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Abstract

We present a sufficient criterion for the Bernstein-Bezier (BB) form of a trivariate polynomial within a tetrahedron, such that the real zero contour of the polynomial defines a smooth and single sheeted algebraic surface patch. We call this an A-patch. We present algorithms to build a mesh of cubic A-patches to interpolate a given set of scattered point data in three dimensions, respecting the topology of any surface triangulation T of the given point set. In these algorithms we first specify "normals" on the data points, then build a simplicial hull consisting of tetrahedra surrounding the surface triangulation T and finally construct cubic A-patches within each tetrahedron. The resulting surface constructed is C^1 (tangent plane) continuous and single sheeted in each of the tetrahedra. We also show how to adjust the free parameters of the A-patches to achieve both local and global shape control.

1 Introduction

The importance of implicit surface representation in modeling geometric objects or reconstructing the image to scattered data have been described in various papers (see for e.g. [2, 6, 8, 10, 15]). The main advantages of implicit surface over its parametric counterpart are: (1) the set of algebraic surfaces are closed under basic modeling operations such as offset and intersection, often required in a solid modeling system. For example, the offset of a parametric surface may not be parametric but is always algebraic and has an implicit representation. (2) For the same polynomial of degree n, implicit algebraic surfaces have more degrees of freedom ($= \binom{n+3}{3} - 1$) compared with rational parametric surface($\leq 4 \binom{n+2}{2} - 1$) surface of the same degree. Hence implicit algebraic surfaces are more flexible to approximate a complicated surface with fewer number of pieces or to achieve higher order of smoothness. However, the main shortcoming held against the popular use of implicit

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surfaces is that the representation being multivalued may cause the real zero contour surface to have multiple sheets, self-intersections and several other undesirable singularities.

In section 3 of this paper, we present a sufficient criterion for the Bernstein-Bezier (BB) form of a trivariate polynomial within a tetrahedron such that the real zero-contour of the polynomial is smooth (non-singular) and a single sheeted algebraic surface. We call this an A-patch. In section 4, we describe how to build a simplicial hull consisting of tetrahedra surrounding a surface triangulation T of the set of scattered data points in 3D. We then show in section 5 how a mesh of cubic A-patches can be used to construct a C^1 interpolatory surface, respecting the topology of the surface triangulation T. In section 6, we show how to adjust the free parameters of the A-patches to achieve both local and global shape control. This C^1 cubic A-patch fitting algorithm is quite appropriate for free form design. In analogy to the final smoothing of an artist's rough sketches, complicated smooth models can be directly formed by first creating a rough polyhedral model of the desired object and then using the fitting algorithms to produce a C^1 smooth solid with extra local and global parameters for fine shape control. Proofs of all theorems and lemmas are given in the Appendix.

Related Prior Work:

The work of characterizing the BB form of polynomials within a tetrahedron such that the zero contour of the polynomial is a single sheeted surface within the tetrahedron, has been attempted in the past. In [15], Sederberg showed that if the coefficients of the BB form of the trivariate polynomial on the lines that parallel one edge, say L, of the tetrahedron, all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the zero contour algebraic surface patch at most once. In [8], Guo treats the same problem by enforcing monotonicity conditions on a cubic polynomial along the direction from one vertex to a point of the opposite face of the vertex. From this he derives a condition $a_{\lambda-e_1+e_4} - a_\lambda \geq 0$ for all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ with $\lambda_1 \geq 1$, where a_λ are the coefficients of the cubic in BB form and e_i is the i-th unit vector. This condition is difficult to satisfy in general, and even if this condition is satisfied, one still cannot avoid singularities on the zero contour. Our condition of a smooth, single sheeted zero contour in Theorem 3.2 of §3 generalizes Sederberg's condition and provides us with an efficient way of generating A-patches.

The second problem we consider is how to join a collection of A-patches to form a C^1 smooth surface interpolating scattered data points and respecting the topology of a given surface triangulation T of the points. For this problem, prior approaches have been given by [5] using quadric patches, [6, 8, 9] using cubic patches and [3] using quintic for convex triangulations and degree seven patches for arbitrary surface triangulations T. All these papers provide heuristics to overcome the multiple sheeted and singularity problems of implicit patches. In this paper our cubic A-patches are guaranteed to be nonsingular and single sheeted within each tetrahedron.

While the details of the methods of [6] and [9] differ somewhat, they both use the scheme of [5] of building a surrounding simplicial hull (consisting of a series of tetrahedra) of the given triangulation T. Such a simplicial hull is nontrivial to construct for triangulations and neither of the papers [5, 6, 8, 9] enumerate the different exceptional cases (possible even for convex triangulations) nor provide solutions to overcoming them. We too use the simplicial hull approach in this paper but enumerate the exceptional situations and provide some heuristic strategies for rectifying them.

In [9], Guo uses a Clough-Tocher split[4] and subdivides each face tetrahedron of the simplicial hull, hence utilizing three patches per face of T. In this paper, we consider the computed "normals" at the given data points, and distinguish between "convex" and "non-convex" faces and edges of the triangulation. These concepts are formally defined in section 4. We use a single cubic A-patch per face of T except for the following two special cases. For a non-convex face, if additionally the three inner products of the face normal and its three adjacent face normals have different signs, then in this case one needs to subdivide the face using a single Clough-Tocher split, yielding C^1 continuity with the help of three cubic A-patches for that face. Furthermore for coplanar adjacent faces of T, we show that the C^1 conditions cannot be met using a single cubic A-patch for each face. Hence for this case we again use Clough-Tocher splits for the pair of coplanar faces yielding C^1 continuity with the help of three cubic A-patches per face. See also the examples and figures in section 7 where the savings in patches becomes evident.

Related papers which approximate scattered data using implicit algebraic patches are [1, 10, 11] and a classification of data fitting using parametric surface patches is given in [13].

2 Notation and Preliminary Details

Problem Given a list of data points $P = \{p_1, \ldots p_k\} \in I\!R^3$ and a surface triangulation T of these points, construct a mesh of low degree algebraic surfaces such that the composite surface is single sheeted C^1 continuous and has the same topology as T.

Convex Hull, Affine Hull: Let $\{p_1, \ldots, p_j\} \in I\!R^3$ with $j \leq 4$. Then the *convex hull* of these points is defined by $[p_1p_2...p_j] = \{p \in I\!R^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i \geq 0, \sum_{i=1}^{j} \alpha_i = 1\}$ and the *affine hull* is defined by $\langle p_1p_2...p_j \rangle = \{p \in I\!R^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \sum_{i=1}^{j} \alpha_i = 1\}$. The interior of the convex hull $[p_1p_2...p_j]$ is denoted by $(p_1p_2...p_j) = \{p \in I\!R^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i > 0, \sum_{i=1}^{j} \alpha_i = 1\}$.

Bernstein-Bezier (BB) Form: Let $p_1,\ p_2,\ p_3,\ p_4 \in I\!R^3$ be affine independent. Then the tetrahedron with vertices $p_1,\ p_2,\ p_3$, and p_4, is $V = [p_1p_2p_3p_4]$. For any $p = \sum_{i=1}^{4} \alpha_i p_i \in V$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Let $p = (x, y, z)^T$, $p_i = (x_i, y_i, z_i)^T$. Then the barycentric coordinates relate to the Cartesian coordinates via the following relation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{2.1}$$

Any polynomial $f(p)$ of degree n can be expressed as Bernstein-Bezier(BB) form over V as $f(p) = \sum_{|\lambda|=n} b_\lambda\ B_\lambda^n(\alpha),\ \lambda \in Z_+^4$, where $B_\lambda^n(\alpha) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\ \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$ is Bernstein polynomial, $|\lambda| = \sum_{i=1}^{4} \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = \sum_{i=1}^{4} \alpha_i e_i$ is barycentric coordinate of p, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$(as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called control points, and Z_+^4 stands for the set of all four dimensional vectors with nonnegative integer components. The following basic facts about the BB form will be used in this paper. The first is derived from the directional derivative formulas(see [7]).

Lemma 2.1. *If $f(p) = \sum_{|\lambda|=n}\ b_\lambda B_\lambda^n(\alpha)$, then*

$$b_{(n-1)e_i+e_j} = b_{ne_i} + \frac{1}{n}(p_j - p_i)^T \nabla f(p_i), \quad j = 1, 2, 3, 4; \quad j \neq i \tag{2.2}$$

where $\nabla f(p) = [\frac{\partial f(p)}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial f(p)}{\partial z}]^T$ $f(p)$

Formula (2.2) will be used to determine the control points around a vertex from the given normal at that vertex.

Lemma 2.2 ([7]). *Let $f(p) = \sum_{|\lambda|=n} a_\lambda B^n_\lambda(\alpha)$ and $g(p) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ be two polynomials defined on two tetrahedra $[p_1p_2p_3p_4]$ and $[p'_1p_2p_3p_4]$, respectively. Then*
(i) f and g are C^0 continuous at the common face $[p_2p_3p_4]$ if and only if

$$a_\lambda = b_\lambda, \quad for \quad any \quad \lambda = 0\lambda_2\lambda_3\lambda_4, \quad |\lambda| = n \tag{2.3}$$

(ii) f and g are C^1 continuous at the common face $[p_2p_3p_4]$ if and only if (2.3) holds and

$$b_{1\lambda_2\lambda_3\lambda_4} = \beta_1 a_{1\lambda_2\lambda_3\lambda_4} + \beta_2 a_{0\lambda_2\lambda_3\lambda_4+0100} + \beta_3 a_{0\lambda_2\lambda_3\lambda_4+0010} + \beta_4 a_{0\lambda_2\lambda_3\lambda_4+0001} \tag{2.4}$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ are defined by the relation $p'_1 = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4$, $|\beta| = 1$.
Relation (2.4) will be called coplanar condition.

Degree Elevation. The polynomial $f(p) = \sum_{|\lambda|=n} b_\lambda \, B^n_\lambda(\alpha)$ can be written as one of degree $n+1$ (see e.g. [7]): $f(p) = \sum_{|\lambda|=n+1} (Eb)_\lambda \, B^{n+1}_\lambda(\alpha)$, $\quad \lambda \in Z^4_+$, where $(Eb)_\lambda = \frac{1}{n+1}\sum_{i=1}^4 \lambda_i b_{\lambda-e_i}$.

Variation Diminishing Property ([7],p.54). Let $y(t) = \sum_{i=0}^n b_i B^n_i(t)$, then $y(t)$ has no more intersections(counting the multiplicities) with any line than the polygon $\{\frac{i}{n}, b_i\}_{i=0}^n$ in $[0,1]$.

Transformation: Since $\sum_{i=1}^4 \alpha_k = 1$, we have from (2.1) that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_2 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_2 - z_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix} \tag{2.5}$$

Let $f(x,y,z) = g(\alpha_1, \alpha_2, \alpha_3)$, then it is easy to check that

$$\nabla f(x,y,z) = (A^{-1})^T \nabla g(\alpha_1, \alpha_2, \alpha_3) \tag{2.6}$$

Therefore, the surface $f(x,y,z) = 0$ is smooth (i.e., $\nabla f(x,y,z) \neq 0$) iff the surface $g(\alpha_1, \alpha_2, \alpha_3) = 0$ is smooth (i.e., $\nabla g(\alpha_1, \alpha_2, \alpha_3) \neq 0$). This means that the smoothness problem of the surface $f(x,y,z) = 0$ can be treated directly in its barycentric form.

3 Sufficient Conditions of an A-Patch

Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ be a given polynomial of degree n on the simplex(tetrahedron) $S = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in I\!R^4 : \sum_{i=1}^4 \alpha_i = 1, \ \alpha_i \geq 0\}$. The surface patch within the simplex is defined by $S_F \subset S : \ F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$. The following two conditions on the trivariate BB-form will be used in this paper.
Smooth vertices condition. For each $i (1 \leq i \leq 4)$, there is at least one non-zero $b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ for $\lambda_i \geq n-1$.
Smooth edges condition. For each pair $(i,j)(1 \leq i,j \leq 4, i \neq j)$, there is either at least one non-zero $b_{me_i+(n-m)e_j}$ for $m = 0, 1, \cdots, n$, or the polynomials $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_k} B^{n-1}_m(t)$ and $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_l} B^{n-1}_m(t)$ have no common zero in $[0,1]$, for distinct i, j, k, l.

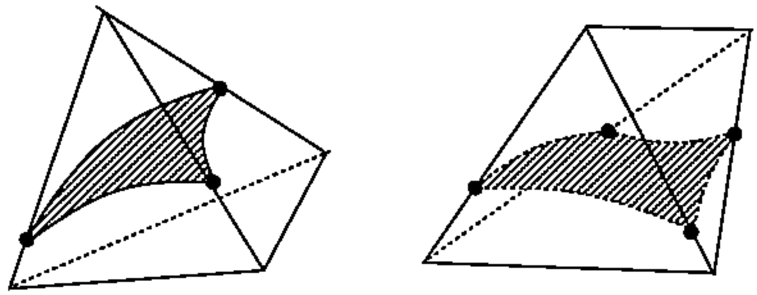


Figure 3.1: Three Sided and Four sided Patches

If the surface S_F contains a vertex/edge, then it is easy to show by the formulas of directional derivatives(see [7], p. 312) that the surface is smooth there if the smooth vertex/edge conditions above are satisfied.

Definition 3.1. *Three-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. If any open line segment (e_j, α^*) with $\alpha^* \in S_j = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \alpha_j = 0, \alpha_i > 0, \sum_{i \neq j} \alpha_i = 1\}$ intersects S_F at most once(counting multiplicities), then we call S_F a *three-sided j-patch* (see Figure 3.1).

Definition 3.2. *Four-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. Let (i, j, k, ℓ) be a permutation of (1, 2, 3, 4). If any open line segment (α^*, β^*) with $\alpha^* \in (e_i e_j)$ and $\beta^* \in (e_k e_\ell)$ intersects S_F at most once(counting multiplicities), then we call S_F a *four-sided ij-kℓ-patch* (see Figure 3.1).

It is easy to see that if S_F is a four-sided ij-$k\ell$-patch, it is then also a ji-ℓk-patch, a ℓk-ji-patch, and so on. The Appendix contains proofs of the lemmas and theorems below.

Lemma 3.1. *The three-sided j-patch and the four-sided ij-kℓ-patch are smooth (non-singular).*

Theorem 3.2. *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ satisfy the smooth vertex and smooth edge conditions and $j (1 \leq j \leq 4)$ be a given integer. If there exists an integer $k(0 \leq k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \geq 0, \qquad \lambda_j = 0, 1, \ldots, k-1, \tag{3.1}$$

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \leq 0, \qquad \lambda_j = k+1, \ldots, n \tag{3.2}$$

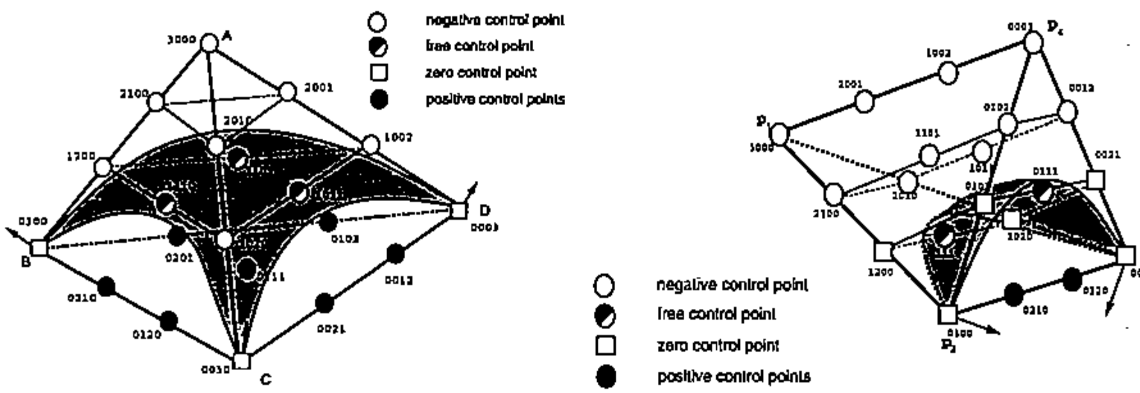
and $\sum_{\substack{|\lambda|=n \\ \lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n \\ \lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \leq n)$, then S_F is a three-sided j-patch.

Theorem 3.3. *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ satisfy the smooth vertex and smooth edge conditions and (i, j, k, ℓ) be a permutation of (1, 2, 3, 4). If there exists an integer $k(0 \leq k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \geq 0; \qquad \lambda_i + \lambda_j = 0, 1, \ldots, k-1, \tag{3.3}$$

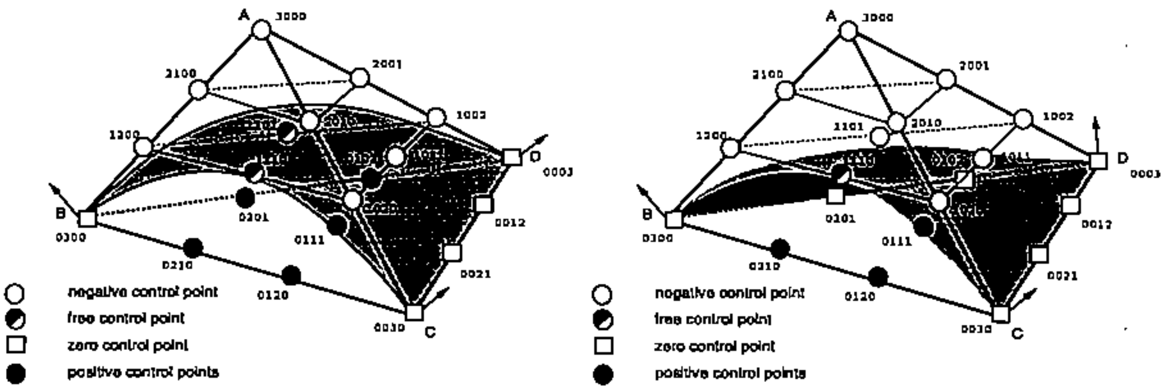
$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \leq 0; \qquad \lambda_i + \lambda_j = k+1, \ldots, n \tag{3.4}$$

and $\sum_{\substack{|\lambda|=n \\ \lambda_i+\lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n \\ \lambda_i+\lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \leq n)$, then S_F is four-sided ij-kℓ-patch.



(a) (b)

Figure 3.2: (a) A three sided patch tangent at p_1, p_2, p_3 (b) A degenerate four sided patch tangent to face $[p_1 p_2 p_4]$ at p_2 and $[p_1 p_3 p_4]$ at p_3



(a) (b)

Figure 3.3: (a) A three sided patch interpolating the edge $[p_2 p_3]$ (b) A three sided patch interpolating edges $[p_2 p_3]$ and $[p_1 p_3]$

Note. The conditions on the coefficients b_λ in Theorems 3.2 and 3.3 are sufficient but not necessary. For example if we want some $B_l < 0$, it is not necessary to let every $b_\lambda < 0$, for $|\lambda| = n, \lambda_4 = \ell$.

Some properties of *A*-patches.

a. *For a three-sided j-patch, if* $b_\lambda = 0$ *for* $\lambda = (n-\ell)e_m + \ell e_j$, $\ell = 0, 1, \ldots, k (m \neq j, k < n)$, *and* $b_\lambda \neq 0$ *for* $\lambda = (n-1)e_m + e_s$, $s \neq j, m$, *then the edge* $[e_j e_m]$ *is tangent with* S_F *at* e_m *with multiplicities k. See also Figure 3.2 (a).*

b. *For a four-sided ij-kℓ-patch, if* $b_\lambda = 0$ *for* $\lambda = (n-q_1-q_2)e_k + q_1 e_i + q_2 e_j$, $q_1 + q_2 = 0, 1, \ldots, s$; *and* $b_\lambda \neq 0$ *for* $\lambda = (n-1)e_k + e_\ell$, *then* S_F *is tangent s times with face* $[e_i e_j e_k]$ *at* e_k.

Note that a four sided patch may degenerate into a two sided patch. See Figure 3.2 (b). However, we do not need to treat the degenerate patches any different and consider it to be a special four sided patch.

c. *For a three-sided j-patch, if* $b_\lambda = 0$ *for* $\lambda = (n-m)e_i + m e_k$, $m = 0, 1, \ldots, n$, *then* S_F

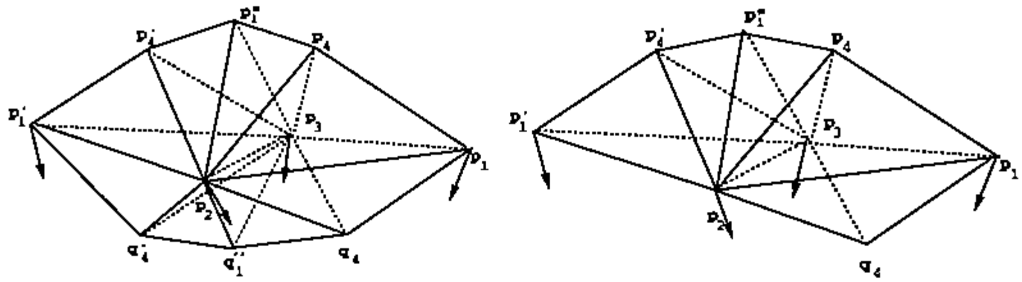


Figure 4.1: The Construction of Tetrahedra for Adjacent Non-Convex/Non-Convex Faces and Convex/Non-Convex Faces

contains the edge $[e_i, e_k]$. *If further,* $b_\lambda = 0$, *for* $\lambda = (n-m-1)e_i + me_k + e_j$, $m = 0, 1, \ldots, n-1$, *then the* S_F *is tangent with the face* $[e_ie_je_k]$. *See also Figure 3.3 (a), (b).*

4 Normals and the Simplicial Hull

For the given point set $P = \{p_1, \ldots, p_k\} \in \mathbb{R}^3$ and their surface triangulation T, we first construct a normal set $N = \{n_1, \ldots, n_k\} \in \mathbb{R}^3$ for P. That is, for each point p_i, we associate a normal n_i. We will force the constructed surface to interpolate points p_i and at each point have a normal n_i for $i = 1, \cdots, k$. These normals therefore also provide a mechanism to control the shape of the C^1 interpolating surface. Common approaches to construct these normals at a point p_i include (a) an average of the face normals of the incident faces (b) the gradient of a local spherical fit to the surface triangulation at each vertex. Computing an optimal normal assignment is yet an unsolved problem and we are experimenting with different local and global normal selections schemes [1, 14, 12]. Of course at times the data set can have prespecified normals and this too can be the input of the C^1 fitting algorithm.

Without loss of generality we assume that the assigned normals all point to the same side of T. If T is a closed surface triangulation (a simplicial polyhedron) then we assume the normals all point to the exterior.

Definition 4.1. *Convex edge, non-convex edge.*

Let $[p_ip_j]$ be an edge of T. If $(p_j - p_i)^T n_i \; (p_i - p_j)^T n_j \geq 0$ and at least one of $(p_j - p_i)^T n_i$ and $(p_i - p_j)^T n_j$ is positive, then we say the edge $[p_ip_j]$ is *positive convex*. If both the numbers are zero then we say it is *zero convex*. A *negative convex* edge is similarly defined. If $(p_j - p_i)^T n_i \; (p_i - p_j)^T n_j < 0$, then we say the edge is *non-convex*.

Definition 4.2. *Convex face, non-convex face.*

Let $[p_ip_jp_k]$ be a face of T. If its three edges are nonnegative (positive or zero) convex and at least one of them is positive convex, then we say the face $[p_ip_jp_k]$ is *positive convex*. If all the three edges are zero convex then we label the face as *zero convex*. A *negative convex* face is similarly defined. All the other cases $[p_ip_jp_k]$ are labeled as *non-convex*.

Note, that here we are overloading the term *convex* to characterize the relations between the normals and edges of faces. We distinguish between convex and non-convex faces in the simplicial hull below where we build one tetrahedron for convex faces and double tetrahedra for non-convex faces.

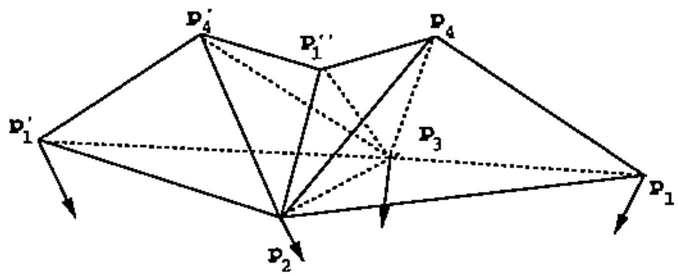


Figure 4.2: The Construction of Tetrahedra for Adjacent Convex/Convex Faces

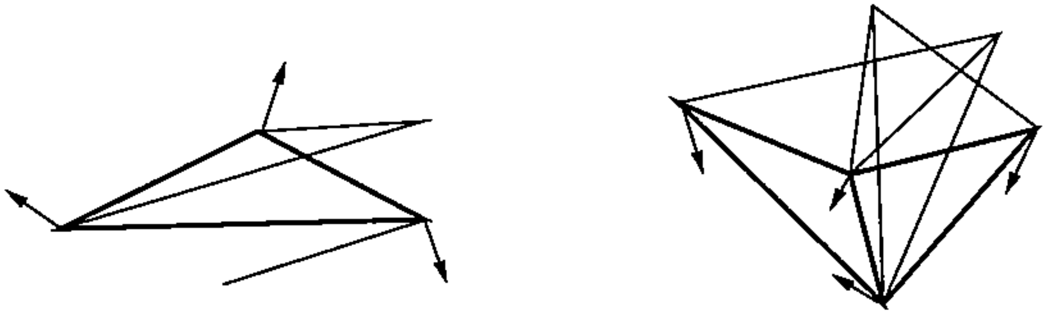
Definition 4.3. *Simplicial hull.*

A simplicial hull of T, denoted by $\sum$, is a collection of non-degenerate tetrahedra which satisfies:
(1) Each tetrahedron in $\sum$ has either a single edge of T(then it will be called an *edge tetrahedron*) or a single face of T(then it will be called a *face tetrahedron*).
(2) For each face of T there is/are only one/two face tetrahedron/tetrahedra in $\sum$ if the face is convex/non-convex.
(3) Two face tetrahedra that share a common edge do not intersect anywhere else. This condition is referred to in this paper as *non-intersection.*
(4) For each edge there is/are only one/two pair/pairs of common face sharing edge tetrahedra in $\sum$ if the edge is convex/non-convex such that the pair/pairs fills the region between the two adjacent face tetrahedra in the same side of T.
(5) For each vertex, the tangent plane defined by the vertex normal is contained in all the tetrahedra containing the vertex. This condition is called *tangent plane containment.*

It should be noted that, for a given surface triangulation and normals assignment, T there may exist infinitely many simplicial hulls or no simplicial hull may exist. We now describe a scheme for constructing a simplicial hull for the surface triangulation T and prescribed vertex normal assignment. We also enumerate the exceptional configurations where a simplicial hull of T is not possible and then provide a solution for constructing the simplicial hull for a locally modified T.
1. **Build Face Tetrahedra.** For each face $F = [p_1p_2p_3]$ of T, let L be a straight line that is perpendicular to the face F and passes through the center of the inscribed circle of F. Then choose points p_4 and/or q_4 off each side of F to be the farthermost intersection points between L and the tangent planes of the vertices of the face. If F is a non-convex face, two face tetrahedra $[p_1p_2p_3p_4]$ and $[p_1p_2p_3q_4]$ are formed. If F is positive convex, then p_4 is chosen on the side opposite to the direction of the normals, and a single face tetrahedron $[p_1p_2p_3p_4]$ is formed. If F is negative convex, then q_4 is chosen on the same side as the normals and again the single face tetrahedron $[p_1p_2p_3q_4]$ is formed. Figure 4.2 shows the case where both faces are convex and Figure 4.1 shows the cases where at least one of the two adjacent faces is non-convex.

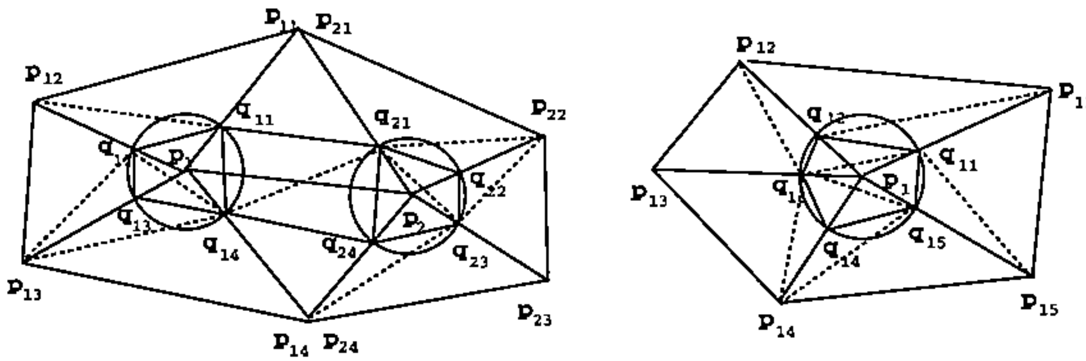
A sufficient condition for constructing face tetrahedra with tangent plane containment is that the angle of the assigned normal n_i at each vertex p_i with each of the surrounding face's normals is less than $\pi/2$. If this condition is not met then an exception occurs and we term the vertex as *sharp.* See Figure 4.3 (a).

A sufficient condition for adjacent face tetrahedra to be non-intersecting is as follows. For two adjacent faces $F = [p_1p_2p_3]$ and $F' = [p'_1p_2p_3]$, the angle between them, denoted as $\angle FF'$, is defined as the outer dihedral angle if the edge between F and F' is negatively convex and inner dihedral



(a) (b)

Figure 4.3: (a) No Tangent Plane Containment (b) Self-Intersecting Tetrahedra



(a) (b)

Figure 4.4: The re-triangulation of (a) sharp edge (b) and sharp vertex

angle otherwise. For $[p_2p_3]$ the common edge between F and F', let $[p_1p_2p_3p_4]$ and $[p_1'p_2p_3p_4']$ be the face tetrahedra respectively. Then the two tetrahedra are non-intersecting if the angles $\angle[p_4p_2p_3][p_1p_2p_3] < \frac{1}{2}\angle FF'$ and $\angle[p_4'p_2p_3][p_1'p_2p_3] < \frac{1}{2}\angle FF'$. If this condition is not met then an exception may occur and we term the common edge $[p_2p_3]$ as *sharp*. See Figure 4.3 (b).

A heuristic strategy rectifies the *sharp* edge and *sharp* vertex configurations is a local retriangulation of the original surface triangulation T. This strategy has worked well in several of the smoothing examples we have performed.

(i) **Sharp edge problem.** Let $[p_1p_2]$ be a sharp edge(see Figure 4.4(a)), and let $[p_ip_{ij}]$ $(i = 1,2;\ j = 1,2,\cdots k_i)$ be the remaining surrounding edges of p_i in adjacency order. Take two spheres $S(p_i, r_i)$ with centers p_i and radius r_i, where r_i are positive numbers that are less than the half of the surrounding edge's lengthes $\|p_i - p_{ij}\|$. The sharper one wants the constructed smooth surface around the edge $[p_1p_2]$, the smaller we take r_i. Let q_{ij} be the intersection points of $S(p_i, r_i)$ and $[p_ip_{ij}]$. Then $q_{i1}, q_{i2}, ..., q_{ik_i}$ form two closed polygons, and $p_{ij}, p_{ij+1}, q_{ij+1}, q_{ij}$ forms a four sided closed polygons and finally, $q_{11}, q_{21}, q_{2k_2}, q_{1k_1}$ forms another four sided closed polygon. Triangulate these polygons (the dotted line in Figure 4.4(a)) by connecting adjacent edges of the polygons in the least inner angle order.

(ii) Sharp vertex problem. Let p_1 be a sharp vertex(see Figure 4.4(b)), and let $[p_1 p_{1j}]$ $(j = 1, 2, \cdots k)$ be the surrounding edges of p_1 in adjacency order. Take a sphere $S(p_1, r)$ with center p_1 and radius r, where r is positive number that is less than the half of the surrounding edge's lengthes $\|p_1 - p_{1j}\|$. The sharper one wants the constructed smooth surface around the vertex p_1, the smaller we take r . Let q_{1j} be the intersection points of $S(p_1, r)$ and $[p_1 p_{1j}]$. Then $q_{11}, q_{12}, ..., q_{1k}$ form a closed polygon, and $p_{1j}, p_{1j+1}, q_{1j+1}, q_{1j}$ forms a four sided closed polygon. Triangulate these polygons (the dotted line in Figure 4.4(b)) by connecting the adjacent edges of the polygon in the least inner angle.

2. Build Edge Tetrahedra. Let $[p_2 p_3]$ be an edge of T and $[p_1 p_2 p_3]$ and $[p'_1 p_2 p_3]$ be the two adjacent faces. Let $[p_1 p_2 p_3 p_4]$ and/or $[p_1 p_2 p_3 q_4]$, and $[p'_1 p_2 p_3 p'_4]$ and/or $[p'_1 p_2 p_3 q'_4]$ be the face tetrahedra built for the faces $[p_1 p_2 p_3]$ and $[p'_1 p_2 p_3]$, respectively. Then if the edge $[p_2 p_3]$ is non-convex, two pairs of tetrahedra need to be constructed. The first pair $[p''_1 p_2 p_3 p_4]$ and $[p''_1 p_2 p_3 p'_4]$ are between $[p'_1 p_2 p_3 p'_4]$ and $[p_1 p_2 p_3 p_4]$. The second pair $[q''_1 p_2 p_3 q_4]$ and $[q''_1 p_2 p_3 q'_4]$ are between $[p'_1 p_2 p_3 q'_4]$ and $[p_1 p_2 p_3 q_4]$. Here $p''_1 \in (p_4 p'_4)$ or is above (p_4, p'_4), say

$$p''_1 = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(p'_4 + p_4), \quad t \geq 1$$

so that p''_1 is above plane $[p_1 p_2 p_3]$ and plane $[p'_1 p_2 p_3]$. Similarly, $q''_1 \in (q_4 q'_4)$ or is below (q_4, q'_4), say

$$q''_1 = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(q'_4 + q_4), \quad t \geq 1$$

so that q''_1 is below plane $[p_1 p_2 p_3]$ and plane $[p'_1 p_2 p_3]$. If the edge $[p_2 p_3]$ is positive/negative convex, only the first/second pair above are needed. If the edge $[p_2 p_3]$ is zero convex, no tetrahedron is needed here. It should be noted that p_4 and p'_4(q_4 and q'_4) are always visible.

5 Construction of a C^1 Interpolatory Surface using Cubic A-Patches

Having established a simplicial hull $\sum$ for the given surface triangulation T and a set of vertex normals N, we now construct a C^1 function f on the hull $\sum$ such that

$$f(p_i) = 0, \quad \nabla f(p_i) = n_i, \quad i = 1, 2, \ldots, k \tag{5.1}$$

and the zero contour of f within $\sum$ forms a C^1 continuous single sheeted surface with the same topology as T.

5.1 The Construction of a Piecewise C^1 Cubic Function

The construction of the function f over two adjacent faces of T is divided into the following three cases:

(a). Both the faces are non-convex;

(b). Both the faces are convex;

(c). One of them is convex and the other is non-convex.

(a). Both the faces are non-convex

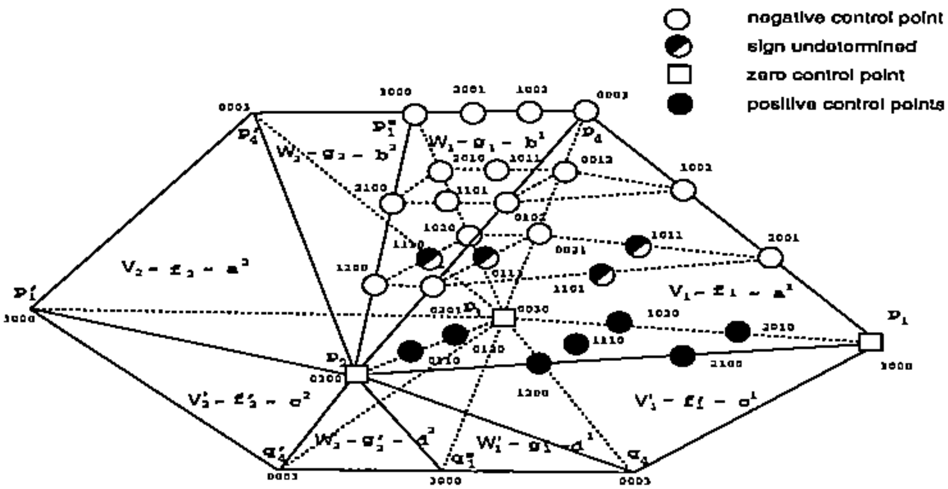


Figure 5.1: Adjacent Tetrahedra, Functions and Control Points for two Non-Convex Adjacent Faces

Let $F = [p_1p_2p_3]$ and $F' = [p_1'p_2p_3]$ be two adjacent non-convex faces. Then we have double tetrahedra $[p_1p_2p_3p_4]$ and $[p_1p_2p_3q_4]$ for F and double tetrahedra $[p_1'p_2p_3p_4']$ and $[p_1'p_2p_3q_4']$ for F'(see Figure 5.1). Let

$$V_1 = [p_1p_2p_3p_4],\ \ V_2 = [p_1'p_2p_3p_4'],\ \ W_1 = [p_1''p_2p_3p_4],\ \ W_2 = [p_1''p_2p_3p_4']$$

$$V_1' = [p_1p_2p_3q_4],\ \ V_2' = [p_1'p_2p_3q_4'],\ \ W_1' = [q_1''p_2p_3q_4],\ \ W_2' = [q_1''p_2p_3q_4']$$

and the cubic polynomials f_i over V_i, g_i over W_i, f_i' over V_i' and g_i' over W_i' be expressed in Bernstein-Bezier forms with coefficients a_λ^i, b_λ^i, c_λ^i, and d_λ^i, $i = 1, 2$, respectively. Now we shall determine these coefficients.

C⁰ Continuity: If two tetrahedra share a common face, we equate the control points of the associated cubic polynomials on the common face(see Lemma 2.2):

$$a^i_{\lambda_1\lambda_2\lambda_30} = c^i_{\lambda_1\lambda_2\lambda_30},\ \ a^i_{0\lambda_2\lambda_3\lambda_4} = b^i_{0\lambda_2\lambda_3\lambda_4},\ \ b^1_{\lambda_1\lambda_2\lambda_30} = b^2_{\lambda_1\lambda_2\lambda_30}$$

$$c^i_{0\lambda_2\lambda_3\lambda_4} = d^i_{0\lambda_2\lambda_3\lambda_4},\ \ d^1_{\lambda_1\lambda_2\lambda_30} = d^2_{\lambda_1\lambda_2\lambda_30}$$

Interpolation: Since zero contours of f_i f_i' and g_i and g_i' pass through p_2 and p_3, $a_\lambda^i = b_\lambda^i = c_\lambda^i = d_\lambda^i = 0$ for $i = 1, 2$ and $\lambda = 0300, 0030$.

Normal Condition: From (5.1) and (2.2) we have, for $j = 2, 3$

$$\begin{aligned}
a^1_{2e_j+e_1} &= \tfrac{1}{3}(p_1 - p_j)^T n_j, & a^2_{2e_j+e_1} &= \tfrac{1}{3}(p_1' - p_j)^T n_j \\
a^1_{2e_j+e_4} &= \tfrac{1}{3}(p_4 - p_j)^T n_j, & a^2_{2e_j+e_4} &= \tfrac{1}{3}(p_4' - p_j)^T n_j, \\
b^1_{2e_j+e_1} &= \tfrac{1}{3}(p_1'' - p_j)^T n_j, & d^1_{2e_j+e_1} &= \tfrac{1}{3}(q_1'' - p_j)^T n_j, \\
c^1_{2e_j+e_4} &= \tfrac{1}{3}(q_4 - p_j)^T n_j, & c^2_{2e_j+e_4} &= \tfrac{1}{3}(q_4' - p_j)^T n_j
\end{aligned} \tag{5.2}$$

C¹ Conditions: At present, set $a^i_{2e_4+e_j}$, $c^i_{2e_4+e_j}$, $j = 1, 2, 3, 4$, b^i_{2001}, and d^i_{2001} to any value(free parameters) and determine the other control points

1. Interface of $[p_2p_3p_4]$ and $[p_2p_3p_4']$. Suppose

$$\begin{array}{rcll} p_1'' &=& \beta_1^1 p_1 + \beta_2^1 p_2 + \beta_3^1 p_3 + \beta_4^1 p_4, & \beta_1^1+\beta_2^1+\beta_3^1+\beta_4^1 = 1 \\ p_1'' &=& \beta_1^2 p_1' + \beta_2^2 p_2 + \beta_3^2 p_3 + \beta_4^2 p_4', & \beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2 = 1 \end{array} \tag{5.3}$$

Then, the C^1 conditions require(see Lemma 2.2)

$$b^i_{1\lambda_2\lambda_3\lambda_4} = \beta_1^i a^i_{1\lambda_2\lambda_3\lambda_4} + \beta_2^i a^i_{0\lambda_2\lambda_3\lambda_4+0100} + \beta_3^i a^i_{0\lambda_2\lambda_3\lambda_4+0010} + \beta_4^i a^i_{0\lambda_2\lambda_3\lambda_4+0001} \tag{5.4}$$

for $\lambda_2\lambda_3\lambda_4 = 002, 101, 011, 110$. Hence b^i_{1002}, b^i_{1101}, and b^i_{1011} are defined, leaving a^i_{1011} and a^i_{1101} to be determined. Equation (5.4) for $\lambda_2\lambda_3\lambda_4 = 110$ will be treated later.

2. Interface at $[p_2p_3p_1'']$. Let

$$p_1'' = \mu_1 p_4 + \mu_2 p_4' + \mu_3 p_2 + \mu_4 p_3, \quad \mu_1+\mu_2+\mu_3+\mu_4 = 1 \tag{5.5}$$

then C^1 conditions require

$$b^i_{\lambda_1\lambda_2\lambda_3 0+1000} = \mu_1 b^1_{\lambda_1\lambda_2\lambda_3 1} + \mu_2 b^2_{\lambda_1\lambda_2\lambda_3 1} + \mu_3 b^i_{\lambda_1\lambda_2\lambda_3 0+0100} + \mu_4 b^i_{\lambda_1\lambda_2\lambda_3 0+0010} \tag{5.6}$$

for $\lambda_1\lambda_2\lambda_3 = 200, 110, 101, 011$. Hence b^i_{3000}, b^i_{2100}, and b^i_{2010}. are defined. The equation for $\lambda_1\lambda_2\lambda_3 = 011$ will be treated later together with (5.4).

3. Interface between $[p_2p_3q_4]$, $[p_2p_3q_1'']$ and $[p_2p_3q_4']$. All control points of g_i' and some of the control points of f_i' can be fixed as f_i and g_i. That is, the relations (5.4)–(5.6) hold when the quantities $a's$, $b's$, $\beta's, \mu's$ are substituted by $c's$, $d's$, $\gamma's, \eta's$ respectively. The two untreated equations left are

$$d^i_{1110} = \gamma_1^i a^i_{1110} + \gamma_2^i a^i_{0210} + \gamma_3^i a^i_{0120} + \gamma_4^i c^i_{0111} \tag{5.7}$$

$$d^i_{1110} = \eta_1 c^1_{0111} + \eta_2 c^2_{0111} + \eta_3 a^i_{0210} + \eta_4 a^i_{0120} \tag{5.8}$$

where the coefficients γ_i and η_i are defined by

$$\begin{array}{rcll} q_1'' &=& \gamma_1^1 p_1 + \gamma_2^1 p_2 + \gamma_3^1 p_3 + \gamma_4^1 q_4, & \gamma_1^1+\gamma_2^1+\gamma_3^1+\gamma_4^1 = 1 \\ q_1'' &=& \gamma_1^2 p_1' + \gamma_2^2 p_2 + \gamma_3^2 p_3 + \gamma_4^2 q_4', & \gamma_1^2+\gamma_2^2+\gamma_3^2+\gamma_4^2 = 1 \\ q_1'' &=& \eta_1 q_4 + \eta_2 q_4' + \eta_3 p_2 + \eta_4 p_3, & \eta_1+\eta_2+\eta_3+\eta_4 = 1 \end{array} \tag{5.9}$$

4. Interface between $[p_1p_2p_3]$ and $[p_1'p_2p_3]$. Let

$$\begin{array}{rcll} q_4 &=& \alpha_1^1 p_1 + \alpha_2^1 p_2 + \alpha_3^1 p_3 + \alpha_4^1 p_4, & \alpha_1^1+\alpha_2^1+\alpha_3^1+\alpha_4^1 = 1 \\ q_4' &=& \alpha_1^2 p_1' + \alpha_2^2 p_2 + \alpha_3^2 p_3 + \alpha_4^2 p_4', & \alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2 = 1 \end{array} \tag{5.10}$$

Then we have

$$c^i_{0111} = \alpha_1^i a^i_{1110} + \alpha_2^i a^i_{0210} + \alpha_3^i a^i_{0120} + \alpha_4^i a^i_{0111} \tag{5.11}$$

Now we treat the equations (5.4), (5.6), (5.7), (5.8) and (5.11). It follows from (5.4), (5.6), (5.7) and (5.8) that

$$\mu_1 a^1_{0111} + \mu_2 a^2_{0111} + \mu_3 a^i_{0210} + \mu_4 a^i_{0120} = \beta_1^i a^i_{1110} + \beta_2^i a^i_{0210} + \beta_3^i a^i_{0120} + \beta_4^i a^i_{0111} \tag{5.12}$$

$$\eta_1 c^1_{0111} + \eta_2 c^2_{0111} + \eta_3 a^i_{0210} + \eta_4 a^i_{0120} = \gamma^i_1 a^i_{1110} + \gamma^i_2 a^i_{0210} + \gamma^i_3 a^i_{0120} + \gamma^i_4 c^i_{0111} \tag{5.13}$$

Therefore, (5.11)–(5.13) form a linear system with six equations and six unknowns a^i_{0111}, a^i_{1110}, c^i_{0111} for $i = 1, 2$. It is important to point out that this is not an independent system(see Theorem 5.1 for the solvability of the system). It has 4 independent equations and has infinitely many solutions. In fact, if we assume p_1, p_2, p_3, p'_1 are not coplanar and then denote

$$\begin{array}{rclll} p_4 & = & \theta^1_1 p_1 + \theta^1_2 p_2 + \theta^1_3 p_3 + \theta^1_4 p'_1, & \theta^1_1 + \theta^1_2 + \theta^1_3 + \theta^1_4 = 1 \\ p'_4 & = & \theta^2_1 p_1 + \theta^2_2 p_2 + \theta^2_3 p_3 + \theta^2_4 p'_1, & \theta^2_1 + \theta^2_2 + \theta^2_3 + \theta^2_4 = 1 \\ q_4 & = & \vartheta^1_1 p_1 + \vartheta^1_2 p_2 + \vartheta^1_3 p_3 + \vartheta^1_4 p'_1, & \vartheta^1_1 + \vartheta^1_2 + \vartheta^1_3 + \vartheta^1_4 = 1 \\ q'_4 & = & \vartheta^2_1 p_1 + \vartheta^2_2 p_2 + \vartheta^2_3 p_3 + \vartheta^2_4 p'_1, & \vartheta^2_1 + \vartheta^2_2 + \vartheta^2_3 + \vartheta^2_4 = 1 \end{array} \tag{5.14}$$

then we can derive from (5.12) and (5.13) that

$$a^i_{0111} = \theta^i_1 a^1_{1110} + \theta^i_2 a^i_{0210} + \theta^i_3 a^i_{0120} + \theta^i_4 a^2_{1110} \tag{5.15}$$

$$c^i_{0111} = \vartheta^i_1 a^1_{1110} + \vartheta^i_2 a^i_{0210} + \vartheta^i_3 a^i_{0120} + \vartheta^i_4 a^2_{1110} \tag{5.16}$$

If the edge $[p_2p_3]$ is nonnegative (or non-positive) convex, a^i_{1110}(or c^i_{1110}) are free and equation (5.16) (or (5.15)) is removed, since we do not need the function g'_1 and g'_2(or g_1 and g_2). The free parameters a^i_{1110}(or c^i_{1110}) may be determined by approximating a quadratic(see §6 or [6]).

b. Both faces are convex.

(b1). Both faces are nonnegative (or non-positive) convex.

Following the discussion of (a), the scheme for determining the control points are as before, except for the following:

1. Only half the control points are needed. That is, we need a^i_λ, b^i_λ for functions f_i and g_i if F and F' are nonnegative convex, or c^i_λ, d^i_λ for functions f'_i and g'_i if F and F' are non-positive convex.

2. a^i_{1110} (or c^i_{1110}) can be determined freely. One way to choose a^i_{1110} (or c^i_{1110}) is to make the cubic approximate a quadratic (see §6). In particular, $a^i_{1110} = 0$ (or $c^i_{1110} = 0$) if the face is zero convex.

3. We now need only (5.15) for unknowns a^1_{0111} and a^2_{0111} if the edge $[p_2p_3]$ is nonnegative convex, or (5.16) for unknowns c^1_{0111} and c^2_{0111} if the edge $[p_2p_3]$ is non-positive convex.

(b2). One positive convex face and one negative convex face.

In this case, the common edge must be zero convex. Suppose F is positive convex and F' is negative convex. All the control points are determined as before except for the following:

1. We only need to construct f_i, g_i and f'_2, that is, c^1_λ, d^i_λ are not needed. The functions g_i and f_2 have no contribution to the surface, and are used for smooth transition from f_1 to f'_2.

2. $a^1_{1110} \geq 0$ and $c^2_{1110} \leq 0$ can be determined freely(see §6).

3. we need only have (5.11) for $i = 2$ and (5.15) for unknowns a^1_{0111}, a^2_{0111} and c^2_{0111}.

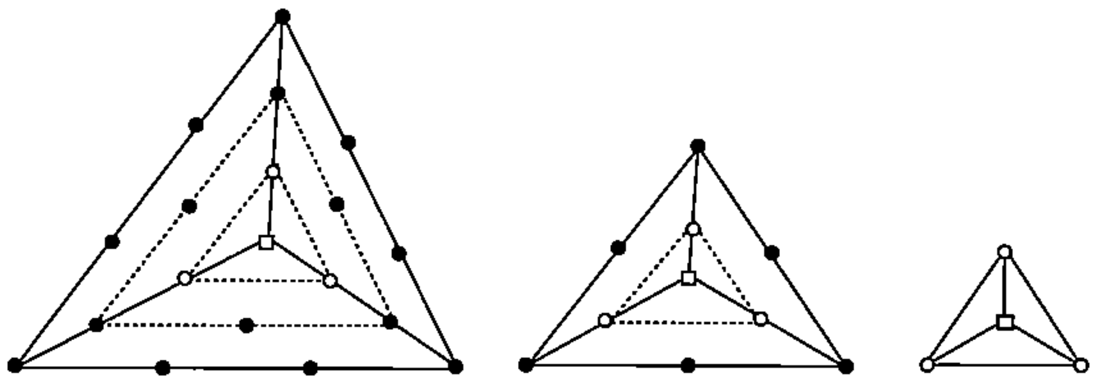


Figure 5.2: The Control Points of 0-th,1st and 2nd Layers

(b3). Both faces are zero convex.

This case in fact is included in case **(b1)**. The surface is defined directly as the planar faces of the surface triangulation. No function needs to be constructed.

c. One convex face and one non-convex face.

Suppose $[p_1p_2p_3]$ is convex, $[p_1'p_2p_3]$ is non-convex. The following are the exceptions:

1. The function f_1' and g_i' and their control points c_λ^1, d_λ^i are not needed if F is nonnegative convex. The function f_1 and g_i and their control points a_λ^1, b_λ^i are not needed if F is non-positive convex.

2. $a_{1110}^1 \geq 0$ (or $c_{1110}^1 \leq 0$) and a_{1110}^2 (c_{1110}^2) can be determined freely as in case (b). In particular, $a_{1110}^1 = 0$ (or $c_{1110}^1 = 0$) if $[p_1p_2p_3]$ is zero convex.

3. For the treatment of equations (5.11)–(5.13), we need only have (5.11) for $i = 2$ and (5.15) for unknowns a_{0111}^1, a_{0111}^2 and c_{0111}^2 if the edge $[p_2p_3]$ is nonnegative convex, or solve (5.11) for $i = 2$ and (5.13) for unknowns c_{0111}^1, c_{0111}^2 and a_{0111}^2 if the edge $[p_2p_3]$ is non-positive convex(see Theorem 5.1 (ii) for the solvability of the system).

d. Coplanarity of adjacent faces

In the discussions above, we have assumed that p_1, p_1', p_2, p_3 are affine independent. If p_1, p_1', p_2, p_3 are coplanar, then the coefficient matrices of the linear systems (5.12) and (5.13) are singular. However, the system (5.11)–(5.13) are still solvable(see Theorem 5.1) taking a_{0111}^i or c_{0111}^i as free parameters. The other unknowns are given directly by these equations. Since the parameters $a_{1110}^i, i = 1, 2$ become now dependent, they are overly determined and a solution may be not possible. In this case we split the involved tetrahedron into sub-tetrahedra by subdividing the triangles $[p_1p_2p_3]$ and $[p_1'p_2p_3]$ into three subtriangles at their center points w and w' (a Clough-Tocher split). A solution is now possible where the coefficients are specified as before by regarding w as p_1 and w' as p_1'.

We then need to determine the remaining coefficients over the sub-tetrahedra $U_1 = [p_2p_3p_4w]$, $U_2 = [p_1p_3p_4w]$, and $U_3 = [p_1p_2p_4w]$ such that the C^1 condition is satisfied. In fact, since $w \in [p_1p_2p_3]$, the coefficients on the same layer are C^1 related. For the 0-th layer (see Figure 5.2), the control points labeled $\bullet$ are thus already determined. The control points $\circ$ are determined by a coplanar condition with surrounding $\bullet$. Finally, the point $\square$ is determined from the surrounding three points $\circ$ by the coplanar condition.

For the 1st layer (see Figure 5.2), the control points labeled $\circ$ and $\square$ are similarly determined as the 0-th layer. For the 2nd layer (see Figure 5.2), the control points $\circ$ are arbitrarily chosen and $\square$ is determined by the coplanar condition. Finally, the 3rd layer coefficient is free.

5.2 The Solvability of the Related System

Concerning the solvability of the system (5.11)–(5.13) and its sub-system, we have the following result. The proof is given in the Appendix.

Theorem 5.1 *Given two affine independent point sets (p_2, p_3, p_4', p_4) and (p_2, p_3, q_4', q_4) as in Figure 5.1. (i) The system (5.11)–(5.13) has four independent equations. If (p_1, p_1', p_2, p_3) is affine independent, then (5.12) and (5.13) are four independent equations for the unknowns a^i_{0111} and c^i_{0111} for $i = 1, 2$.*

(ii) Let $\{r_1, \cdots, r_6\} = \{p_1, p_1', p_4, p_4', q_4', q_4\}$, $\{x_1, \cdots, x_6\} = \{a^1_{1110}, a^2_{1110}, a^1_{0111} a^2_{0111}, c^1_{0111}, c^2_{0111}\}$. For any $1 \le i < j \le 6$, if r_i, r_j, p_2, p_3 are affine independent, then

$$x_k = \phi^k_1 x_i + \phi^k_2 x_j + \phi^k_3 a^1_{0210} + \phi^k_4 a^1_{0120}, \quad k \neq i, j \tag{5.17}$$

where ϕ^k_i are defined by $r_k = \phi^k_1 r_i + \phi^k_2 r_j + \phi^k_3 p_2 + \phi^k_4 p_3$, $\phi^k_1 + \phi^k_2 + \phi^k_3 + \phi^k_4 = 1$.

5.3 Construction of Single Sheeted A-Patches

Having built C^1 cubics with some free control points, we now illustrate how to determine these free control points such that the zero-contours are three-sided or four-sided A-patches (smooth and single sheeted).

We assume (without loss of generality) that all the normals point to the same side of the surface triangulation T. That is the side on which q_4 and q_4' lie(see Figure 5.1). Under this assumption, it follows from Definition 4.1 and equation (5.2) that, the control points on the edge, say a^i_{0210}, a^i_{0120} on edge $[p_2 p_3]$(see Figure 5.1), are non-negative if the edge is non-negative convex, and non-positive if the edge is non-positive convex. Now we can divide all the control points into 7 groups called layers. The 0-th layer consists of the control points that are "on" the faces of T. The 1st layer is next to the 0-th layer but opposite to the normal direction, followed by the 2nd and 3rd layers. Next to the 0-th layer and on the same side as the normal, is the -1st layer, then the -2nd and -3rd layers. Now we show that, we can set all the control points on the 2nd and 3rd layer negative and the control points on the -2nd and -3rd layers positive.

For the face-tetrahedra, it is always possible to make the 2nd and 3rd layers control points negative, because these control points are free under the C^0 condition. For the control points on the edge-tetrahedra, it follows from (5.4) that the 2nd and 3rd layers control points can be negative only if the 2nd layer control points on the neighbor face-tetrahedra are small enough. This is achieved since β^i_4 in (5.4) is positive(see the proof of Proposition 5.3 for details). Similarly, the control points on the -2nd and -3rd layers can be chosen to be positive. Furthermore, all these control points can be chosen as large as one needs in absolute value in order to get single sheeted patches.

Since the control points around the vertices of T are determined by the normals, the smooth vertex condition is obviously satisfied. If the surface contains the edge $[p_2p_3]$(see Figure 5.1), then since a^i_{1110}(or a^i_{0111}) is freely chosen, the smooth edge condition is easily satisfied(see the proof of Proposition 5.3). Referring to Figure 5.1, we prove in the following that the patches constructed over V_1 and W_1 are single sheeted. The other patches are similar.

Proposition 5.2. *If the face $[p_1p_2p_3]$ is non-negative convex, then the control points can be determined so that the surface over V_1 is a three-sided 4-patch.*

Proposition 5.3. *If the edge $[p_2p_3]$ is non-negative convex, then the control points can be determined such that the surface over W_1 is a four-sided 14-23-patch.*

Subdivision. For any face of $T = [p_1, p_2, p_3]$, if it is non-convex and if the three inner products of the face normal and its three adjacent face normals have different signs, then subdivide the double face tetrahedra into 6 subtetrahedra by adding a vertex at the center w of the face (a Clough-Tocher split). The coefficients are specified as before by regarding w as p_1(see Figure 5.1).

Proposition 5.4. *If the above subdivision procedure above is performed, then the control points can be chosen so that the surface over V_1 is a three-sided 4-patch, and the surface over W_1 is a four-sided 14-23-patch.*

These propositions guarantee that the surface constructed are single sheeted.

6 Shape Control

From the discussion of §5, there are several parameters that can influence the shape of the constructed C^1 surface. These parameters include (a) the length of the normal if its orientation is fixed, (b) a^i_{1110}, and (c) $a^i_{0102} < 0$, $a^i_{1002} < 0$ $a^i_{0012} < 0$, $a^i_{0003} < 0$ and $b^i_{2001} < 0$ for $i = 1, 2$.

(a). Interactive Shape Control

The influence of the length of a normal at a vertex is as follows: if the normal becomes longer then the surface becomes flatter at this point. Parameter a_{1110} lifts the surface upwards to the top vertex of the tetrahedron, while others push the surface downwards toward the bottom of the tetrahedron. In order to get a desirable surface, one may specify some additional data points in the tetrahedron considered, then approximate these points in the least square sense.

(b). Default Shape Control

Here we only consider the effect of the free parameters, that is, suppose the normals are fixed. The aim of the default choice of these parameters is to avoid producing bumpy surfaces. The commonly used method is to keep the surface patch close to a quadric patch([1, 6]).

By least squares approximation of the coefficients of a quadric ([6]), one can derive that

$$a_{1110} = \frac{1}{4}(a_{1200} + a_{2100} + a_{2010} + a_{1020} + a_{0210} + a_{0120})$$

Using the same idea, the other parameters can also be determined. For example, a_λ for $\lambda_4 > 1$ can be determined by the degree elevation formula

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4} \lambda_i x_{\lambda - e_i}, \quad |\lambda| = 3, \quad \lambda_4 > 1 \tag{6.1}$$

where $x_{\lambda-e_i}$ is the solution of the following equations in the least squares sense

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4}\lambda_i x_{\lambda-e_i}, \quad |\lambda| = 3, \quad \lambda_4 = 0, 1$$

In the same way, b_{2001} can be determined. Therefore, under the C^1 conditions, we can define two sets of control points $\{a_\lambda^s\}$ and $\{a_\lambda^q\}$ over V_1, where $\{a_\lambda^s\}$ is yielded from the single sheeted consideration(see Proposition 5.2–5.6), and $\{a_\lambda^q\}$ comes from approximating a simple(quadratic) surface. Note that the surface defined by $\{a_\lambda^s\}$ above may not be desirable in shape, while the surface defined by $\{a_\lambda^q\}$ above may not be single sheeted. In our implementation we take a finite sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ and consider $\{a_\lambda^{(i)}\} = \{(1-t_i)a_\lambda^q + t_i a_\lambda^s\}, \quad i = 0, 1, \cdots, m$ selecting the single sheeted surface defined by $\{a_\lambda^{(i)}\}$ for smallest index i. Experiments show that this approach works well and a desirable surface is obtained with $t_i < 0.5$. Examples are shown in Figure 7.5.

7 Examples

Examples of the simplicial hull construction and C^1 smoothed triangulations using cubic A-patches are shown in Figures 7.1, 7.2, 7.3, 7.4 and 7.5. Color pictures of Figures 7.3 and 7.4 are also provided at the end of the paper. Note in these figures how the "convex" faces are smoothed by a single cubic A-patch per face, while a Clough-Tocher splitting occurs for co-planar faces and some "non-convex" faces, as determined by the vertex normals assignment and the adjacent faces.

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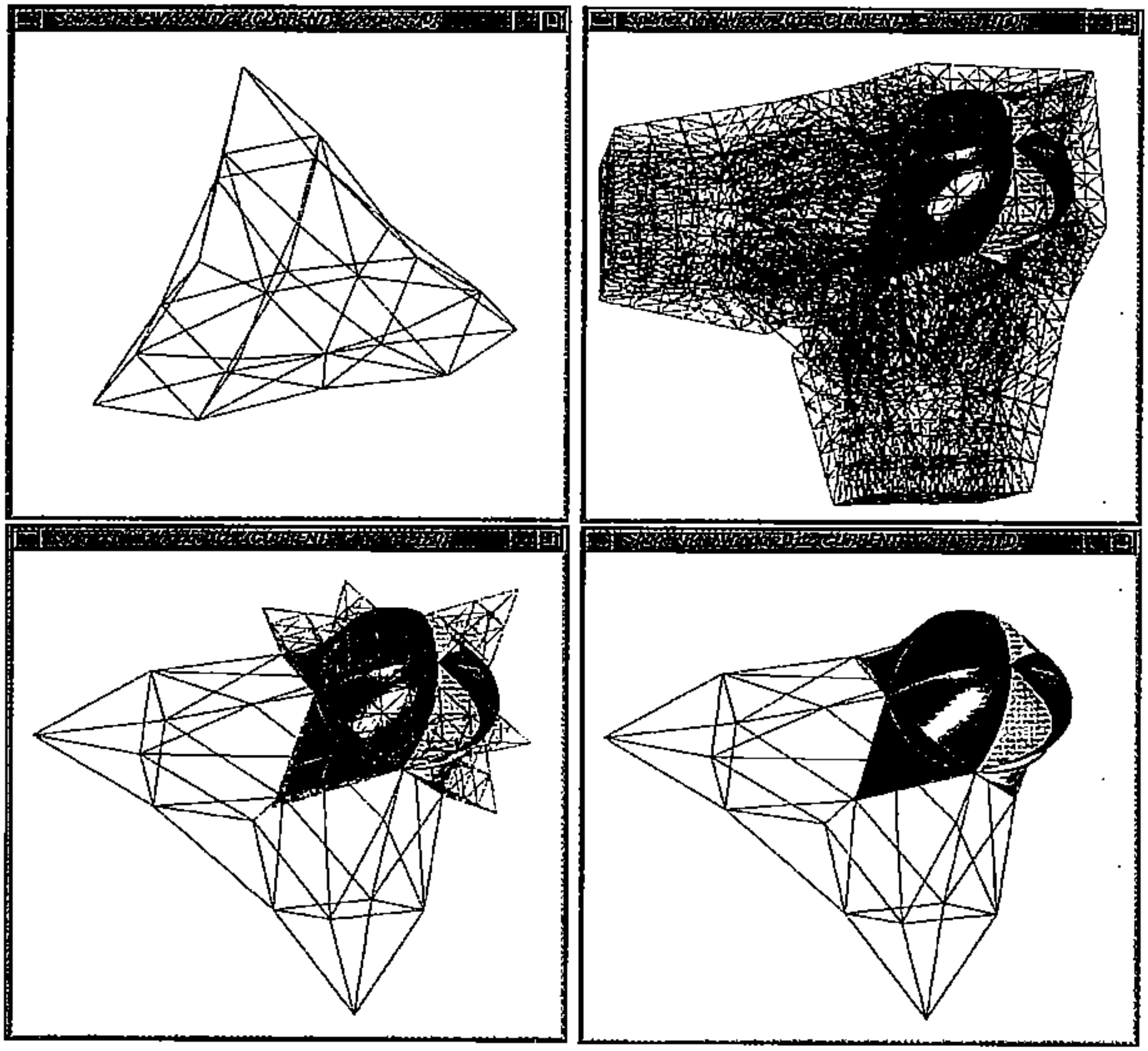

Figure 7.1: A Surface Triangulation, the Simplicial Hull and some of the interpolatory C^1 Cubic A-Patches

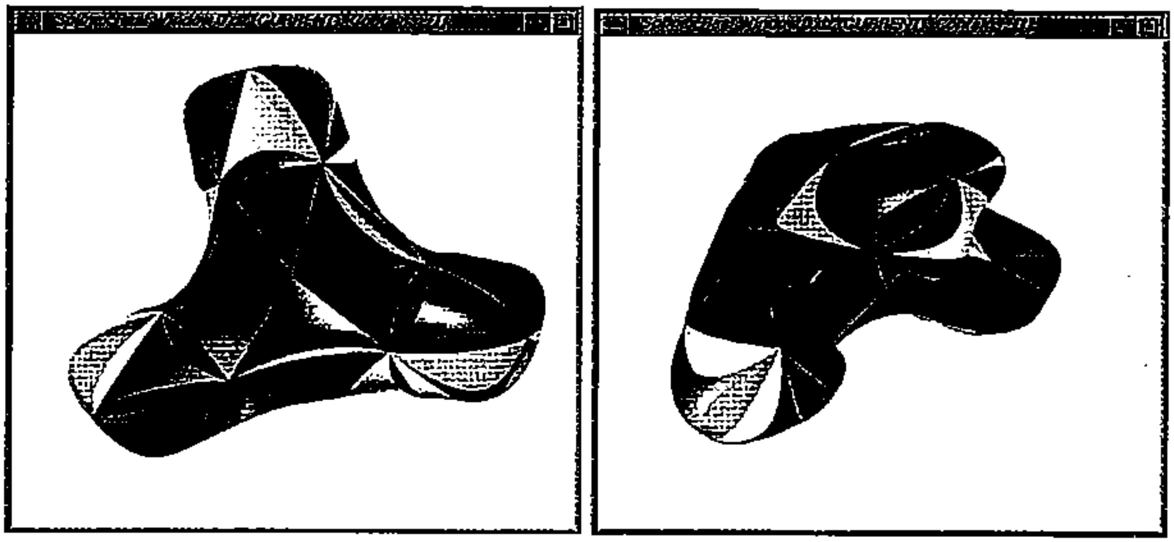

Figure 7.2: Different Smoothings of the Surface Triangulation using C^1 Cubic A-Patches

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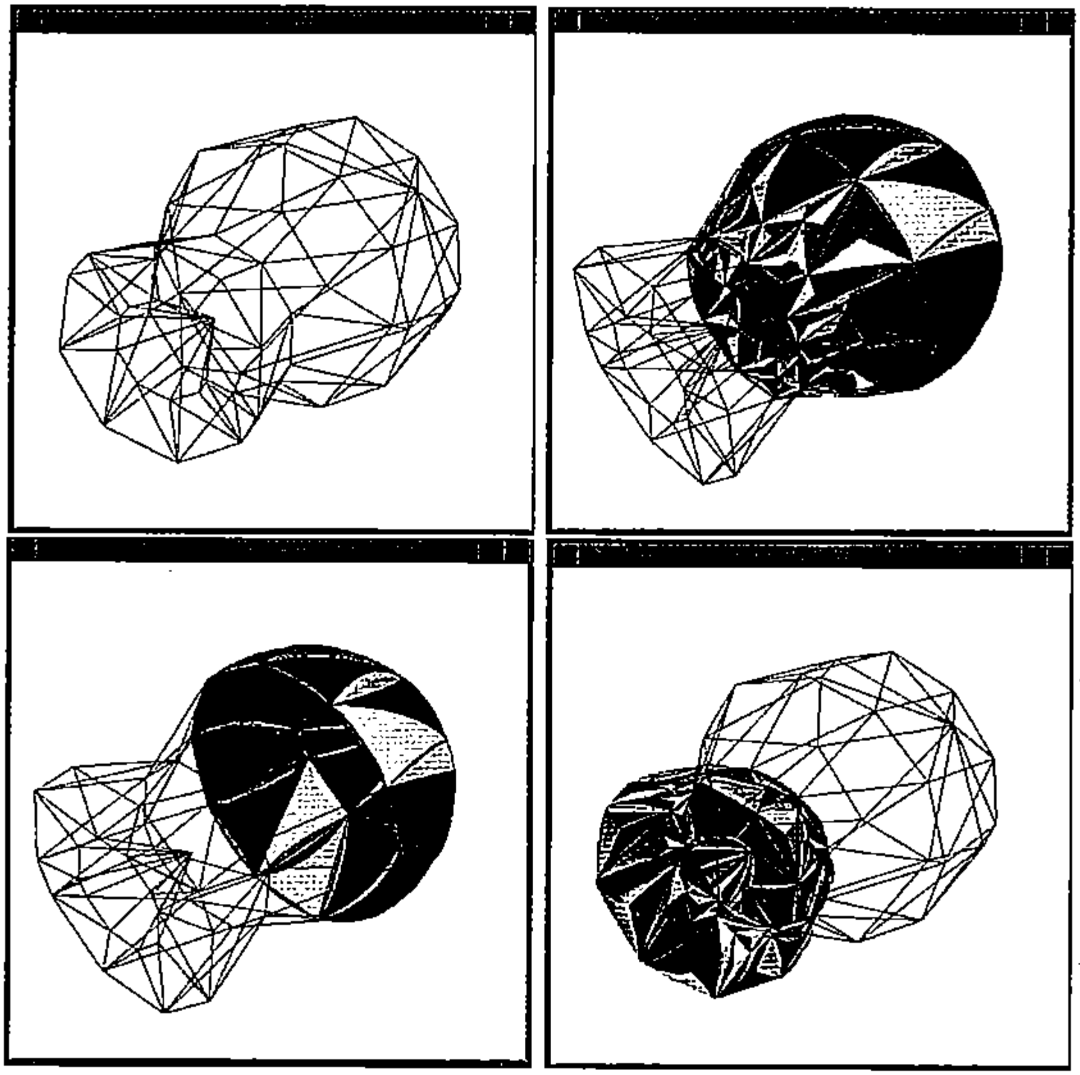


Figure 7.3: A Surface Triangulation and some of the interpolatory C^1 Cubic A-Patches

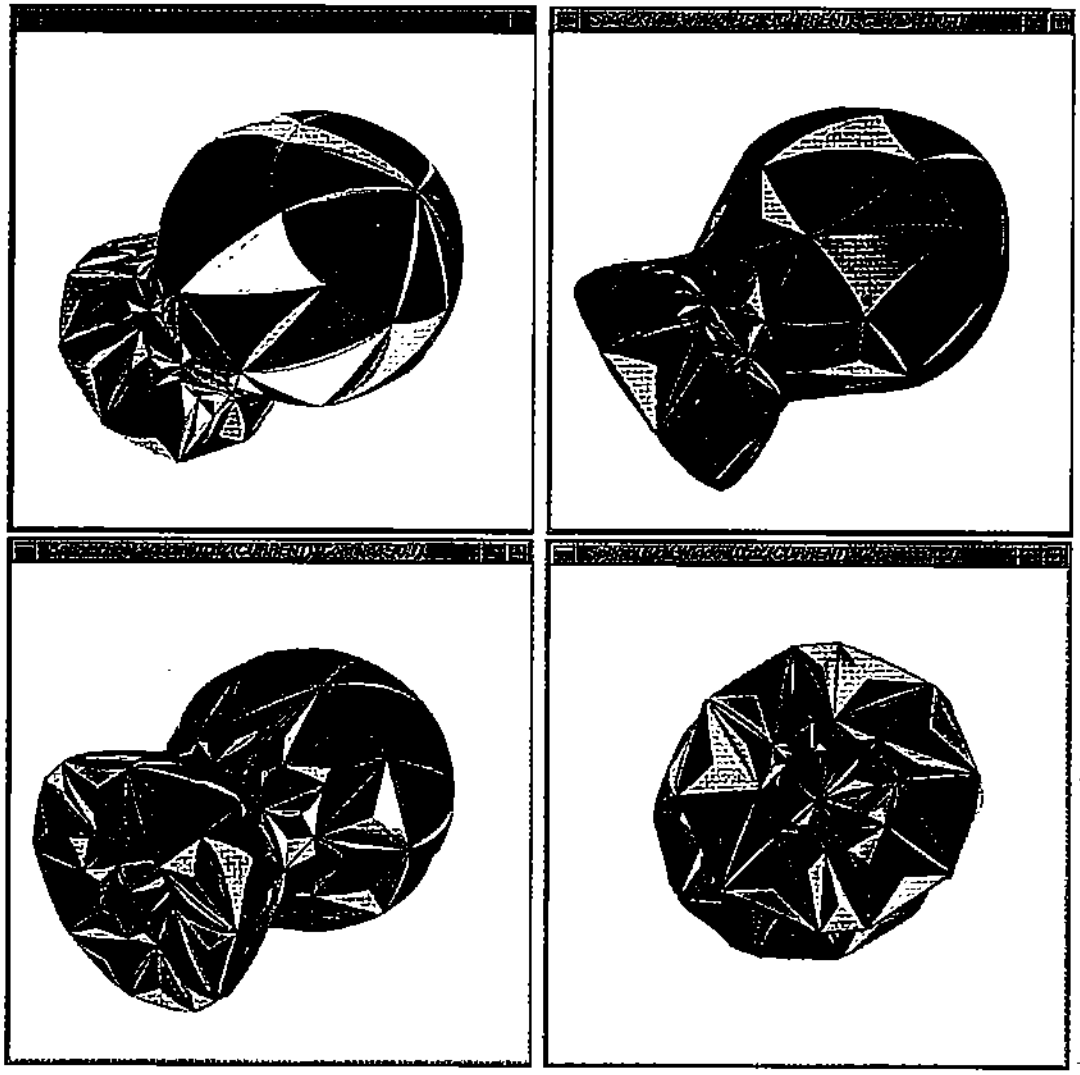
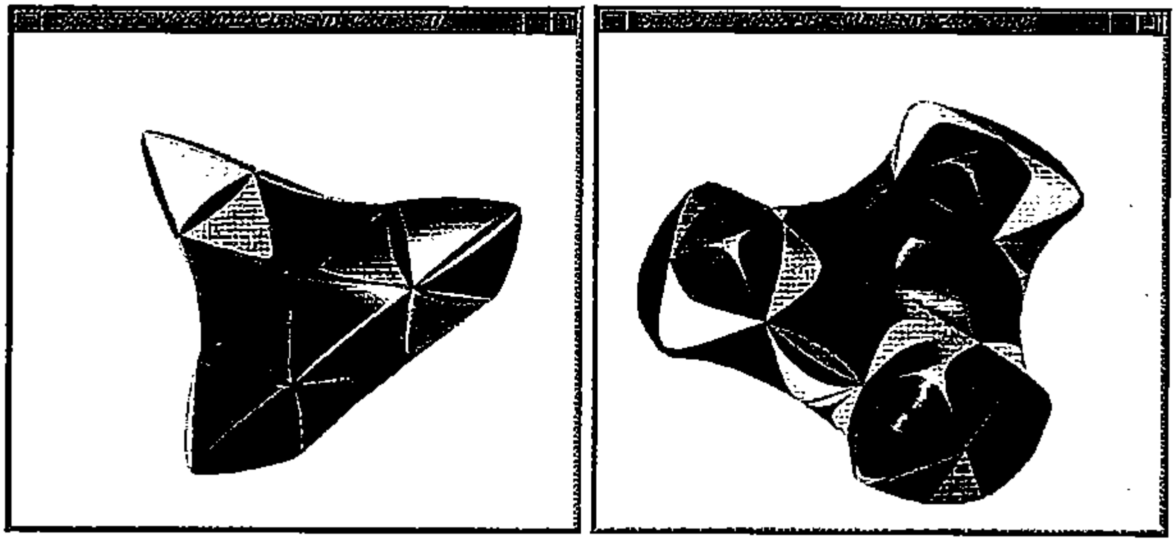


Figure 7.4: The Complete Smoothing of the Surface Triangulation using C^1 Cubic A-Patches



(a) (b)

Figure 7.5: Shape Modifications of the C^1 Mesh of Cubic A-patches

8 Appendix

The proof of Lemma 3.1. Let $g(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1, \alpha_2, \alpha_3, 1-\alpha_1-\alpha_2-\alpha_3)$. The smoothness of the surface patch S_F requires that $\nabla g(\alpha_1, \alpha_2, \alpha_3) \neq 0$ for every $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ on S_F. We prove only the smoothness of the three-sided j-patch. The proof of smoothness of the four-sided patch is similar.

Suppose the three-sided j-patch is not smooth. There will then be a point $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)^T \in S_F$ in the interior of S such that $\nabla g = 0$. Since $\frac{\partial g}{\partial \alpha_i} = \frac{\partial F}{\partial \alpha_i} - \frac{\partial F}{\partial \alpha_4}$, $i = 1,2,3$, we have $\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \ldots = \frac{\partial F}{\partial \alpha_4}$. Using Euler's formula[16] for homogeneous polynomials $\sum_{i=1}^4 \alpha_i \frac{\partial F}{\partial \alpha_i} = 4F$ and $\sum_{i=1}^{4} \alpha_i = 1$, we have $\frac{\partial F}{\partial \alpha_i} = 0$, $i = 1, \ldots, 4$. Let $p_1 \in S_j$ and $t = t^* \in (0,1)$ such that $\alpha^* = t^* e_j + (1-t^*)p_1 = \alpha(t^*)$. That is $F(\alpha(t^*)) = 0$. And further $\frac{\partial F(\alpha(t))}{\partial t}\Big|_{t=t^*} = \sum_{i=1}^4 \frac{\partial F}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} = 0$. This implies that t^* is a double zero of $F(\alpha(t))$, a contradiction to the definition of the three-sided patch. ◇

The proof of Theorem 3.2. For the sake of simplicity, we assume $j = 4$. Let $p = (y_1, y_2, y_3, 0)^T \in S_4$ (i.e., $y_i > 0, \sum_{i=1}^3 y_i = 1$),

$$\alpha(t) \;=\; te_4 + (1-t)p = ((1-t)y_1, (1-t)y_2, (1-t)y_3, t)^T$$

for $t \in (0,1)$. Then

$$\begin{aligned} F(\alpha(t)) &= \sum_{|\lambda|=n} \frac{b_\lambda n!}{\lambda!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} (1-t)^{\lambda_1+\lambda_2+\lambda_3} t^{\lambda_4} \\ &= \sum_{|\lambda|=n} \frac{b_\lambda (\lambda_1+\lambda_2+\lambda_3)!}{\lambda_1!\lambda_2!\lambda_3!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} \frac{n!}{(\lambda_1+\lambda_2+\lambda_3)!\lambda_4!} (1-t)^{\lambda_1+\lambda_2+\lambda_3} t^{\lambda_4} \\ &= \sum_{\ell=0}^{n} \left(\sum_{\substack{|\lambda|=n \\ \lambda_4=\ell}} b_\lambda B^{n-\ell}_{\lambda_1\lambda_2\lambda_3}(y_1,y_2,y_3) \right) B^n_\ell(t) \\ &= \sum_{\ell=0}^{n} B_\ell(y_1,y_2,y_3) B^n_\ell(t). \end{aligned} \tag{8.1}$$

By (3.1) and (3.2), $B_0 > 0$ if $k > 0$, $B_\ell \geq 0$, for $\ell = 1, \ldots, k-1$, $B_\ell \leq 0$, for $\ell = k+1, \ldots, n$. If $B_n = \ldots = B_{n-m+1} = 0$; $B_{n-m} < 0$ for some m with $0 \leq m \leq n-k-1$, then $F(\alpha(t))$ can be written as

$$F(\alpha(t)) = (1-t)^m \sum_{\ell=0}^{n-m} C_\ell(y_1,y_2,y_3) B^{n-m}_\ell(t) \tag{8.2}$$

where $C_0 > 0$ if $k > 0$, $C_{n-m} < 0$, and the sequence $C_0, C_1 \ldots C_{n-m}$ has at most one sign change. By the variation diminishing property of the functional BB form, the equation $F(\alpha(t))$ has at most one root in (0,1). Finally, we need to show the surface at the boundary of the tetrahedron is smooth. In the proof above, if we allow the intersection to occur at the boundary, then there may be an intersection of higher multiplicity at $t = 0$ or $t = 1$.That is, the surface contains vertices or edges of the tetrahedron. Here the smooth vertex and smooth edge conditions in the theorem guarantee that the surface is also smooth on the boundary of S. ◇

The proof of Theorem 3.3. Without loss of generality, we assume $(i,j,k,\ell) = (1,2,3,4)$. Then the edge $[e_1e_2]$ and $[e_3e_4]$ can be expressed as

$$[e_1e_2] = \{p : p = ue_1 + (1-u)e_2,\ u \in [0,1]\}$$

$$[e_3e_4] = \{p : p = ve_3 + (1-v)e_4,\ v \in [0,1]\}$$

and the line segment passing through the two edges is

$$\alpha(t) = t[e_1e_2] + (1-t)[e_3e_4] = (ut, (1-u)t, v(1-t), (1-v)(1-t))^T$$

for $t \in (0,1)$. Hence

$$\begin{aligned} F(\alpha(t)) &= \sum_{|\lambda|=n} \frac{b_\lambda n!}{\lambda!} u^{\lambda_1}(1-u)^{\lambda_2} v^{\lambda_3} (1-v)^{\lambda_4} t^{\lambda_1+\lambda_2} (1-t)^{\lambda_3+\lambda_4} \\ &= \sum_{\ell=0}^{n} \left(\sum_{\substack{|\lambda|=n \\ \lambda_1+\lambda_2=\ell}} \frac{b_\lambda (\lambda_1+\lambda_2)!(\lambda_3+\lambda_4)!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!} u^{\lambda_1}(1-u)^{\lambda_2} v^{\lambda_3}(1-v)^{\lambda_4} \right) \frac{n!}{\ell!(n-\ell)!} t^\ell (1-t)^{n-\ell} \\ &= \sum_{\ell=0}^{n} B_\ell(u,v) B^n_\ell(t). \end{aligned}$$

It follows from (3.3) and (3.4) that $F(\alpha(t))$ has at most one zero in (0,1). Again, the smooth vertex and smooth edge conditions in the theorem guarantee that the surface is smooth on the boundary of S. ◇

The proofs of the properties of A-patches. Property **(a)** can be verified by re-considering the proof of Theorem 3.2. For example, if $m = 1$, $j = 4$, $(y_1, y_2, y_3) = (1, 0, 0)$ at e_m. Hence

$$B_\ell(y_1, y_2, y_3) = b_{(n-\ell)e_1+\ell e_4} = 0, \qquad \ell = 0, 1, \ldots, k.$$

Therefore $t = 0$ is the root of $F(\alpha(t))$ with multiplicity $k + 1$. On the other hand, e_m is not a singular point of S_F, since $b_\lambda \neq 0$ for $\lambda = (n-1)e_m + e_s$. $\diamond$

We illustrate Property **(b)** by showing that any line passing through edge $[e_i e_j]$ and vertex e_k is tangent to S_F with multiplicity s. In fact, if we take $v = 1$ in the proof of Theorem 3.3, we have

$$B_\ell(u, v) = B_\ell(u, 1) = b_{\lambda_i e_i + \lambda_j e_j + (n-\ell)e_k} = 0.$$

Hence $t = 0$ is a root of $F(\alpha(t))$ with multiplicity $s + 1$. Again, e_k is not a singular point. $\diamond$

The proof of Property **(c)** is similar to **(a)**.

The proof of Theorem 5.1. (i). The system (5.11)-(5.13) can be written as $XA = -[a^i_{0210}\, a^i_{0120}]\, B$, where

$$A = \begin{bmatrix} \alpha_1^1 & 0 & \beta_1^1 & 0 & \gamma_1^1 & 0 \\ 0 & \alpha_1^2 & 0 & \beta_1^2 & 0 & \gamma_1^2 \\ \alpha_4^1 & 0 & \beta_4^1 - \mu_1 & -\mu_1 & 0 & 0 \\ 0 & \alpha_4^2 & -\mu_2 & \beta_4^2 - \mu_2 & 0 & 0 \\ -1 & 0 & 0 & 0 & \gamma_4^1 - \eta_1 & -\eta_1 \\ 0 & -1 & 0 & 0 & -\eta_2 & \gamma_4^2 - \eta_2 \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha_2^1 & \alpha_2^2 & \beta_2^1 - \mu_3 & \beta_2^2 - \mu_3 & \gamma_2^1 - \eta_3 & \gamma_2^2 - \eta_3 \\ \alpha_3^1 & \alpha_3^2 & \beta_3^1 - \mu_4 & \beta_3^2 - \mu_4 & \gamma_3^1 - \eta_4 & \gamma_3^2 - \eta_4 \end{bmatrix}$$

It follows from (5.3),(5.5), (5.9) and (5.10) that

$$\begin{bmatrix} p_1 & p_1' & p_4 & p_4' & q_4 & q_4' & p_2 & p_3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Hence the rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is at most four, that is, the matrix A is singular. Since $\beta_1^1 \neq 0, \beta_1^2 \neq 0$, the first two rows and the last two rows of A are independent. That is, matrix A has rank four. Hence the system (5.11)-(5.13) has four independent equations. Now we show that if (p_1, p_1', p_2, p_3) is affine independent, then the sub-matrices A_1 and A_2 are nonsingular, where

$$A_1 = \begin{bmatrix} \beta_4^1 - \mu_1 & -\mu_1 \\ -\mu_2 & \beta_4^2 - \mu_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \gamma_4^1 - \eta_1 & -\eta_1 \\ -\eta_2 & \gamma_4^2 - \eta_2 \end{bmatrix}$$

are the sub-matrices of A and they are the coefficient matrices of the equations (5.12) and (5.13) respectively. This implies that (5.12) and (5.13) are four independent equations for unknowns a^i_{0111} and c^i_{0111}, $i = 1, 2$. In fact, the affine independency of (p_1, p_1', p_2, p_3) is the necessary and sufficient condition for the nonsingularity of A_1 and A_2. It follows from (5.3),(5.5) that

$$\begin{bmatrix} p_4 & p_4' \\ 1 & 1 \end{bmatrix} A_1 = -\begin{bmatrix} p_1 & p_1' & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} B_1, \quad B_1 = \begin{bmatrix} \beta_1^1 & 0 \\ 0 & \beta_1^2 \\ \beta_2^1 - \mu_3 & \beta_2^2 - \mu_3 \\ \beta_3^1 - \mu_4 & \beta_3^2 - \mu_4 \end{bmatrix} \tag{8.3}$$

Since $p_4 \neq p'_4$ and $\beta_1^1 \neq 0, \beta_1^2 \neq 0$, matrix A_1 is of full rank if the matrix $\begin{bmatrix} p_1 & p'_1 & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is nonsingular. On the other hand, if this matrix is singular, that is p_1, p'_1, p_2, p_3 are coplanar, then the matrix A_1 is also singular. Otherwise, p_4, p'_4 will lie on the the plane $\langle p_1 p'_1 p_2 p_3 \rangle$ by (8.3), which yields a contradiction. Similarly, A_2 is nonsingular iff $\begin{bmatrix} p_1 & p'_1 & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is nonsingular.

(ii). (a). If p_1, p'_1, p_2, p_3 are affine independent, then by (8.3) we know that $a^i_{0111}, i = 1, 2$ can be expressed as an affine combination of a^i_{1110} and a^1_{0210}, a^1_{0120}. By (5.11), $c^i_{0111}, i = 1, 2$ can also be expressed as an affine combination of a^i_{1110} and a^1_{0210}, a^1_{0120}.

(b). If we take p_4, p'_4, p_2, p_3 or q_4, q'_4, p_2, p_3 to be the affine independent set, then the equations (5.11)-(5.13) are already in the form (5.17).

(c). Any other cases can be derived from one of the above cases.

◇

The proof of Proposition 5.2. Since the 0-th layer control points are non-negative, and the second and third layers control points can be set negative, the defined surface is then a three-sided 4-patch (see Theorem 3.2). ◇

The proof of Proposition 5.3. First, the 0-th layer control points are non-negative. Now we show that the second and third layer control points can be set negative. Since p'' is above the planes $\langle p_1 p_2 p_3 \rangle$ and $\langle p'_1 p_2 p_3 \rangle$ (i.e., it is at the same side as p_4 of the planes), then $\beta^i_4 > 0$. Hence from (5.4) and (5.5), b_{1101} and b_{1011} can be set negative if a_{0102} and a_{0012} are chosen small enough. Similarly, by (5.3), $b_{1002} < 0$ if a_{0003} is chosen small enough. Also b_{2001} can be set negative since it is free. Now it follows from (5.8)–(5.10) and $\mu_1 > 0$, $\mu_2 > 0$ that b_{2100}, b_{2010} and b_{3000} can be set negative. Therefore the surface defined in this way is a four-sided 14-23-patch over W_1 if $[p_2 p_3]$ is positive convex. If $[p_2 p_3]$ is zero convex, that is $a^i_{0210} = a^i_{0120} = 0$, then by (5.12) we can make $a^i_{0111} < 0$ and $b^i_{1110} < 0$ by choosing the free parameter $a^i_{1110}, i = 1, 2$. Hence the 1st layer control points are non-positive. Hence here the patch over W_1 degenerates to the edge $[p_2 p_3]$ and the smooth edges condition is satisfied. However, if the parameter a^i_{1110} are over determined, then a subdivision as in the coplanar case is needed. ◇

The proof of Proposition 5.4. (i). $[p'_1 p_2 p_3]$ is non-convex face(see Figure 5.1). We show that all the 1st layer's control points over V_i and $W_i, i = 1, 2$ can be set non-positive, and the -1st layer's control points over $V'_i, i = 1, 2$ and $W'_i, i = 1, 2$ can be set non-negative. If p_1, p'_1, p_2, p_3 are affine independent, then we use the equalities (5.15) and (5.16). Since both p_4 and p'_4 are at the same side of the surface triangulation T, $\theta^i_1 \theta^i_4 > 0$ for $i = 1, 2$. Assume, without loss of generality, that $\theta^1_1 > 0, \theta^1_4 > 0$ and $\theta^2_1 > 0, \theta^2_4 > 0$ and then $\vartheta^1_1 < 0, \vartheta^1_4 < 0$ and $\vartheta^2_1 < 0, \vartheta^2_4 < 0$. Then by (5.15) and (5.16), we can take a^i_{1110} small enough such that $a^i_{0111} < 0$ and $c^i_{0111} > 0$, and furthermore, their absolute value can be larger than any specified value. Since the 1st and -1st layer's control points that are determined by the normals are non-positive and non-negative, respectively, all the the 1st and -1st layer's control points can be set non-positive and non-negative, respectively. Therefore, the surfaces over V_i and V'_i are three-sided 4-patches, and the surfaces over W_i and W'_i are four-sided 14-23-patches(see Theorem 3.2 and 3.3). If p_1, p'_1, p_2, p_3 are coplanar(not affine independent), then by Theorem 5.1, all the unknowns can be expressed linearly by $a^i_{0111}, i = 1, 2$(or $c^i_{0111}, i = 1, 2$). It is easy to see that, we can take $a^i_{0111} < 0$(or $c^i_{0111} > 0$) small(or big) enough so that $c^i_{0111} > 0$(or $a^i_{0111} < 0$).

(ii). If $[p_1' p_2 p_3]$ is convex, then the edge $[p_2 p_3]$ is convex also. Then Proposition 5.2 and 5.3 can be used for this face and edge. As for the face $[p_1 p_2 p_3]$, the discussion above can be used.

Finally, we point out why the splitting is necessary. Consider the face $[p_1 p_2 p_3]$ as an example(see Figure 5.1). In order to have $a^1_{0111}, a^1_{1101}, a^1_{1011}$ less than zero, a^1_{1110} has to be determined three times by the three C^1 constraints if no splitting is performed. Therefore, in general a solution is impossible without splitting. Also note that, if the three inner products between the face normal and its neighbor's face normals have the same sign(positive or negative), then a^1_{1110} can be determined so that $a^1_{0111}, a^1_{1101}, a^1_{1011}$ are less than zero. Hence here we do not need to split the face. $\diamond$