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Abstract

We consider surfaces defined with the dimensionality paradigm as the natural projection of a 2-manifold in n-space into a subspace of three dimensions. Such surfaces can represent exactly many operations of interest in geometric modeling and its applications. We show that this class of surfaces is closed under the operations of offsetting, bisecting, and blending (using rolling-ball blends), and analyze the growth behavior of the number of variables needed. Furthermore, we present several techniques how to evaluate trimmed patches of these surfaces and demonstrate the utility of these techniques.

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1 Introduction

It is well-known that the traditional parametric representation of surfaces in CAGD cannot directly support certain desirable geometric operations including the derivation of offsets and rolling-ball blends. Given parametric base surfaces, these derived surfaces are in general not parametric. While implicit surfaces are, in principle, closed under these operations, this fact does not necessarily help in practice because the derivation of the implicit equation of the derived surface may rest on a symbolic computation that is often beyond the current state of the art. In a sequence of papers [1, 4, 3, 5] it has been argued, therefore, that such constrained surfaces should be represented using the *dimensionality paradigm*, that is, as the natural projection of a 2-manifold in n-space. This manifold is represented by a set of m equations in n variables that express, intuitively and straightforwardly, the geometric constraints that define the derived surface from the given ones. In general, such a manifold is defined by the following system of nonlinear equations:

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$$\begin{aligned} f_1(x_1, x_2, ..., x_n) &= 0 \\ f_2(x_1, x_2, ..., x_n) &= 0 \\ &\vdots \\ f_m(x_1, x_2, ..., x_n) &= 0 \end{aligned} \tag{1}$$

In past work, we have assumed that the given *base* surfaces are parametric or implicit, and that their evaluation does not require trimming. Both assumptions need not apply in practice: We may wish to iterate these surface operations, for example blending an offset surface with an equidistance surface. Moreover, it may be necessary to trim the derived surfaces, for example retaining only a part of a blending surface. In this paper we address both of these problems.

In Section 2, we show how to iterate the operations of forming offset surfaces, equidistance surfaces, and constant- and variable-radius blending surfaces. Roughly speaking, iterating these operations requires the derivation of closed-form expressions for the surface normal at a point. Such expressions exist, of course, for implicit and parametric surfaces. We show here that such expressions can also be found in case the base surfaces are given as systems of nonlinear equations that are formed to express the surface operations under consideration. We also give an analysis of the growth of the number of variables when iterating the surface operations.

In the subsequent sections we address the trimming problem and show how the surfaces defined using our approach may be trimmed back to the parts that are of interest in a geometric operation. Several strategies for trimming are explored. It is possible in many cases to alter Chuang's surface evaluation algorithm [2] slightly so that one set of variables is used for the projection, while a different set is used for trimming. This approach can be supplemented by introducing special variables that define the area to be trimmed functionally. We also explore a direct trimming strategy in which the trimming computation augments the surface exploration.

2 Closure under Surface Operations

In order to avoid redundancy, we restrict attention to surface operations only. The analogous operations on curves follow the same pattern, and the corresponding closure theorems are easily proved. We consider the following operations:

1. *Offsetting*
   Given a base surface f and an offset distance r, represent the offset surface

of f by r.

2. *Bisecting*
   Given two base surfaces f and g, represent the surface of points that have equal normal distance from both f and g.

3. *Blending*
   Given two surfaces f and g, represent the rolling-ball blend that connects the two surfaces, either by a constant-radius blend of given radius r, or by a variable-radius blend where the radius variation is prescribed by a reference surface.

We show that these three operations can be freely iterated and mixed; that is, that a base surface can be any surface obtained from a number of implicit and/or parametric surfaces by a sequence of the three surface operations. We also discuss the size of the systems of nonlinear equations that are so obtained.

In essence, our result applies to any surface operation that can be constructed from reference points on the surface and two linearly independent tangent directions at those reference points, as long as from them expressions can be derived that give, at a generic surface point, two linearly independent tangent directions. We will show that this is the case for the above surface operations. We will use the following lemma:

Lemma 1
Let (a, b, c) be a surface normal at a nonsingular surface point. Then there are expressions for two linearly independent tangent directions to the surface at that point. Conversely, assume that $\mathbf{t}_1$ and $\mathbf{t}_2$ are two linearly independent tangent directions at a nonsingular surface point. Then there is an expression for the surface normal at that point.

Proof
Since the point is not singular, the surface normal does not vanish. Clearly the three vectors

$$\begin{aligned}\mathbf{t}_1 &= (0, -c, b)\\ \mathbf{t}_2 &= (c, 0, -a)\\ \mathbf{t}_3 &= (-b, a, 0)\end{aligned}$$

are perpendicular to

$$\mathbf{n} = (a, b, c)$$

and so the $\mathbf{t}_i$ are tangents to the surface. Moreover, there must be at least two among them that are linearly independent. Furthermore, if $\mathbf{t}_1$ and $\mathbf{t}_2$ are linearly independent tangent vectors, then $\mathbf{t}_1 \times \mathbf{t}_2$ is a normal vector. □

The lemma justifies that in each case it suffices to know either two linearly independent tangent vectors or the normal vector. Note that we may not be able to decide in general which of the three vectors $\mathbf{t}_1 = (0, -c, b)$, $\mathbf{t}_2 = (c, 0, -a)$, and $\mathbf{t}_3 = (-b, a, 0)$ are linearly independent. Since the vector components are usually expressions or variable names, at those points of the surface that lie on the intersection with, say, the surface $a = 0$, the vectors $\mathbf{t}_2$ and $\mathbf{t}_3$ are linearly dependent, but elsewhere they are not. Similarly, the other two pairs of vectors are linearly dependent on certain other curves. In [3, 6] we have therefore advocated working with redundant equation systems.

To ensure arbitrary iterations of offsetting, bisecting and blending operations on constrained surfaces, we give three theorems for normal expressions after each of the surface operations. We will use the following definition in these theorems.

Definition *Foot Point*
Given a point $\mathbf{p}$, its *foot point*, $\mathbf{p}_0$, on a surface f is a point of f that has shortest distance to $\mathbf{p}$.

Theorem 1: *Normal of an Offset Surface*
The normal $\mathbf{n}$ of an offset surface at a nonsingular point $\mathbf{p}$ coincides with the normal $\mathbf{n}_0$ at its foot point $\mathbf{p}_0$. In other words, $\mathbf{n} = \mathbf{n}_0$.

Proof
Suppose that two normal vectors $\mathbf{n}$ and $\mathbf{n}_0$ do not coincide, as shown in Figure 1. There must be some tangent vectors which are not perpendicular to $\mathbf{n}_0$ since $\mathbf{n}$ is different from $\mathbf{n}_0$. Then in a small neighborhood around $\mathbf{p}$ on the offset surface there exists a point which is closer to $\mathbf{p}_0$ than the offset distance. We conclude from this contradiction that $\mathbf{p}$ is not the offset point to $\mathbf{p}_0$. □

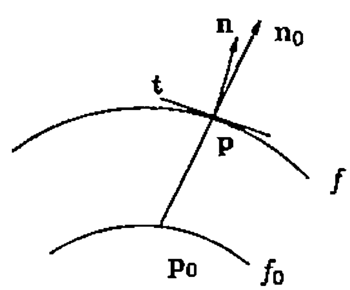


Figure 1: Normal of an offset surface: $\mathbf{n}$ and $\mathbf{n}_0$ should be coincident

Theorem 2: *Normal of a Bisecting Surface*
The normal $\mathbf{n}$ of a bisecting surface at a nonsingular point $\mathbf{p}$ is parallel to the straight line connecting the foot points $\mathbf{p}_f$ and $\mathbf{p}_g$ on the two base surfaces f

and g. In other words, $\mathbf{n} = \pm(\mathbf{p}_f - \mathbf{p}_g)$.

Proof

Suppose that the normal $\mathbf{n}$ is not parallel to $(\mathbf{p}_f - \mathbf{p}_g)$ as shown in Figure 2. We approximate the neighborhoods of $\mathbf{p}_f$ and $\mathbf{p}_g$ by their tangent planes. There must be some tangent vectors at $\mathbf{p}$ which are not perpendicular to $(\mathbf{p}_f - \mathbf{p}_g)$. Then in a small neighborhood around $\mathbf{p}$ there exists a point which has different distances from the two approximating planes. We conclude from the contradiction that $\mathbf{p}$ does not bisect the two base surfaces. □

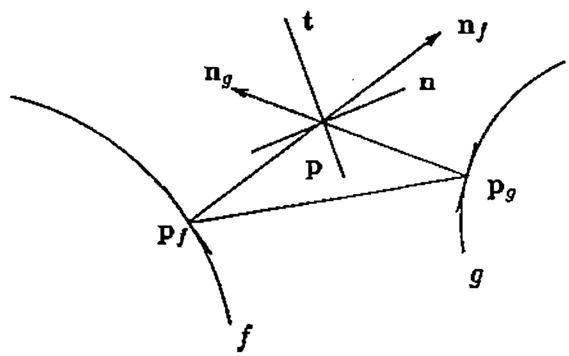


Figure 2: Normal of a bisecting surface: $\mathbf{n}$ should be parallel to $\mathbf{p}_f - \mathbf{p}_g$

Theorem 3: *Normal of a Blending Surface*

The normal $\mathbf{n}$ of a blending surface at a nonsingular point $\mathbf{p}$ is the straight line from the center $\mathbf{c}$ of its corresponding rolling ball instance at this point. In other words, $\mathbf{n} = \mathbf{p} - \mathbf{c}$.

Proof

This immediately follows from the fact that a blending surface is the envelope of rolling balls. □

Corollary

The surface operations of offsetting, bisecting and blending can be iterated.

Proof

In the above three theorems we have given a normal expression for each of these surface operations. The corollary therefore follows from Lemma 1 and the constructions of [4, 6]. □

2.1 Number of Variables

Next, we discuss the number of variables needed in each operation. To simplify the formulations, we will use

$$\mathbf{F}(x_f, y_f, z_f, ...) = 0 \quad (2)$$
$$\mathbf{G}(x_g, y_g, z_g, ...) = 0 \quad (3)$$
$$\mathbf{H}(x_h, y_h, z_h, ...) = 0 \quad (4)$$

to abbreviate the systems of equations such as (1) that define the surfaces f, g and h. The surfaces are the projections into the subspaces spanned by (x_f, y_f, z_f), (x_g, y_g, z_g) and (x_h, y_h, z_h), respectively. Let (a_f, b_f, c_f), (a_g, b_g, c_g) and (a_h, b_h, c_h) be the normals at generic points, of the three surfaces.We review the equational formulation of the surface derivations thereby establishing the needed number of variables.

2.1.1 Offset Surfaces

Consider offsetting a base surface f by a distance r. Let (x_f, y_f, z_f) on f be a foot point, and (x, y, z) be its offset point on the derived surface. Then an offset surface must satisfy the following constraints [3]:

1. The foot point is on the base surface.
2. The distance between the foot point and its offset point equals r.
3. The vector connecting the foot point and its offset point is collinear with the normal of the base surface f, or by Lemma 1, perpendicular to any surface tangent vector.

Translating these constraints into equations, and omitting the faithfulness condition equations derived in [6], we obtain

$$\mathbf{F}(x_f, y_f, z_f, ...) = 0 \quad (5)$$
$$(x - x_f)^2 + (y - y_f)^2 + (z - z_f)^2 = r^2 \quad (6)$$
$$(x - x_f, y - y_f, z - z_f) \times (a_f, b_f, c_f) = (0, 0, 0) \quad (7)$$

where (a_f, b_f, c_f) is the normal of f at the foot point. As justified in [6], the system of equations includes a redundancy.

2.1.2 Bisecting Surfaces

Consider two base surfaces f and g. A bisecting surface or equi-distance surface is comprised of the intersection curves of pairs offset surfaces from f and g, by the same offset r. Intuitively,

1. The two foot points are on the base surfaces.
2. The distances of a point on the bisecting surface from the corresponding foot points are equal (to r, a variable).
3. The two vectors, from the surface point to its corresponding foot points, are normals of the two base surfaces respectively.

These constraints define the bisecting surface with the following equations:

$$\begin{aligned}
\mathrm{F}(x_f, y_f, z_f, \ldots) &= 0 && (8)\\
\mathrm{G}(x_g, y_g, z_g, \ldots) &= 0 && (9)\\
(x - x_f)^2 + (y - y_f)^2 + (z - z_f)^2 &= r^2 && (10)\\
(x - y_g)^2 + (y - y_g)^2 + (z - z_g)^2 &= r^2 && (11)\\
(x - x_f, y - y_f, z - z_f) \times (a_f, b_f, c_f) &= (0, 0, 0) && (12)\\
(x - x_g, y - y_g, z - z_g) \times (a_g, b_g, c_g) &= (0, 0, 0) && (13)
\end{aligned}$$

2.1.3 Blending Surfaces

Since constant-radius blends are a special case of variable-radius blending surfaces, we begin with the general case.

To construct a variable-radius blending surface, we consider a curve, referred to as the *spine*, that lies on a surface bisecting two base surfaces f and g (see Figure 3). At each point on the curve we place a sphere with radius the distance of the point from the base surfaces. Then the envelope of these spheres is a variable-radius blending surface of the two base surfaces.

In general, the spine curve is obtained by intersecting the equidistance surface with a reference surface h [1].

When a ball rolls over two base surfaces, only a circle on each sphere in the family contributes generally to the envelope. The plane containing this circle must be perpendicular to the tangent to the spine at the center of the sphere, and must contain the foot points, [4]; see also Figure 3. These considerations lead to the following constraints

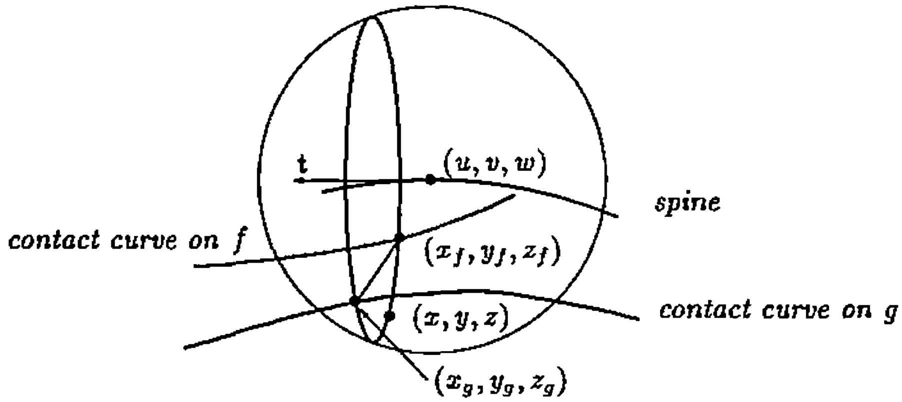


Figure 3: Formulation of a blending surface

1. The foot points are on the respective base surfaces.
2. The spine is on the reference surface.
3. The radius of the sphere is r, the distance of the center from the base surfaces.
4. The two vectors, from the center of the sphere to the two foot points, are normal vectors of the base surfaces.
5. The circle of the sphere that contributes to the blending surface contains the foot points.
6. The plane containing this circle is perpendicular to the tangent of the spine at the center of the sphere.

To convert these constraints into equations, we let (u, v, w) be the center of a sphere, a point on the spine. We let (t_u, t_v, t_w) be the spine tangent vector at (u, v, w), and (x, y, z) be a point on the blending surface. We obtain:

$$\mathbf{F}(x_f, y_f, z_f, ...) = 0 \tag{14}$$

$$\mathbf{G}(x_g, y_g, z_g, ...) = 0 \tag{15}$$

$$\mathbf{H}(u, v, w, ...) = 0 \tag{16}$$

$$(u - x_f)^2 + (v - y_f)^2 + (w - z_f)^2 = r^2 \tag{17}$$

$$(u - y_g)^2 + (v - y_g)^2 + (w - z_g)^2 = r^2 \tag{18}$$

$$(u - x_f, v - y_f, w - z_f) \times (a_f, b_f, c_f) = (0,0,0) \tag{19}$$

$$(u - x_g, v - y_g, w - z_g) \times (a_g, b_g, c_g) = (0,0,0) \tag{20}$$

$$(x-u)^2+(y-v)^2+(z-w)^2 = r^2 \quad (21)$$
$$(x_f-x_g, y_f-y_g, z_f-z_g)\times(a_h,b_h,c_h) = (t_u,t_v,t_w) \quad (22)$$
$$(x-x_f, y-y_f, z-z_f)\cdot(t_u,t_v,t_w) = 0 \quad (23)$$
$$(x-x_g, y-y_g, z-z_g)\cdot(t_u,t_v,t_w) = 0 \quad (24)$$

A constant-radius blending surface is a special case of a variable-radius blending surface where the spine is the intersection curve of two offset surfaces[7] with the same fixed offset r. Since the radii of spheres in the family are same, the contributing circle must be the largest circle on the sphere and the spine tangent is the normal of the plane containing that circle.

$$\mathrm{F}(x_f,y_f,z_f,\ldots) = 0 \quad (25)$$
$$\mathrm{G}(x_g,y_g,z_g,\ldots) = 0 \quad (26)$$
$$(u-x_f)^2+(v-y_f)^2+(w-z_f)^2 = r^2 \quad (27)$$
$$(u-y_g)^2+(v-y_g)^2+(w-z_g)^2 = r^2 \quad (28)$$
$$(u-x_f, v-y_f, w-z_f)\times(a_f,b_f,c_f) = (0,0,0) \quad (29)$$
$$(u-x_g, v-y_g, w-z_g)\times(a_g,b_g,c_g) = (0,0,0) \quad (30)$$
$$(x-u)^2+(y-v)^2+(z-w)^2 = r^2 \quad (31)$$
$$\begin{vmatrix} x & y & z & 1 \\ u & v & w & 1 \\ x_f & y_f & z_f & 1 \\ x_g & y_g & z_g & 1 \end{vmatrix} = 0 \quad (32)$$

2.1.4 Variable Counts

Let n_f, n_g and n_h be the numbers of variables needed to define the surfaces f, g and h. Note that n_f is 3 for an implicit surface and 5 for a parametric surface. Table 1 counts the number of variables needed for each operation.

operation	offset	bisect	constant-R blend	variable-R blend
variables	n_f+3	n_f+n_g+3	n_f+n_g+6	$n_f+n_g+n_h+3$

Table 1: The number of variables after each operation

In this table, we have already excluded those variables which can be eliminated from surface formulations without symbolic computation, such as r in a bisecting surface.

2.1.5 Reducing the Number of Variables

The geometric properties of the surface operations allow us to reduce the number of variables in many cases. For example, since the offset of an offset is the offset of the base surface by the combined distance, we need to use only $n_f + 3$ instead of $n_f + 6$ variables. The following operation combinations show how to take advantage of this and analogous observations.

1. *Offsetting an offset surface*:

   Only one offset operation is needed, by the sum of the distances, properly signed[1].

2. *Offsetting a blending surface*:

   Since the blending surface is the envelope of a family of spheres, its offset is obtained by suitably enlarging every sphere of the family. In particular, replace equation (21) in the case of variable-radius blending and equation (31) in the case of constant-radius blending with the equation

   $$(x-u)^2 + (y-v)^2 + (z-w)^2 = (r + \mathit{offset})^2 \tag{33}$$

   where *offset* is the offset distance.

3. *Bisecting two offset surfaces*:

   Assume that d_f and d_g are offset distances from base surfaces f and g, and that we want the equidistance surface of the two offsets. We directly derive the bisecting surface from the base surfaces, by modifying equations (8)–(13); see also Figure 4. In equation (10) we replace r with $(r + d_f)$, and in equation (11) we replace r with $(r + d_g)$. The other equations are unchanged because the normals of base surface and offsets agree at corresponding points.

   The normal of the new surface is obtained from the foot points q_f and q_g on the two offsets with help of Theorem 2. Although the offsets were not explicitly formulated, the foot points are easily found from $\mathbf{p}_f$ and $\mathbf{p}_g$ and the normals of the base surfaces: Let $(a_f^{(1)}, b_f^{(1)}, c_f^{(1)})$ and $(a_g^{(1)}, b_g^{(1)}, c_g^{(1)})$ be unit normal vectors for surfaces f and g. Then

   $$\begin{aligned} \mathbf{q}_f: &\quad (x_f + d_f a_f^{(1)},\ \ y_f + d_f b_f^{(1)},\ \ z_f + d_f c_f^{(1)}) \\ \mathbf{q}_g: &\quad (x_g + d_g a_g^{(1)},\ \ y_g + d_g b_g^{(1)},\ \ z_g + d_g c_g^{(1)}) \end{aligned}$$

[1] We assume that the local offset is taken. For example, offsetting first by $+r$ and then by $-r$ results in the original surface. This is in contrast to global offset in the sense of Rossignac and Requicha[7].

The normal expression for the bisecting surface is now $\mathbf{n} = \pm(\mathbf{q}_f - \mathbf{q}_g)$.

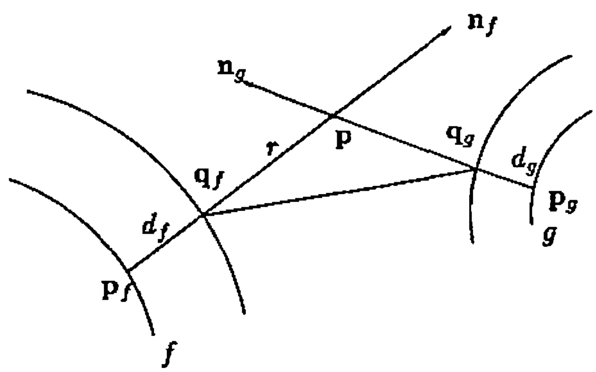


Figure 4: Construct a bisecting surface from two offset surfaces

4. *Blending two offset surfaces*:

   Since the formulation of a blending surface contains that of a bisecting surface, a similar reduction applies. We omit the details.

3 Trimming Surfaces in Higher Dimension

We consider evaluating a restricted area of a surface defined with the dimensionality paradigm, i.e., defined by the system of equations (1). We assume that the surface we want is the projection of the manifold in n-space into the (x_1, x_2, x_3) subspace.

In [2], an algorithm has been described for evaluating the projected surface in a domain defined by the range of x_1, x_2, and x_3; that is, the surface area inside the volume

$$a_1 \leq x_1 \leq b_1$$
$$a_2 \leq x_2 \leq b_2$$
$$a_3 \leq x_3 \leq b_3$$

is found. In many cases, the surface area of interest cannot be so defined. For example, consider the offset of a rectangular parametric surface patch. Although the offset patch would be logically a rectangular area, the patch is not bounded in 3-space by three pairs of parallel planes.

3.1 Rectangular Area Evaluation

Consider first evaluating a surface area restricted to a rectangular area in a different subspace. For example, the offset of a rectangular parametric surface patch would be the set of all solutions of the system for which the parametric variables are in the domain of the patch. Since the parametric variables occur among the variables of the system, the simplest situation is:

> *Rectangular Area Projections*
> Given the system (1), evaluate the projection of the manifold into the subspace spanned by x_1, x_2, and x_3, in the domain
>
> $$a_i \leq x_i \leq b_i, \quad a_j \leq x_j \leq b_j$$

Our algorithm is based on Chuang's method, [2], and likewise requires an initial starting point.

We evaluate the surface projection on the $N_1 \times N_2$ grid of points where

$$\begin{aligned} x_i &\in [a_i, a_i + h_1, ..., b_i] \\ x_j &\in [a_j, a_j + h_2, ..., b_j] \end{aligned}$$

with $h_1 = (b_i - a_i)/N_1$ and $h_2 = (b_j - a_j)/N_2$, beginning with the point $\mathbf{p} = (a_1, ..., a_i, ..., a_j, ...a_n)$:

1. At the point $\mathbf{p} = (y_1, ..., y_n)$ on the surface, construct a local approximant $\Phi = (\phi_1, ..., \phi_n)$ such that
$$x_k = \phi_k(s, t)$$
and
$$\phi_k(0, 0) = y_k$$
for $1 \leq k \leq n$.

2. Let a neighboring point $\mathbf{q} = (z_1, ..., z_n)$ be one at which $z_i = y_i \pm h_1$ and $z_j = y_j$, or $z_i = y_i$ and $z_j = y_j \pm h_2$. Find the neighboring points not yet determined as follows:

   (a) Solve
   $$\begin{aligned} \phi_i(s, t) &= z_i \\ \phi_j(s, t) &= z_j \end{aligned}$$
   for s and t. For example, the ϕ_k could be linear in s and t, in which case Φ is essentially the tangent plane.

(b) Recover an estimate of the remaining coordinates by evaluating $\phi_k(s,t)$ at the values found for s and t, where $k = 1, ..., n$, $k \neq i$ and $k \neq j$.

(c) Refine the estimate with Newton iteration adjoining to the system the equations

$$x_i = z_i \qquad x_j = z_j$$

3. Repeat the above steps until all grid points have been determined.

From the traced grid points the trimmed surface is obtained in (x_1, x_2, x_3) space. As a simple example, Figure 5 shows an offset of a NURB surface. Clearly our algorithm can be generalized to evaluate triangular areas.

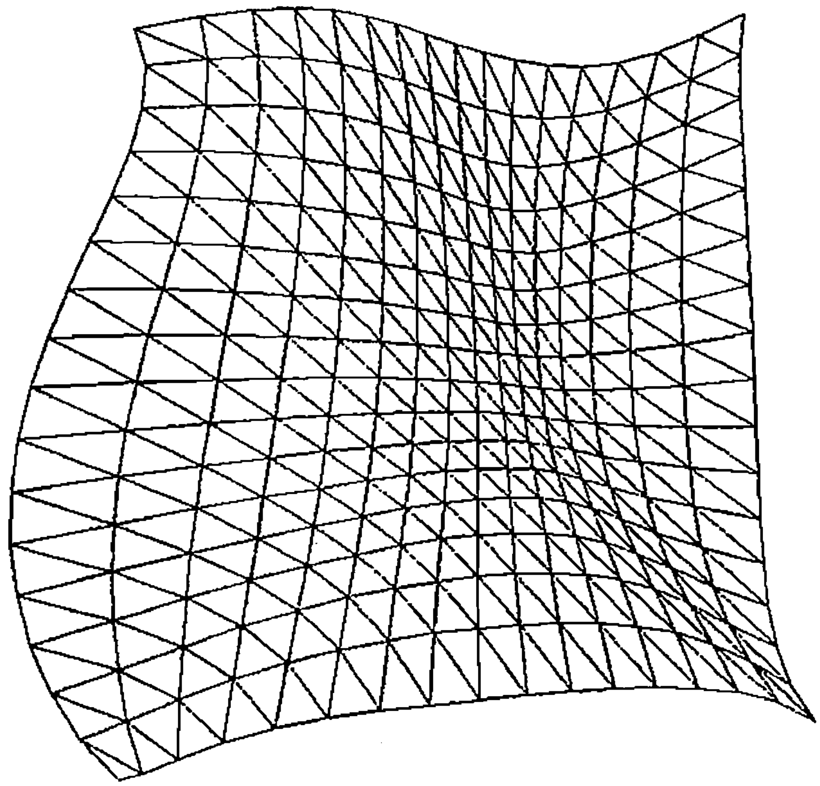


Figure 5: Offset surface of a NURB surface

3.2 Logically Rectangular Area Evaluation

In some cases, the surface area of interest is logically a rectangle but cannot be defined as the restriction of two variables that occur naturally in the defining system of equations. For example, with a rolling ball blend the surface area of interest is bordered by two contact curves that are not plane sections of the manifold in some subspace. We introduce two mechanisms for dealing with this situation. First, we can introduce auxiliary variables that express a geometric relationship which defines a logical boundary. Second, we can introduce virtual variables, such as arc length on a curve, that are not defined equationally but are introduced procedurally in the mechanics of the surface evaluation algorithm.

Consider a cross-section of a blending surface which contains the circle a rolling ball contributes to the surface, as shown in Figure 6. The line segment connecting the two foot-points $\overline{\mathbf{p}_f\mathbf{p}_g}$ separates the circle into two parts, an arc which contains $\mathbf{p}$, a point on the blending surface, and the other which contains $\mathbf{p}'$, a point not on the blending surface. We will keep the arc containing the

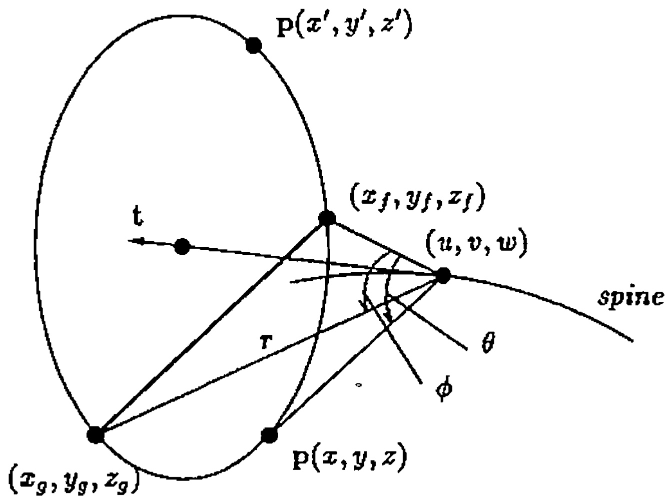


Figure 6: Trimming a blending surface

point $\mathbf{p}$ in Figure 6, and parameterize it so the parameter value ranges from 0 to 1.

There are several ways to parameterize the arc we are interested in. For example, we can use the chord length d, assuming for simplicity that the arc of interest does not exceed a half circle. Adding the equations

$$(x - x_f)^2 + (y - y_f)^2 + (z - z_f)^2 = d \quad (34)$$

$$(x_g - x_f)^2 + (y_g - y_f)^2 + (z_g - z_f)^2 = d_{max} \quad (35)$$

$$\alpha d_{max} = d \quad (36)$$

we note that the parameter α ranges from 0 to 1 as $\mathbf{p}$ moves from $\mathbf{p}_f$ to $\mathbf{p}_g$ along the arc. To distinguish the lower from the upper arc, we may add the condition that the determinant

$$D = \begin{vmatrix} t_x & t_y & t_z & 0 \\ x & y & z & 1 \\ x_f & y_f & z_f & 1 \\ x_g & y_g & z_g & 1 \end{vmatrix}$$

be nonnegative. This can be expressed by the equation

$$D = e^2 \tag{37}$$

if the surface is evaluated in real space. So, by adding equations (34–37), we have expressed the area of interest in one dimension by the new variable α. Other parameterizations might be based on trigonometric functions and angles, and would lead to more uniform parameter speed.

Ideally, the second domain dimension would be expressed by a variable expressing the length of the subtended spine. We do not have an attractive technique to express this quantity as a variable defined by suitably adjoined equations. Instead, we obtain this quantity implicitly as part of the algorithm. We call this implicit quantity a *virtual parameter*. We will have to characterize the curve on the surface that fixes the virtual parameter. In the case of rolling ball blends the curve is characterized by the fact that the associated spine point is fixed.

To obtain surface points with suitably spaced virtual parameter values, we first must evaluate the spine as a curve and compute a subset of points that are uniformly spaced. Each point is associated with a particular value of the virtual parameter β. We would like to evaluate for each spine point the corresponding surface section in the range of the parameter α as discussed before. One way would be to construct a point on the section based on the geometry of the rolling ball blend. For example, for constant-radius blends, the normal plane to the spine contains the segment we are interested in.

A more general way is as follows: Having evaluated a particular section, we pick a point on it and determine, from the local surface approximant, an estimate for a neighboring point in the direction perpendicular to the section curve tangent. Then, adjoining the condition fixing the virtual parameter, the estimate is refined to a surface point on the corresponding section curve which is now evaluated in turn.

In summary, the constrained surfaces is trimmed by a suitable parameterization that constructs a rectangular domain. The parameterization can be considered in a very loose sense: We do not require rational expressions as is the case for classical parametric surfaces. Instead, implicit variables with trimming ranges can be used. Furthermore, we may have virtual parameters that are evaluated procedurally. We complete this section with an example. Figure 7 illustrates a trimmed blending surface for two cylinders defined by

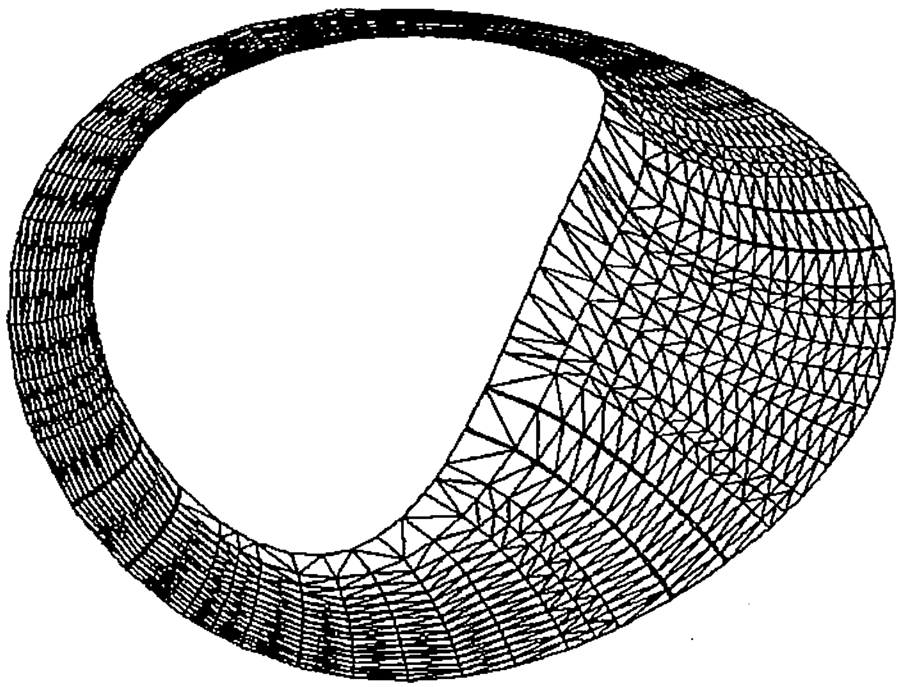


Figure 7: Trimmed blending surface

$$cylinder_1: \quad x^2 + y^2 = 9$$
$$cylinder_2: \quad y^2 + z^2 = 25$$

where the reference surface is given by

$$reference: (x+1)^2 + y^2 = 25$$

In this example, we parameterize the blending arc which is one side of the logically rectangular area, and trace this area simultaneously with its blending spine. The spine points are traced over a grid:

$$x_{min} \leq x \leq x_{max}$$
$$y_{min} \leq y \leq y_{max}$$

Taking only a part of the blending surface and parameterizing the spine of this particular case, we obtain Figure 8 by evaluating a rectangular area. Note that the traversal grid lines are circuler arcs.

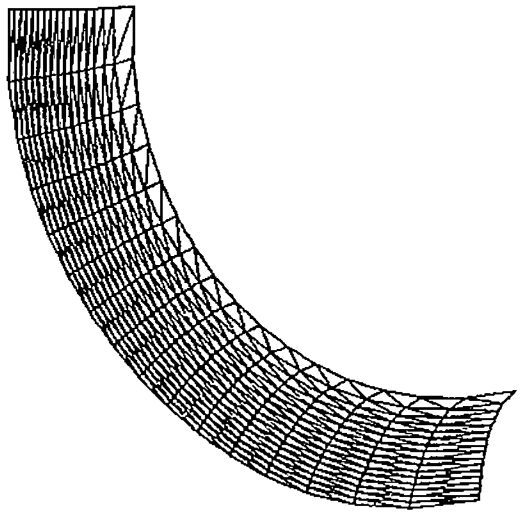


Figure 8: A part of the trimmed blending surface

4 General Trimming

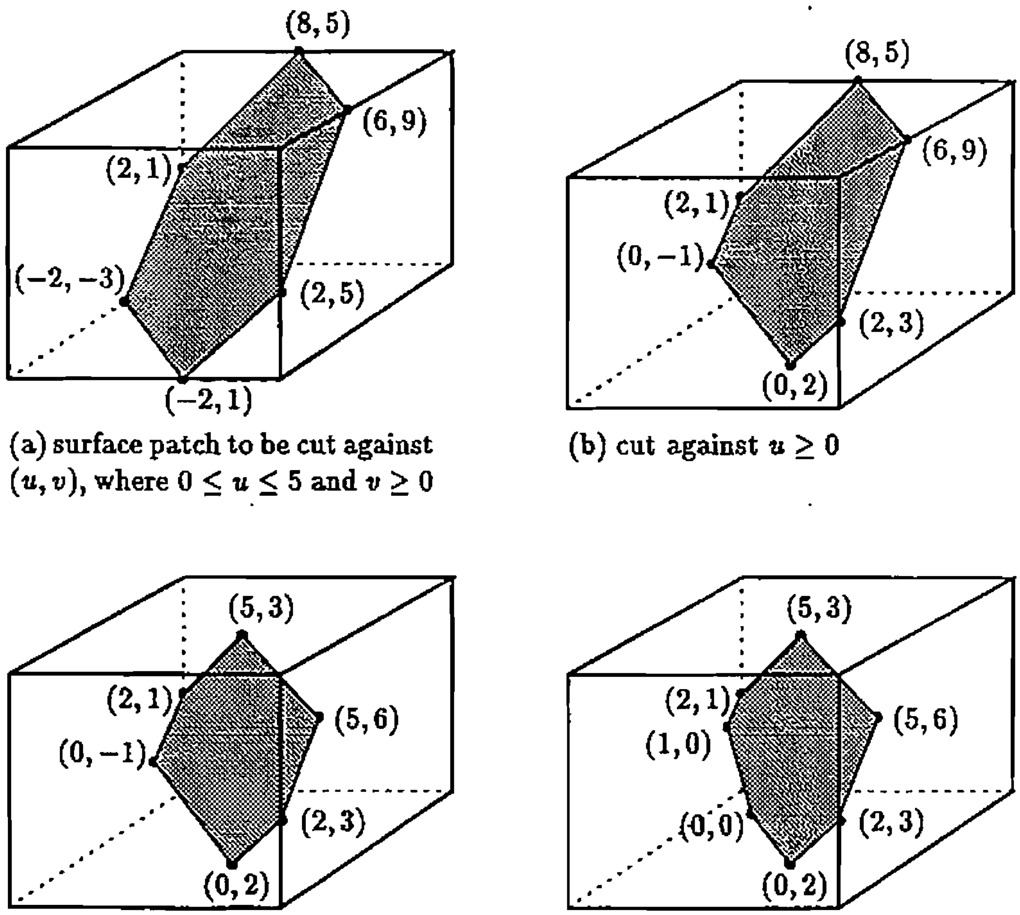
The techniques described before solve those surface trimming problems that logically can be reduced to evaluating a rectangular area. However, such reduction is not always possible and so a more general algorithm to trim constrained surfaces is needed. In the simplest form, the general trimming problem can be stated as follows:

> *General Trimming for Constrained Surfaces*
> Given the system (1), evaluate the projection of the manifold in the domain
> $$a_i \leq x_i \leq b_i \qquad \text{where} \qquad j \leq i \leq k$$
> into the subspace spanned by x_1, x_2, and x_3.

Compared with the rectangular area evaluation, the general problem is easier to express, but more difficult to solve. We call the variables x_i, $j \leq i \leq k$, *trimming parameters*. Note that they could be auxiliary variables that are expressly introduced for the purpose of trimming. For instance, when evaluating a blending surface, we treat the determinant D as a trimming parameter so the blending surface is divided into two parts along two contact curves, one corresponding to $D < 0$, the other to $D \geq 0$. As before, the part with $D \geq 0$ is the desired blending part.

To find a solution to the general trimming problem, we revisit Chuang's algorithm and consider a decomposition in the (x_1, x_2, x_3)-subspace induced by a regular, rectangular grid. When a particular cube is explored, we evaluate the trimming parameters at the intersection of the approximated surface projection with the boundary of the cube.

Consider a surface facet inside a cube, as shown in Figure 9 (a), before trimming. The facet is a convex polygon. We will trim the facet by considering each trimming parameter in turn. Figure 9 (b), (c) and (d) illustrate the trimming process in a cube for two trimming parameters: $0 \leq u \leq 5$ and $v > 0$. Note that we can trim each cube separately.



(a) surface patch to be cut against (u, v), where $0 \leq u \leq 5$ and $v \geq 0$

(b) cut against $u \geq 0$

(c) cut against $u \leq 5$

(d) cut against $v \geq 0$

Figure 9: Variable trimming inside a cube

If we have constructed a linear approximant, all variables x_i, $i > 3$, vary linearly in this cube. A linear approximant $\mathcal{H}$ is expressed as

$$x_i = h_i(s,t) \qquad \text{where} \qquad 1 \le i \le n$$

and every h_i is a linear function of s and t. $\mathcal{H}$ is also viewed as a linear parametrization of a plane. The projection of the plane in 3D space is a plane $\mathcal{M}$:

$$\begin{aligned} x_1 &= h_1(s,t) \\ x_2 &= h_2(s,t) \\ x_3 &= h_3(s,t) \end{aligned}$$

Each face of the cube defines a half space so the plane $\mathcal{M}$ is truncated by six half spaces to a convex polygon. Suppose the given cube is bounded from (c_1, c_2, c_3) to (d_1, d_2, d_3), then the convex polygon is represented by

$$\begin{aligned} c_1 &\le h_1(s,t) \le d_1 \\ c_2 &\le h_2(s,t) \le d_2 \\ c_3 &\le h_3(s,t) \le d_3 \end{aligned}$$

Similarly, any trimming variable, say $x_i \ge a_i$, also defines a half space in higher dimensions. Then the convex polygon in $\mathcal{M}$ is further truncated by a half plane projected from this half space. We add the condition

$$x_i \ge a_i$$

to further trim the convex polygon for every trimming variable. In practice, we evaluate the values of each trimming variable at the vertices of the convex polygon, from the function h_i, and linearly interpolate s and t with respect to the trimming conditions. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vertices in the convex polygon, s_1, t_1 and s_2, t_2 be their coordinates of $\mathcal{H}$. Assume that we have

$$c_i = h_i(s_1, t_1) \le x_i \le h_i(s_2, t_2) = d_i$$

If $c_i < a_i < d_i$, we delete the vertex $\mathbf{v}_1$ and insert a new vertex, $\mathbf{v}$, with the linearly interpolated parameters

$$\begin{aligned} s &= \frac{d_i - a_i}{d_i - c_i} s_1 + \frac{a_i - c_i}{d_i - c_i} s_2 \\ t &= \frac{d_i - a_i}{d_i - c_i} t_1 + \frac{a_i - c_i}{d_i - c_i} t_2 \end{aligned}$$

The trimming for $x_i \le b_i$ is done analogously.

We illustrate the process with an example of corner blending; Figure 10. The corner is bounded by a circular cylinder, an elliptic cylinder and a Bezier surface. The cylinder is defined as

$$y^2 + z^2 = 25$$

and the elliptic cylinder is given by

$$(x + 0.15z)^2 + y^2 = 9$$

Finally, the Bezier surface is defined by 16 control points.

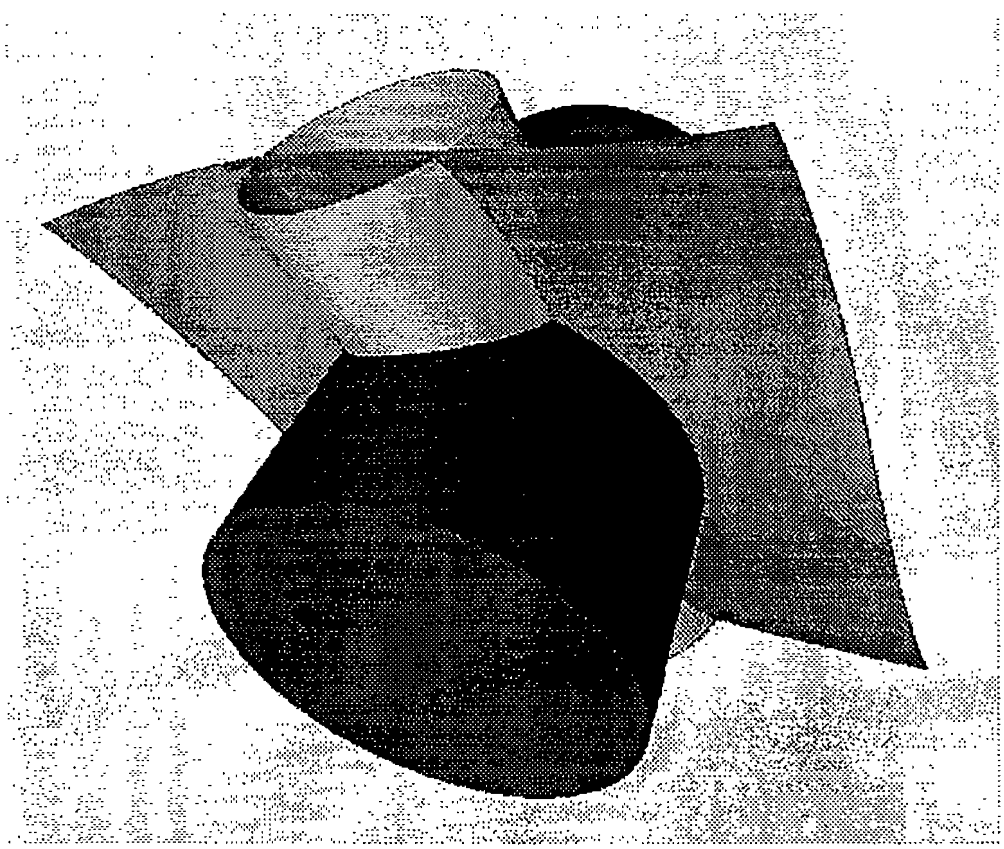


Figure 10: Three surfaces to be blended at a corner

We construct blending surfaces of between each surface pair and add a spherical patch as corner blend. Figure 11 shows the result.

Figure 11: Corner blending of three surfaces

5 Summary

We have studied two issues in this paper. First, we showed that the class of constrained surfaces defined with the dimensionality paradigm is closed under offsetting, bisecting and blending operations. The result also applies to curves. Moreover, we have clarified the growth behavior of the number of variables involved, and given some techniques for reducing the growth. Second, we have explored the surface trimming problem and presented several methods. In analogy to parametric surface evaluation over rectangular domains, we examined trimming constrained surfaces by a rectangular area defined by two variables of the surface definition and generalized the method to an evaluation of a logically rectangular area using virtual parameters. We then generalized trimming by restricting any number of variables. We solve this problem in Section 4 by modifying Chuang's surface evaluation algorithm.

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