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by

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Abstract

A special class of meshes for outlining a surface, vertex-degree bounded, polyhedral meshes, is shown to have a simple smoothing algorithm that generates a small number of low degree polynomial pieces per cell. In particular, the input mesh need not be refined to obtain the Bernstein-Bézier control points as averages of the mesh points. Thus vertex-degree bounded, polyhedral meshes are the simplest and immediate generalization of the regular (tensor-product or box-spline) mesh. A mesh is vertex-degree bounded if at most four cells meet at each vertex, and polyhedral if all cells with more than four edges are planar.

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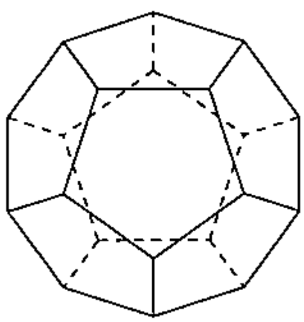
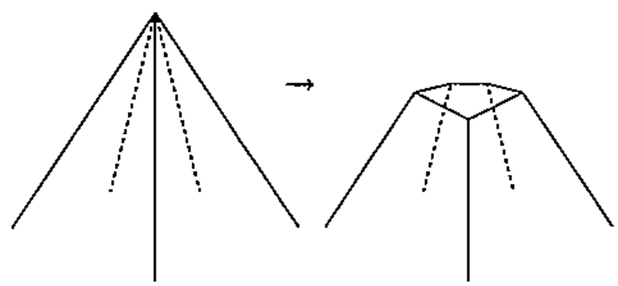
1. Introduction

Recent advances in the theory of splines over irregular meshes suggest that the degree and number of polynomial pieces necessary to smooth a mesh of points are related both to the combinatorial structure and the geometric variation of the mesh. Mesh cells are considered to have low variation if they are either planar or quadrilateral, and the combinatorial structure of a mesh is simpler where the vertex degree, the number of neighbors to the mesh point, is uniform or four. Thus, we have the following classification. The mesh type of least complexity is the well-known *regular mesh*. In a regular mesh each mesh point is surrounded by four, possibly triangulated, quadrilateral cells. Such a mesh can be viewed as the control mesh of a tensor product spline or a four direction box spline yielding one biquadratic piece in the case of a C^1 tensor-product construction and four quadratic pieces per cell in the case of a C^1 box spline surface (see e.g. [1]). At the other end of the spectrum are *irregular meshes*. Irregular meshes may have any number of not necessarily planar cells surrounding mesh points and an arbitrary number of vertices per cell.

To meet the primary goal, low polynomial degree, various approaches adopt a refinement strategy to partially regularize the irregular mesh and triangulate it to reduce the geometric variation. For example, [13] and [14] triangulate and split the mesh to fit piecewise quartic surfaces and [8] restricts the combinatorial structure of the input mesh to fit S-patches. The construction in [9] (see also [10] and [7] for a similar construction based on three-sided pieces) refines the mesh into a simpler mesh to obtain on the average 16 biquadratic or bicubic polynomial pieces per original cell. In [11] it was observed that *degree-bounded meshes* are almost as general as irregular meshes but can be smoothed using only one refinement and hence 4 pieces per facet. An irregular mesh is degree-bounded if cells with more than four edges have no vertices with more than four neighbors and, symmetrically, vertices with more than four neighbors are surrounded by cells with at most four edges.

This paper completes the 'complexity scale of meshes' by showing that vertex-degree bounded polyhedral meshes can be smoothed without further refinement. The proof is an algorithm that constructs the surface using four total degree cubic patches per facet. (An analogous tensor product scheme would be akin to [5] and [12].) *Vertex-degree bounded polyhedral meshes* have at most four cells meet at each vertex and cells with more than four edges are planar. The planarity requirement for cells with more than four edges is a major restriction since it prevents us in general from simply using the dual of a triangulation. Yet, there are important classes of polyhedral meshes. First the Doo-Sabin [3] or Catmull-Clark [2] refinement used to smooth irregular meshes turns them into special vertex-degree bounded meshes, that become polyhedral after local affine projection. Secondly, meshes may be modified to have the required structure; for example by chopping off convex vertices. Finally, buckminsterfullerene geodesic structures found in nature [15], like the dodecahedron below obey the constraint (and partially motivated this study).

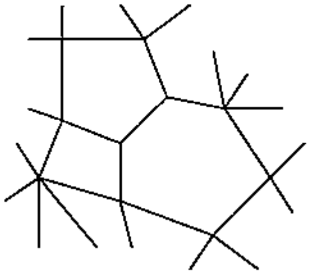




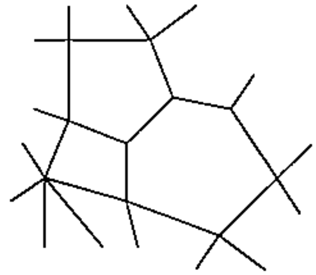
The planarity of the cells allows both a simple construction and characterization of the surface. The algorithm in Section 3 uses the planar mesh cells as tangent planes and allows the user to choose an interpolation point per cell. Should the mesh additionally be regular, then the C^1 surface becomes the quadratic defined by the tangent planes.

2. A hierarchy of meshes

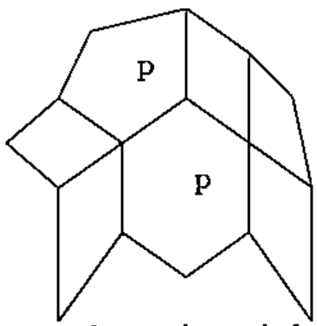
The relationships between the mesh types are as follows. An irregular mesh can be transformed into a degree-bounded mesh by triangulating offending cells. A degree-bounded mesh transforms to a vertex-regular polyhedral mesh after one Doo-Sabin refinement step [3] followed by a projection. Similarly, two subdivisions may be used to refine



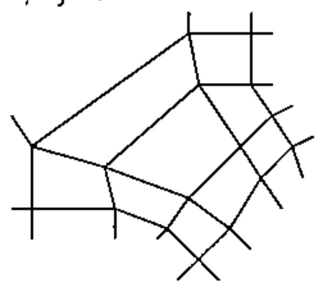
irregular, m-n



irregular, degree-bounded
$\min\{m,n\} < 5$



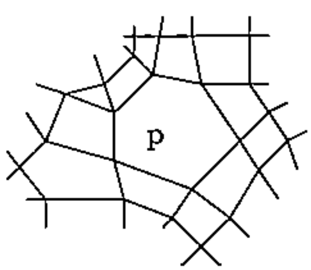
vertex-degree bounded, polyhedral
$n < 5$, cell is planar if $m > 4$



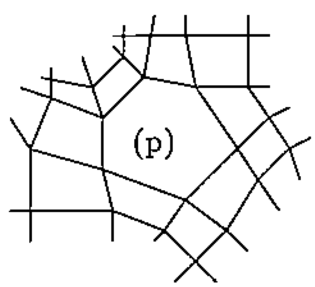
regular, 4-4
(tensor product or box spline mesh)

Mesh types arranged in descending order of complexity.

an irregular mesh into a vertex-regular mesh, where irregular cells with $m \neq 4$ edges are separated from one another. (The p indicating planarity is placed in parenthesis since the algorithm in [10] also works for non-polyhedral meshes.) Vertex-regular polyhedral meshes in turn form a subclass of vertex-degree bounded polyhedral meshes. Regular meshes stand at the bottom of the complexity scale as the mesh type with the least modeling capability.



vertex-regular, polyhedral
$n = 4$, cell is planar if $m > 4$

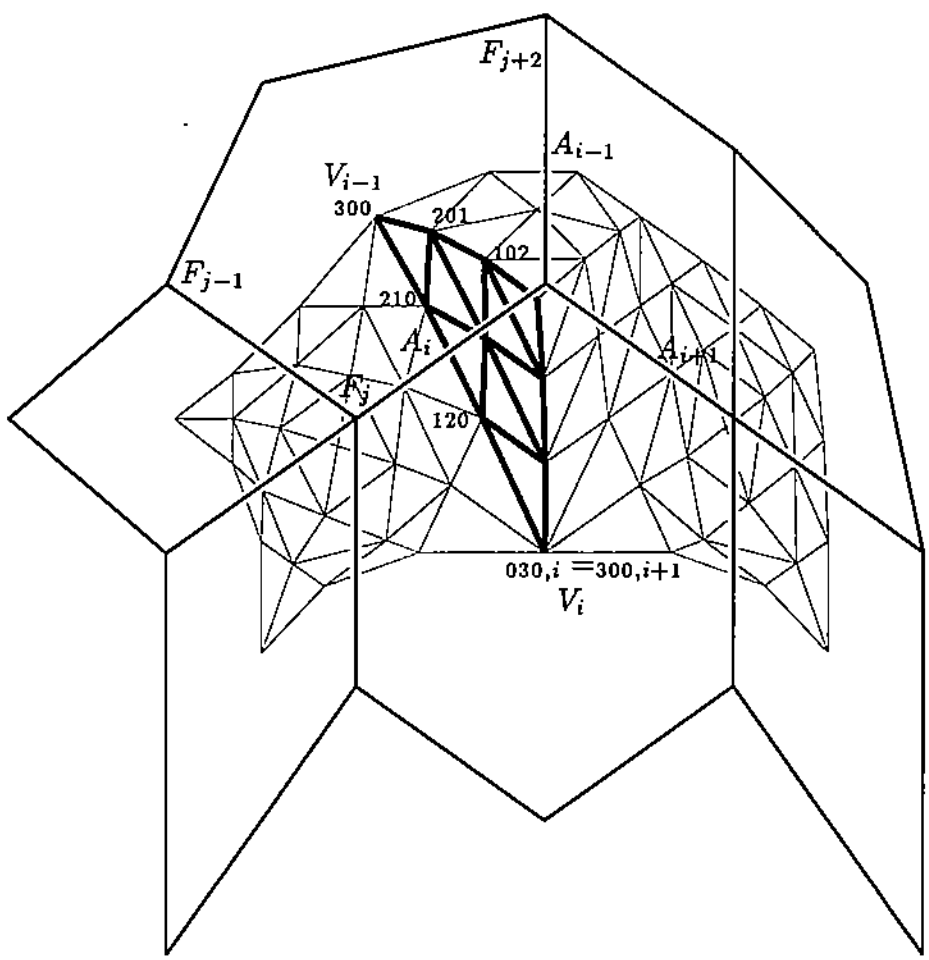


vertex-regular, separated
$n = 4$, cell is isolated if $m > 4$

Special vertex-degree bounded, polyhedral meshes.

3. An algorithm for smoothing a vertex-degree bounded polyhedron

The *input* to the algorithm are the vertices and connectivity information of a mesh that has at most four cells meeting at any (interior) mesh point and such that all cells with more than four edges are planar. The *output* is the Bernstein-Bézier representation of a C^1 surface that follows the outline of the input mesh and consists of $2e$ three-sided patches, where e is the number of edges of the input mesh.



1. On each facet, choose a point V and on each edge choose a point A. For nonplanar (hence quadrilateral) cells with vertices F_j, V has to be the centroid and A_j an edge midpoint:

$$V := \frac{1}{n}\sum F_j \qquad A_j := (F_j + F_{j+1})/2.$$

2. Each edge emanating from a mesh point F gives rise to a patch as follows. Let A_{i-1}, A_i and A_{i+1} be the points on consecutive edges, and V_{i-1}, V_i the points in the faces that share A_i. Then the ith patch corresponding to the vertex has the following coefficients.

$$P_{300,i} = V_{i-1} \qquad P_{210,i} = (2A_i + V_{i-1})/3 \qquad P_{120,i} = (2A_i + V_i)/3 \qquad P_{030,i} = V_i$$

$$\begin{array}{lcc}
\text{if } F \text{ has} & n = 3 \text{ neighbors} & n = 4 \text{ neighbors} \\
P_{201,i} = & (P_{300,i} + P_{210,i} + P_{120,i-1})/3 & (P_{210,i} + P_{120,i-1})/2 \\
P_{111,i} = & (A_{i-1} + A_{i+1} + 5A_i + V_{i-1} + V_i & (A_{i-1} + A_{i+1} + 4A_i \\
 & +3\ell(V_{i-1} + V_i - 2A_i))/9 & +2\ell(V_{i-1} + V_i - 2A_i))/6 \\
P_{021,i} = & (P_{030,i} + P_{120,i} + P_{210,i-1})/3 & (P_{120,i} + P_{210,i-1})/2 \\
P_{102,i} = & (P_{201,i} + P_{111,i} + P_{111,i-1})/3 & (P_{111,i} + P_{111,i-1})/2 \\
P_{012,i} = & (P_{021,i} + P_{111,i} + P_{111,i+1})/3 & (P_{111,i} + P_{111,i+1})/2
\end{array}$$

$$P_{003,i} = \frac{1}{n}\sum_{i=1}^{n} P_{012,i}$$

Let Q be the patch abutting P labeled so that $Q_{ij0} = P_{ij0}$. Then $\ell := l_0 - l_1$, where l_0 and l_1 solve

$$l_0\frac{3}{2}(P_{210} - P_{300}) = m_0(P_{201} - P_{300}) + (1 - m_0)(Q_{201} - Q_{300}) \qquad (*)$$

$$l_1\frac{3}{2}(P_{030} - P_{120}) = m_1(P_{021} - P_{120}) + (1 - m_1)(Q_{021} - Q_{120}).$$

That is, l_i is the relative length of the projection of two transversal tangents onto the versal tangent.

Remarks. Since the construction averages the input, it is desirable that the planar cells of the input mesh be convex polygons. In the absence of explicit boundary conditions, a surface with boundary can be defined by contraction analogous to the construction of B-spline based surfaces; the contraction is evident in the computed Bernstein-Bézier net shown above.

4. Properties of the smoothed surface

As a vertex point of the patches, $V = P_{300}$ is interpolated. This shows that the surface interpolates an arbitrary point on each planar cell. All A_i of a cell lie in the same plane, either because the cell is planar or because the centroid and the edge midpoints are chosen. Thus all planar cells are tangent planes of the surface.

Theorem. *The piecewise cubic surface generated by the algorithm is C^1.*

Proof Oriented tangent plane continuity is characterized as the agreement of the derivatives of two maps p and q from $\mathbb{R}^2$ to $\mathbb{R}^n$ after reparametrization by a map φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ that connects the domains Ω_p and Ω_q of p and q:

$$p = q \circ \varphi \quad \text{and} \quad D_1 p = D_1(q \circ \varphi) \quad \text{along } E_p$$

where $\varphi(E_p) = E_q$, E_p and E_q are edges of Ω_p and Ω_q respectively (see e.g. [Gregory 91]). D_1 denotes differentiation in the direction perpendicular to E_p and φ maps interior points of Ω_q to exterior points of Ω_p thus avoiding cusps. The derivative of the connecting-map used by the algorithm is

$$D\varphi := \begin{bmatrix} -\mu/\nu \\ \lambda/\nu \end{bmatrix}$$

where

$$\mu := (1-t)am_0 + tm_1 \qquad \nu := n_1$$

$$\lambda := (1-t)^2 al_0 + 2(l_0 + al_1)t(1-t) + t^2 l_1,$$

m_0, m_1, l_0 and l_1 are the solutions of Equations $(*)$ and $n_0 := 1 - m_0$, $n_1 := 1 - m_1$, $a := n_1/n_0$. To show that $\nu D_1 p + \mu D_1 q - \lambda D_2 q$ vanishes it suffices to show that the four coefficients of this cubic polynomial are zero. The first and the fourth coefficient are zero by the choice of l_i and m_i in Equations $(*)$. The remaining two equations in the unknowns P_{111} and Q_{111} have a one parameter family of solutions

$$3P_{111,0} := \frac{k}{r}A_{i-1} + krA_{i+1} + ((l_1 - 1 + k)r^2 + (2-\ell)r + (k - l_0))/rA_i$$
$$+ ((1-l_1)r + l_0 - 2k)/rV_{i-1} + (1 - l_1 - 2k)r + l_0)V_i$$

where $k := \begin{cases} 1/3 & \text{if } n = 3 \\ 1/2 & \text{if } n = 4 \end{cases}$. The particular solution used in the algorithm corresponds to $r = 1$. ■

Corollary. *If $n = 4$ and $\ell = 0$, then the patch generated by the algorithm is quadratic.*

Proof Since the boundary curve is a degree-raised quadratic and

$$6P_{111} := A_{i-1} + A_{i+1} + 4A_i + 2\ell(V_{i-1} + V_i - 2A_i),$$

the construction yields a quadratic polynomial patch with coefficients

$$\begin{array}{ccccc} & & \frac{1}{4}\sum A_j & & \\ & \frac{A_{i-1}+A_i}{2} & & \frac{A_{i+1}+A_i}{2} & \\ V_{i-1} & & A_i & & V_i \end{array}$$

■

To check the formulae, I implemented the algorithm and generated an open and a closed surface, a corner and a Christmas ornament if you like.

References

[1] C. W. de Boor, K. Höllig, S. Riemenschneider, *Box splines*, Springer Verlag, NY, to appear.

[2] E. Catmull, J. Clark, Recursively generated B-spline surfaces on arbitrary topological meshes, *Computer Aided Design* 10, No 6, 1978: 350–355.

[3] D. Doo, A subdivision algorithm for smoothing down irregularly shaped polyhedrons, *Proceedings on interactive techniques in computer aided design*, Bologna, 1978: 157–165.

[4] G. Farin, Curves and surfaces for computer aided geometric design, Academic Press, 1990.

[5] T.N.T. Goodman, Closed surfaces defined from biquadratic splines, Constructive Approximation 7 1991, 149-160.

[6] J.A. Gregory, Smooth parametric surfaces and n-sided patches, *Computation of curves and Surfaces*, W. Dahmen, M. Gasca and C.A. Micchelli, eds., Kluwer Academic Publishers, Dordrecht, 1990: 457–498.

[7] C. T. Loop, Smooth low degree polynomial spline surfaces over irregular meshes, preprint, Jan 1993.

[8] C. T. Loop, T. DeRose, A multisided generalization of Bézier surfaces, ACM TOG 8(3): 204–234.

[9] J. Peters, Constructing C^1 surfaces of arbitrary topology using biquadratic and bicubic splines, May 1992, in *Designing fair curves and surfaces*, N. Sapidis (ed.), to appear.

[10] J. Peters, Smooth free-form surfaces over irregular meshes generalizing quadratic splines, Sept 1993, CAGD 10, 1993:347–361.

[11] J. Peters, C^1 Free-form surface splines, *SIAM J. of Num. Anal.* x, xx–xx.

[12] A.R.M. Piah, Construction of smooth surfaces by piecewise tensor product polynomials, *Report CS 91-04, University of Dundee.* Feb 1991.

[13] B. Piper, Visually smooth interpolation with triangular Bézier patches, in: Geometric Modelling: Algorithms and trends, G.E. Farin (ed.) 221–233. SIAM, 1987.

[14] L. Shirman, C. Séquin, Local surface interpolation with Bézier patches, Computer Aided Geometric Design 4 : 279–295. Errata Computer Aided Geometric Design 6.

[15] R. E. Smalley, The third form of Carbon, *Naval Research Reviews*, **3** 1991.

Examples