(15,20)
      REAL N,N1,ZZ(4)

```

```

      COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
      COMMON/FUN/PHIE,DELT
      COMMON/POIS/ZZ

```

```

      FOU=ZZ(II)/(1.0-ZZ(II))
      DO 201 J=2,NN
      PP=J-1

```

```

DO 201 K=1,MM
VV=K-1
CD=(LAM(J,K)**2-1.0)/(LAM(J,K)**2-BETA**2)
DCD=CLOG(CD)/N
CE=(1.0/(CC*C))*(CEXP(DCD))
CCE=REAL(CE)-AIMAG(CE)*PO
PEN=REAL(EI(J,K))-AIMAG(EI(J,K))*PO
BC=1.0/((2.0*3.141593)*(1.0-BETA))
BCB=BC*PO
DD=CLOG((LAM(J,K)+BETA)/(LAM(J,K)+1.0))
EE=CLOG((LAM(J,K)-1.0)/(LAM(J,K)-BETA))
R=(1.0-BETA)/((LAM(J,K)+1.0)*(LAM(J,K)+BETA))
RR=(1.0-BETA)/((LAM(J,K)-1.0)*(LAM(J,K)-BETA))

```

C PHI FUNCTION

C -----

```

IF(N.LE.4.0) GO TO 202
PHI1=2.0*3.141593*(1.0/N)*PO
PHI2=2.0*3.141593*(-1.0/N)*PO
PHI3=CEXP(PHI1)
PHI9=CEXP(PHI2)
GO TO 203
202 PHI1=2.0*3.141593*(-1.0/N)*PO
PHI2=2.0*3.141593*(1.0/N)*PO
PHI3=-CEXP(PHI1)
PHI9=-CEXP(PHI2)
203 PHI4=(-BCB/2.0)*(DD+LAM(J,K)*DD-BETA+1.0)
PHI5=(-BCB/2.0)*(EE-LAM(J,K)*EE+BETA-1.0)
PHI6=(-PO/(4.0*3.141593))*CLOG((LAM(J,K)-BETA)/(LAM(J,K)+BETA))
PHI7=(PHI4*PHI3+PHI5*PHI9+PHI6)*(1.0-FOU)
PHI8=(PHI4+PHI5+PHI6)*(1.0+FOU)

PHIE(J,K)=PHI7+PHI8

```

C PSI FUNCTION

C -----

```

DELT1=(-BCB/2.0)*(R+DD+LAM(J,K)*R)
DELT2=(-BCB/2.0)*(RR-EE-LAM(J,K)*RR)
DELT3=(-PO/(2.0*3.141593))*(BETA/(LAM(J,K)**2-BETA**2))
DELT4=(DELT1*PHI3+DELT2*PHI9+DELT3)*(1.0-FOU)
DELT5=(DELT1+DELT2+DELT3)*(1.0+FOU)
DELT6=-PHIE(J,K)*CCE/CE
IF(N.LE.4.0) GO TO 204
DELT7=CEXP(-PHI1)
DELT8=CEXP(-PHI2)
GO TO 205
204 DELT7=-CEXP(-PHI1)
DELT8=-CEXP(-PHI2)
205 DELT9=(PHI4*DELT7+PHI5*DELT8+PHI6)*(1.0-FOU)

```

```
DELT11=(PHI4+PHI5+PHI6)*(1.0+FOU)
DELT12=(DELT9+DELT11)*(CCE/CE)
```

```
DELT(J,K)=(DELT4+DELT5)*((PEN-EIC(J,K))/CE)+DELT6+DELT12
201 CONTINUE
```

```
RETURN
END
```

```
$IBFTC SUB3
```

```
SUBROUTINE STRESS(II)
```

```
C THE CALCULATION OF THE STRESSES
```

```
C -----
```

```
COMPLEX DLJ,DLK,LAM(15,20),EI(15,20),EIC(15,20),PO,PHIE(15,20),DEL
1T(15,20)
REAL ZZ(4)
```

```
COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
COMMON/FUN/PHIE,DELT
COMMON/STR/A(15,20),B(15,20),S(15,20),BMAX(15,20),AMAX(15,20),SMIN
1(15,20),ALFA1(15,20),ALFA2(15,20)
COMMON/POIS/ZZ
```

```
FOU=ZZ(II)/(1.0-ZZ(II))
```

```
DO 301 J=2,NN
```

```
DO 301 K=1,MM
```

```
IF(AIMAG(EI(J,K)).LE.0.0) GO TO 302
```

```
A(J,K)=2.0*REAL(PHIE(J,K))+REAL(DELT(J,K))
```

```
B(J,K)=AIMAG(DELT(J,K))
```

```
S(J,K)=2.0*REAL(PHIE(J,K))-REAL(DELT(J,K))
```

```
CAR=(A(J,K)+S(J,K))/2.0
```

```
CHIP=(A(J,K)-S(J,K))/2.0
```

```
BMAX(J,K)=SQRT(CHIP**2+B(J,K)**2)
```

```
AMAX(J,K)=CAR+BMAX(J,K)
```

```
SMIN(J,K)=CAR-BMAX(J,K)
```

```
GA=B(J,K)/CHIP
```

```
ALFA1(J,K)=(ATAN(GA)/2.0)*(180.0/3.141593)+90.0
```

```
ALFA2(J,K)=ALFA1(J,K)-90.0
```

```
GO TO 301
```

```
302 A(J,K)=2.0*REAL(PHIE(J,K))+REAL(DELT(J,K))+AIMAG(EI(J,K))
```

```
B(J,K)=AIMAG(DELT(J,K))
```

```
S(J,K)=2.0*REAL(PHIE(J,K))-REAL(DELT(J,K))+AIMAG(EI(J,K))*FOU
```

```
CAR=(A(J,K)+S(J,K))/2.0
```

```
CHIP=(A(J,K)-S(J,K))/2.0
```

```
BMAX(J,K)=SQRT(CHIP**2+B(J,K)**2)
```

```
AMAX(J,K)=CAR+BMAX(J,K)
```

```
SMIN(J,K)=CAR-BMAX(J,K)
```

```
GA=B(J,K)/CHIP
```



```

      ALFA1(J,K)=(ATAN(GA)/2.0)*(180.0/3.141593)+90.0
      ALFA2(J,K)=ALFA1(J,K)-90.0
301  CONTINUE
      RETURN
      END

$IBFTC SUB4
      SUBROUTINE PHI(II)

C      THE INTEGRAL OF PHIE FUNCTION
C      -----

      COMPLEX DLJ,DLK,LAM(15,20),CAM(15,20),EI(15,20),EIC(15,20),PO,PHI1
1,PHI2,PHI3,PHI9,BCB,SPHI1,SPHI2,SPHI3,SPHI4,SPHI5,SPHI6,SPHI7,SPHI
28,SPHI9,SPHI10,SPHI11,SPHI12,SPHI13,SPHI14,SPHI15,SPHIA,SPHIB,SPHI
32I,SPHI22,SPHI23,SPHI24,SPHI25,SPHI26,SPHI27,SPHI28,SPHI29,SPHI30,
4SPHI31,SPHI32,SPHI33,SPHI34,SPHI35,SPHIC,SPHID,SPHI(15,20),SHI(15,
520)
      REAL N,ZZ(4)

      COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
      COMMON/PH/SPHI,SHI
      COMMON/POIS/ZZ

      FOU=ZZ(II)/(1.0-ZZ(II))
      IF(N.LE.4.0) GO TO 402
      PHI1=2.0*3.141593*(1.0/N)*PO
      PHI2=2.0*3.141593*(-1.0/N)*PO
      PHI3=CEXP(PHI1)
      PHI9=CEXP(PHI2)
      GO TO 403
402  PHI1=2.0*3.141593*(-1.0/N)*PO
      PHI2=2.0*3.141593*(1.0/N)*PO
      PHI3=-CEXP(PHI1)
      PHI9=-CEXP(PHI2)
403  BC=1.0/((2.0*3.141593)*(1.0-BETA))
      BCB=BC*PO
      DO 401 J=1,NN
      DO 401 K=1,MM
      IF(J.EQ.1.AND.K.NE.1) GO TO 401
      SPHI1=LAM(J,K)+BETA
      SPHI2=LAM(J,K)+1.0
      SPHI3=LAM(J,K)-1.0
      SPHI4=LAM(J,K)-BETA
      SPHI5=SPHI1*CLOG(SPHI1)
      SPHI6=SPHI2*CLOG(SPHI2)
      SPHI7=SPHI3*CLOG(SPHI3)
      SPHI8=SPHI4*CLOG(SPHI4)
      SPHI9=SPHI5-SPHI6+1.0-BETA
      SPHI10=SPHI7-SPHI8+1.0-BETA
      SPHI11=(SPHI1/2.0)*SPHI5-(SPHI1**2)/4.0-BETA*SPHI5+BETA*SPHI1-(SPH
1I2/2.0)*SPHI6+(SPHI2**2)/4.0+SPHI6-SPHI2

```



```

    SPHI12=(SPHI3/2.0)*SPHI7-(SPHI3**2)/4.0+SPHI7-SPHI3-(SPHI4/2.0)*SP
    LHI8+(SPHI4**2)/4.0-BETA*SPHI8+BETA*SPHI4
    SPHI13=(-BCB/2.0)*(SPHI9+SPHI11+(1.0-BETA)*LAM(J,K))
    SPHI14=(-BCB/2.0)*(SPHI10-SPHI12+(BETA-1.0)*LAM(J,K))
    SPHI15=(-PO/(4.0*3.141593))*(SPHI8-SPHI5+2.0*BETA)
    SPHIA=(SPHI13+SPHI14+SPHI15)*(1.0+FOU)
    SPHIB=(SPHI13*PHI3+SPHI14*PHI9+SPHI15)*(1.0-FOU)
    CAM(J,K)=REAL(LAM(J,K))-AIMAG(LAM(J,K))*PO
    SPHI21=CAM(J,K)+BETA
    SPHI22=CAM(J,K)+1.0
    SPHI23=CAM(J,K)-1.0
    SPHI24=CAM(J,K)-BETA
    SPHI25=SPHI21*CLOG(SPHI21)
    SPHI26=SPHI22*CLOG(SPHI22)
    SPHI27=SPHI23*CLOG(SPHI23)
    SPHI28=SPHI24*CLOG(SPHI24)
    SPHI29=SPHI25-SPHI26+1.0-BETA
    SPHI30=SPHI27-SPHI28+1.0-BETA
    SPHI31=(SPHI21/2.0)*SPHI25-(SPHI21**2)/4.0-BETA*SPHI25+BETA*SPHI21
    1-(SPHI22/2.0)*SPHI26+(SPHI22**2)/4.0+SPHI26-SPHI22
    SPHI32=(SPHI23/2.0)*SPHI27-(SPHI23**2)/4.0+SPHI27-SPHI23-(SPHI24/2
    1.0)*SPHI28+(SPHI24**2)/4.0-BETA*SPHI28+BETA*SPHI24
    SPHI33=(-BCB/2.0)*(SPHI29+SPHI31+(1.0-BETA)*CAM(J,K))
    SPHI34=(-BCB/2.0)*(SPHI30-SPHI32+(BETA-1.0)*CAM(J,K))
    SPHI35=(-PO/(4.0*3.141593))*(SPHI28-SPHI25+2.0*BETA)
    SPHIC=(SPHI33+SPHI34+SPHI35)*(1.0+FOU)
    SPHID=(SPHI33*PHI3+SPHI34*PHI9+SPHI35)*(1.0-FOU)
    SHI(J,K)=SPHIC+SPHID
    SPHI(J,K)=SPHIA+SPHIB
    IF(J,EQ.1) GO TO 401
    SHI(J,K)=SHI(J,K)-SHI(1,1)
    SPHI(J,K)=SPHI(J,K)-SPHI(1,1)
401 CONTINUE
    RETURN
    END

```

\$IBFTC SUB5

SUBROUTINE PSI(II)

C THE MODIFICATION OF PHI FUNCTION FOR THE DISPLACEMENT

C -----

```

    COMPLEX DLJ,DLK,LAM(15,20),EI(15,20),EIC(15,20),PO,GI(15,20),ADCD,
    1CAM(15,20),ACD,ACE,FEN,BCB,ADD,AEE,PHI1,PHI2,PHI3,PHI9,APHI4,APHI5
    2,APHI6,APHI7,APHI8,APHI8
    REAL N,ZZ(4)

```

```

    COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
    COMMON/GES/GI
    COMMON/POIS/ZZ

```

```

    FOU=ZZ(II)/(1.0-ZZ(II))

```

```

IF(N.LE.4.0) GO TO 502
PHI1=2.0*3.141593*(1.0/N)*PO
PHI2=2.0*3.141593*(-1.0/N)*PO
PHI3=CEXP(PHI1)
PHI9=CEXP(PHI2)
GO TO 503
502 PHI1=2.0*3.141593*(-1.0/N)*PO
PHI2=2.0*3.141593*(1.0/N)*PO
PHI3=-CEXP(PHI1)
PHI9=-CEXP(PHI2)
503 BC=1.0/((2.0*3.141593)*(1.0-BETA))
BCB=BC*PO
DO 501 J=1,NN
DO 501 K=1,MM
IF(J.EQ.1.AND.K.NE.1) GO TO 501
CAM(J,K)=REAL(LAM(J,K))-AIMAG(LAM(J,K))*PO
ACD=(CAM(J,K)**2-1.0)/(CAM(J,K)**2-BETA**2)
ADCD=CLOG(ACD)/N
ACE=(1.0/(CC*C))*CEXP(ADCD)
PEN=REAL(EI(J,K))-AIMAG(EI(J,K))*PO
ADD=CLOG((CAM(J,K)+BETA)/(CAM(J,K)+1.0))
AEE=CLOG((CAM(J,K)-1.0)/(CAM(J,K)-BETA))
APHI4=(-BCB/2.0)*(ADD+CAM(J,K)*ADD-BETA+1.0)
APHI5=(-BCB/2.0)*(AEE-CAM(J,K)*AEE+BETA-1.0)
APHI6=(-PO/(4.0*3.141593))*CLOG((CAM(J,K)-BETA)/(CAM(J,K)+BETA))
APHI7=(APHI4*PHI3+APHI5*PHI9+APHI6)*(1.0-FOU)
APHI8=(APHI4+APHI5+APHI6)*(1.0+FOU)
APHIE=(APHI7+APHI8)/ACE
GI(J,K)=(REAL(APHIE)-AIMAG(APHIE)*PO)*(EI(J,K)-PEN)
IF(J.EQ.1) GO TO 501
GI(J,K)=GI(J,K)-GI(1,1)
501 CONTINUE
RETURN
END

```

\$IBFTC SUB6

SUBROUTINE DISPLA(II)

C THE ELASTIC DISPLACEMENT
C -----

COMPLEX SPHI(15,20),SHI(15,20),GI(15,20),DLJ,DLK,LAM(15,20),EI(15,
120),EIC(15,20),PO,DI
REAL N,ZZ(4)

COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
COMMON/PH/SPHI,SHI
COMMON/GES/GI
COMMON/DIS/U(15,20),V(15,20)
COMMON/POIS/ZZ
COMMON/CHIF/ADJ

APPENDIX D

```

FOU=ZZ(II)/(1.0-ZZ(II))
DO 601 J=2,NN
DO 601 K=1,MM
IF(AIMAG(EI(J,K)).LE.0.0) GO TO 602
DI=((3.0-4.0*ZZ(II))*SPHI(J,K)+SHI(J,K)-GI(J,K))*(1.0+ZZ(II))
GO TO 603
602 DI=((3.0-4.0*ZZ(II))*SPHI(J,K)+SHI(J,K)-GI(J,K)+PO*(1.0-FOU)*AIMAG
1(EI(J,K))*2/2.0)*(1.0+ZZ(II))
603 U(J,K)=REAL(DI)
V(J,K)=ADJ- AIMAG(DI)
601 CONTINUE
RETURN
END

```

\$IBFTC SUB7

SUBROUTINE VISCO(II)

C THE VISCOELASTIC DISPLACEMENT

C -----

COMPLEX DLJ,DLK,LAM(15,20),EI(15,20),EIC(15,20),PO
REAL N,ZZ(4),AT(18),VT(18),UV(18),VV(18)

COMMON/TRA/NN,MM,DLJ,DLK,N,C,BETA,CC,LAM,EI,EIC,PO,PR,DEJ,DEK
COMMON/DIS/U(15,20),V(15,20)
COMMON/VET/NL,DT,DET

```

C *****
C * UV(LK)= THE VISCOELASTIC DISPLACEMENT IN HORIZ. DIRECTION *
C * -----*
C * VV(LK)=THE VISCOELASTIC DISPLACEMENT IN VERTICAL DIRECTION*
C * -----*
C * AT(LK)= TIME IN HOURS *
C * -----*
C * VT(LK)= TIME IN DAYS *
C * -----*
C * E1=ER/E *
C *****

```

```

DO 701 J=2,NN
DO 701 K=1,MM
DO 705 LK=1,NL
P=(LK+1)/2-1
V=LK/2-1
IF((LK-1)/2.EQ.LK/2) GO TO 702
AT(LK)=DT*10.0**V
VT(LK)=AT(LK)/24.0
GO TO 703
702 AT(LK)=DET*10.0**P
VT(LK)=AT(LK)/24.0
703 E1=8358.0/((11112.0)*(AT(LK)**0.025953))

```

```
UV(LK)=U(J,K)/E1  
VV(LK)=V(J,K)/E1  
705 CONTINUE  
WRITE(6,27) EI(J,K)  
WRITE(6,28) (AT(LK),VT(LK),UV(LK),VV(LK),LK=1,NL)  
27 FORMAT(///1H0,4HX/H=,F9.5,7X,4HY/H=,F9.5)  
28 FORMAT(1H0,10H TIME(HRS),3X,10HTIME(DAYS),6X,5HUV(X),5X,5HVV(Y)///  
1(LX,E9.2,F12.5,3X,2F10.5))  
701 CONTINUE  
RETURN  
END
```

\$DATA

APPENDIX D

Influence Diagrams

This Appendix contains influence diagrams for vertical normal stress and horizontal normal stress along vertical sections for $\mu = 0.3$, $\alpha = 15^\circ$, 30° , 45° , 60° and 75° , and various embankment shapes. Influence diagrams for horizontal and vertical shear stress for $\alpha = 45^\circ$, $\mu = 0.3$ and various embankment shapes are also given.

The diagrams indicate the stress due to the embankment weight alone. Stresses due to the weight of material underlying the embankment must be superimposed to obtain the total stress. The stresses are expressed in dimensionless form as $\sigma_y/\gamma H$, $\sigma_x/\gamma H$, or $\tau_{xy}/\gamma H$. The coordinates are also in dimensionless form. For convenience in the semi-logarithmic plot, the depth is measured from the top of the embankment, and designated $\bar{y}/H$. This is in contrast to the discussion in the body of the paper where the vertical distances are measured from the base of the embankment and designated y/H .

Each of the four diagrams in a given figure refers to a particular vertical section, shown schematically on the diagram. The upper left diagram indicates stresses along the center line; the upper right diagram indicates stresses along a vertical section midway between the center line and the toe of the slope; the lower left diagram indicates stresses along a vertical section through the toe of the slope; the lower right diagram indicates stresses along a vertical section located a distance from center line equal to 1.5 times the distance from the center line to the toe of the slope.

Figure D1. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 15^\circ$, $\mu = 0.3$

Figure D2 . Influence Diagrams for Vertical Normal Stress
Along Selected Vertical Sections for $\alpha = 30^\circ$, $\mu = 0.3$

Figure D3. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 45^\circ$, $\mu = 0.3$

Figure D4. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 60^\circ$, $\mu = 0.3$

Figure D5. Influence Diagrams for Vertical Normal Stress Along Selected Vertical Sections for $\alpha = 75^\circ$, $\mu = 0.3$

Figure D6. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 15^\circ$, $\mu = 0.3$

Figure D7. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 30^\circ$, $\mu = 0.3$

Figure D8. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 45^\circ$, $\mu = 0.3$

Figure D9. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 60^\circ$, $\mu = 0.3$

Figure D10. Influence Diagrams for Horizontal Normal Stress Along Selected Vertical Sections for $\alpha = 75^\circ$, $\mu = 0.3$

VITA

VITA

George Y. Baladi was born in Aleppo, Syria on February 8, 1937 and is presently a citizen of the Syrian Arab Republic. He attended Aleppo University from September, 1957 through June, 1961, receiving a Bachelor degree in Civil Engineering. He then taught soil mechanics, strength of materials, and mathematics for two years at Aleppo University.

In September, 1963, he started his graduate work at Purdue University, and graduated in June, 1965 with a Master of Science in Civil Engineering, majoring in soil mechanics. For the next two years and a half, he worked on his Ph.D. at Purdue University.

Mr. Baladi is an Associate Member of the American Society of Civil Engineers, and a member of the Syrian Society of Civil Engineers.

In 1963, Mr. Baladi was married to Layla Zeito and they are the parents of Zena and Joseph.