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Abstract

Suppose that A is an $n \times n$ nonsingular irreducible p-cyclic H-matrix and let J^A, $\mathcal{L}_\omega^A$ and S_ω^A denote, respectively, the Jacobi, the SOR and the SSOR iteration matrices associated with A. In an earlier paper [14] the second of the present authors showed that if the spectral radius $\rho(|J^A|) \in (0, r_3)$, where $r_3 \approx 0.418192802$ is the unique real root of the cubic $17r^3 + r^2 - r - 1$ in the interval $(0,1)$, then there exists a neighborhood $\Omega_{\omega(A)}$ of $\omega(A) := 2/(1+\rho(|J^A|)$ such that $\rho(S_\omega^A) < |\omega - 1| \leq \rho(\mathcal{L}_\omega^A)$, $\forall\ \omega \in \Omega_{\omega(A)}$. Our recent work on domains of convergence of the p-cyclic SSOR method ([5], [6], and [7]) provided us with the necessary tools required to generalize and solve the problem treated in [14] for any $p \geq 2$. The complete solution to this problem is obtained as a by-product of the analysis to determine the region in the $(\rho(|J^A|), \omega)$-plane in which the spectral radius of a certain majorizer of a matrix similar to the SSOR iteration matrix is strictly less than $|\omega - 1|$.

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1 Introduction

Let A be an $n \times n$ complex nonsingular irreducible block p-cyclic H-matrix which, without loss of generality, may be assumed to have the following block form

$$A := I - L - U =: \begin{bmatrix} I_1 & -B_1 & & & & \\ & I_2 & -B_2 & & & \\ & & I_3 & -B_3 & & \\ & & & \ddots & & \\ & & & & I_{p-1} & -B_{p-1} \\ -B_p & & & & & I_p \end{bmatrix}. \tag{1.1}$$

In (1.1) L and U are the strictly lower and strictly upper triangular parts of A, respectively, and I_j, $j = 1(1)p$, is the unit matrix of order n_j with $\sum_{j=1}^{n} n_j = n$. The block Jacobi iteration matrix associated with A is then

$$J^A := \begin{bmatrix} O_1 & B_1 & & & & \\ & O_2 & B_2 & & & \\ & & O_3 & B_3 & & \\ & & & \ddots & & \\ & & & & O_{p-1} & B_{p-1} \\ B_p & & & & & O_p \end{bmatrix}, \tag{1.2}$$

where O_j is the square nullmatrix of order n_j, $j = 1(1)p$. As is known, the SOR and the SSOR iteration matrices associated with A in (1.1), denoted by $\mathcal{L}_\omega^A$ and $\mathcal{S}_\omega^A$, respectively, are given by the following matrix expressions

$$\mathcal{L}_\omega^A := (I - \omega L)[(1-\omega)I + \omega U] \tag{1.3}$$

and

$$\mathcal{S}_\omega^A := (I - \omega U)\tilde{\mathcal{S}}_\omega^A (I - \omega U)^{-1}. \tag{1.4}$$

In (1.4)

$$\tilde{\mathcal{S}}_\omega^A := M(L)M(U), \tag{1.5}$$

where $M(C)$ denotes the matrix product

$$M(C) := (I - \omega C)^{-1}[(1-\omega)I + \omega C]. \tag{1.6}$$

Two results on the spectral radii of $\mathcal{L}_\omega^A$ and $\mathcal{S}_\omega^A$ are well-known. Specifically,

$$\left.\begin{matrix} \rho(\mathcal{L}_\omega^A) \\ \rho(\mathcal{S}_\omega^A) \end{matrix}\right\} \leq |1-\omega| + \omega\rho(|J^A|), \qquad \forall\, \omega \in (0, \omega(A)], \tag{1.7}$$

with

$$\left.\begin{array}{l}\rho(\mathcal{L}_\omega^A)\\ \rho(S_\omega^A)\end{array}\right\} < 1, \qquad \forall\, \omega \in (0, \omega(A)), \tag{1.8}$$

where

$$\omega(A) := \frac{2}{1+\rho(|J^A|)}. \tag{1.9}$$

The result on $\rho(\mathcal{L}_\omega^A)$ is due to Kahan for M-matrices [9] and to Kulisch for H-matrices [10] while the one on $\rho(S_\omega^A)$ is due to Alefeld and Varga [1]. Based on previous works on the SOR and SSOR iteration matrices for the class of H-matrices A (see, e.g., [13], [12], [17], [18]) Neumaier and Varga [11] gave the exact domain of convergence, in the $(\rho(|J^A|), \omega)$-plane, for the SSOR iterative method for the class of nonsingular H-matrices. An open question regarding convergence on one of the boundaries of the region in [11] was settled later in [4].

The discovery of the functional equation, that connects the eigenvalue spectra $\sigma(S_\omega^A)$ and $\sigma(J^A)$ for p-cyclic matrices A, by Varga, Niethammer and Cai [18], extended later to cover the class of generalized p-cyclic consistently ordered matrices by Chong and Cai [3], allowed the present authors to determine exact regions of convergence, in the $(\rho(J^A), \omega)$-plane, for the p-cyclic and the generalized p-cyclic SSOR iteration method ([5], [6]). Also, in a work that was developed later than this present one [7], the same authors determined exact regions of convergence, in the $(\rho(|J^A|), \omega)$-plane, for the p-cyclic SSOR so that $\rho(S_\omega^A) < |\omega - 1|$. Obviously, this inequality implies dominant convergence of the SSOR method over the SOR one for p-cyclic matrices since from Kahan's work it is $|\omega - 1| \leq \rho(\mathcal{L}_\omega^A)$.

In this paper we adopt the classical terminology and notation which is most common and can be found in the excellent textbooks by Varga [16], Young [19] and Berman and Plemmons [2]. The manuscript is organized as follows. In Section 2 we derive the analog of the Varga-Niethammer-Cai functional equation that connects the eigenvalue spectra of the absolute value of the Jacobi iteration matrix $|J^A|$ and of a certain majorizer Q_ω^A of the $\tilde{S}_\omega^A$ matrix, similar to the SSOR iteration matrix S_ω^A. In this way, equation (2.27) (or (2.33)) of [14] is generalized. In Section 3 we determine the regions in the $(\rho(|J^A|), \omega)$-plane such that $\rho(Q_\omega^A) < |\omega - 1|$. Finally, in Section 4 we determine the neighborhoods of dominant convergence $\Omega_{\omega(A)}$ such that the following series of relationships holds

$$\rho(S_{\omega(A)}^A) = \rho(\tilde{S}_{\omega(A)}^A) \leq \rho(|\tilde{S}_{\omega(A)}^A|) \leq \rho(Q_{\omega(A)}^A) < \omega(A) - 1 \leq \rho(\mathcal{L}_{\omega(A)}^A) = 1. \tag{1.10}$$

2 Derivation of the Varga-Niethammer-Cai's (VNC) Functional Equation Analog

We begin with the derivation of VNC equation as in [18] since its analog is to differ from it at a number of certain points only. First, from (1.1) and (1.6) it is obtained that

$$M(L) = \begin{bmatrix} (1-\omega)I_1 & & & & & \\ & (1-\omega)I_2 & & & \mathbf{0} & \\ & & (1-\omega)I_3 & & & \\ & \mathbf{0} & & \ddots & & \\ & & & & (1-\omega)I_{p-1} & \\ \omega(2-\omega)B_p & & & & & (1-\omega)I_p \end{bmatrix}, \tag{2.1}$$

and

$$M(U) = \begin{bmatrix} (1-\omega)I_1 & \omega(2-\omega)B_1 & \omega^2(2-\omega)B_1B_2 & \dots & \omega^{p-1}(2-\omega)B_1B_2\dots B_{p-1} \\ & (1-\omega)I_2 & \omega(2-\omega)B_2 & \dots & \omega^{p-2}(2-\omega)B_2\dots B_{p-1} \\ & & (1-\omega)I_3 & \dots & \omega^{p-3}(2-\omega)B_3\dots B_{p-1} \\ & \mathbf{0} & \ddots & & \\ & & & (1-\omega)I_{p-1} & \omega(2-\omega)B_{p-1} \\ & & & & (1-\omega)I_p \end{bmatrix}. \tag{2.2}$$

Using (2.1) and (2.2) in (1.5) yields

$$\tilde{S}_\omega^A = (1-\omega)^2 I + \begin{bmatrix} O_1 & \omega\sigma_1 B_1 & \omega^2\sigma B_1B_2 & \omega^3\sigma B_1B_2B_3 & \dots & \omega^{p-1}\sigma B_1B_2\dots B_{p-1} \\ & O_2 & \omega\sigma B_2 & \omega^2\sigma B_2B_3 & \dots & \omega^{p-2}\sigma B_2B_3\dots B_{p-1} \\ & & O_3 & \omega\sigma B_3 & \dots & \omega^{p-3}B_3B_4\dots B_{p-1} \\ & & & \ddots & & \\ & & & & O_{p-1} & \omega\sigma B_{p-1} \\ \omega\sigma B_p & \omega^2\mu^2 B_pB_1 & \omega^3\mu^2 B_pB_1B_2 & \dots & & \omega^p\mu^2 B_pB_1\dots B_{p-1} \end{bmatrix}, \tag{2.3}$$

where

$$\sigma = (1-\omega)(2-\omega), \quad \mu = 2-\omega. \tag{2.4}$$

Let $\lambda \in \sigma(\tilde{S}_\omega^A)$, with $\lambda \neq (1-\omega)^2$, and $\tau = \lambda - (1-\omega)^2 \in \sigma(T_\omega)$, where

$$T_\omega = \tilde{S}_\omega^A - (1-\omega)^2 I. \tag{2.5}$$

Let also $X = [X_1^T,\ X_2^T, \dots, X_{p-1}^T,\ X_p^T]^T \in \mathbb{C}^n$ be the eigenvector of T_ω corresponding to τ with $X_j \in \mathbb{C}^{n_j}$, $j = 1(1)p$. There will be

$$T_\omega X = \tau X. \tag{2.6}$$

So, using (2.4) and (2.5) in (2.6) it can be obtained

$$
\begin{aligned}
\sigma\omega B_1X_2+\sigma\omega^2B_1B_2X_3+\sigma\omega^3B_1B_2B_3X_4+\ldots+\sigma\omega^{p-1}B_1B_2\ldots B_{p-1}X_p &= \tau X_1\\
\sigma\omega B_2X_3+\sigma\omega^2B_2B_3X_4+\ldots+\sigma\omega^{p-2}B_2B_3\ldots B_{p-1}X_p &= \tau X_2\\
\sigma\omega B_3X_4+\ldots+\sigma\omega^{p-3}B_3B_4\ldots B_{p-1}X_p &= \tau X_3\\
\ddots \qquad\qquad\qquad & \\
\sigma\omega B_{p-2}X_{p-1}+\sigma\omega^2B_{p-2}B_{p-1}X_p &= \tau X_{p-2}\\
\sigma\omega B_{p-1}X_p &= \tau X_{p-1}\\
\omega\sigma B_pX_1+\omega^2\mu^2B_pB_1X_2+\omega^3\mu^2B_pB_1B_2X_3+\ldots+\omega^p\mu^2B_pB_1B_2\ldots B_{p-1}X_p &= \tau X_p
\end{aligned}
\tag{2.7}
$$

Solving now the $(p-1)^{\text{st}}$ equation of (2.7) for X_{p-1} and using the result obtained in the $(p-2)^{\text{nd}}$ one; then solving for X_{p-2} and so on up to the first equation which is solved for X_1, and finally substituting into the last equation we have that

$$
\begin{aligned}
\tau X_p &= \big[\omega\sigma\tfrac{\sigma}{\tau}\omega^{p-1}(\tfrac{\sigma}{\tau}+1)^{p-2}+\omega^2\mu^2\tfrac{\sigma}{\tau}\omega^{p-2}(\tfrac{\sigma}{\tau}+1)^{p-3}+\ldots\\
&+\omega^{p-2}\mu^2\tfrac{\sigma}{\tau}\omega^2(\tfrac{\sigma}{\tau}+1)+\omega^{p-1}\mu^2\tfrac{\sigma}{\tau}\omega+\omega^p\mu^2\big]\,B_pB_1\ldots B_{p-1}X_p\\
&= \tfrac{1}{\tau^{p-1}}(\mu^2\tau+\sigma^2)(\tau+\sigma)^{p-2}\omega^pB_pB_1\ldots B_{p-1}X_p.
\end{aligned}
$$

From this we obtain

$$
(\mu^2\tau+\sigma^2)(\tau+\sigma)^{p-2}\omega^pB_pB_1\ldots B_{p-1}X_p=\tau^pX_p. \tag{2.8}
$$

Since $\tau\neq 0$ it follows that $X_p\neq 0$ (for if $X_p=0$ then from (2.7) $X=0$) is an eigenvector of $B_pB_1\ldots B_{p-1}$ with corresponding eigenvalue $\tau^p/\left((\mu^2\tau+\sigma^2)(\tau+\sigma)^{p-2}\omega^p\right)$. On the other hand, from (1.2) it is

$$
(J^A)^p=\begin{bmatrix}
B_1B_2\ldots B_{p-1}B_p & & & \\
& B_2B_3\ldots B_pB_1 & & \\
& & \ddots & \\
& & & B_pB_1\ldots B_{p-2}B_{p-1}
\end{bmatrix}. \tag{2.9}
$$

Thus if $0\neq b\in\sigma(J^A)$ there exists $Y_p\in\mathbb{C}^{n_p}\setminus\{0\}$ such that

$$
B_pB_1\ldots B_{p-2}B_{p-1}Y_p=b^pY_p. \tag{2.10}
$$

Therefore from (2.8) and (2.10) it is implied that

$$\tau^p = (\mu^2\tau + \sigma^2)(\tau + \sigma)^{p-2}\omega^p b^p, \tag{2.11}$$

which is nothing but VNC functional equation [18].

To derive the analog of (2.11) we are interested in we note that from (2.3) and (2.5) we have

$$|\tilde{S}_\omega^A| = (1-\omega)^2 I + |T_\omega| \leq (1-\omega)^2 I + R_\omega^A =: Q_\omega^A \tag{2.12}$$

where

$$R_\omega^A := \begin{bmatrix} O_1 & \omega\delta C_1 & \omega^2\delta C_1 C_2 & \dots & \omega^{p-1}\delta C_1 C_2 \dots C_{p-1} \\ & O_2 & \omega\delta C_2 & \dots & \omega^{p-2}\delta C_2 C_3 \dots C_{p-1} \\ & & O_3 & \dots & \omega^{p-3}\delta C_3 C_4 \dots C_{p-1} \\ & & & \ddots & \\ & & & O_{p-1} & \omega\delta C_{p-1} \\ \omega\delta C_p & \omega^2\mu^2 C_p C_1 & \omega^3\mu^2 C_p C_1 C_2 & \dots & \omega^p\mu^2 C_p C_1 \dots C_{p-1} \end{bmatrix} \tag{2.13}$$

with

$$\omega \in (0,2),\ \delta = |\sigma| = |1-\omega|(2-\omega),\ C_j = |B_j|,\ j = 1(1)p. \tag{2.14}$$

An analysis similar to the previous one for the derivation of VNC functional equation (2.11) leads to the following conclusion.

Theorem 2.1. Let A be the block p-cyclic matrix in (1.1) and J^A in (1.2) be the Jacobi iteration matrix associated with A. Let also that for any $\omega \in (0,2)$, μ and δ be defined by (2.4) and (2.14), respectively. Then, if $\tau \in \sigma(R_\omega^A) \setminus \{0\}$, with R_ω^A being defined in (2.13) – (2.14), and c satisfies

$$\tau^p = (\mu^2\tau + \delta^2)(\tau + \delta)^{p-2}\omega^p c^p \tag{2.15}$$

then $c \in \sigma(|J^A|)$. Conversely, if $c \in \sigma(|J^A|)$ and if $\tau \neq 0$ satisfies (2.15) then $\tau \in \sigma(R_\omega^A)$.

Remarks: i) It is pointed out that the validity of the theorem does to require A to be irreducible or H-matrix. ii) (2.15) differs from VNC equation (2.11) in three points: a) It has $\delta = |1-\omega|(2-\omega)$ instead of $\sigma = (1-\omega)(2-\omega)$. b) It has $c \in \sigma(|J^A|)$ instead of $b \in \sigma(J^A)$, and c) It is $\tau \in \sigma(R_\omega^A)$ instead of $\tau \in \sigma(T_\omega)$, where $|T_\omega| \leq R_\omega^A$. iii) Based on the differences that are pointed out in (ii) previously one could note that (2.15) would be identically (2.11) if $\omega \in (0,1)$ and $|J^A| = J^A$. The latter holds, e.g., if A is an M-matrix.

3 Regions of Dominant Convergence

As has already been mentioned, in this section we shall try to determine regions in the $((\rho(|J^A|),\omega))$-plane, $\rho(|J^A|) \in [0,1)$ and $\omega \in (0,2)$, such that

$$\rho(Q_\omega^A) < |\omega - 1| \leq \rho(\mathcal{L}_\omega^A), \tag{3.1}$$

where Q_ω^A is the majorizer of $\tilde{S}_\omega^A$ (see (2.12)). It is reminded that $\tilde{S}_\omega^A$ is similar to the SSOR iteration matrix S_ω^A. For the leftmost inequality in (3.1) to hold use of (2.15) will be made and Rouché's theorem (see [15], [8]) will be applied. We note that this particular way of attacking our present problem led us later on to more relaxed assumptions and therefore to the stricter regions of dominant convergence derived in the corresponding section of [7]. We also point out that the regions regarding the majorizer Q_ω^A obtained in this section are derived here for the first time.

Using (2.12) the first inequality in (3.1) becomes

$$\rho(R_\omega^A) < |\omega - 1|(1 - |\omega - 1|). \tag{3.2}$$

Since $\rho(R_\omega^A) = \max|\tilde{\tau}|$, with $\tau \neq 0$ satisfying (2.15), we set

$$\tau = |\omega - 1|(1 - |\omega - 1|)\varphi \tag{3.3}$$

and require, equivalently, that

$$|\varphi| = \frac{|\tau|}{|\omega - 1|(1 - |\omega - 1|)} < 1. \tag{3.4}$$

Using the simple transformation

$$t := \omega - 1, \quad \omega \in (0,2) \setminus \{1\} \tag{3.5}$$

and substituting τ from (3.3) into (2.15) yields, after some simple manipulation,

$$f(\varphi) := t\varphi^p - [(1-t)\varphi + t](\varphi + 1)^{p-2}(1+t)^p c^p = 0. \tag{3.6}$$

Since the function

$$g(\varphi) := t\varphi^p, \quad t \in (-1,1) \setminus \{0\} \tag{3.7}$$

has all its zeros equal to $0 < 1$, then according to Rouché's theorem ([15], [8]), $f(\varphi)$ will have all its zeros strictly within the unit circle, i.e., $|\varphi| < 1$, if and only if

$$|f(\varphi) - g(\varphi)| < |g(\varphi)|, \quad \varphi \in \partial\Omega, \tag{3.8}$$

where $\partial\Omega$ denotes the unit circle. In view of (3.6) and (3.7), (3.8) is equivalent to

$$|(1-t)\varphi + t||\varphi + 1|^{p-2}(1+t)^p|c|^p < t|\varphi|^p, \quad \varphi \in \partial\Omega \tag{3.9}$$

where

$$\varphi = x + iy, \quad x, y \in \mathbb{R}, \quad x^2 + y^2 = 1. \tag{3.10}$$

Using (3.10) in (3.9) it can be obtained that

$$\{[(1-t)x+t]^2+[(1-t)y]^2\}^{1/2}\,[(x+1)^2+y^2]^{p/2-1}(1+t)^p|c|^p<|t|$$

or

$$\frac{(1+t)^p|c|^p}{|t|}<\frac{1}{\max_{-1\leq x\leq 1}\{[(1-t)^2+t^2+2t(1-t)x]^{1/2}(2+2x)^{p/2-1}\}}. \tag{3.11}$$

To find the maximum in the denominator of the right hand side of (3.11) we introduce the function

$$h(x,t):=[(1-t)^2+t^2+2t(1-t)x]^{1/2}(2+2x)^{p/2-1}, \tag{3.12}$$

where $x\in[-1,1]$ and $t\in(-1,1)\setminus\{0\}$. In the sequel we shall use the symbol "$\sim$" to denote equality in sign between two expressions. Thus by differentiating (3.12) with respect to x we obtain that

$$\frac{\partial h}{\partial x}\sim t\left\{x+\frac{2t(1-t)+(p-2)[(1-t)^2+t^2]}{2(p-1)(1-t)t}\right\}. \tag{3.13}$$

We distinguish two cases:

Case I: $t\in(-1,0)$. In this case the inequality

$$s(t,p):=-\frac{2t(1-t)+(p-2)[(1-t)^2+t^2]}{2(p-1)(1-t)t}>-1 \tag{3.14}$$

always holds, since it can be proved that it is equivalent to $(1-2t)^2>0$. On the other hand the inequality

$$s(t,p)<1 \tag{3.15}$$

is equivalent to

$$4t^2-4t-(p-2)>0.$$

The latter inequality holds if and only if $t<\frac{1-\sqrt{p-1}}{2}$, which in turn can hold if and only if $p<10$. From the analysis so far it follows that: i) For $p<10$. a) If $-1<t<\frac{1-\sqrt{p-1}}{2}$ then for $1\geq x\geq s(t,p)$ and in view of (3.13) it is $\frac{\partial h}{\partial x}\leq 0$, therefore $h(x,t)$ strictly decreases as a function of x, while for $-1\leq x\leq s(t,p)$, $\frac{\partial h}{\partial x}\geq 0$ and $h(x,t)$ strictly increases with x. In either case it is

$$\max_{-1\leq x\leq 1}h(x,t)=h(s(t,p),t). \tag{3.16}$$

b) If $\frac{1-\sqrt{p-1}}{2}\leq t<0$ then $s(t,p)\geq 1\geq x\geq -1$, $\frac{\partial h}{\partial x}\geq 0$, $h(x,t)$ strictly increases with x and

$$\max_{-1\leq x\leq 1}h(x,t)=h(1,t). \tag{3.17}$$

ii) For $p\geq 10$ it is always

$$s(t,p) \geq 1 \tag{3.18}$$

implying that $\frac{\partial h}{\partial x} \geq 0$ and again (3.17) holds.

Case II: $t \in (0,1)$. In this case it is always

$$s(t,p) \leq -1$$

and therefore (3.17) holds.

The analysis so far and especially the conclusions from the two cases examined effectively prove the following theorem.

Theorem 3.1: Let A be the nonsingular block p-cyclic consistently ordered matrix in (1.1) and J^A, $\mathcal{L}_\omega^A$, and S_ω^A be the block Jacobi, block SOR and block SSOR iteration matrices associated with A and defined in (1.2), (1.3) and (1.4), respectively. Let also, the relaxation factor satisfy $\omega \in (0,2)$ and let $\nu := \rho(|J^A|)$. Set

$$\omega_p := \frac{3-\sqrt{p-1}}{2}. \tag{3.19}$$

Then for all points (ν,ω) in the domain $R(p)$, where

$$R(p) := \begin{cases} \underline{\text{For } 3 \leq p \leq 9} & \begin{cases} 0 < \omega < \omega_p, & 0 \leq \nu < \frac{(p-1)^{1/2-1/2p}}{(p-2)^{1/2-1/p}} \cdot \frac{(1-\omega)^{1/2}(2-\omega)^{1/2-1/p}}{\omega(3-2\omega)^{1-1/p}} \\ \omega_p \leq \omega_{\neq 1} < 2, & 0 \leq \nu < \frac{|\omega-1|^{1/p}}{2^{1-2/p}\omega} \end{cases} \\ \underline{\text{For } p \geq 10} & \quad 0 < \omega_{\neq 1} < 2, \quad 0 \leq \nu < \frac{|\omega-1|^{1/p}}{2^{1-2/p}\omega} \end{cases} \tag{3.20}$$

the spectral radii $\rho(S_\omega^A)$, $\rho(Q_\omega^A)$ and $\rho(\mathcal{L}_\omega^A)$ satisfy the series of inequalities

$$\rho(S_\omega^A) = \rho(\tilde{S}_\omega^A) \leq \rho(|\tilde{S}_\omega^A|) \leq \rho(Q_\omega^A) < |\omega-1| \leq \rho(\mathcal{L}_\omega^A). \tag{3.21}$$

In (3.21), Q_ω^A is the majorizer of $\tilde{S}_\omega^A$ defined in (2.12) and $\tilde{S}_\omega^A$ is the similar S_ω^A matrix defined in (1.4).

Proof: In Case (IIa) examined previously which holds for $3 \leq p \leq 9$ and $0 < \omega < \omega_p$, equality (3.16) was deduced. If in the right hand side of (3.16), (3.12) is used, where $s(t,p)$ is replaced by its equivalent expression from (3.14), next in the resulting expression $\omega - 1$ is used for t and then the expression yielded is substituted into (3.11), one can solve for $|c|$, recalling that $c \in \sigma(|J^A|)$, to obtain

$$\nu = \rho(|J^A|) < \frac{(p-1)^{1/2-1/2p}}{(p-2)^{1/2-1/p}} \cdot \frac{(1-\omega)^{1/2}(2-\omega)^{1/2-1/p}}{\omega(3-2\omega)^{1-1/p}}. \tag{3.22}$$

In Case (IIb), where $\omega_p \leq \omega < 1$ we work in exactly the same way, except that (3.17) is used instead of (3.16), to finally obtain that

$$\nu = \rho(|J^A|) < \frac{|\omega - 1|^{1/p}}{2^{1-2/p}\omega}. \tag{3.23}$$

In all other remaining cases as was seen in the analysis that preceded the present theorem it turns out that (3.17) holds so, in turn, (3.23) will hold too. Obviously, for all $(\nu, \omega) \in R(p)$ (3.1) will hold which together with (2.12) and (1.4) prove the validity of (3.20).□

Remark: A direct consequence of (3.20) is that for any $(\nu, \omega) \in R(p)$ the SSOR method does converge and, what is more, converges faster than the SOR method, provided that the later method converges too.

We note that so far A has been assumed to be nonsingular and block p-cyclic consistently ordered only. In the next section A will be assumed to be an irreducible H-matrix as well.

4 Neighborhoods of Dominant Convergence

In this section we shall try to determine the largest intervals $(0, r_p) \subseteq (0,1)$ in r_p such that

$$\rho(\bar{S}^A_{\omega(A)}) < \omega(A) - 1 \tag{4.1}$$

for all nonsingular irreducible p-cyclic matrices with $\nu = \rho(|J^A|) \in (0, r_p)$ where, it is reminded that, $\omega(A)$ is given by (1.9). Although we may work on without the further assumptions of irreducibility and of the H-matrix character of A we shall make both these assumptions to be consistent with the generalization of $p = 3$, studied in [14], to any $p \geq 3$.

To begin our analysis we consider relationships (1.10). However, the three relationships on the left hold for any $\omega \in (0,2)$ as was already proved in (3.21). The second relationship from the right also holds for any $\omega \in (0,2)$ while the rightmost one is the one due to Kahan [9]. In fact only

$$\rho(Q^A_{\omega(A)}) < \omega(A) - 1 \tag{4.2}$$

should be considered which, in view of (3.21), will certainly imply the validity of (4.1). However, (4.2) is nothing but the corresponding inequality in (3.1) with $\omega = \omega(A) = \frac{2}{1+\nu}$. The analysis in Section 3, however, led us to (3.22) and (3.23). Since $\omega(A) \in (1,2)$ we have to consider only (3.23), namely

$$\nu < \frac{(\omega(A) - 1)^{1/p}}{2^{1-2/p}\omega(A)}. \tag{4.3}$$

Using the expression for $\omega(A)$ in terms of ν we equivalently have

$$f(\nu) := 4^{p-1}\nu^p - (1-\nu)(1+\nu)^{p-1} < 0. \tag{4.4}$$

From the analysis in [14] and bearing in mind (4.2) we have that the Perron root $\rho(R^A_{\omega(A)})$ of $R^A_{\omega(A)}$ given in (2.13) satisfies

$$\rho(R^A_{\omega(A)}) = \rho(Q^A_{\omega(A)}) - (1-\omega(A))^2 < \frac{2\nu(1-\nu)}{(1+\nu)^2}. \tag{4.5}$$

To see that $\rho(R^A_{\omega(A)})$ is indeed the Perron root of $R^A_{\omega(A)}$ we note that A is irreducible. So are then the matrices $|J^A|$ and R^A_ω. Moreover, according to Frobenius (see [16], Thm 2.6) the irreducibility of $|J^A|$ implies that $C := C_pC_1C_2\ldots C_{p-1}$, in (2.13) – (2.14), is primitive and thus

$$\rho(C) = \rho^p(|J^A|) = \nu^p \in \sigma(C).$$

It also follows that $R^A_{\omega(A)}$ too is primitive since it is an irreducible matrix whole cycle lengths have a greatest common divisor equal to 1. So, for $c = \rho(|J^A|) \in \sigma(|J|)$, (2.15) will give among others the Perron root of $R^A_{\omega(A)}$ for $\omega = \omega(A)$. By Descartes' rule of signs it is readily seen that (2.15) possesses a unique real positive root $\tau = \rho(R^A_{\omega(A)})$. From the analysis of the previous section, the discussion so far and the H-matrix character of A it becomes clear that $\nu < 1$. Since inequality (4.4) holds for all ν for which $\nu < r_p \leq 1$ in order to determine r_p it suffices to find the value of $r_p \in (0,1]$ that makes $f(r_p)$ vanish. In other words r_p in the interval in question is the real positive number that satisfies

$$f(r_p) := 4^{p-1}r_p^p - (1-r_p)(1+r_p)^{p-1} = 0. \tag{4.6}$$

For $p = 3$, (4.6) turns out to be

$$17r_3^3 + r_3^2 - r_3 - 1 = 0 \tag{4.7}$$

which is the equation (2.31) of [14].

To simplify the notation in the study of the function f in (4.6) we shall drop the index p from r_p; we shall use it only if it is absolutely necessary. Also, since (4.6) trivially holds for $p = 2$ in the subsequent analysis we shall be considering $p \geq 2$ despite the fact that so far we have been dealing with $p \geq 3$. Our very first result of this section is stated and proved in the theorem below.

Theorem 4.1: Under the assumptions of Theorem 3.1 and the additional assumption that A is an irreducible H-matrix, the Perron root τ of $R^A_{\omega(A)}$ equals $\frac{2r(1-r)}{(1+r)^2}$ for a unique value of $r = r_p \in (1/3, 1/2)$.

Proof: It is readily checked, using (2.15) with $\omega = \omega(A)$, that if the Perron root τ of $R^A_{\omega(A)}$ equals $\frac{2r(1-r)}{(1+r)^2}$ then r will satisfy equation (4.6), with $r \in (0,1)$. To study the function $f(r)$ in (0,1) we define

$$g(r) := 4^{p-1}r^p, \quad h(r) := (1-r)(1+r)^{p-1} \quad r \in (0,1] \tag{4.8}$$

and form their ratio

$$k(r) := \frac{h(r)}{g(r)} = \frac{(1-r)(1+r)^{p-1}}{4^{p-1}r^p}. \tag{4.9}$$

We set $s := 1/r$, $s \in [1, \infty)$, and consider

$$\ell(s) := k(1/s) = \frac{1}{4^{p-1}}(s-1)(s+1)^{p-1}, \quad s \in [1, \infty) \tag{4.10}$$

where

$$\ell(1) = 0 \quad \lim_{s \to \infty} \ell(s) = \infty. \tag{4.11}$$

Differentiating (4.10) with respect to s we obtain

$$\frac{\partial \ell(s)}{\partial s} \sim p(s-1) + 2. \tag{4.12}$$

The right hand side of (4.12) is positive for every $s \in [1, \infty)$ and every $p \geq 2$. So, $\ell(s)$ is a strictly increasing function of $s \in [1, \infty)$. However, by virtue of (4.11) there will exist a unique value of $s \in (1, \infty)$, denoted by s_p, such that $\ell(s_p) = 1$. In view of (4.9) and (4.10) it is implied that there exists a unique value of $r \in (0, 1)$ denoted by $r_p = 1/s_p$ for which $k(r)$ assumes the value 1 or, equivalently, because of (4.6) and (4.7), $f(r)$ becomes zero. Also, since from (4.10) $\ell(2) = (\frac{3}{4})^{p-1} < 1$ and $\ell(3) = 2 > 1$ it is implied that $s_p = 1/r_p \in (2, 3)$ which proves that $r_p \in (1/3, 1/2)$. □

The value of r_p, as a function of $p \geq 2$, strictly decreases with respect to p. More specifically we have the following valid statement.

Theorem 4.2. Under the assumptions of Theorem 4.1 the unique positive real root r_p of (4.6) is a strictly decreasing function of $p \geq 2$. Specifically

$$\begin{aligned} r_2 &= \tfrac{1}{\sqrt{5}} \approx 0.447213596, \\ r_3 &\approx 0.418192802, \\ r_4 &\approx 0.400511732, \end{aligned} \tag{4.13}$$

and

$$\lim_{p \to \infty} r_p = 1/3.$$

Proof: First we embed the set of integers $p \in [2, \infty)$ into the set of real numbers $p \in [2, \infty)$ and consider $r_p = r(p)$, the unique real positive root of (4.6) in $(1/3, 1/2)$, as a function of the real $p \in [2, \infty)$. Differentiating (4.6) with respect to p we obtain

$$\begin{gathered} (4r)^p \left[\ell n(4r) + \tfrac{p}{r}\tfrac{\partial r}{\partial p}\right] + 4(1+r)^{p-1}\tfrac{\partial r}{\partial p} \\ -4(1-r)(1+r)^{p-1}\left[\ell n(1+r) + \tfrac{(p-1)}{(1+r)}\tfrac{\partial r}{\partial p}\right] = 0 \end{gathered}$$

We substitute in the above equation $(4r)^p$ from (4.6) and after some manipulation we have

$$\left[\frac{(1-r)(p+r)}{r(1+r)} + 1\right]\frac{\partial r}{\partial p} = (1-r)\ell n\left(\frac{1+r}{4r}\right). \tag{4.14}$$

Since from Theorem 4.1 it is $\tau = \tau_p \in (1/3, 1/2)$ we have $1 - \tau > 0$ and $\frac{1+\tau}{4\tau} < 1$, implying from (4.14) that $\frac{\partial \tau}{\partial p} < 0$. Therefore τ_p is a strictly decreasing function with respect to p. Since it is also bounded from below by $1/3$, τ_p will possess a limit as $p \to \infty$. To find this limit we take (4.6), solve for $4^{p-1}\tau^p$, extract p^{th} roots and take limits of both sides. Having in mind that $2^{-1/p} < (1-\tau)^{1/p} < (2/3)^{1/p}$, $(4/3)^{-1/p} < (1+\tau)^{-1/p} < 3/2$ and that $\lim_{p\to\infty} x^{1/p} = 1$, for every $x > 0$, we obtain

$$4 \lim_{p\to\infty} \tau = 1 + \lim_{p\to\infty} \tau$$

from which $\lim_{p\to\infty} \tau = 1/3$ follows. □

We close this section with the statement and proof of the generalization of Lemma 2.1(i) of [14].

Theorem 4.3: Under the assumptions of Theorem 4.1 the Perron root τ of $R^A_{\omega(A)}$ is a strictly increasing function of ν for all $\nu \in [0, 1]$.

Proof: The Perron root τ satisfies (2.15) with $\omega = \omega(A) = \frac{2}{1+\nu}$. Using (2.14), (2.15) can be rewritten as

$$\psi(\tau) := \tau^p - \frac{(2\nu)^{p+2}}{(1+\nu)^{3p}}[(1+\nu)^2\tau + (1-\nu)^2][(1+\nu)^2\tau + 2\nu(1-\nu)]^{p-2} = 0. \tag{4.15}$$

Now we differentiate (4.15) with respect to ν, multiply both members by τ, replace τ^p in the resulting equation using (4.15) and, after some simplifications and rearrangement of terms, end up with

$$a(\nu)\frac{\partial \tau}{\partial \nu} = b(\nu), \tag{4.16}$$

where

$$\begin{aligned} a(\nu) \ :=\ & \nu \{p[(1+\nu)^2\tau + (1-\nu)^2]2\nu(1-\nu) \\ & +\tau(1+\nu)^2[(1+\nu)^2\tau + 2(1-\nu)(1-2\nu)]\}. \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} b(\nu) := \tfrac{\tau}{(1+\nu)} \quad & \{[p(1-2\nu) + 2(1+\nu)][(1+\nu)^2\tau + (1-\nu)^2][(1+\nu)^2\tau + 2\nu(1-\nu)] \\ & +2\nu(1+\nu)\,[(1+\nu)\tau[(1+\nu)^2\tau + 2\nu(1-\nu)] - (1-\nu)[(1+\nu)^2\tau + 2\nu(1-\nu)] \\ & +(p-2)[(1+\nu)^2\tau + (1-\nu)^2][(1+\nu)\tau + (1-2\nu)]]\}. \end{aligned} \tag{4.18}$$

The expression in the braces in the right hand side of (4.17) is a quadratic in τ of the form

$$\alpha(\nu)\tau^2 + \beta(\nu)\tau + \gamma(\nu)$$

where

$$
\begin{aligned}
\alpha(\nu) &= (1+\nu)^4 > 0 \\
\beta(\nu) &= 2(1-\nu)(1+\nu)^2[(p-2)\nu+1] > 0 \\
\gamma(\nu) &= 2p\nu(1-\nu)^3 > 0
\end{aligned}
$$

for all $\nu \in (0,1)$. Since $\tau > 0$, it is implied that

$$a(\nu) > 0, \quad \nu \in (0,1). \tag{4.19}$$

On the other hand, the expression in the braces in the right hand side of (4.18) can be rewritten as follows

$$
\begin{aligned}
&(p-2)[(1+\nu)^2\tau + (1-\nu)^2][(1+\nu)^2\tau + 4\nu(1-2\nu)] \\
&+[(1+\nu)^2\tau + 2\nu(1-\nu)][4(1+\nu)^2\tau + 4(1-\nu)(1-2\nu)].
\end{aligned} \tag{4.20}
$$

Obviously, for $\nu \in (0,1/2]$ the expression (4.20) is positive because $p \geq 2$, $\tau > 0$ and $2\nu - 1 \geq 0$. For $\nu \in (1/2,1)$ we will show that each of the last factors in the two terms of products appearing in (4.20) are positive. Or, since $0 < \frac{(1-\nu)(2\nu-1)}{(1+\nu)^2} < \frac{4\nu(2\nu-1)}{(1+\nu)^2}$, as is readily checked, we will prove that

$$q(\nu) := \frac{4\nu(2\nu-1)}{(1+\nu)^2} < \tau \tag{4.21}$$

holds for all $\nu \in (1/2,1)$. We observe that from (4.15) it is $\psi(0) < 0$. Since the Perron root τ is the only positive real zero of $\psi(\tau)$ it suffices then to prove that $\psi(q(\nu)) < 0$. Indeed, using $q(\nu)$, from (4.21), for τ in (4.15) we obtain after some algebra that

$$\psi(q(\nu)) = \left(\frac{4\nu}{(1+\nu)^2}\right)^p \left[(2\nu-1)^p - \left(\frac{\nu(3\nu-1)}{1+\nu}\right)^p\right]. \tag{4.22}$$

However, for $\nu \in (1/2,1)$ the expression in the brackets in the right hand side of (4.22) and the difference

$$(2\nu-1) - \frac{\nu(3\nu-1)}{1+\nu} = -\frac{(1-\nu)^2}{1+\nu} < 0$$

are of the same sign. Therefore $\psi(q(\nu)) < 0$ implying that the expression (4.20) is positive for all $\nu \in (0,1)$ and so is $b(\nu)$ in (4.18) and (4.16). From (4.16), since $a(\nu)$ and $b(\nu)$ are positive for all $\nu \in (0,1)$, it is concluded that $\frac{\partial\tau}{\partial\nu} > 0$. This result implies that τ strictly increases with $\nu \in [0,1]$ which concludes the proof of the theorem. □

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