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A-Splines: Local Interpolation and Approximation using C^k-Continuous Piecewise Real Algebraic Curves

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Abstract

We present a concise characterization of the Bernstein-Bezier (BB) form of an implicitly defined bivariate polynomial over a triangle, such that the zero contour of the polynomial defines a smooth and single sheeted real algebraic curve segment. We call a piecewise C^k-continuous chain of such real algebraic curve segments in BB-form as an A-spline (short for algebraic spline). We show how to construct quadratic, cubic and quartic A-splines to locally interpolate and/or approximate the vertices of an arbitrary planar polygon with up to C^1, C^3 and C^5 continuity, respectively. Quadratic A-splines are always locally convex. We also prove that our C^1, C^2 or C^3 cubic A-splines are always locally convex. Additionally, we derive evaluation formulas for the efficient display of all these A-splines and present examples of their use in geometric modeling by constructing surfaces of revolution and generalized swept cylinders.

1 Introduction

Designing fonts with piecewise smooth curves or fitting curves to scattered data for image reconstruction are just two of the diverse applications of spline curve constructions. In this paper, we generalize past curve fitting schemes with conics [10],[22],[24], [26] and parametric spline fitting [12],[19],[27] achieving fits with fewer number of pieces or with higher order of smoothness/continuity. We exhibit efficient techniques to deal with higher degree implicitly defined algebraic curves, $f(x, y) = 0$, with $f(x, y)$ a bivariate real polynomial. The spline techniques of this paper simplify and extend all prior approaches and applications of algebraic curves to problems in geometric modeling [5],[7], [8], [9], [20], [21], [28], [29], [30]. The main advantages of the implicitly defined algebraic curve over the functional and parametric curve are: (1) the class of algebraic curve is closed under several geometric operations (intersections, union, offset, etc.), often required in a solid modeling system. For example, the offset of a parametric curve may not be parametric but is always algebraic and has an implicit representation. (2) For the same polynomial of degree n, implicit algebraic curves have more degrees of freedom

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($= \frac{(n+2)(n+1)}{2} - 1 = \frac{n(n+3)}{2}$) compared with rational function ($= 2n+1$) and rational parametric ($= 3n-1$) curves of the same degree. Hence implicit algebraic curves are more flexible to approximate a complicated curve with fewer number of pieces or to achieve higher order of smoothness. For example, by local interpolation, implicit algebraic curves have the potential to achieve C^k continuity with $k \leq \frac{n(n+3)}{4} - 1$, while functional or parametric curve can achieve $C^{k'}$ continuity with $k' = n-1$. However, the primary drawback for the widespread use of the implicit algebraic curve is that the curve can have singularities (see [32]) and can possess several components. In this paper we show how to isolate a non-singular and single sheeted segment of implicit algebraic curves and furthermore how to stitch these segments together to form splines with continuity as high as C^5 using only degree 4 curve pieces. Note that the degrees of freedom for the rational parametric is consistent with the well known theorem that rational parametric curves are exactly the irreducible implicit algebraic curves of genus 0[32]. An irreducible implicit curve of genus 0 and degree n possesses the maximum number of singularities an irreducible curve can have, viz., $\frac{(n-1)(n-2)}{2}$ and this is exactly the difference between the degrees of freedom of arbitrary implicit algebraic curves and rational parametric algebraic curves of the same degree n.

Main results: We present a simple characterization of the Bernstein-Bezier (BB) form of an implicitly defined bivariate polynomial over a triangle, such that the zero contour of the polynomial defines a smooth and single sheeted real algebraic curve segment. We call a piecewise C^k-continuous chain of such real algebraic curve segments in BB-form as an A-spline (short for algebraic spline). We present algorithms to construct quadratic, cubic and quartic A-splines to locally interpolate and/or approximate the vertices of an arbitrary planar polygon with up to C^1, C^3 and C^5 continuity, respectively. Note that parametric cubic and quartic splines can only achieve local C^2 and C^4 continuity respectively [13]. Our quadratic A-splines are always locally convex. We also prove that our C^1, C^2 or C^3 cubic A-splines are always locally convex. Additionally, we derive evaluation formulas for the efficient display of all these A-splines and present examples of their use in geometric modeling by constructing surfaces of revolution and generalized swept cylinders.

Related Prior Work:

Since 1960's, considerable work on polynomial spline interpolation and approximation has been done(see [13] for a bibliography). In general, spline interpolation has been viewed as a global fitting problem to arbitrary scattered data[10],[12],[19],[22],[24], [26], [27]. Here we consider local interpolation to an ordered set of points, defining an arbitrary polygon. Local interpolation by polynomials and rational functions is rather an old and simple technique that trace back to Hermite and Cauchy[11]. However, local interpolation by the zero sets of algebraic polynomials (implicit algebraic curves, surfaces etc.) is relatively new[5],[20], [21],[28]. We lay emphasis in this paper on using a single sheet and non-singular segment of implicit algebraic curves. In this aspect, Sederberg, in[28], has characterized the coefficients of the BB form of an implicitly defined bivariate polynomials on a triangle in such a way that if the coefficients on the lines that parallel to one side, say L, of the triangle all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the algebraic curve segment at most once. Our characterization in Theorem 2.1 is more general, with Sederberg's condition forming a special case. In [30], Sederberg, Zhao and Zundel gave another similar characterization which guarantee the single sheeted property of their TPAC by requiring that $b_{i0} \geq 0$, that $b_{0i}, b_{m-1,i} \leq 0$ and that the directional derivative of PAC(*piecewise algebraic curves*) with respect to any direction $s = \alpha u$ be non-zero within the triangle domain, here b_{ij} denotes the Bezier coefficient. This condition is also much more restrictive than the characterization we present here.

A recent related paper which constructs a family of C^1 continuous cubic algebraic splines is given by Paluszny and Patterson [20]. They use the following reduced form of the cubic

$$F(s,t,u) = as^2u + bsu^2 - cst^2 - dt^2u + estu,$$

with $a > 0, b > 0, c > 0, d > 0$, and (s,t,u) in BB-coordinates over a triangle and guarantee that the segment of the curve inside the triangle is convex. This result is a special case of the present paper as we consider the general implicit cubic. Further, the above family of $F(s,t,u) = 0$ curves cannot achieve C^2 continuity (see the discussion of §4). Our cubic A-splines can always achieve C^3-continuity, and even C^4-continuity for some special cases. Furthermore, our results generalize to higher degree curves. For example, we show how to construct quartic $A-splines$ with C^5 continuity, and an $n-2$ parameter family of tangent continuous, degree n algebraic splines.

2 Notation and Mathematical Preliminaries

Let $f(x,y)$ be a bivariate polynomial of degree n with real coefficients, and $p_i = (x_i, y_i)^T$, $i = 0,1,2$ be three affine independent points in the xy-plane. Then the transform

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 & x_1 & x_0 \\ y_2 & y_1 & y_0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{2.1}$$

maps $f(x,y)$ into its barycentric coordinate form $g(\alpha_0,\alpha_1,\alpha_2) = f(x,y)$, where $0 \le \alpha_0 \le 1$, $0 \le \alpha_1 \le 1$, and $\alpha_0 + \alpha_1 + \alpha_2 = 1$, Note that $g(\alpha_0,\alpha_1,\alpha_2)$ can also be written as $F(\alpha_0,\alpha_1) = g(\alpha_0,\alpha_1,1-\alpha_0-\alpha_1)$, where (x,y) and (α_0,α_1) are related by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 - x_0 & x_1 - x_0 \\ y_2 - y_0 & y_1 - y_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{2.2}$$

The inverse transform of (2.2) is

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} y_1 - y_0 & x_0 - x_1 \\ y_0 - y_2 & x_2 - x_0 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} / \Delta(p_0,p_1,p_2), \tag{2.3}$$

where $\Delta(p_0,p_1,p_2) = \det \begin{bmatrix} p_2 & p_1 & p_0 \\ 1 & 1 & 1 \end{bmatrix}$. In the barycentric coordinate system, $g(\alpha_0,\alpha_1,\alpha_2)$ can be expressed as the BB form (see [14, 17]).

$$g(\alpha_0,\alpha_1,\alpha_2) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} b_{ij} \frac{n!}{i!j!(n-i-j)!} \alpha_0^i \alpha_1^j \alpha_2^{n-i-j}.$$

Related to the triangle $p_0p_1p_2$, we also use two local coordinates denoted as $(\mathcal{X},\mathcal{Y})_{(p_0,p_1)}$ and $(\mathcal{X},\mathcal{Y})_{(p_2,p_1)}$, which are defined by shifting the origin of the xy-system to p_0 and p_2 respectively, and then rotating them in such a way that the vectors $p_0 - p_1$ (resp. $p_0 - p_2$) are in the same direction as the new y-axis. That is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\|p_1 - p_i\|} \begin{bmatrix} y_1 - y_i & x_1 - x_i \\ x_i - x_1 & y_1 - y_i \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 0,2 \tag{2.4}$$

Transform (2.4) is orthogonal, it does not change the shape of the curve $f(x,y)=0$. While the transform (2.3) does not change the smoothness of a curve, we may characterize the curve on the standard triangle $P_0P_1P_2$ with $P_0=(0,0)^T$, $P_1=(0,1)^T$ and $P_2=(1,0)^T$ for the function $g(\alpha_0,\alpha_1,\alpha_2)$ or $F(\alpha_0,\alpha_1)$ under the barycentric coordinate system (α_0,α_1) in the BB form and study the smooth join problem under the local system $(\mathcal{X},\mathcal{Y})$ by considering the derivative agreements at the join points. The local systems connect the the barycentric coordinate system in the following way:

$$\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \frac{1}{\|p_1-p_0\|} \begin{bmatrix} \Delta(p_0,p_1,p_2) & 0 \\ \langle p_1-p_0, p_2-p_0\rangle & \|p_1-p_0\|^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \quad \text{at } p_0 \tag{2.5}$$

and

$$\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \frac{1}{\|p_1-p_2\|} \begin{bmatrix} \Delta(p_0,p_1,p_2) & \Delta(p_0,p_1,p_2) \\ \langle p_1-p_2, p_2-p_0\rangle & \langle p_1-p_2, p_1-p_0\rangle \end{bmatrix} \begin{bmatrix} \alpha_0-1 \\ \alpha_1 \end{bmatrix}, \quad \text{at } p_2 \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ stands for inner product of two vectors.

Let $\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ and $\mathcal{F}_2(\mathcal{X},\mathcal{Y})$ be $F(\alpha_0,\alpha_1)$ under the transforms (2.5) and (2.6), respectively, and let $\mathcal{F}_i(0,0)=0$ and $\frac{\partial\mathcal{F}_i(0,0)}{\partial\mathcal{X}}\neq 0$ for $i=0,2$. Then each $\mathcal{F}_i(\mathcal{X},\mathcal{Y})=0$ can be expressed locally at p_i as a function of $\mathcal{Y}$, denoted $\mathcal{X}_{(p_i,p_1)}(\mathcal{Y})$. The k-th local derivatives, denoted by $\mathcal{X}^{(k)}_{(p_i,p_1)}=\mathcal{X}^{(k)}_{(p_i,p_1)}(0)$, are also well defined. Here the subscript (p_i,p_1) is to emphasize that $\mathcal{X}$ is related to the local system $(\mathcal{X},\mathcal{Y})_{(p_i,p_1)}$ that is defined by p_i and p_1. Correspondingly, $\alpha^{(k)}_{0i}=\alpha^{(k)}_{0i}(0)$ defined by $F(\alpha_0,\alpha_1)=0$ are also well defined at P_i for $i=0,2$. Now, suppose $\mathcal{X}^{(1)}_{(p_i,p_1)}=0$, that is, the curves $\mathcal{F}_i(\mathcal{X},\mathcal{Y})$ are tangent with y-axis(this is always the case in this paper), then we can establish, by differentiating $\mathcal{X}=\mathcal{X}_{(p_i,p_1)}(\mathcal{Y})$, the following relations among these derivatives.

$$\alpha^{(1)}_{00}=0 \qquad \alpha^{(1)}_{02}=-1$$

$$\alpha^{(2)}_{0i}=\frac{\|p_1-p_i\|^3\mathcal{X}^{(2)}_{(p_i,p_1)}}{\Delta(p_0,p_1,p_2)} \tag{2.7}$$

$$\alpha^{(3)}_{0i}=\frac{\|p_1-p_i\|^4\mathcal{X}^{(3)}_{(p_i,p_1)}}{\Delta(p_0,p_1,p_2)}+\frac{3\|p_1-p_i\|^4\langle p_1-p_i,p_2-p_0\rangle(\mathcal{X}^{(2)}_{(p_i,p_1)})^2}{\Delta^2(p_0,p_1,p_2)} \tag{2.8}$$

and if $\mathcal{X}^{(2)}_{(p_i,p_1)}=0$,

$$\alpha^{(4)}_{0i}=\frac{\|p_1-p_i\|^5\mathcal{X}^{(4)}_{(p_i,p_1)}}{\Delta(p_0,p_1,p_2)} \tag{2.9}$$

$$\alpha^{(5)}_{0i}=\frac{\|p_1-p_i\|^6\mathcal{X}^{(5)}_{(p_i,p_1)}}{\Delta(p_0,p_1,p_2)}+\frac{10\|p_1-p_i\|^6\langle p_1-p_i,p_2-p_0\rangle(\mathcal{X}^{(3)}_{(p_i,p_1)})^2}{\Delta^2(p_0,p_1,p_2)} \tag{2.10}$$

Note the orientation of the local systems determined by p_1. An alternative way is to use the opposite orientation system $(\mathcal{X}_1,\mathcal{Y}_1)=(-\mathcal{X},-\mathcal{Y})$ at the same point. Then the derivatives of one function under the opposite orientation systems have the following relation

$$\mathcal{X}^{(k)}=(-1)^{k+1}\mathcal{X}^{(k)}_1 \tag{2.11}$$

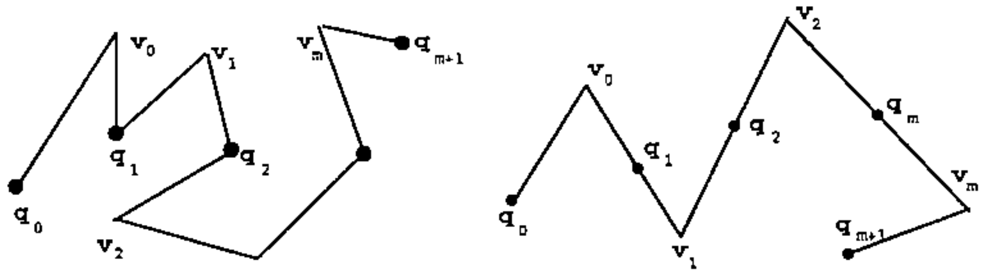

Figure 2.1: A C^0 polygon and a C^1 polygon

In the following section, we shall use both the systems when we join two curve segments smoothly at one point, since the two local systems that relate to two triangles and incident at a common point are always opposite oriented.

Let p_0, p_1 and p_2 be three affine independent points in the xy-plane. Then we consider the two line segments $[p_0, p_1]$ and $[p_1, p_2]$ as a segment of a polygon, denoted by $p_0\widehat{p_1}p_2$. We shall consider p_1 as a controller and p_0 and p_2 as interpolation points. An arbitrary polygon chain(or polygon for brevity) consists of a sequence of consecutive polygon segments denoted by $\{q_i\widehat{v_i q_{i+1}}\}_{i=0}^{m}$. A polygon is said to be of C^1 if

$$(v_i - q_{i+1}) = \alpha_i(v_{i+1} - q_{i+1}), \ \alpha_i < 0, \quad \text{for} \ \ i = 0, \cdots, m \tag{2.12}$$

otherwise, it is of type C^0(see Fig. 2.1). If $q_0 = q_{m+1}$, then the polygon is closed. Note that a C^1 polygon can be trivially constructed from an arbitrary polygon by inserting one vertex per edge of the polygon.

3 The Characterization of A-Splines

Let

$$F(\alpha_0, \alpha_1) = \sum_{j=0}^{n}\sum_{i=0}^{n-j} b_{ij}\frac{n!}{i!j!(n-i-j)!}\alpha_0^i\alpha_1^j(1-\alpha_0-\alpha_1)^{n-i-j}. \tag{3.1}$$

be the BB form(see [14, 17]) of $F(\alpha_0, \alpha_1)$. Since there is constant multiplier to the equation $F(\alpha_0, \alpha_1) = 0$. We may assume $b_{0n} = -1$ if $b_{0n} \neq 0$.

Theorem 3.1 *For the given polynomial $F(\alpha_0, \alpha_1)$ defined as (3.1), if there exists an integer $k(0 < k < n)$ such that(see Fig. 3.1 for $n = 3$ and $k = 1$)*

$$b_{ij} \geq 0, \quad \text{for} \ \ i = 0, 1, \ldots n-j; \ \ j = 0, 1, \ldots, k-1 \tag{3.2}$$

$$b_{ij} \leq 0 \ \ \text{for} \ \ i = 0, 1, \ldots n-j; \ \ j = k+1, \ldots, n \tag{3.3}$$

and $\sum_{i=0}^{n} b_{i0} > 0$, $\sum_{i=0}^{n-j} b_{ij} < 0$ for at least one j $(k < j \leq n)$, then

(i) for any β that $0 < \beta < 1$, the straight line $\alpha_0 = \beta(1-\alpha_1)$, that pass through P_1 and the line segment (P_0, P_2), intersect the curve $F(\alpha_0, \alpha_1) = 0$ one and only one time(counting multiplicity) in the interior of the triangle $P_0P_1P_2$.

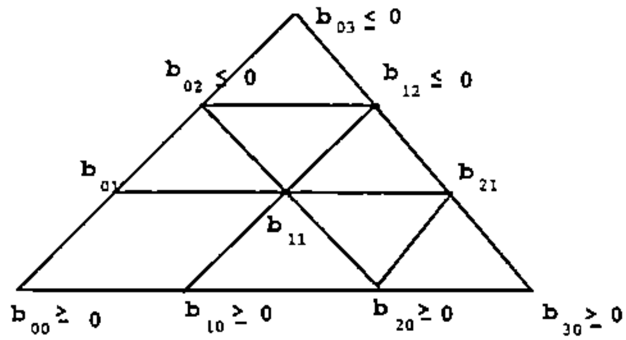

Figure 3.1: Bezier Coefficients of Cubic

(ii) The value α_1 determined by $B_\beta(\alpha_1) = F(\beta(1-\alpha_1),\alpha_1) = 0$ in the interior of the triangle is an analytic function of β.

Proof. (i). Substitute $\alpha_0 = \beta(1-\alpha_1)$ into $F(\alpha_0,\alpha_1)$ we have

$$\begin{aligned} B_\beta(\alpha_1) &= F(\beta(1-\alpha_1),\alpha_1) \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-j} b_{ij}\frac{n!}{i!j!(n-i-j)!}\beta^i(1-\beta)^{n-j-i}\alpha_1^j(1-\alpha_1)^{n-j} \\ &= \sum_{j=0}^{n}\Big(\sum_{i=0}^{n-j} b_{ij}\frac{(n-j)!}{i!(n-i-j)!}\beta^i(1-\beta)^{n-j-i}\Big)\frac{n!}{j!(n-j)!}\alpha_1^j(1-\alpha_1)^{n-j} \\ &= \sum_{j=0}^{n} b_j(\beta)B_j^n(\alpha_1) \end{aligned}$$

where

$$b_j(\beta) = \sum_{i=0}^{n-j} b_{ij}B_i^{n-j}(\beta), \quad B_j^n(\alpha_1) = \frac{n!}{j!(n-j)!}\alpha_1^j(1-\alpha_1)^{n-j}$$

It follows from (3.2)–(3.3) that $b_0(\beta) > 0,\ b_j(\beta) \geq 0,\ j = 1,\ldots,k-1,\ b_j(\beta) \leq 0,\ j = k+1,\ldots,n$. Let $b_n(\beta) = \cdots = b_{n-l+1}(\beta) = 0;\ \ b_{n-l}(\beta) < 0$ for some l with $0 \leq l \leq n-k-1$. Then $B_\beta(\alpha_1)$ can be written as

$$B_\beta(\alpha_1) = (1-\alpha_1)^l\sum_{j=0}^{n-l} c_j(\beta)B_j^{n-l}(\alpha_1)$$

where $c_0 > 0, c_{n-l} < 0$ and the sequence $c_0, c_1, \cdots, c_{n-l}$ has one sign change. By variation diminishing property, the equation $B_\beta(\alpha_1) = 0$ has exactly one root in $(0,1)$.

(ii). Since $\alpha_1 = \alpha_1(\beta)$ is a simple zero of $B_\beta(\alpha_1) = 0$ in $(0,1)$, i.e., $B'_\beta(\alpha_1(\beta)) \neq 0$, the function $\alpha_1(\beta)$ is well defined and analytic by implicit function theorem. ◇

This theorem guarantees that there is one and only one segment of $F(\alpha_0,\alpha_1) = 0$ within the standard triangle. The term "algebraic spline" that we have used in this paper is a chain of such curve segments with some continuity at the join points. We should mention that the curve $F(\alpha_0,\alpha_1) = 0$ pass through $P_1 = (1,0)^T$ if $b_{0n} = 0$. However, we do not use this part of the curve. In our application in §4, we take b_{0n} to be -1. The next theorem goes further about the smoothness of the curve and the properties on the boundary of the triangle.

Theorem 3.2. *Let $F(\alpha_0, \alpha_1)$ be defined as Theorem 3.1, then*

(i) The curve $F(\alpha_0, \alpha_1) = 0$ is smooth in the interior of the triangle $P_0P_1P_2$.

(ii) If we further assume $b_{0j} = 0$ for $j = 0, ..., k$ and $b_{0,k+1} < 0$, then the curve passes through P_0, tangent with the line $\alpha_0 = 0$ with multiplicity $k+1$ at P_0 and no other intersection with $\alpha_0 = 0$ for $0 < \alpha_1 < 1$. Similarly, if $b_{j,n-j} = 0$ for $j = 0, ..., k$ and $b_{k+1,n-(k+1)} < 0$, then the curve pass through P_2, tangent with the line $\alpha_0 + \alpha_1 - 1 = 0$ with multiplicity $k+1$ at P_2 and no other intersection with $\alpha_0 + \alpha_1 - 1 = 0$ for $\alpha_0 > 0, \alpha_1 > 0$.

(iii) If $b_{00} = b_{01} = b_{10} = 0$, then P_0 is a singular point of the curve. Similarly, if $b_{0n} = b_{0,n-1} = b_{1,n-1} = 0$, then P_2 is a singular point of the curve.

Proof. (i). If (α_0^*, α_1^*) is a singular point of $F(\alpha_0, \alpha_1) = 0$, i.e., $\frac{\partial F}{\partial \alpha_0} = \frac{\partial F}{\partial \alpha_1} = 0$ at (α_0^*, α_1^*). Let $\beta^* = \frac{\alpha_0^*}{1-\alpha_1^*}$. Then $B_{\beta^*}(\alpha_1^*) = 0$ and $B'_{\beta^*}(\alpha_1^*) = -\frac{\partial F}{\partial \alpha_0}\beta^* + \frac{\partial F}{\partial \alpha_1} = 0$. That is, α_1^* is a double zero of $B_{\beta^*}(\alpha_1)$. A contradiction to Theorem 3.1.

(ii). Since $\alpha_0 = 0$ corresponds to $\beta = 0$ in the proof of Theorem 3.1. Hence in the given case

$$B_0(\alpha_1) = \sum_{j=0}^{n} b_{0j} B_j^n(\alpha_1) = \sum_{j=k+1}^{n} b_{0j} B_j^n(\alpha_1) = \alpha_1^{k+1} G(\alpha_1)$$

where $G(\alpha_1) = \sum_{j=k+1}^{n} b_{0j} \frac{n!}{j!(n-j)!} \alpha_1^{j-(k+1)} (1-\alpha_1)^{n-j}$ has no zero on $[0,1)$ because its coefficients have same sign and the first coefficient is negative. That is, $\alpha_1 = 0$ is the only zero of $B_0(\alpha_1)$ on $[0,1)$ and has multiplicity $k+1$. The second conclusion in this item can be similarly proven.

(iii). If $b_{00} = b_{01} = b_{10} = 0$, then $\alpha_1 = 0$ is a double zero of $F(0, \alpha_1)$ and $\alpha_0 = 0$ is a double zero of $F(\alpha_0, 0)$. Hence $\frac{\partial F}{\partial \alpha_0} = \frac{\partial F}{\partial \alpha_1} = 0$ at $(0,0)^T$. That is, P_0 is a singular point of the curve. At $P_2 = (1,0)$ the same conclusion holds. $\diamond$

Since it is obvious that the quadratic algebraic spline is convex, we consider the convexity of cubic splines. At present, the general case for arbitrary degree curve is left as an open problem. Even for the cubic case, the convexity is not always guaranteed. Only if the curve tangent with the sides of the triangle at P_0 and P_2, i.e., the case of Theorem 3.2.ii, then the curve segment is convex. This is of course a special case, but it is the most important case for polygon approximation(see §4).

To prove the convexity conclusion, we need the following lemma.

Lemma 3.3. *If P is an inflection point of the cubic algebraic curve $f(x,y) = 0$ and $L^*(x,y) = ax + by + c = 0$ is the tangent line passing through P. Then L^* separates the curve into two parts, one part located in the half space $L^*(x,y) < 0$, the other part located in the half space $L^*(x,y) > 0$. That is, P is the only intersection point between L and the curve.*

Proof. Suppose L^* can be written as $x = ky + b$ and $P = (x^*, y^*)$. Then by the definition(see [32],p71) of inflection point we know that y^* is a triple zero of $f(ky+b, y)$, i.e., $f(ky+b, y) = a(y-y^*)^3$ for some non-zero constant a. This means that the curve locates on the both sides of L^* and the curve can not intersect with line L^* at any other point by Bezout Theorem[32]. $\diamond$

Theorem 3.4. *The cubic algebraic spline defined in Theorem 3.2.ii has no inflection point inside the reference triangle.*

Proof. We prove this theorem with the aid of geometric intuition(see Fig. 3.2) although it is easy to translate it into analysis language. Suppose on the contrary, then there is at least one inflection point in the triangle $P_0P_1P_2$. Let $P = (\alpha_0^*, \alpha_1^*)$ be the first one, i.e., its β-coordinate is minimal. Now the curve in the triangle is divided into two parts. The first part, say C_1,

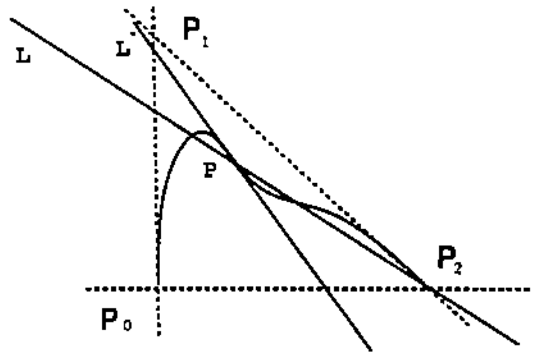

Figure 3.2: L will intersect with the curve four times if it has inflection point

corresponds to $[0, \beta^*]$ and second part, C_2, corresponds to $[\beta^*, 1]$ with $\beta^* = \alpha_0^*/(1-\alpha_1^*)$(see Theorem 3.1). Let L^* be the tangent line through this inflection point. By Lemma 3.3, the line L^* can not intersect with both line segments (P_0, P_1) and (P_1, P_2). Otherwise, the curve segment can not pass through both vertices P_0 and P_2. The only cases are that L^* intersects with either $(P_0, P_1]$ and $[P_0, P_2]$ or $(P_2, P_1]$ and $[P_0, P_2]$. Without loss of generality, we assume that L^* intersects with $(P_0, P_1]$ and $[P_0, P_2]$. In this case, the slope of L^* is less than zero and also L^* is not a vertical line. It follows from Lemma 3.3 that C_1 is below the line L^* since P_0 is so, and similarly C_2 is above that line. Now, let L be another line that pass through P and P_2. If L coincide with L^*, then a contradiction is yield by Bezout theorem because L has four intersection points with the cubic. So L does not coincide with L^*. By the fact that the slope of L is greater than the slope of L^*, we have that C_1 will intersect with line L(except point P). By the same reason and the fact that $[P_1, P_2]$ is tangent to the curve, we can get that segment C_2 intersect with L in addition to P and P_2. So L intersect with our cubic algebraic curve four times. This is contradict with Bezout theorem. So the segment inside the triangle is convex. ◇

4 C^k Approximation of Polygons by A-Splines

For simplicity we assume that we are given a polygon in the plane, that is we have a point set $\{q_i\}_{i=0}^{m+1}$, and also a vertex set $\{v_i\}_{i=0}^{m}$, (see Figure 2.1), such that the three points q_i, v_i and q_{i+1} are affine independent. Several schemes exist which produce a desired polygon chain from scattered data [15],[16],[31]. To produce a C^1 polygon from a polygonal chain is trivial and amounts to inserting a single additional vertex per polygon edge. In this section, we intend to first construct a piece of algebraic spline for each segment of the polygon $q_i\widehat{v_i q_i}_{+1}$ and then join these curve segments with the desired smoothness. Although it is possible to develop A-splines of arbitrary degree based on previous section, we pay attention here on simple algebraic spline, i.e., quadratic, cubic, quartic and $n-2$ parameter family. The scheme used in the following sub-sections are similar, but we develop them separately since the derivations are quite different.

4.1 Quadratic A-Splines(Conic Splines)

Since a quadratic algebraic polynomial has 5 degrees of freedom, by accounting the interpolation conditions at each q_i, C^1 smoothness curve can be reached at first glance by local interpolation.

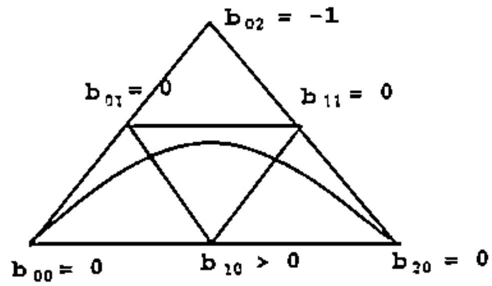

Figure 4.1: Quadratic A-Splines

This C^1 interpolation costs 4 degrees of freedom. The 5th degree of freedom can be used to approximate additional data or to achieve C^2 continuity by choosing it properly. However, we shall show that it is in generally impossible to obtain a C^2 approximation, while C^1 continuity is guaranteed.

As a general case, we take one segment of the polygon as $p_0\widehat{p_1}p_2$ with $p_i = (x_i, y_i)^T$. Then the transformation (2.1) maps a polynomial $f(x,y)$ into its barycentric form on the triangle $p_0p_1p_2$ and it maps p_0, p_1 and p_2 to $P_0 = (0,0)^T$, $P_1 = (0,1)^T$ and $P_2 = (1,0)^T$ respectively. Let

$$\begin{aligned} f(x,y) &\equiv F(\alpha_0, \alpha_1) \\ &= -\alpha_1^2 + b_{10}(1-\alpha_0-\alpha_1)\alpha_0 \end{aligned} \tag{4.1}$$

be a quadratic algebraic polynomial that pass p_0 and p_2 and tangent the line p_0p_1 at p_0 and p_2p_1 at p_2, respectively, for any given $b_{10} > 0$(see Fig 4.1).

4.1.1 Approximation of Additional Data

First we compute the *k-th* derivative $\alpha_{0i}^{(k)} = \alpha_{0i}^{(k)}(0)$, where $\alpha_{0i}(\alpha_1)$ is a function of α_1 obtained by expression $F(\alpha_0, \alpha_1) = 0$ locally at P_i. Differentiating (4.1) about α_1, we immediately obtain the condition

$$\alpha_{00}^{(1)} = 0, \quad \alpha_{02}^{(1)} = -1 \tag{4.2}$$

$$\alpha_{00}^{(2)} = \tfrac{2}{b_{10}} \quad \alpha_{02}^{(2)} = -\tfrac{2}{b_{10}} \tag{4.3}$$

Given an additional set of points $\{q_i\}_{i=0}^m$ in the triangle $p_0p_1p_2$ with at least one point in the interior of the triangle. Let $Q_i = (\alpha_{0i}, \alpha_{1i})^T$ be the map of q_i by transform (2.3). Then $\alpha_{0i} \geq 0$, $\alpha_{1i} \geq 0$ and $\alpha_{0i} + \alpha_{1i} \leq 1$, and there is some i such that the three inequalities are strict. Then there exist positive b_{10} such that $\sum_{i=0}^m F(\alpha_{0i}, \alpha_{1i})^2 = \min$. In fact, we have the following least square problem

$$b_{10}(1-\alpha_{0i}-\alpha_{1i})\alpha_{0i} = \alpha_{1i}^2, \quad i = 0, 1, \ldots, m$$

Hence the least square solution is $b_{10} = \dfrac{\sum_{i=0}^{m} \alpha_{1i}^2(1-\alpha_{0i}-\alpha_{1i})\alpha_{0i}}{\sum_{i=0}^{m}(1-\alpha_{0i}-\alpha_{1i})^2\alpha_{0i}^2} > 0$

Notes:

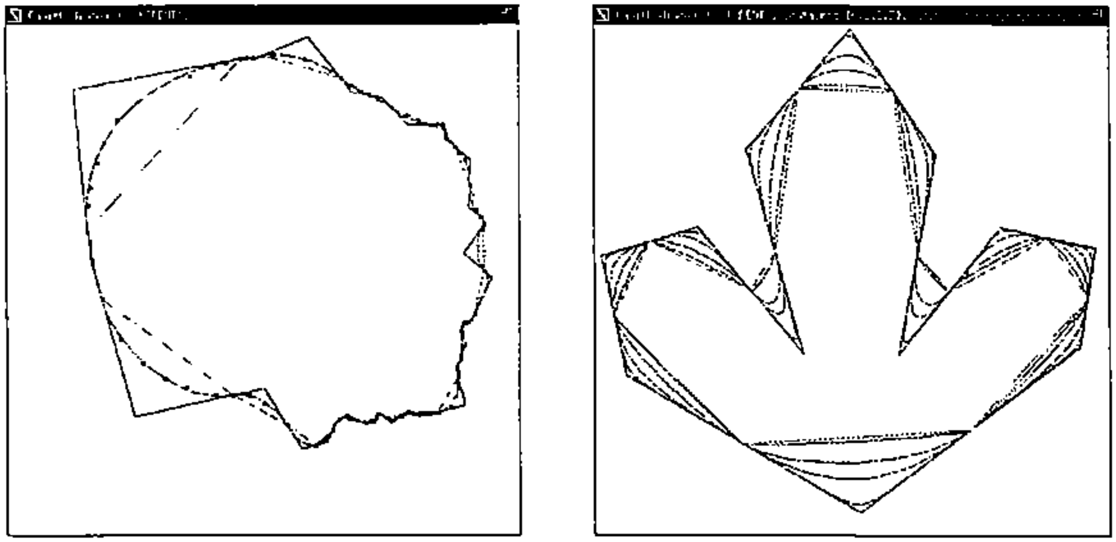

Figure 4.2: C^1 Quadratic A-Splines(Conic Splines)

1. If $m = 0$, then $b_{10} = \dfrac{\alpha_{10}^2}{(1-\alpha_{00}-\alpha_{10})\alpha_{00}}$ and the curve $F(\alpha_0, \alpha_1) = 0$ passes through $Q_0 = (\alpha_{00}, \alpha_{10})^T$.
2. If $m < 0$, i.e., no point is given, a default value is to choose $Q_0 = (1/3, 1/3)^T$, i.e., $b_{10} = 1$.

The left figure of Figure 4.2 shows an example of quadratic least square C^1 A-spline approximation to the scattered data of one slice of a human head. The right figure shows that the quadratic C^1 A-spline can pass through any given (one) point in each triangle.

4.1.2 C^2 Continuity?

Suppose we are given a C^1 polygon, then, if the quadratic algebraic curve $f(x,y) = 0$ is defined by (4.1) on each segment of the polygon, the composite curve is C^1 continuous automatically for any $b_{10} > 0$. The parameter b_{10} can be used to interpolate the second order derivative at the end points locally or globally.

A. Interpolate second order derivatives locally

We consider one segment $p_0\widehat{p_1}p_2$ of the polygon and interpolate second order derivatives at $\mathbf{p_0}$ and $\mathbf{p_2}$. Since the second order derivatives at the end-points of the polygon segment are so specified that independent on the second order derivatives at the neighbour segments, the composite curve in general can not achieve C^2 continuity.

It follows from (4.3) that

$$\alpha_{00}^{(2)} > 0, \quad \alpha_{02}^{(2)} < 0 \quad \text{and} \quad \alpha_{00}^{(2)} = -\alpha_{02}^{(2)} \tag{4.4}$$

Then from (2.7), we have

$$\mathcal{X}_{(p_0,p_1)}^{(2)} \Delta(p_0, p_1, p_2) > 0 \tag{4.5}$$

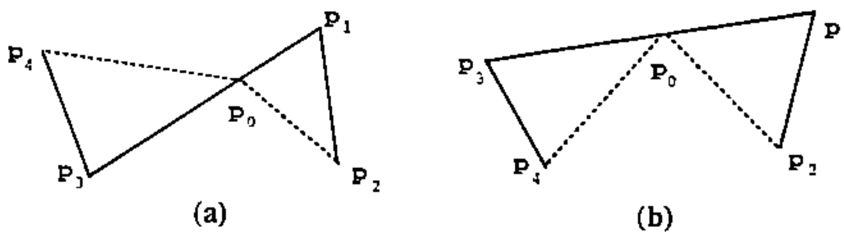

Figure 4.3: The two different cases of C^1 join polygon segments

$$\mathcal{X}^{(2)}_{(p_0,p_1)}\|p_1 - p_0\|^3 + \mathcal{X}^{(2)}_{(p_2,p_1)}\|p_1 - p_2\|^3 = 0 \tag{4.6}$$

Lemma 4.1 *Given the second derivatives $\mathcal{X}^{(2)}_{(p_0,p_1)}$ and $\mathcal{X}^{(2)}_{(p_2,p_1)}$ at p_0 and p_2, such that (4.5) and (4.6) hold, then the curve $f(x,y) = -\alpha_1^2 + b_{10}(1-\alpha_0-\alpha_1)\alpha_0 = 0$, with $b_{10} > 0$ defined by (4.3) and (2.7), interpolate p_0 and p_2 with the given second derivative values $\mathcal{X}^{(2)}_{(p_0,p_1)}$ and $\mathcal{X}^{(2)}_{(p_2,p_1)}$.*

Proof. If (4.5) holds, then $\alpha^{(2)}_{00}$ defined by (2.7) is positive and then b_{10}, defined by the first equality of (4.3), is positive. If (4.6) holds, $\alpha^{(2)}_{02}$ defined by the second equality of (2.7) satisfies the second equality of (4.3). Therefore the lemma is obviously true. ◇

B. Interpolate second order derivatives globally

Now we consider the problem of C^2 approximation. Let $p_4\widehat{p_3}p_0$ and $p_0\widehat{p_1}p_2$ be two segments of polygon. They join at p_0. Of course, we assume p_3, p_0 and p_1 are co-linear (see Figure 4.3). We need only consider the smoothness at p_0. Now the problem is: Whether the $\mathcal{X}^{(2)}_{(p_0,p_1)}(= -\mathcal{X}^{(2)}_{(p_0,p_3)})$, which makes $\alpha^{(2)}_{00} > 0$ on triangle $p_0p_1p_2$, makes $\alpha^{(2)}_{00} > 0$(for Case (a) of Figure 4.3) on the triangle $p_0p_3p_4$ and $\alpha^{(2)}_{00} < 0$ (for Case (b) of Figure 4.3) on the triangle $p_4p_3p_0$? We term these two join configuration a Case(a)-join and a Case(b)-join, respectively. Note that for Case(a)-join $n^T(p_2 - p_0)\cdot n^T(p_4 - p_0) < 0$; and for Case(b)-join $n^T(p_2 - p_0)\cdot n^T(p_4 - p_0) > 0$. where $n = (p_1 - p_0)^\perp = (y_1 - y_0, -(x_1 - x_0))^T$ and $n^T(p_2 - p_0)\cdot n^T(p_4 - p_0) \neq 0$ since p_0, p_1 and p_2 as well as p_0, p_3 and p_4 are affine independent.

Case(a)-join. In this case, p_2 and p_4 lie on different sides of the line p_3p_1 (see Figure 4.3.a). From (2.7), we have on the triangle $p_0p_3p_4$ that $\mathcal{X}^{(2)}_{(p_0,p_3)} = \tilde{\alpha}^{(2)}_{00}\frac{\Delta(p_0,p_3,p_4)}{\|p_3-p_0\|^3}$, here $\tilde{\alpha}^{(2)}_{00}$ is $\alpha^{(2)}_{00}$ on the triangle $p_0p_3p_4$. Hence we need to have $\tilde{\alpha}^{(2)}_{00} > 0$(see (4.4)). Then by

$$\tilde{\alpha}^{(2)}_{00}\frac{\Delta(p_0,p_3,p_4)}{\|p_3 - p_0\|^3} = -\alpha^{(2)}_{00}\frac{\Delta(p_0,p_1,p_2)}{\|p_1 - p_0\|^3} \quad (i.e., \;\; \mathcal{X}^{(2)}_{(p_0,p_3)} = -\mathcal{X}^{(2)}_{(p_0,p_1)})$$

we need

$$\Delta(p_0,p_3,p_4)\Delta(p_0,p_1,p_2) < 0 \tag{4.7}$$

Since

$$p_3 - p_0 = \beta(p_1 - p_0), \quad \beta < 0 \tag{4.8}$$

and

$$\begin{aligned}\Delta(p_0,p_3,p_4) &= (x_4 - x_0)(y_3 - y_0) - (x_3 - x_0)(y_4 - y_0) \\ &= \beta n^T(p_4 - p_0)\end{aligned} \tag{4.9}$$

$$\Delta(p_0, p_1, p_2) = (x_2 - x_0)(y_1 - y_0) - (x_1 - x_0)(y_2 - y_0) = n^T(p_2 - p_0) \quad (4.10)$$

we have

$$\Delta(p_0, p_3, p_4)\Delta(p_0, p_1, p_2) = \beta n^T(p_4 - p_0) \cdot n^T(p_2 - p_0) > 0 \quad (4.11)$$

This is a contradiction to (4.7) and therefore it is impossible to make C^2 continuity for a Case(a)-join.

Case(b)-join. In this case p_2 and p_4 lie on the same side of the line p_3p_1 (see Figure 4.3.b). From (2.7), we have on the triangle $p_4p_3p_0$ that $\mathcal{X}^{(2)}_{(p_0,p_3)} = \bar{\alpha}^{(2)}_{00}\frac{\Delta(p_4,p_3,p_0)}{\|p_3-p_0\|^3}$, here $\bar{\alpha}^{(2)}_{00}$ is $\alpha^{(2)}_{02}$ on the triangle $p_4p_3p_0$. Hence we need to have $\bar{\alpha}^{(2)}_{00} < 0$(see (4.4)). Then by

$$\bar{\alpha}^{(2)}_{00}\frac{\Delta(p_4, p_3, p_0)}{\|p_3 - p_0\|^3} = -\alpha^{(2)}_{00}\frac{\Delta(p_0, p_1, p_2)}{\|p_1 - p_0\|^3} \quad (i.e., \quad \mathcal{X}^{(2)}_{(p_0,p_3)} = -\mathcal{X}^{(2)}_{(p_0,p_1)})$$

we need

$$\Delta(p_4, p_3, p_0)\Delta(p_0, p_1, p_2) > 0 \quad (4.12)$$

Since $\Delta(p_4, p_3, p_0) = -\Delta(p_0, p_3, p_4)$, we have by (4.9), (4.10) that (4.12) is true. Therefore, it is possible to produce C^2 continuity for a Case(b)-join.

The above discussion can be summarized in the following theorem.

Theorem 4.2. **A.** *Let $q_0\widehat{v_0}q_1$ and $q_1\widehat{v_1}q_2$ be two C^1 joined polygon segments at q_1. Then if q_1 is of a Case(a)-join, there are no quadratic algebraic curves that meet with C^2 continuity at q_1.*

B. *Let $\{q_i\widehat{v_i q_{i+1}}\}_{i=0}^m$ be a C^1 polygon. Then if q_i is of a Case(b)-join for $i = 1, 2, \ldots, m$, there exist $\mathcal{X}^{(2)}_{(q_i,v_i)} \neq 0$, $i = 0, 1, \ldots, m$, and $\mathcal{X}^{(2)}_{(q_{m+1},v_m)} \neq 0$ such that*

$$\mathcal{X}^{(2)}_{(q_i,v_i)}\Delta(q_i, v_i, q_{i+1}) > 0, \quad i = 0, 1, \ldots, m, \quad \mathcal{X}^{(2)}_{(q_{m+1},v_m)}\Delta(q_m, v_m, q_{m+1}) < 0 \quad (4.13)$$

$$\begin{aligned} &\mathcal{X}^{(2)}_{(q_i,v_i)}\|v_i - q_i\|^3 - \mathcal{X}^{(2)}_{(q_{i+1},v_{i+1})}\|v_i - q_{i+1}\|^3 = 0, \quad i = 0, 1, \ldots, m-1, \\ &\mathcal{X}^{(2)}_{(q_m,v_m)}\|v_m - q_m\|^3 + \mathcal{X}^{(2)}_{(q_{m+1},v_m)}\|v_m - q_{m+1}\|^3 = 0 \end{aligned} \quad (4.14)$$

and the resulting quadratic algebraic curves, defined on each triangle $q_iv_iq_{i+1}$ by (4.1) with b_{10} determined by (4.3) and (2.7) from $\mathcal{X}^{(2)}_{(q_i,v_i)}$, form a C^2 continuous curve.

Notes:

1. The inequalities (4.13) and the linear system (4.14) are homogeneous, then the solution can have a positive constant multiplier. This multiplier can be used to control the shape of the curve globally.
2. If $\mathcal{X}^{(2)}_{(q_0,v_0)}\Delta(q_0, v_0, q_1) > 0$, then (4.14) yields (4.13).
3. The system (4.14) has $m+1$ equations and $m+2$ unknowns. Its solution is one dimensional. However, if the polygon is closed, that is $q_0 = q_{m+1}$, if we want the composite curve to be C^2 continuous at q_0, that is $\mathcal{X}^{(2)}_{(q_0,v_0)} = -\mathcal{X}^{(2)}_{(q_{m+1},v_m)}$, then (4.14) becomes a $(m+1)\times(m+1)$ system. In order to have no trivial solution, its coefficient matrix must be singular. That is, the determinant of coefficient matrix, which is $\prod_{i=0}^{m}\|v_i - q_i\|^3 - \prod_{i=0}^{m}\|v_i - q_{i+1}\|^3$, must be zero. i.e. $\prod_{i=0}^{m}\|v_i - q_i\| = \prod_{i=0}^{m}\|v_i - q_{i+1}\|$.

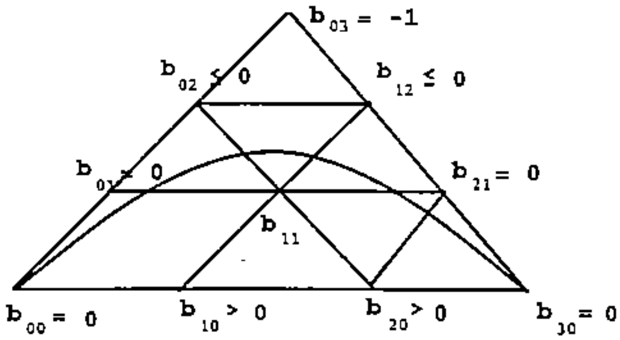

Figure 4.4: Cubic A-Splines Characterization

4.2 Cubic A-Splines

Since quadratic A-spline cannot always achieve C^2 approximation of a general C^1 polygon, we consider the use of cubic algebraic splines to achieve this and higher order of smoothness. We shall show that the cubic A-spline can always achieve C^3 smoothness by local interpolation and even C^4 smoothness by adding a few global conditions. Note that a parametric cubic spline can only achieve local C^2-continuity. Algebraic cubics have 9 degrees of freedom which drops to 5 for a C^1 curve at the endpoints of the polygon segment.

As in §4.1, we still use transform (2.1)–(2.3). Consider a cubic algebraic curve segment defined over a triangle $p_1p_1p_2$(see Fig. 4.4).

$$\begin{aligned} F(\alpha_0,\alpha_1) &= -\alpha_1^3 + b_{10}\alpha_0(1-\alpha_0-\alpha_1)^2 + b_{20}\alpha_0^2(1-\alpha_0-\alpha_1) \\ &\quad + b_{02}\alpha_1^2(1-\alpha_0-\alpha_1) + b_{12}\alpha_0\alpha_1^2 + b_{11}\alpha_0\alpha_1(1-\alpha_0-\alpha_1) \end{aligned} \tag{4.15}$$

with

$$b_{10} > 0, \quad b_{20} > 0, \quad b_{02} \le 0, \quad b_{12} \le 0 \tag{4.16}$$

Following the scheme of §4.1, we first compute $\alpha_{0i}^{(k)} = \alpha_{0i}^{(k)}(0)$ by differentiating $F(\alpha_0,\alpha_1) = 0$ about α_1 as follows:

$$\alpha_{00}^{(1)} = 0 \qquad \alpha_{02}^{(1)} = -1$$

$$\frac{\alpha_{00}^{(2)}}{2!} = -\frac{b_{02}}{b_{10}}, \qquad \frac{\alpha_{02}^{(2)}}{2!} = \frac{b_{12}}{b_{20}}, \tag{4.17}$$

$$\frac{\alpha_{00}^{(3)}}{3!} = \frac{b_{10} - b_{10}b_{02} + b_{11}b_{02}}{b_{10}^2}, \qquad \frac{\alpha_{02}^{(3)}}{3!} = \frac{-b_{20} + b_{20}b_{12} - b_{11}b_{12}}{b_{20}^2} \tag{4.18}$$

$$\frac{\alpha_{00}^{(4)}}{4!} = -\frac{b_{11} - 2b_{10}}{b_{10}^2} \ (\text{if } b_{02} = 0), \quad \frac{\alpha_{02}^{(4)}}{4!} = -\frac{2b_{20} - b_{11}}{b_{20}^2} \ (\text{if } b_{12} = 0) \tag{4.19}$$

4.2.1 C^2 Continuity

If the given polygon is C^1, then for any b_{10}, b_{20}, b_{11}, b_{02} and b_{12} that satisfy (4.16), the composite curve is C^1 continuous. Now we consider the problem of C^2 continuity.

Again, we consider the join of two curve segments on polygon $p_4\widehat{p_3}p_0$ and $p_0\widehat{p_1}p_2$ (see §4.1 and Figure 4.3). First, we require from (4.16) and (4.17) that

$$\alpha_{00}^{(2)} \geq 0, \qquad \alpha_{02}^{(2)} \leq 0 \tag{4.20}$$

Then by (2.7), we need to have

$$\mathcal{X}_{(p_0,p_1)}^{(2)}\Delta(p_0,p_1,p_2) \geq 0, \quad \mathcal{X}_{(p_2,p_1)}^{(2)}\Delta(p_0,p_1,p_2) \leq 0 \tag{4.21}$$

That is, for each segment of polygon, we have two inequalities. We must show that these inequalities are consistent. In other words, for each point at which we want to join two segments of curve, we have two inequalities. Both of them are related to the same second derivative value. These two inequalities must be consistent. Now we give the conditions under which the inequalities are consistent.

I. If p_0 is of a Case(a)-join, then (4.16), (4.17) and (2.7) require that

$$-\mathcal{X}_{(p_0,p_1)}^{(2)}\Delta(p_0,p_3,p_4) \geq 0 \tag{4.22}$$

If $\mathcal{X}_{(p_0,p_1)}^{(2)} \neq 0$, then by (4.21) and (4.22) we need to have $\Delta(p_0,p_1,p_2)\Delta(p_0,p_3,p_4) \leq 0$. Unfortunately, this is not true (see (4.11)). Therefore (4.21) and (4.22) hold iff $\mathcal{X}_{(p_0,p_1)}^{(2)} = 0$.

II. If p_0 is of a Case(b)-join, $n^T(p_2-p_0)\cdot n^T(p_4-p_0) > 0$, then (4.16), (4.17) and (2.7) require that $-\mathcal{X}_{(p_0,p_1)}^{(2)}\Delta(p_4,p_3,p_0) \leq 0$. Then by (4.21) we need to have $\Delta(p_0,p_1,p_2)\Delta(p_4,p_3,p_0) \geq 0$. This is always true(see (4.12).

Theorem 4.3 *Let $\{q_i\widehat{v_i}q_{i+1}\}_{i=0}^m$ form a C^1 polygon. If we specify the second derivative values such that $\mathcal{X}_{(q_i,v_i)}^{(2)} = 0$ if q_i is of a Case(a)-join, or $\mathcal{X}_{(q_i,v_i)}^{(2)}\Delta(q_i,v_i,q_{i+1}) \geq 0$ if q_i is of a Case(b)-join for $i = 1,2,\ldots,m$, and $\mathcal{X}_{(q_0,v_0)}^{(2)}\Delta(q_0,v_0,q_{0+1}) \geq 0$, $\mathcal{X}_{(q_{m+1},v_m)}^{(2)}\Delta(q_m,v_m,q_{m+1}) \leq 0$. Then if b_{02} and b_{12} are determined by (4.17) and (2.7), the composite curve is C^2 smooth with the given the second derivative values for any given $b_{10} > 0$, $b_{20} > 0$ and any b_{11}.*

From this theorem, we know that there are still three degrees of freedom to be used. These freedoms can be used to achieve C^3 continuity (see §4.2.2) or interpolate the given scattered data.

The left figure of Figure 4.5 shows that the A-spline can be used to fonts design and the right figure shows the different features of the C^2 cubic A-spline when the free parameters are changed.

4.2.2 C^3 Continuity

Under the C^2 join condition, now we go further to achieve C^3 continuity. Let $p_4\widehat{p_3}p_0$ and $p_0\widehat{p_1}p_2$ be two segments of polygon. They C^1 join at p_0.

A. If p_0 is of a Case(a)-join, then by Theorem 4.3 we must have $\mathcal{X}_{(p_0,p_1)}^{(2)} = 0$. From (4.18), (2.8) we have

$$\alpha_{00}^{(3)} = \frac{6}{b_{10}}, \qquad \mathcal{X}_{(p_0,p_1)}^{(3)} = \alpha_{00}^{(3)}\frac{\Delta(p_0,p_1,p_2)}{\|p_1-p_0\|^4}$$

This requires that

$$\mathcal{X}_{(p_0,p_1)}^{(3)}\Delta(p_0,p_1,p_2) > 0 \tag{4.23}$$

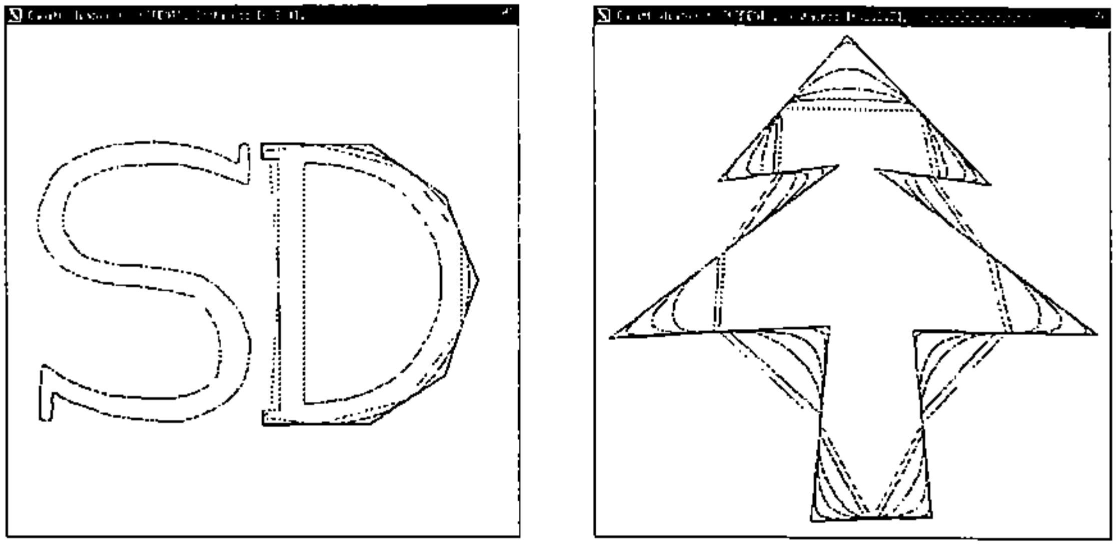

Figure 4.5: C^2 Cubic Interpolation and Approximation

The other inequality related to the point p_0 is $\mathcal{X}^{(3)}_{(p_0,p_3)}\Delta(p_0,p_3,p_4) > 0$. That is, we need to have $\Delta(p_0,p_1,p_2)\Delta(p_0,p_3,p_4) > 0$ because $\mathcal{X}^{(3)}_{(p_0,p_1)} = \mathcal{X}^{(3)}_{(p_0,p_3)}$. This is just true(see (4.11)). Therefore C^3 continuity is guaranteed by giving $\mathcal{X}^{(3)}_{(p_0,p_1)}$ as (4.23) in this case.

B. If p_0 is of a Case(b)-join, we first show that it is impossible to achieve C^3 by taking $\mathcal{X}^{(2)}_{(p_0,p_1)} = 0$. Because in this case, we require from (4.18) and (2.8) that $\mathcal{X}^{(3)}_{(p_0,p_3)}\Delta(p_4,p_3,p_0) < 0$ in addition to (4.23). Hence $\Delta(p_0,p_1,p_2)\Delta(p_4,p_3,p_0) < 0$ a contradiction to (4.12). Now suppose $\mathcal{X}^{(2)}_{(p_0,p_1)} \neq 0$, i.e., by (4.17),

$$\mathcal{X}^{(2)}_{(p_0,p_1)}\Delta(p_0,p_1,p_2) > 0 \tag{4.24}$$

then by (4.12), $-\mathcal{X}^{(2)}_{(p_0,p_1)}\Delta(p_4,p_3,p_0) < 0$. Hence C^2 continuity is guaranteed at p_0. From (4.17) and (4.18), we have

$$\left(\frac{\alpha^{(3)}_{00}}{6} - \frac{\alpha^{(2)}_{00}}{2}\right) b_{10} = 1 - b_{11}\frac{\alpha^{(2)}_{00}}{2}, \quad \left(\frac{\alpha^{(3)}_{02}}{6} - \frac{\alpha^{(2)}_{02}}{2}\right) b_{20} = -1 - b_{11}\frac{\alpha^{(2)}_{02}}{2} \tag{4.25}$$

In order to make b_{10} and b_{20} positive, we have the following inequalities

$$\pm\left\{\begin{array}{l} 1 - b_{11}\frac{\alpha^{(2)}_{00}}{2} > 0 \\ \frac{\alpha^{(3)}_{00}}{6} - \frac{\alpha^{(2)}_{00}}{2} > 0, \end{array}\right. \qquad \pm\left\{\begin{array}{l} -1 - b_{11}\frac{\alpha^{(2)}_{02}}{2} > 0 \\ \frac{\alpha^{(3)}_{02}}{6} - \frac{\alpha^{(2)}_{02}}{2} > 0 \end{array}\right. \tag{4.26}$$

Therefore, on the triangle $p_0p_1p_2$ and at point p_0, we have the inequalities.

$$\pm\left\{\begin{array}{l} 1 - \frac{\|p_1-p_0\|^3 b_{11} D_2}{\Delta(p_0,p_1,p_2)} > 0 \\ \frac{\|p_1-p_0\|^4 D_3}{\Delta(p_0,p_1,p_2)} + 2\left(\frac{D_2}{\Delta(p_0,p_1,p_2)}\right)^2 \|p_1-p_0\|^4 \langle p_1-p_0, p_2-p_0\rangle - \frac{\|p_1-p_0\|^3 D_2}{\Delta(p_0,p_1,p_2)} > 0 \end{array}\right. \tag{4.27}$$

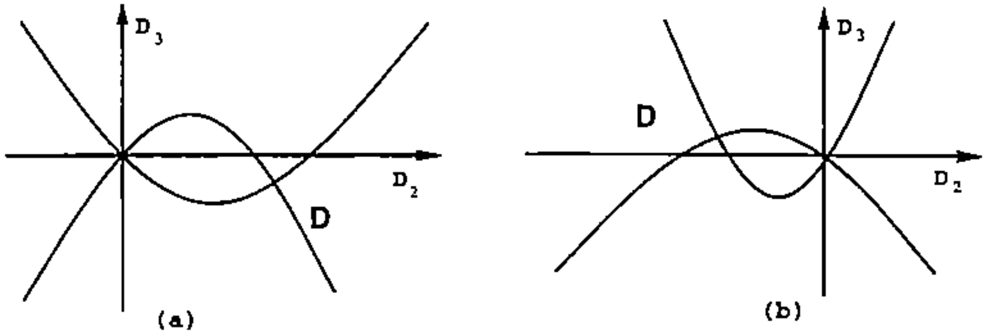

Figure 4.6: The feasible domain(denote as D for D_2 and D_3)

where $D_k = \frac{X^{(k)}_{(p_0,p_1)}}{k!}$, and at p_2

$$\pm\begin{cases} -1 - \frac{\|p_1-p_2\|^3 b_{11} D_2}{\Delta(p_0,p_1,p_2)} > 0 \\ \frac{\|p_1-p_2\|^4 D_3}{\Delta(p_0,p_1,p_2)} + 2\left(\frac{D_2}{\Delta(p_0,p_1,p_2)}\right)^2 \|p_1-p_2\|^4 \langle p_1-p_2, p_2-p_0\rangle - \frac{\|p_1-p_2\|^3 D_2}{\Delta(p_0,p_1,p_2)} > 0 \end{cases} \tag{4.28}$$

where $D_k = \frac{X^{(k)}_{(p_2,p_1)}}{k!}$. For $p_4\widehat{p_3}p_0$ at p_0, we have

$$\pm\begin{cases} -1 + \frac{\|p_3-p_0\|^3 b_{11} D_2}{\Delta(p_4,p_3,p_0)} > 0 \\ \frac{\|p_3-p_0\|^4 D_3}{\Delta(p_4,p_3,p_0)} + 2\left(\frac{D_2}{\Delta(p_4,p_3,p_0)}\right)^2 \|p_3-p_0\|^4 \langle p_3-p_0, p_0-p_4\rangle + \frac{\|p_3-p_0\|^3 D_2}{\Delta(p_4,p_3,p_0)} > 0 \end{cases} \tag{4.29}$$

For each segment of polygon, there is one parameter b_{11} to be determined. This parameter can be used to interpolate some point in the least square sense within the triangle, or to achieve C^4 continuity.

If we take $b_{11} \le 0$, then by (4.24), we know that the first inequality of $+$(4.27)(take $+$ sign) and the first inequality of $-$(4.29) (take $-$ sign) always true. Hence the feasible domain for D_2 and D_3 is determined at p_0 by

$$sign\ \Delta(p_0,p_1,p_2)\begin{cases} D_3 - a_1 D_2^2 - b_1 D_2 > 0 \\ D_3 - a_2 D_2^2 + b_2 D_2 < 0 \end{cases}$$

where $a_1 = \frac{2\langle p_1-p_0, p_0-p_2\rangle}{\Delta(p_0,p_1,p_2)}$, $b_1 = \frac{1}{\|p_1-p_0\|}$, $a_2 = \frac{2\langle p_3-p_0, p_4-p_0\rangle}{\Delta(p_4,p_3,p_0)}$, $b_2 = \frac{1}{\|p_3-p_0\|}$. Let $\mathbf{D} = \mathbf{D}(p_0,p_1,p_2,p_3,p_4)$ be the feasible domain for D_2 and D_3 which depends on p_0, p_1, p_2, p_3 and p_4. Then only for

$$\langle p_1-p_0, p_2-p_0\rangle > 0, \quad \langle p_3-p_0, p_4-p_0\rangle > 0$$

$\mathbf{D}$ is no-empty set. For $\Delta(p_0,p_1,p_2) > 0$(hence $D_2 > 0$) and $\Delta(p_0,p_1,p_2) < 0$(hence $D_2 < 0$), the feasible domains $\mathbf{D}$ for D_2 and D_3 are shown as Fig. 4.6.a and Fig. 4.6.b, respectively.

The discussion above can be summarized as the following theorem

Theorem 4.4 *Let $\{q_i\widehat{v_i q_{i+1}}\}_{i=0}^m$ form a C^1 polygon and assume $\langle v_i - q_i, q_{i+1} - q_i\rangle > 0$, $\langle v_{i-1} - q_i, q_{i-1} - q_i\rangle > 0$ if $1 \le i \le m$. At each point q_i $(i = 0, 1, \ldots, m+1)$, if we specify the second and third order derivatives as follows(regard q_i, v_i, q_{i+1} as p_0, p_1, p_2 for $i \ge 0$ and q_{i-1}, v_{i-1}, q_i as p_4, p_3, p_0 for $i \le m+1$):*

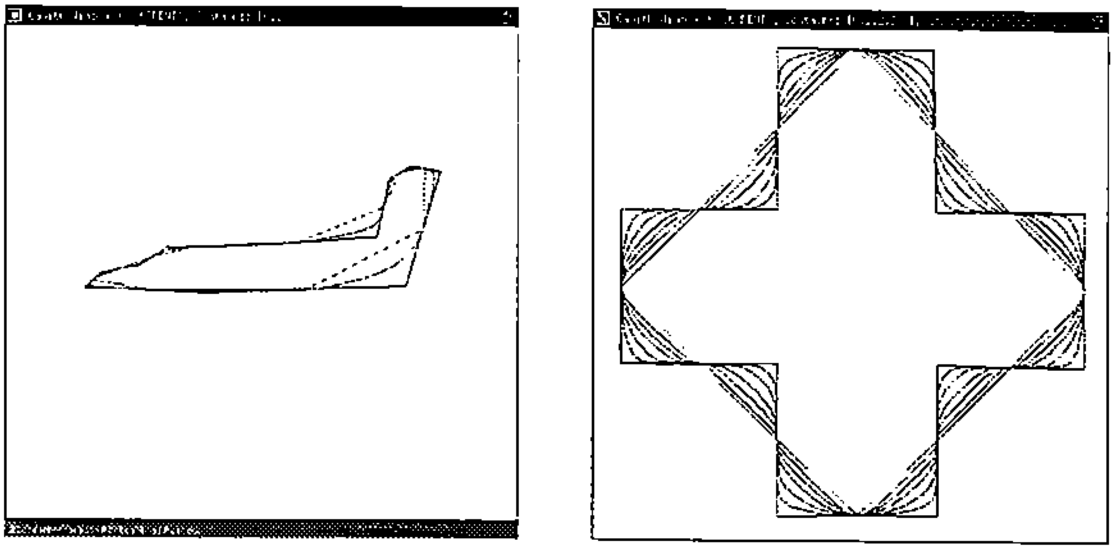

Figure 4.7: C^3 Local Interpolation and Approximation with Cubic A-splines

(a) $\mathcal{X}^{(2)}_{(q_i,v_i)} = 0$, $\mathcal{X}^{(3)}_{(q_i,v_i)}$ *satisfy (4.23) if* q_i *is of a Case(a)-join and* $1 \le i \le m$.

(b) $\mathcal{X}^{(2)}_{(q_i,v_i)}\Delta(q_i, v_i, q_{i+1}) \ge 0$, $\mathcal{X}^{(2)}_{(q_i,v_i)}$ *and* $\mathcal{X}^{(3)}_{(q_i,v_i)}$ *satisfy both* $+(4.27)$ *and* $-(4.29)$ *i.e.,* $(D_2, D_3) \in \mathbf{D}(q_i, v_i, q_{i+1}, v_{i-1}, q_{i-1})$ *if* q_i *is of a Case(b)-join and* $1 \le i \le m$.

(c) *For* $i = 0$ *and* $i = m+1$, $\mathcal{X}^{(2)}_{(q_0,v_0)}\Delta(q_0, v_0, q_{0+1}) \ge 0$, $\mathcal{X}^{(2)}_{(q_{m+1},v_m)}\Delta(q_m, v_m, q_{m+1}) \le 0$, *and* $\mathcal{X}^{(3)}_{(q_0,v_0)}$ *and* $\mathcal{X}^{(3)}_{(q_{m+1},v_m)}$ *satisfy* $+(4.27)$ *and* $-(4.28)$*, respectively.*
Then for any $b_{11} \le 0$*, if* b_{10} *and* b_{20} *are determined by (4.25);* b_{02} *and* b_{12} *are determined by (4.17), the resulted curve is of* C^3 *smooth.*

Figure 4.7 shows some examples of C^3 cubic A-spline. These splines are displayed by evaluating the formulas of Section 5.

4.2.3 C^4 Continuity ?

At present, we consider C^4 smoothness, under the C^3 discussion above, only for polygons that join according to Case A. In this case, we have $\mathcal{X}^{(2)}_{(q_i,v_i)} = 0$, then $b_{02} = b_{12} = 0$, and b_{10} and b_{20} are determined by (4.25) from the third derivatives. Then parameter b_{11} is totally free. By choosing b_{11} properly, C^4 continuity can be achieved. It follows from (4.19) that $b_{10}^2\alpha_{00}^{(4)} + b_{20}^2\alpha_{02}^{(4)} = 48(b_{10} - b_{20})$, i.e.,

$$b_{10}^2 \frac{\|p_1 - p_0\|^5}{\Delta(p_0, p_1, p_2)} \mathcal{X}^{(4)}_{(p_0,p_1)} + b_{20}^2 \frac{\|p_1 - p_2\|^5}{\Delta(p_0, p_1, p_2)} \mathcal{X}^{(4)}_{(p_2,p_1)} = 48(b_{10} - b_{20})$$

Let $\{q_i\widehat{v_iq_i}_{+1}\}_{i=0}^m$ be the given C^1 polygon. Then for each $q_i\widehat{v_iq_i}_{+1}$ we have an equation

$$\begin{aligned} &\delta_i \mathcal{X}^{(4)}_{(q_i,v_i)} + \beta_i \mathcal{X}^{(4)}_{(q_{i+1},v_{i+1})} = \gamma_i, \quad i = 0, 1, \ldots, m-1 \\ &\delta_m \mathcal{X}^{(4)}_{(q_m,v_m)} - \beta_m \mathcal{X}^{(4)}_{(q_{m+1},v_m)} = \gamma_m \end{aligned} \tag{4.30}$$

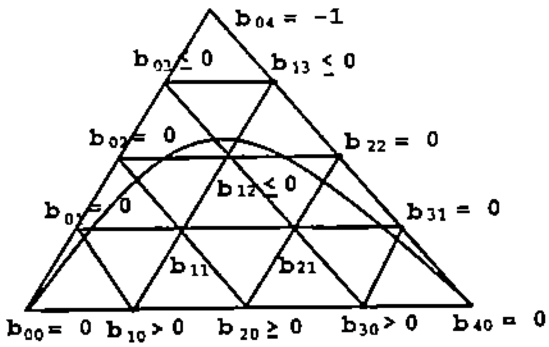

Figure 4.8: Quartic A-Splines Characterization

This is a system that has $m+1$ equations and $m+2$ unknowns. Since $\delta_i \neq 0$, $\beta_i \neq 0$, system (4.30) always has solution and the solution is one dimensional. If the polygon is closed, that is $q_0 = q_{m+1}$ and $\mathcal{X}^{(4)}_{(q_0,v_0)} = -\mathcal{X}^{(4)}_{(q_{m+1},v_m)}$, then the determinant of the coefficient matrix is $\prod_{i=0}^{m} \delta_i + (-1)^m \prod_{i=0}^{m} \beta_i$. If it is non-zero. The system has unique solution.

4.3 Quartic A-Splines, C^5 Continuity

In order to achieve even higher order of smoothness, we consider quartic A-splines local interpolation and approximations. Quartic has 14 degrees of freedom which is reduced to 8 by forcing the curve to be C^2 continuous at the end-points of the polygon(see Fig. 4.8). Let

$$\begin{aligned} F(\alpha_0,\alpha_1) &= -\alpha_1^4 + b_{03}\alpha_1^3(1-\alpha_0-\alpha_1) + b_{13}\alpha_0\alpha_1^3 + b_{12}\alpha_0\alpha_1^2(1-\alpha_0-\alpha_1) \\ &+ b_{11}\alpha_0\alpha_1(1-\alpha_0-\alpha_1)^2 + b_{21}\alpha_0^2\alpha_1(1-\alpha_0-\alpha_1) \\ &+ b_{10}\alpha_0(1-\alpha_0-\alpha_1)^3 + b_{20}\alpha_0^2(1-\alpha_0-\alpha_1)^2 + b_{30}\alpha_0^3(1-\alpha_0-\alpha_1) \end{aligned} \tag{4.31}$$

with

$$b_{10} > 0, \quad b_{20} \geq 0, \quad b_{30} > 0, \quad b_{12} \leq 0, \quad b_{03} \leq 0, \quad b_{13} \leq 0 \tag{4.32}$$

As before, we can compute the derivatives as follows: $\alpha_{00}^{(1)} = 0$, $\alpha_{02}^{(1)} = -1$, $\alpha_{00}^{(2)} = 0$, $\alpha_{02}^{(2)} = 0$ and

$$\frac{\alpha_{00}^{(3)}}{3!} = -\frac{b_{03}}{b_{10}}, \quad \frac{\alpha_{02}^{(3)}}{3!} = \frac{b_{13}}{b_{30}} \tag{4.33}$$

$$\frac{\alpha_{00}^{(4)}}{4!} = \frac{b_{10} + b_{11}b_{03} - 2b_{10}b_{03}}{b_{10}^2}, \quad \frac{\alpha_{02}^{(4)}}{4!} = \frac{-b_{30} - b_{13}b_{21} + 2b_{13}b_{30}}{b_{30}^2} \tag{4.34}$$

$$\frac{\alpha_{00}^{(5)}}{5!} = -\frac{(3b_{10} + b_{12} - 2b_{11})\frac{\alpha_{00}^{(3)}}{3!} + (b_{11} - 3b_{10})\frac{\alpha_{00}^{(4)}}{4!}}{b_{10}} \tag{4.35}$$

$$\frac{\alpha_{02}^{(5)}}{5!} = -\frac{(3b_{30} + b_{12} - 2b_{21})\frac{\alpha_{02}^{(3)}}{3!} + (b_{21} - 3b_{30})\frac{\alpha_{02}^{(4)}}{4!}}{b_{30}}, \tag{4.36}$$

Now we establish the conditions under which (4.32) are satisfied. It follows from (4.33) and (4.34) that

$$b_{03} = -b_{10}\frac{\alpha_{00}^{(3)}}{3!}, \quad b_{13} = b_{30}\frac{\alpha_{02}^{(3)}}{3!}, \tag{4.37}$$

with $\frac{\alpha_{00}^{(3)}}{3!} \geq 0$ and $\frac{\alpha_{02}^{(3)}}{3!} \leq 0$

$$b_{11} = \frac{1 + 2b_{10}\frac{\alpha_{00}^{(3)}}{3!} - b_{10}\frac{\alpha_{00}^{(4)}}{4!}}{\frac{\alpha_{00}^{(3)}}{3!}}, \quad b_{21} = \frac{-1 + 2b_{30}\frac{\alpha_{02}^{(3)}}{3!} - b_{30}\frac{\alpha_{02}^{(4)}}{4!}}{\frac{\alpha_{02}^{(3)}}{3!}} \tag{4.38}$$

Substitute (4.37) and (4.38) into (4.35) and (4.36) respectively, we have

$$b_{10} = \frac{2\frac{\alpha_{00}^{(3)}}{3!} - \frac{\alpha_{00}^{(4)}}{4!} - \left(\frac{\alpha_{00}^{(3)}}{3!}\right)^2 b_{12}}{\frac{\alpha_{00}^{(5)}}{5!}\frac{\alpha_{00}^{(3)}}{3!} + \frac{\alpha_{00}^{(4)}}{4!}\frac{\alpha_{00}^{(3)}}{3!} - \left(\frac{\alpha_{00}^{(3)}}{3!}\right)^2 - \left(\frac{\alpha_{00}^{(4)}}{4!}\right)^2} \tag{4.39}$$

$$b_{30} = \frac{-2\frac{\alpha_{02}^{(3)}}{3!} + \frac{\alpha_{02}^{(4)}}{4!} - \left(\frac{\alpha_{02}^{(3)}}{3!}\right)^2 b_{12}}{\frac{\alpha_{02}^{(5)}}{5!}\frac{\alpha_{02}^{(3)}}{3!} + \frac{\alpha_{02}^{(4)}}{4!}\frac{\alpha_{02}^{(3)}}{3!} - \left(\frac{\alpha_{02}^{(3)}}{3!}\right)^2 - \left(\frac{\alpha_{02}^{(4)}}{4!}\right)^2} \tag{4.40}$$

where $b_{12} \leq 0$ suppose to be given. In order to have $b_{10} > 0$, we must have

$$\pm\begin{cases} 2\frac{\alpha_{00}^{(3)}}{3!} - \frac{\alpha_{00}^{(4)}}{4!} - \left(\frac{\alpha_{00}^{(3)}}{3!}\right)^2 b_{12} > 0 \\ \frac{\alpha_{00}^{(5)}}{5!}\frac{\alpha_{00}^{(3)}}{3!} + \frac{\alpha_{00}^{(4)}}{4!}\frac{\alpha_{00}^{(3)}}{3!} - \left(\frac{\alpha_{00}^{(3)}}{3!}\right)^2 - \left(\frac{\alpha_{00}^{(4)}}{4!}\right)^2 > 0 \end{cases} \tag{4.41}$$

By using relations (2.8)-(2.10) we know that the derivatives at p_0 should satisfied the following inequalities.

$$\Delta(p_0, p_1, p_2)\mathcal{X}_{(p_0,p_1)}^{(3)} \geq 0 \tag{4.42}$$

$$\pm\begin{cases} 2\Delta(p_0,p_1,p_2)D_3 - \|p_1 - p_0\|\Delta(p_0,p_1,p_2)D_4 - b_{12}\|p_1 - p_0\|^4 D_3^2 > 0 \\ D_5 D_3 + \frac{3\langle p_1 - p_0, p_2 - p_0\rangle D_3^3}{\Delta(p_0,p_1,p_2)} - \frac{D_3^2}{\|p_1 - p_0\|^2} + \frac{D_4 D_3}{\|p_1 - p_0\|} - D_4^2 > 0 \end{cases} \tag{4.43}$$

where $D_k = \frac{\mathcal{X}_{(p_0,p_1)}^{(k)}}{k!}$. Similarly, at p_2 we have

$$\Delta(p_0, p_1, p_2)\mathcal{X}_{(p_2,p_1)}^{(3)} \leq 0 \tag{4.44}$$

$$\pm\begin{cases} -2\Delta(p_0,p_1,p_2)D_3 + \|p_1 - p_2\|\Delta(p_0,p_1,p_2)D_4 - b_{12}\|p_1 - p_2\|^4 D_3^2 > 0 \\ D_5 D_3 + \frac{3\langle p_1 - p_2, p_2 - p_0\rangle D_3^3}{\Delta(p_0,p_1,p_2)} - \frac{D_3^2}{\|p_1 - p_2\|^2} + \frac{D_4 D_3}{\|p_1 - p_2\|} - D_4^2 > 0 \end{cases} \tag{4.45}$$

where $D_k = \frac{\mathcal{X}_{(p_2,p_1)}^{(k)}}{k!}$. Therefore, we have got the following

Lemma 4.5 *Let the derivatives $\mathcal{X}^{(k)}_{(p_i,p_1)}$ be given for $i=0,2,\ k=3,4,5$ such that conditions (4.42)-(4.45) are satisfied. Then for any given $b_{12}\le 0$, the BB coefficients b_{03}, b_{13}, b_{11}, b_{21}, b_{10} and b_{30} defined by (4.37)-(4.40), respectively, satisfy the conditions (4.32). The coefficient b_{20} left to be free.*

Now we consider the join of two curve segments that are defined on two C^1 polygon $p_4\widehat{p_3}p_0$ and $p_0\widehat{p_1}p_2$ (see §4.1 and Figure 4.3). Let $f_1(x,y)=0$ and $f_2(x,y)=0$ be two quartic algebraic polynomials whose BB forms on the triangles $p_4p_3p_0$ and $p_0p_1p_2$ are as (4.31), where $b_{12}\le 0$ suppose to be given. Then the two curves C^2 join at p_0 automatically with the zero first and second derivatives under local coordinates. Now we show how to specify the third, fourth and fifth derivatives at p_0 such that BB coefficients satisfy (4.32) and the two curves C^5 join at p_0 with the given derivatives.

Case A. p_0 is of a Case(a)-join. At first, the third derivative $\mathcal{X}^{(3)}_{(p_0,p_1)}$ should be given such that the inequality (4.42) holds strictly ($\mathcal{X}^{(3)}_{(p_0,p_1)}=0$ will make (4.43) and (4.47) inconsistent). This $\mathcal{X}^{(3)}_{(p_0,p_1)}$ will make $b_{03}<0$ by (4.37) on the triangle $p_0p_1p_2$. Sine $\Delta(p_0,p_1,p_2)\Delta(p_0,p_3,p_4)>0$ (see (4.11)), then

$$\mathcal{X}^{(3)}_{(p_0,p_3)}\Delta(p_0,p_3,p_4)>0 \tag{4.46}$$

This implies by (4.37) that $b_{03}<0$ on the triangle $p_0p_3p_4$. For the fourth and fifth derivatives, apart from inequalities (4.43) on the triangle $p_0p_1p_2$, we need also to have the inequalities

$$\pm\begin{cases} 2\Delta(p_0,p_3,p_4)D_3+\|p_3-p_0\|\Delta(p_0,p_3,p_4)D_4-\bar{B}_{12}\|p_3-p_0\|^4D_3^2>0 \\[2ex] D_5D_3+\dfrac{3(p_3-p_0,p_4-p_0)D_3^3}{\Delta(p_0,p_3,p_4)}-\dfrac{D_3^2}{\|p_3-p_0\|^2}-\dfrac{D_4D_3}{\|p_3-p_0\|}-D_4^2>0\end{cases} \tag{4.47}$$

on $p_0p_3p_4$, where $\bar{B}_{12}$ is the BB coefficient b_{21} on $p_0p_3p_4$

Suppose the third derivative $\mathcal{X}^{(3)}_{(p_0,p_1)}$ has been chosen, then we regard D_4 and D_5 as unknowns in the (4.43) and (4.47). We note that constant terms of the first inequalities of (4.43) and (4.47) are greater than zero for any $b_{12}\le 0$, the coefficients of D_4 have opposite sign; and further D_5 and D_4^2 in the second inequalities of them have the same sign coefficients and D_4 has opposite sign coefficients. These observations suggest that

(i) only +(4.43) and +(4.47) are consistent

(ii) the feasible domain $\mathrm{D}(b_{12})$ of D_4 and D_5, which depends on b_{12}, is as in the Fig.4.9 for $\Delta(p_0,p_1,p_2)>0$

(iii) $\mathrm{D}(s)\subset\mathrm{D}(t)$, if $0\ge s>t$.

Case B. p_0 is of a Case(b)-join. Firstly, we must choose

$$\mathcal{X}^{(3)}_{(p_0,p_1)}=0,\quad \text{then}\quad b_{03}=0 \tag{4.48}$$

otherwise, both (4.42) and (4.46) can not be satisfied because of $\Delta(p_0,p_1,p_2)\Delta(p_0,p_3,p_4)<0$. Secondly, choose $\mathcal{X}^{(4)}_{(p_0,p_1)}$ such that

$$\Delta(p_0,p_1,p_2)\mathcal{X}^{(4)}_{(p_0,p_1)}>0 \tag{4.49}$$

This will make

$$b_{10}=\frac{4!\Delta(p_0,p_1,p_2)}{\|p_1-p_0\|^5\mathcal{X}^{(4)}_{(p_0,p_1)}}>0 \tag{4.50}$$

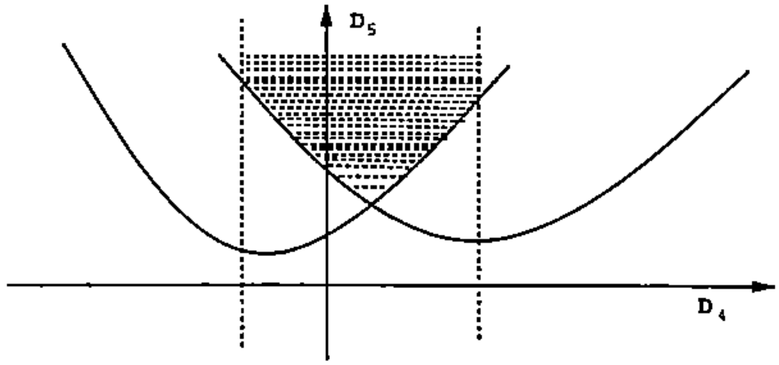

Figure 4.9: The feasible domain $D(b_{12})$(the shaded part)

For the same purpose on the triangle $p_0p_3p_4$, we must have $\Delta(p_0,p_3,p_4)\mathcal{X}^{(4)}_{(p_0,p_3)} > 0$. This is just true by (4.12). Finally, since b_{11} can be any number, $\mathcal{X}^{(5)}_{(p_0,p_1)}$ can be arbitrarily given and

$$b_{11} = 3b_{10} - \frac{4!\|p_1 - p_0\|b_{10}\mathcal{X}^{(5)}_{(p_0,p_1)}}{5!\mathcal{X}^{(4)}_{(p_0,p_1)}} \tag{4.51}$$

From the discussion above, we notes that for the given b_{12}, the other three BB coefficients b_{03}, b_{10} and b_{11} are determined by the derivatives $\mathcal{X}^{(3)}_{(p_0,p_1)}$, $\mathcal{X}^{(4)}_{(p_0,p_1)}$ and $\mathcal{X}^{(5)}_{(p_0,p_1)}$. Similarly, b_{13}, b_{30} and b_{21} are determined by the derivatives $\mathcal{X}^{(3)}_{(p_2,p_1)}$, $\mathcal{X}^{(4)}_{(p_2,p_1)}$ and $\mathcal{X}^{(5)}_{(p_2,p_1)}$. The parameter b_{20} is left to be free subject to nonnegative. This freedom can be used to control the shape of the curve. For example, we can interpolate one point inside the triangle. Therefore we have the following

Theorem 4.6. *Let $\{q_i\widehat{v_iq_{i+1}}\}_{i=0}^m$ form a C^1 polygon. At each q_i, if we specify the derivative values as follows (regard q_i, v_i, q_{i+1} as p_0, p_1, p_2 for $i \geq 0$ and q_{i-1}, v_{i-1}, q_i as p_4, p_3, p_0 for $i \leq m+1$):*

(a) *If q_i is of a Case(a)-join (Case A) and $1 \leq i \leq m$, specify $\mathcal{X}^{(k)}_{(q_i,v_i)}(k = 3,4,5)$ according to the strict inequalities (4.42), +(4.43), (4.46) and +(4.47).*

(b) *If q_i is of a Case(b)-join (Case B) and $1 \leq i \leq m$, specify the derivative values $\mathcal{X}^{(k)}_{(q_i,v_i)}(k = 3,4)$ according to (4.48), (4.49) and $\mathcal{X}^{(5)}_{(q_i,v_i)}$ arbitrarily.*

(c) *specify $\mathcal{X}^{(k)}_{(q_0,v_0)}(k = 3,4,5)$ according to (4.42), (4.43) and specify $\mathcal{X}^{(k)}_{(q_{m+1},v_m)}(k = 3,4,5)$ according to (4.44), (4.45).*

Then, if the BB coefficients are determined by (4.37)–(4.40) in Case A or by (4.48), (4.50) and (4.51) in Case B, the composite curve is C^5 continuous.

Figure 4.10 shows examples of C^5 quartic A-splines.

4.4 An $n-2$-Parameter Family of Degree n A-Splines

Work out higher order smoothness by using higher order A-spline is a rather difficult task by the previous scheme. But for C^1 approximation, that is automatically achieved for C^1 polygon

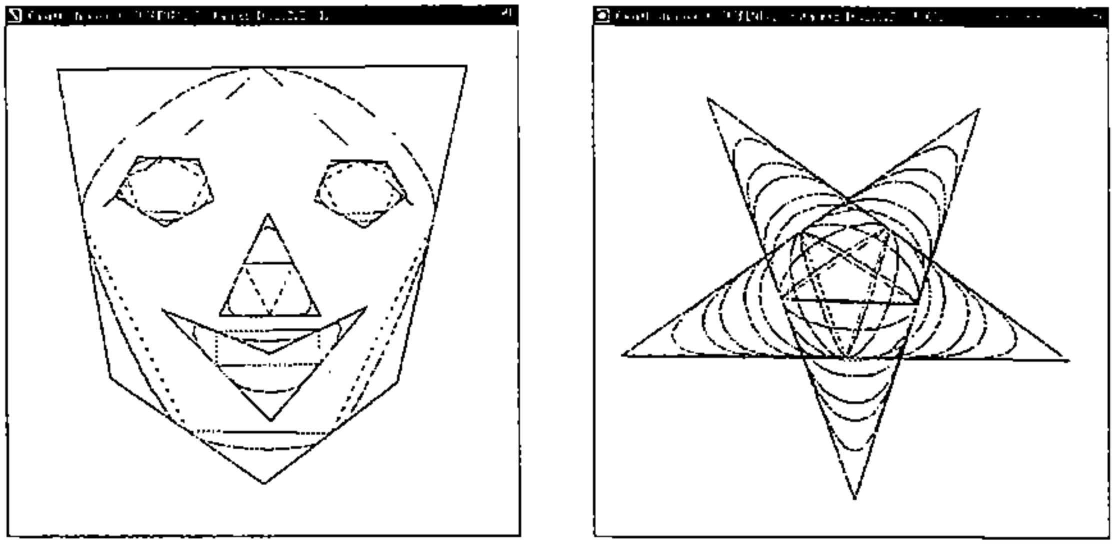

Figure 4.10: C^5 Quartic Approximation

by our A-splines, a special class of higher order A-splines can be used. Let

$$
\begin{aligned}
F(\alpha_0, \alpha_1) &= \sum_{i=1}^{n-2} b_{i1} \frac{n!}{i!(n-i-1)!} \alpha_0^i \alpha_1 (1-\alpha_0-\alpha_1)^{n-i-1} \\
&+ \sum_{i=1}^{n-1} b_{i0} \frac{n!}{i!(n-i)!} \alpha_0^i (1-\alpha_0-\alpha_1)^{n-i} \\
&+ \sum_{i=0}^{n-2} b_{i2} \frac{n!}{i!2!(n-i-2)!} \alpha_0^i \alpha_1^2 (1-\alpha_0-\alpha_1)^{n-i-2}
\end{aligned}
\tag{4.52}
$$

where $b_{10} > 0, b_{n-1,0} > 0$ $b_{i,0} \geq 0$ for $i = 2, \cdots, n-2$; and $b_{02} < 0, b_{n-2,2} < 0$ and $b_{i2} \leq 0$ for $i = 1, \cdots n-3$ are suppose to be given(see Fig. 4.11). A simple case is to choose all of them to be zeros except $b_{10} = b_{n-1,0} = -b_{02} = -b_{n-2,2} = 1$. So we have $n-2$ parameters curve family $F(\alpha_0, \alpha_1) = 0$ and all of them pass through $P_0 = (0,0)^T$ and $P_2 = (1,0)^T$ and tangent with the line $\alpha_0 = 0$ and $\alpha_0 + \alpha_1 - 1 = 0$. Now we use these $n-2$ parameters $b_{i1}, i = 1, \cdots n-2$ to interpolate $n-2$ points within the triangle. It it easy to get the following result about solvability of the interpolation problem.

Theorem 4.7. *Let $n > 2$ and $Q_j = (\alpha_{0j}, \alpha_{1j})^T, j = 1, 2, \cdots, n-2$ be given such that $\alpha_{0j} > 0$, $\alpha_{1j} > 0$, $\alpha_{0j} + \alpha_{1j} < 1$.*

(i). Then there exists a set of $b_{i1}, i = 1, \cdots n-2$ uniquely such that $F(\alpha_{0j}, \alpha_{1j}) = 0$ iff $\beta_1, \cdots, \beta_{n-2}$ are distinct, here $\beta_j = \frac{\alpha_{0j}}{1-\alpha_{1j}}$.

(ii). If $\beta_1 < \beta_2 < \cdots < \beta_{n-2}$, then the curve $F(\alpha_0, \alpha_1) = 0$ passes through the points $Q_1, Q_2, \cdots, Q_{n-2}$ one after another starting from P_0.

Proof. (i). Divide $F(\alpha_0, \alpha_1)$ by $\alpha_0 \alpha_1 (1-\alpha_0-\alpha_1)^{n-2}$ and let $t = \frac{\alpha_0}{1-\alpha_0-\alpha_1}$, we are lead to a

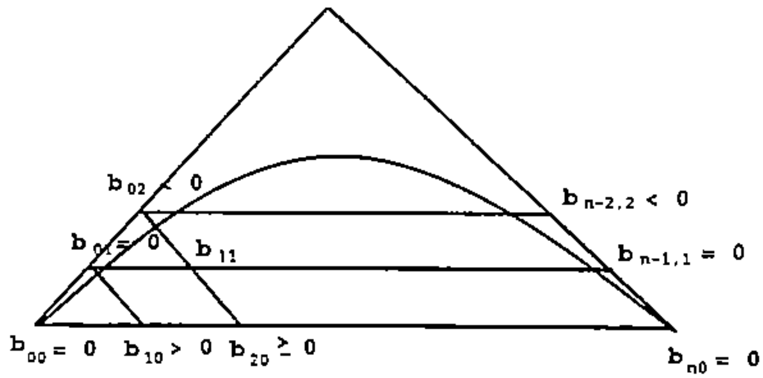

Figure 4.11: Quartic A-Splines

classical polynomial interpolation problem: Find $b_{i1}, i = 1, \cdots n-2$ such that

$$\sum_{i=1}^{n-2} b_{i1} \frac{n!}{i!(n-i-1)!} t_j^{i-1} = \gamma_j, \quad j = 1, \cdots, n-2$$

where $t_j = \frac{\alpha_{0j}}{1-\alpha_{0j}-\alpha_{1j}}$ and γ_j are constants. This problem has unique solution iff t_j are distinct. It is easy to see that t_j are distinct iff β_j are so.

(ii). It follows from Theorem 3.1 that $(\alpha_0, \alpha_1)^T$, if $(\alpha_0, \alpha_1)^T$ on the curve $F(\alpha_0, \alpha_1) = 0$, is a function of β and $(\alpha_0(0), \alpha_1(0))^T = P_0$, $(\alpha_0(1), \alpha_1(1))^T = P_2$. Therefore, the curve passes through the point first that has smaller β. $\diamond$

Like Lagrange polynomial interpolation problem, the coefficient can be computed by any classical method. Unlike Lagrange interpolation problem, the curve here always lies in the given triangle. Hence the error is easy to control. The second statement of the theorem says that the route that the curve passes through these points is known by the position of the points. However, for a general interpolation problem by implicit algebraic polynomial, it is difficult to know a priori that the curve may pass through within a given region.

If the number of data points is larger than $n-2$, the interpolation problem can be solved in the least square sense.

5 Evaluation Formulas of A-Splines for Display

For easy of evaluation or display of the quadratic, cubic, quartic and $n-2$-parameter splines discussed in this paper, we derive some formulas of parameterization that are different from the previously published rational parameterization (see [1],[2],[3],[4] and [25] for more references). These formulas are valid for the single A-spline segment within the given triangle. In general, separating the different segments of the algebraic curve for curve segment display is a difficult problem [5]. All these formulas are in explicit closed form, and provide one possible quick approach for displaying the splines. Of course, more enhanced techniques based on subdivision or similar to integer forward differencing [18] can also be developed for these A-splines. We leave that as an open problem for now.

5.1 Parameterization of Quadratic A-Splines

Let $F(\alpha_0, \alpha_1)$ as (4.1), then by intersect $F(\alpha_0, \alpha_1) = 0$ with the line $\alpha_0 = \beta(1-\alpha_1)$, we get the equation $-\alpha_1^2 + b_{10}\beta(1-\beta)(1-\alpha_1)^2 = 0$. Therefore we have

$$\alpha_1(\beta) = \frac{\sqrt{b_{10}(1-\beta)\beta}}{1+\sqrt{b_{10}(1-\beta)\beta}}, \quad \alpha_0(\beta) = \beta(1-\alpha_1(\beta))$$

for

$$\beta = \frac{1-\cos\beta}{2}, \quad 0 \leq \beta \leq \pi \tag{5.1}$$

Since the curve $F(\alpha_0, \alpha_1) = 0$ tangent with the line $\alpha_0 = 0$ and $\alpha_0 + \alpha_1 - 1 = 0$ at points $(0,0)^T$ and $(1,0)^T$, respectively, we need more points in drawing the curve when β is near to 0 or 1. This is the reason that we use (5.1) rather than use β as a parameter directly.

5.2 Parameterization of Cubic A-Splines

Let $F(\alpha_0, \alpha_1)$ as (4.15) and let $\alpha_0 = \beta(1-\alpha_1)$. Then we have $At^3 + Bt^2 - Ct - 1 = 0$, where $t = \frac{1-\alpha_1}{\alpha_1}$ and

$$A = \beta(1-\beta)[(1-\beta)b_{10} + \beta b_{20}], \quad B = b_{11}\beta(1-\beta), \quad C = -b_{02}(1-\beta) - \beta b_{12},$$

and $A \geq 0$, $C \geq 0$. Therefore, if $\Delta \geq 0$, by noting $t(\beta) = \frac{1-\beta_1(\beta)}{\alpha_1(\beta)}$ is a real root of the above equation we have

$$t(\beta) = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} - \frac{B}{3A}, \tag{5.2}$$

where

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \quad p = -\frac{B^2 + 3AC}{3A^2}, \quad q = \frac{2B^3 + 9ABC - 27A^2}{27A^3}$$

If $\Delta < 0$, then by $0 \leq t \leq \infty$ and $p < 0$ we have

$$t(\beta) = 2r^{\frac{1}{3}}\cos\theta - \frac{B}{3A}, \tag{5.3}$$

where

$$r = \sqrt{-\left(\frac{p}{3}\right)^3}, \quad \theta = \frac{1}{3}\arccos\left(-\frac{q}{2r}\right)$$

Therefore

$$\alpha_1(\beta) = \frac{1}{1+t(\beta)}, \quad \alpha_0(\beta) = \beta(1-\alpha_1(\beta)),$$

where $t(\beta)$ is defined by (5.2) or (5.3) and β is given as (5.1).

5.3 Parameterization of Quartic A-Splines

Let $F(\alpha_0, \alpha_1)$ as (4.31) and $\alpha_0 = \beta(1-\alpha_1)$ with $0 < \beta < 1$. Then by $F\alpha_0, \alpha_1) = 0$ we get the following equation

$$At^4 + Bt^3 + Ct^2 + Dt - 1 = 0 \tag{5.4}$$

with

$$A = b_{10}\beta(1-\beta)^3 + b_{20}\beta^2(1-\beta)^2 + b_{30}\beta^3(1-\beta) > 0, \quad D = b_{03}(1-\beta) + b_{13}\beta \leq 0$$

$$B = b_{11}\beta(1-\beta)^2 + b_{21}\beta^2(1-\beta), \quad C = b_{12}\beta(1-\beta) \leq 0$$

and $t = \frac{1-\alpha_1}{\alpha_1}$. Equation (5.4) has one solution in $(0, \infty)$ and this solution $t(\beta)$ can be found by the formula of quartic equation. Then

$$\alpha_0(\beta) = \beta(1 - \alpha_1(\beta)), \quad \alpha_1(\beta) = \frac{1}{1 + t(\beta)}$$

5.4 Parameterization of the $n-2$-Parameter Family of Degree n A-Splines

Let $F(\alpha_0, \alpha_1)$ as (4.52) and $\alpha_0 = \beta(1-\alpha_1)$ with $0 < \beta < 1$. Then by $F(\alpha_0, \alpha_1) = 0$ we get the equation $At^2 + Bt + C = 0$, where $t = \frac{1-\alpha_1}{\alpha_1}$ and

$$A(\beta) = \sum_{i=1}^{n-1} b_{i0} \frac{n!}{i!(n-i)!} \beta^i (1-\beta)^{n-i} > 0$$

$$B(\beta) = \sum_{i=1}^{n-2} b_{i1} \frac{n!}{i!(n-i-1)!} \beta^i (1-\beta)^{n-i-1}$$

$$C(\beta) = \sum_{i=0}^{n-2} b_{i2} \frac{n!}{i!2!(n-i-2)!} \beta^i (1-\beta)^{n-i-2} < 0$$

Hence

$$\alpha_1(\beta) = \frac{2A(\beta)}{2A(\beta) - B(\beta) + \sqrt{B(\beta)^2 - 4A(\beta)C(\beta)}}, \quad \alpha_0(\beta) = \beta(1 - \alpha_1(\beta))$$

6 Geometric Model Construction

We exhibit some examples of geometric models with piecewise C^k smooth algebraic surfaces generated in a straight forward fashion from A-splines of the earlier sections. One approach is to rotate an A-spline curve around some axis, say y-axis. If the composite curve $f(x, y) = 0$ is C^k continuous, then the rotating surface $f(\pm\sqrt{x^2 + z^2}, y) = 0$ is also C^k continuous except for the point on the curve that passes through y-axis. This point is obviously on the surface and is a singular point. Fig.6.1 shows the mesh and shaded surface generated from rotating a simple quadratic C^1 curve that has a square control polygon as shown in the figure. Fig.6.2 shows three C^2 cubic A-splines and their revolution surfaces. The cup in the Fig.6.3 is produced from a C^3 cubic A-spline curve. Finally, Fig.6.4 shows the revolution surface from a C^5 quartic A-spline. The surface has a singular point since the curve passes through y-axis.

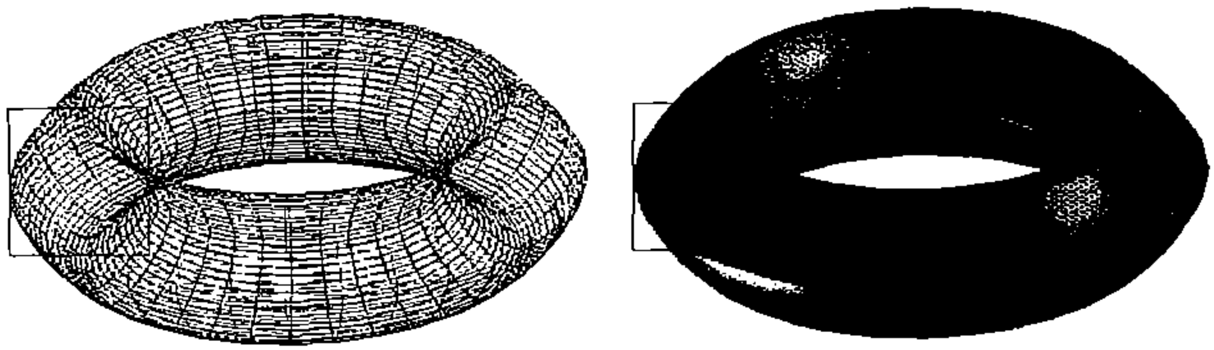

Figure 6.1: A C^1 A-spline and Revolution Surface

Figure 6.2: A C^2 A-spline and Revolution Surface

Another approach to produce surface by A-splines is to sweep the curve along a space curve. Fig. 6.5 shows an A-spline curve and the sweeping surface. The space curve used in this example is $(\frac{2t^2}{1+t^2}, 0, \frac{2t^3}{1+t^2})$.

All implementations are made in the X-11 program called GANITH [6].

7 Conclusions

We have presented a characterization of the BB form of bivariate algebraic such that the zero contour of the polynomials define a single sheeted real curve segment in the given triangle. The quadratic, cubic and quartic A-splines and the $n-2$ parameter family of C^1 algebraic degree n splines are carefully analyzed resulting smoothness conclusion derived for local interpolation and approximation.

Quadratic A-splines can C^1 approximate a polygon with one free parameter b_{10} for each triangle. Fig.4.2 shows that this parameter can be used to interpolate any point within the triangle. If the polygon is of a Case(b)-join at each interpolation point, then quadratic splines can C^2 approximate the polygon by satisfying some global conditions.

Cubic A-splines can C^2 approximate a polygon with three free parameters $b_{10} > 0$, $b_{20} > 0$

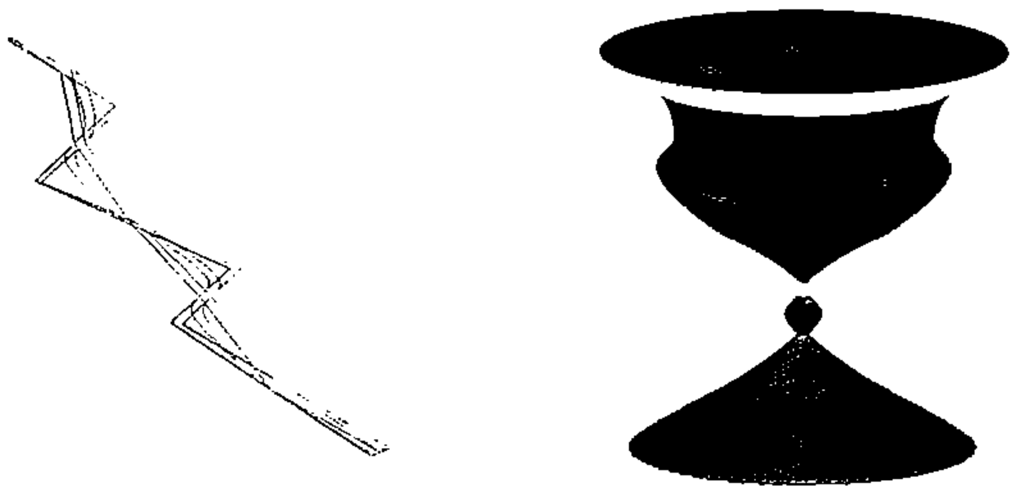

Figure 6.3: A C^3 A-spline and Revolution Surface

Figure 6.4: A C^5 A-spline and Revolution Surface

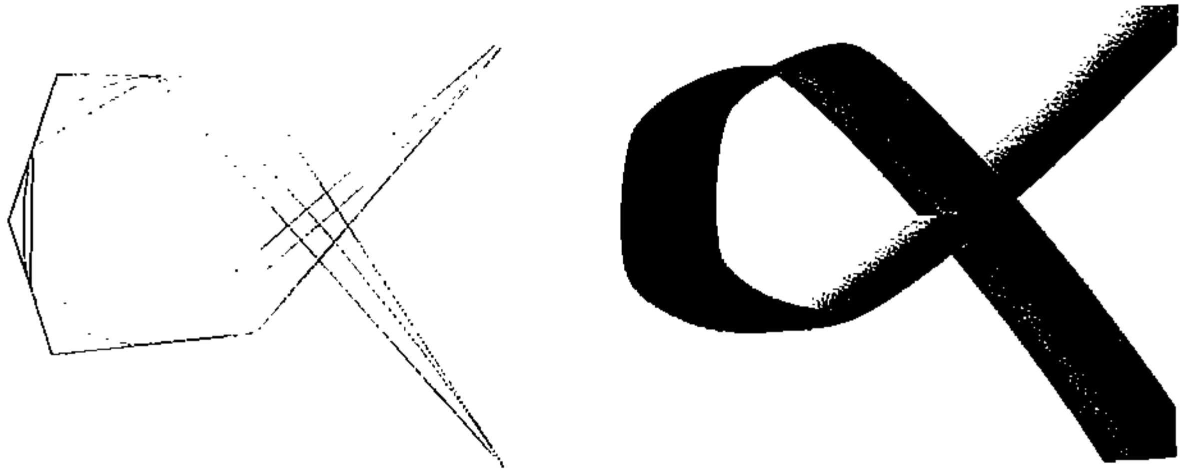

Figure 6.5: A C^1 A-spline and Sweeping Surface

and b_{11} by giving second derivative at the interpolating points properly. Fig.4.5 use b_{11} to interpolate a point in the triangle. Besides, cubic spline can C^3 approximate a polygon with one free parameter $b_{11} \leq 0$. This parameter can be used to control the shape of the curve, but its influence on the curve is of limited when the derivatives at the end points are fixed. However, with a change of derivatives at the end points, desirable shape of the curve can be obtained (see Fig.4.7). Furthermore, if the polygon is of a Case(a)-join, then a C^4 smooth curve can also be constructed.

For quartic A-splines, we can achieve C^5 approximation with two free parameters $b_{20} \geq 0$ and $b_{12} \leq 0$. Parameter b_{20} lifts the curve upwards toward the top vertex of the triangle, while b_{12} pushes the curve downwards toward the bottom of the triangle. So the position of the curve inside the triangle can be controlled by a combination of the two parameters(see Fig.4.10).

The $n-2$ parameter family of C^1 A-splines is used to interpolate $n-2$ points inside the triangle with global C^1 continuity. If the number of points is great than $n-2$, the interpolation approximation problem can be solved in the least square sense.

Several open problems remain. One is the smoothness criteria for C^k-continuous A-splines of degree n with $n \geq 5$ and and $k > 1$. Secondly, faster and robust methods of graphics based on subdivision or integer forward differencing need to be developed. Finally, applications of these A-splines to problems in image processing, computer graphics, animation and geometric modeling etc. need to be fully explored by extensive experimentations.

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