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CONVERGENCE ANALYSIS OF A NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR ELLIPTIC PDES *

J.R. RICE, E.A. VAVALIS AND D. YANG†

1. Introduction.

2. The formulation of the proposed method.

Given: u_{S_i}, $i = 1, 2, \ldots, ndoms$.
For: $k = 1, 2, \ldots$.

For $i = 1$, $n$doms

$$\begin{array}{lll} Lu_i^{(2k-1)} = f & \text{in} & \Omega_i \\ u_i^{(2k-1)} = O & \text{on} & \partial\Omega \cap \partial\Omega_i \\ u_i^{(2k-1)} = u_{S_i} & \text{on} & S_i \\ u_i^{(2k-1)} = u_{S_{i+1}} & \text{on} & S_{i+1} \end{array} \qquad \text{Dirichlet} - \text{sweep}$$

For $i = 1$, $n$doms-1

$$\begin{array}{lll} u'_{S_i} = \beta_i \frac{\partial u_i^{(2k-1)}}{\partial x} + (1-\beta_i) \frac{\partial u_{i+1}^{(2k-1)}}{\partial u} & \text{on} & S_i \\ u'_{S_{i+1}} = \beta_i \frac{\partial u_i^{(2k-1)}}{\partial x} + (1-\beta_i) \frac{\partial u_{i+1}^{(2k-1)}}{\partial u} & \text{on} & S_i \end{array} \qquad \text{Neumann smoother}$$

For $i = 1$, $n$doms

$$\begin{array}{lll} Lu_i^{(2k)} = f & \text{in} & \Omega_i \\ u_i^{(2k)} = O & \text{on} & \partial\Omega \cap \partial\Omega_i \\ \frac{\partial u_i^{(2k)}}{\partial x} = u'_{S_i} & \text{on} & S_i \\ \frac{\partial u_i^{(2k)}}{\partial x} = u'_{S_{i+1}} & \text{on} & S_{i+1} \end{array} \qquad \text{Neumann} - \text{sweep}$$

For $i = 1$, $n$doms-1

$$\begin{array}{lll} u_{S_i} = \alpha_i u_i^{(2k)} + (1-\alpha_i) u_{i+1}^{(2k)} & \text{on} & S_i \\ u_{S_{i+1}} = \alpha_i u_i^{(2k)} + (1-\alpha_i) u_{i+1}^{(2k)} & \text{on} & S_{i+1} \end{array} \qquad \text{Dirichlet smoother}$$

3. Convergence analysis of the 2-dimensional problem.

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3.1. The two sub-domain case. We consider the PDE problem

$$(1) \qquad -\Delta u + \gamma u = f \quad \text{in} \quad \Omega \equiv [-a,b]\times[-1,1], \quad u=0 \quad \text{on} \quad \partial\Omega,$$

where γ is a positive constant. We split the PDE domain Ω into sub-domains $\Omega_1 \equiv [-a,0]\times[-1,1]$ and $\Omega_1 \equiv [0,b]\times[-1,1]$. If we denote by u_i the approximation of the solution of the associated PDE problem on domain Ω_i at iteration j and by $e_i^{(j)}(x,y) = u(x,y) - u_i^{(j)}(x,y)$ for $(x,y)\in\Omega_i$ it is easy to see that the error function $e_i^{(j)}$ satisfy the following relations

$$(2) \qquad \begin{array}{llll} -\Delta e_1^{(2k+1)} + \gamma e_1^{(2k+1)} = 0 & \text{in } \Omega & -\Delta e_2^{(2k+1)} + \gamma e_2^{(2k+1)} = 0 & \text{in } \Omega_2 \\ e_1^{(2k+1)} = \alpha e_1^{(2k)} + (1-\alpha) e_2^{(2k)} & \text{on } x=0 & e_2^{(2k+1)} = \alpha e_2^{(2k)} + (1-\alpha) e_1^{(2k)} & \text{on } x=0 \end{array}$$

We define the functions

$$\Psi_i(y) = \sin\frac{i\pi}{2}(y+1), \quad \Phi_i(x) = \frac{\sin h(\sqrt{\gamma_i}(x+a)}{\sin h(\sqrt{\gamma_i}a)} \quad \text{and} \quad Z_i(x) = \frac{\sin h\left(\sqrt{\gamma_i}(-x+b)\right)}{\sin h\left(\sqrt{\gamma_i}b\right)}, \qquad (3)$$

(where $\gamma_i = \gamma + \left(\frac{i\pi}{2}\right)^2$) which for $\forall i > 1$ satisfy the ODE's

$$(4) \qquad \Psi_i''(y) + \left(\frac{i\pi}{2}\right)^2 \Psi_i(y) = 0, \quad \text{for} \quad -1 < y < 1 \quad \Psi_i(-1) = \Psi_i(1) = 0,$$

$$(5) - \Phi_i(x) + \left[\gamma + \left(\frac{i\pi}{2}\right)^2\right] \Phi_i(x) = 0, \quad \text{for} \quad -a < x < 0 \quad \Phi_i(-a) = 0, \ \Phi_i(0) = 1$$

and

$$(6) \quad -Z_i''(x) + \left[\gamma + \left(\frac{i\pi}{2}\right)^2\right] Z_i(x) = 0, \quad \text{for} \quad 0 \le x < b \quad Z_i(0) = 1, \quad Z_i(b) = 0.$$

We expand the error functions in each sub-domain in terms of the functions given in 3) as follows

$$(7)\, e_1^{(j)}(x,y) = \sum_{i=1}^{\infty} a_i^{(j)} \Phi_i(x)\Psi_i(y), \qquad e_2^{(j)}(x,y) = \sum_{i=1}^{\infty} b_i^{(j)} Z_i(x)\Psi_i(y) \qquad j = 1,2,\ldots.$$

where the coefficients of the above series are given for $k = 1,2,\ldots$ by

$$(8) \qquad a_i^{(2k+1)} = \int_{-1}^{1} \left[\alpha e_1^{(2k)}(0,y) + (1-\alpha) e_2^{(2k)}(0,y)\right] \Psi_i(y)dy,$$

$$(9) \qquad b_i^{(2k+1)} = \int_{-1}^{1} \left[\alpha e_2^{(2k)}(0,y) + (1-\alpha) e_1^{(2k)}(0,y)\right] \Psi_i(y)dy,$$

$$(10)\quad a_i^{2(k+1)} = \frac{tanh\left(\sqrt{\gamma_i}a\right)}{\sqrt{\gamma_i}} \int_{-1}^{1} \left[\beta\frac{\partial e_1^{(2k+1)}(0,y)}{\partial x} + (1-\beta)\frac{\partial e_2^{(2k+1)}(0,y)}{\partial x} \right] \Psi_i(y)dy,$$

and

$$(11)\quad b_i^{2(k+1)} = -\frac{tanh\left(\sqrt{\gamma_i}b\right)}{\sqrt{\gamma_i}} \int_{-1}^{1} \left[\beta\frac{\partial e_2^{(2k+1)}(0,y)}{\partial x} + (1-\beta)\frac{\partial e_i^{(2k+1)}(0,y)}{\partial x} \right] \Psi_i(y)dy.$$

Using the boundary conditions of (2) equations (10) and (11) become

$$(12)\qquad a_i^{2(k+1)} = \frac{tanh(\sqrt{\gamma_i}a)}{\sqrt{\gamma_i}} \left[\ \beta a_i^{(2k+1)}\Phi_i'(0) + (1-\beta)b_i^{(2k+1)}z_i'(0)\ \right]$$

and

$$(13)\qquad b_i^{2(k+1)} = \frac{tanh(\sqrt{\gamma_i}b)}{\sqrt{\gamma_i}} \left[\ \beta b_i^{(2k+1)}z_i'(0) + (1-\beta)a_i^{(2k+1)}\Phi_i'(0)\ \right]$$

respectively. Adding the squares of the above two equalities and using the fact that

$$\Phi_i'(0) = \sqrt{\gamma_i}\frac{coth(\sqrt{\gamma_i}a)}{\sin h(\sqrt{\gamma_i}a)}, \qquad Z_i'(0) = -\sqrt{\gamma_i}\frac{coth(\sqrt{\gamma_i}b)}{\sin h(\sqrt{\gamma_i}b)}$$

we have that

$$\begin{aligned}\left(\ a_i^{2(k+1)}\ \right)^2 + \left(\ b_i^{2(k+1)}\ \right)^2 \ &= \ tanh^2(\sqrt{\gamma_i}a)\left[\ \beta a_i^{(2k+1)}coth(\sqrt{\gamma_i}a) + (1-\beta)b_i^{(2k+1)}coth(\sqrt{\gamma_i}b)\ \right]^2 \\ &+ \ tanh^2(\sqrt{\gamma_i}b)\left[\ \beta b_i^{(2k+1)}coth(\sqrt{\gamma_i}b) + (1-\beta)a_i^{(2k+1)}coth(\sqrt{\gamma_i}a)\ \right]^2\end{aligned} \tag{14}$$

Similarly relations (8) and (9) give

$$\begin{aligned} a_i^{(2k+1)} &= \alpha a_i^{(2k)}\Psi_i(0) + (1-\alpha)b_i^{(2k)}Z_i(0) &= \alpha a_i^{(2k)} + (1-\alpha)b_i^{(2k)} \\ b_i^{(2k+1)} &= \alpha b_i^{(2k)}Z_i(0) + (1-\alpha)a_i^{(2k)}\Psi_i(0) &= \alpha b_i^{(2k)} + (1-\alpha)a_i^{(2k)}\end{aligned}$$

which lead us to

$$\begin{aligned}\left(\ a_i^{2(k+1)}\ \right)^2 + \left(\ b_i^{2(k+1)}\ \right)^2 \ &= \ tanh^2(\sqrt{\gamma_i}a)\Big\{\ \beta coth(\sqrt{\gamma_i}a)[\alpha a_i^{(2k)} + (1-\alpha)b_i^{(2k)}] \\ &\quad +(1-\beta)coth(\sqrt{\gamma_i}b)[\alpha b_i^{(2k)} + (1-\alpha)a_i^{(2k)}]\ \Big\}^2 + \\ &+ \ tanh^2(\sqrt{\gamma_i}b)\Big\{\beta coth(\sqrt{\gamma_i}b)[\alpha b_i^{(2k+1)} + (1-\alpha)a_i^{2k)}] + \\ &\quad +(1-\beta)coth(\sqrt{\gamma_i}a)[\alpha a_i^{(2k)} + (1-\alpha)b_i^{(2k)}]\ \Big\}^2\end{aligned} \tag{15}$$

Setting $\rho_i = \frac{tanh(\sqrt{\gamma_i}b)}{\tan h(\sqrt{\gamma_i}a)}$ the above equality becomes

$$
\begin{aligned}
(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 &= [\beta a_i^{(2k+1)} + (1-\beta)(-\rho(\gamma_i))^{-1} b_i^{(2k+1)}]^2 + [\beta b_i^{(2k+1)} + (1-\beta)(-\rho(\gamma_i)) a_i^{(2k+} \\
&= \beta^2 (a_i^{(2k+1)})^2 - 2\beta(1-\beta)\rho^{-1} a_i^{(2k+1)} b_i^{(2k+1)} + (1-\beta)^2 \rho^{-2} (b_i^{2k+1})^2 + \\
&\quad \beta^2 (b_i^{(2k+1)})^2 - 2\beta(1-\beta)\rho a_i^{(2k+1)} b_i^{(2k+1)} + (1-\beta)^2 \rho^2 (a_i^{(2k+1)})^2 \\
&= [\beta^2 + (1-\beta)^2\rho^2][\alpha^2 (a_i^{(2k)})^2 + 2\alpha(1-\alpha) a_i^{(2k)} b_i^{(2k)} + (1-\alpha)^2 (b_i^{2k})^2] \\
&\quad + [\beta^2 + (1-\beta)^2 \rho^{-2}][\alpha^2 (b_i^{(2k)})^2 + 2\alpha(1-\alpha) a_i^{(2k)} b_i^{(2k)} + (1-\alpha)^2 (a_i^{(2k)})^2] \\
&\quad - 2\beta(1+\beta)(\rho+\rho^{-1})[(1-\alpha+2\alpha^2) a_i^{92k)} b_i^{(2k)} + \alpha(1-\alpha)(a_i^{(2k)})^2 + \alpha(1-
\end{aligned}
$$

and finaly we have that

$$
\begin{aligned}
(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 &= \{\alpha^2[\beta^2 + (1-\beta)^2\rho^2] + (1-\alpha)^2[\beta^2 + (1-\beta)^2\rho^{-2}] - 2\alpha\beta(1-\alpha)(1-\beta)(\\
&+ \{\alpha^2[\beta^2 + (1-\beta)^2\rho^{-2}] + (1-\alpha)^2[\beta^2 + (1-\beta)^2\rho^2] - 2\alpha\beta(1-\alpha)(1-\beta)(\\
&+ \{2\alpha(1-\alpha)[2\beta^2 + (1-\beta)^2(\rho^2+\rho^{-2})] - 2\beta(1-\beta)(1-2\alpha+2\alpha^2)(\rho+\rho^{-}
\end{aligned}
\tag{16}
$$

Notice that $||e_1^{(2k+1)}|_{p:x=0}||^2_{L^2(p)} = \sum_{i=1}^{\infty} (a_i^{(2k+1)})^2$ and $||e_2^{(2k+1)}|_{p:x=0}||^2_{L^2(p)} = \sum_{i=1}^{\infty} (b_i^{(2k+1)})^2$.

THEOREM 3.1. *Denote* $\sigma_2 = \frac{a+b}{\min\{a,b\}}$. *If* $\alpha = \frac{1}{2}$ *then for the model problem and its decomposition considered in this section:*
(i) The proposed method converges if $\beta \in (0, \frac{2}{3}]$ *and if*

$$(1-\beta)^2\sigma_2^2 - 2\beta(1-\beta)\sigma_2 - 2(1-2\beta) < 1$$

.

(ii) The optimum value of the relaxation parameter β *is* $\beta_{opt} = \frac{\sigma_2^2+\sigma_2-2}{\sigma_2^2+2\sigma_2^2}$ *and the following relation holds*

$$(17) \quad || e_1^{2(i+1)}\Big|_{x=0} ||^2 + || e_2^{2(i+1)}\Big|_{x=0} ||^2 \leq \frac{1}{2}\frac{\sigma_2^2-4}{\sigma_2^2+2\sigma_2}\left[|| e_1^{(2k)}\Big|_{x=0} ||^2 + || e_2^{(2k)}\Big|_{x=0} ||^2\right].$$

Proof:

$$
\begin{aligned}
(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 &= \tfrac{1}{4}[2\beta^2 + (1-\beta)^2(\rho^2+\rho^{-2}) - 2\beta(1-\beta)(\rho+\rho^{-1})](a_i^{(2k)})^2 \\
&+ \tfrac{1}{4}[2\beta^2 + (1-\beta)^2(\rho^2+\rho^{-2}) - 2\beta(1-\beta)(\rho+\rho^{-1})](b_i^{(2k)})^2 \\
&+ \tfrac{1}{2}[2\beta^2 + (1-\beta)^2(\rho^2+\rho^{-2}) - 2\beta(1-\beta)(\rho+\rho^{-1})]a_i^{(2k)}b_i^{(2k)}
\end{aligned}
$$

$$
(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 = \tfrac{1}{4}g(\rho)[(a_i^{(2k)})^2 + (b_i^{(2k)})^2]^2 \le \tfrac{1}{2}g(\rho)[(a_i^{(2k)})^2 + (b_i^{(2k)})^2]
$$

where $g(\rho) = 2\beta^2 + (1-\beta)^2(\rho^2+\rho^{-2}) - 2\beta(1-\beta)(\rho+\rho^{-1})$.

Set $\sigma = \rho + \rho^{-1}$. Then we have that $(2 \le)\sigma_1 \equiv \frac{1+b}{\max\{a,b\}} \le \sigma \le \frac{a+b}{\min\{a,b\}} \equiv \sigma_2$

Define

$$
\overline{g}(\sigma) = (1-\beta)^2\sigma^2 - 2\beta(1-\beta)\sigma + 4\beta - 2
$$

$$
\overline{g}'(\sigma) = 2(1-b)^2\sigma - 2\beta(1-\beta) = 2(1-\beta)[(1-\beta)\sigma - \beta]
$$

$$
\text{if} \quad \beta \le \frac{\sigma_1}{1+\sigma_1} = \frac{a+b}{a+b+\max\{a,b\}} \quad \text{then} \quad \forall\lambda_i \quad (\frac{x}{1+x} \quad \text{increase})
$$

$$
\overline{g}'(\sigma) \ge 0 \quad \text{and}
$$

$$
(a_i^{2(k+1)})^2 + (b_i^{(2(k+1)})^2 \le \frac{1}{2}\overline{g}(\sigma_2)[(a_i^{(2k)})^2 + (b_i^{2k})^2]
$$

$$
\begin{aligned}
\overline{g}(\sigma_2) &= (1-\beta)^2\sigma_2^2 - 2\beta(1-\beta)\sigma_2 - 2(1-2\beta) \\
&= (\sigma_2^2 + 2\sigma_2)\beta^2 - (2\sigma_2^2 + 2\sigma_2 - 4)\beta + \sigma_2^2 - 2 \quad \text{no zeros } \sigma_2 > 2 \quad \text{double zero } \sigma_2 = 2
\end{aligned}
$$

So if $\beta = \dfrac{\sigma_2^2 + \sigma_2 - 2}{\sigma_2^2 + 2\sigma_2}$, $\overline{g}(\sigma_2)$ achieves minimum $\min_{\beta}(\overline{g}(\sigma_2)) = \dfrac{\sigma_2^2 - 4}{\sigma_2^2 + 2\sigma_2}$. □

To depict the rapid convergence of the method we present the follwing corollary.

COROLLARY 3.2. *Assume that $\alpha = \frac{1}{2}$ we have that*

- *For $a = b$ $\beta_{opt} = \frac{1}{2}$ and the method converges after one iteration (one Dirichlet sweep and one Neumann sweep.)*
- *If $a = 2b$ (or $b = 2\alpha$), $\beta_{opt} = \frac{2}{3}$ and*

$$
||e_1^{2(k+1)}|_{x=0}||^2 + ||e_2^{2(k+1)}|_{x=0}||^2 \le \frac{1}{6}[||e_1^{(2k)}|_{x=0}||^2 + ||e_2^{(2k)}| + x = 0||^2
$$

- *If* $a = \frac{3}{2}b$ *(or* $b - \frac{3}{2}a$*),* $\beta_opt = \frac{6.35}{11.25}$ *and*

$$||e_1^{2(k+1)}|_{x=0}||^2 + ||e_2^{2(k+1)}|_{x=0}||^2 \leq \frac{2.25}{22.5}[||e_1^{(2k)}|_{x=0}||^2 + ||e_2^{(2k)}|_{x=0}||^2]$$

THEOREM 3.3. *(long/narrow sub-domains) When* $\alpha = \beta = \frac{1}{2}$ *the proposed method converges if*
(i) in the case that $a > b$ *we have* $b \geq (2 - \sqrt{3})a$ *and*
(ii) in the case that $a < b$ *we have* $b \leq (2 + \sqrt{3})a$.
Proof:

$$(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 \leq \frac{1}{2}[\frac{1}{2} + \frac{1}{4}(\rho^2 + \rho^{-2}) - \frac{1}{2}(\rho + \rho^{-1})][(a_i^{(2k)})^2 + (b_i^{(2k)})^2].$$

we need that $\frac{1}{4}[1 + \frac{1}{2}(\rho^2 + \rho^{-2}) - (\rho + \rho^{-1})] < 1$

$$\rho + \rho^{-1} < 4, \quad 2 - \sqrt{3} < \rho < 2 + \sqrt{3}$$

□

$$2 - \sqrt{3} < \frac{tanh(\sqrt{\gamma_i} b)}{tanh(\sqrt{\gamma_i} a)} < 2 + \sqrt{3}.$$

3.2. The three sub-domain case. In this section we consider the PDE problem given in (1) where $\Omega \equiv [-a, c] \times [-1, 1]]$ and the three sub-domain splitting $\Omega_1 \equiv [-a, o] \times [-1, 1]]$, $\Omega_2 \equiv [a, b] \times [-1, 1]]$ and $\Omega_3 \equiv [b, c] \times [-1, 1]]$. Then we have the expression for the error functions

$$-\Delta e_i^{(2k+1)} + \gamma e_i^{(2k+1)} = 0 \qquad -\Delta e_i^{2(k+1)} + \gamma e_i^{2(k+1)} \; 0$$

$$e_i^{(2k+1)} = \alpha e_i^{(2k)} + (1 - \alpha) e_j^{(2k)} \quad \frac{\partial e_i^{2(k+1)}}{\partial x} = \beta \frac{\partial e_i^{(2k+1)}}{\partial x} + (1 - \beta) \frac{\partial e_j^{(2k+1)}}{\partial x}$$

$$I_i(y) = \sin\left(\frac{i\pi}{2}(y + 1)\right) \quad -I_i''(y) + (\frac{i\pi}{2})^2 I_i(y) = 0 \qquad -1 < y < 1$$
$$I_i(-1) = I_i(1) = 0$$

$$\Phi_i(x) = \frac{\sin h(\sqrt{\gamma_i}(x+a))}{\sin h(\sqrt{\gamma_i} a)} \quad -\Phi_i''(x) + (\gamma + (\frac{i\pi}{2})^2)\Phi_i(x) = 0 \quad -a < x < 0$$
$$\Phi_i(-a) = 0, \quad \Phi_i(0) = 1$$

$$\Psi_i(x) = \frac{\sin h(\sqrt{\gamma_i}(-x+b))}{\sin h(\sqrt{\gamma_i} b)} \quad -\Psi_I''(x) + (\gamma + (\frac{i\pi}{2})^2)\Psi_i(x) = 0 \quad 0 < x < b$$
$$\Psi_i(0) = 1, \quad \Psi_i(b) = 0$$

$$Z_i(x) = \frac{\sin h(\sqrt{\gamma_i} x)}{\sin h(\sqrt{\gamma_i} b)} \quad -Z_i''(x) + (\gamma + (\frac{i\pi}{2})^2) Z_i(x) = 0 \quad 0 < x < b$$
$$Z_i(0) = 0, \quad Z_i(b) = 1$$

$$X_i(x) = \frac{\sin h(\sqrt{\gamma_i}(x+c))}{\sin h(\sqrt{\gamma_i}(-b=c))} \quad X_i''(x) + (\gamma + (\frac{i\pi}{2})^2) X_i(x) = 0 \quad b < x < c$$
$$X_i(b) = 1, \quad X_i(c) = 0$$

where $\gamma_i = \gamma + (\frac{i\pi}{2})^2$

$$e_i^{(n)}(x,y) = \sum_{i=1}^{\infty} b_{1,i}^n \Psi_i(x) I_i(y), \qquad e_3^{(n)} = \sum_{i=1}^{\infty} b_{3,i}^n X_i(x) I_i(y)$$

$$e_2^{(n)}(x,y) = \sum_{i=1}^{\infty} b_{2,1,i}^n \Psi_i(x) I_i(y) + \sum_{i+1}^{\infty} b_{2,2,i}^n X_i(x) I_i(y)$$

$$\begin{aligned}
\Psi_i(0) &= \sqrt{\gamma_i} coth(\sqrt{\gamma_i} a), \qquad \Psi'(0) = -\sqrt{\gamma_i} coth(\sqrt{\gamma_i} b) \\
Z_i'(0) &= \frac{\sqrt{\gamma_i}}{\sin h(\sqrt{\gamma_i} b)}, \\
\Psi_i'(b) &= \frac{\sqrt{\gamma_i}}{\sin h(\sqrt{\gamma_i} b)}, \quad Z_i'(b) = \sqrt{\gamma_i} coth(\sqrt{\gamma_i} b), \quad X_i'(b) = -\sqrt{\gamma_i} coth(\sqrt{\gamma_i}(c-b))
\end{aligned}$$

$$\begin{aligned}
b_{1,i}^{2(k+1)} &= \frac{1}{\Psi_i'(0)} \int_{-1}^{1} \frac{\partial e_i^{2(k+1)}}{\partial x}(0,y) I_i(y) dy = \frac{1}{\Psi_i'(0)} \int_{-1}^{1} [\beta \frac{\partial c_i^{(2k+1)}}{\partial x}(0,6) + (1-\beta) \frac{\partial e_2^{(2k+1)}}{\partial x}(0,y \\
&= \frac{1}{\Psi_i'(0)} [\beta b_{1,i}^{2k+1} \Psi_i'(0) +(1-\beta) b_{2,1,i}^{2k+1} \Psi_k'(0) + (1-\beta) b_{2,2,i}^{2k+1} Z_i'(0)] \\
\left(\rho = \frac{tanh(\sqrt{\gamma_i} a)}{tanh(\sqrt{\gamma_i} b)}\right) &= \beta b_{1,i}^{2k+1} - (1-\beta)\rho b_{2,1,i}^{2k+1} + (1-\beta) \frac{tanh(\sqrt{\gamma_i} a)}{\sin h(\sqrt{\gamma_i} b)} b_{2,2,i}^{2k+1} \\
b_{2,1,i}^{2(k+1)} &= \frac{Z_i'(b) \int_{-1}^{1} \frac{\partial e_1^{2(k+1)}}{\partial x}(0,y) I_1(y) dy - Z_i'(0) \int_{-1}^{1} \frac{\partial e_2^{2(k+1)}}{\partial x}(b,y) I_i(y) dy}{\Psi_i'(0) Z_i'(b) - \Psi_i'(b) Z_i'(0)} \\
&= \frac{Z_i'(b) \int_{-1}^{1} \frac{\partial e_1^{(2k+1)}}{\partial x} + (1-\beta) \frac{\partial e_2^{2k+1)}}{\partial x}] I_i(y) dy - Z_i'(0) \int_{=1}^{1} [\beta \frac{\partial e_2^{(2k+1)}}{\partial x} + (1-\beta) \frac{\partial e_2^{(2k+1)}}{\partial x}] I_i(y,0)}{\Psi_i'(0) Z_i'(b) - \Psi_i'(b) z_i'(0)} \\
&= \frac{Z_i'(b) [\beta b_{1,R}^{2x+1)} [\beta b_{1.R}^{2k+1} \Psi_I'(0) + (1-\beta) b_{2,1,i}^{2x+1} \Psi_i'(0)] - Z_i'(0) [\beta b_{2,2,i}^{2k+1} Z_i'(b) + (1-b) b_{3,i}^{2k+1} X_i^1(b)]}{\Psi_i'(0) Z_i'(b) - \Psi_i'(b) Z_i'(0)}
\end{aligned}$$

4. Numerical experiments.

Numerical experiment that confirm the above given theoretical results and exibit that the proposed method outperforms other existing non-ovelaping domain decomposition methods will be presented.