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EVALUATION AND APPROXIMATE EVALUATION OF THE MULTIVARIATE BERNSTEIN-BEZIER FORM ON A REGULARLY PARTITIONED SIMPLEX

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Evaluation and approximate evaluation of the multivariate * Bernstein-Bézier form on a regularly partitioned simplex

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Abstract

Polynomials of total degree d in m variables have a geometrically intuitive representation in the Bernstein-Bézier form over an m-dimensional simplex. Two algorithms are given that evaluate, respectively approximate the BB form on a large number of points corresponding to a regular partition of the simplex. The first algorithm is an adaptation of isoparametric evaluation to the simplicial domain. The second is a subdivision algorithm that approximately evaluates by averaging coefficients that correspond to adjacent nodes in the partition. The merits of the first are low storage requirement and flexibility in the number and location of evaluation points; the advantages of the second are speed, stability and availability of all derivatives as a simple extension of the evaluation. In contrast to de Casteljau's algorithm, both algorithms have a cost of evaluation per point that is linear in the degree regardless of the number of variables. To demonstrate that both algorithms are practical, implementations in the case of a triangular domain, are compared to generic implementations of six algorithms in the literature.

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1. Introduction

The Bernstein-Bézier form (BB form) is an important tool for representing piecewise polynomials. Many applications in computer aided geometric design benefit from the intuitive geometric meaning of its coefficients and the fact that, like the power or Taylor form, the BB form is capable of representing polynomials of total degree in many variables. It is often counted as a disadvantage of the BB form vis á vis the power form that the natural algorithm for evaluating the BB form, de Casteljau's algorithm [C59], has a higher complexity than the evaluation of the power form by nested multiplication, and a much higher complexity than forward differencing when generating a large number of points. It may therefore surprise and reassure users of the BB form that there exist two simple algorithms that generate points on a regular lattice with a cost per point that is linear in the degree regardless of the number of variables. In fact, the constant associated with the linear term of the theoretical time complexity of the subdivision algorithm decreases with the number of variables and is lower than the constant associated with forward differencing. Numerical experiments, given at the end of this paper, confirm the theoretical complexity.

Polynomials in m variables come both as tensor-product polynomials, defined on an m-dimensional cube, and as total-degree polynomials, defined over an m-dimensional simplex. The stable and efficient evaluation and approximate evaluation of *polynomials of total degree*, in an arbitrary number of variables, and, in particular, in the case of two variables over a triangle, is the concern of this paper. The goal is to generate a *large number* of points corresponding to the BB form on a regular partition of the domain simplex, namely the parameters with barycentric coordinates α/n, where $\alpha \in \mathbb{Z}^{m+1}$ and $0 \leq \alpha_i \leq n$. For example, if the simplex is the right-angled unit triangle, then the parameters are $(i/n, j/n)$, where $0 \leq i + j \leq n$. Approximate evaluation on such a lattice is of interest, e.g. when rendering a polynomial surface. While efficient lattice-oriented algorithms are known for polynomials in tensor-product form [LCR80][LR80], there are with the exception of repeated extrapolation [D87, 3.3.7 p43] no examples of algorithms that take advantage of the regular partition if the domain is a simplex.

Section 2 reviews the multiindex notation used throughout the paper, the definition of the power and BB form in this notation, de Casteljau's algorithm and subdivision by recursive use of de Casteljau's algorithm. Section 3 describes the two new algorithms. In either case, efficiency follows from a specialized version of de Casteljau's algorithm and the nesting of operations both with respect to the degree and the parameter dimension. The first algorithm recursively reduces of the parameter dimension to obtain a sequence of univariate polynomials corresponding to lines parallel to one edge of the simplex. The univariate polynomials can then be evaluated by standard methods, such as univariate forward differencing. The approach uses little space and offers flexibility with respect to the location and number of evaluation points. The second algorithm is a subdivision method that generates points on or close to the polynomial. As with the first algorithm, this is due to the fact that the scheme *reduces the multivariate subdivision to a sequence of univariate subdivisions.* Each of these can be coded efficiently and stably as an averaging of 'adjacent' coefficients into average storage locations. Section 4 compares implementations of the two new algorithms for two variables with generic versions of algorithms from the literature reviewed in the Appendix.

2. Notation, de Casteljau's algorithm and subdivision

The number of variables, the total degree of the polynomial, and the number of points per edge of the domain simplex are denoted by

$$m, \quad d, \quad n$$

respectively. Generating $n \gg d$ points per edge and distributing the parameter values uniformly over the domain with barycentric coordinates α/n, where $\alpha \in \mathbb{Z}^{m+1}$ and $0 \leq \alpha_i \leq n$ results in a total of $\binom{n+m}{m}$ points. For the description of polynomials in many variables, the following multiindex notation is used.

$$\alpha := (\alpha_s, \dots, \alpha_m), \ \xi^\alpha := \xi_s^{\alpha_s} \cdots \xi_m^{\alpha_m}, \ |\alpha| := \alpha_s + \dots + \alpha_m, \ \text{ and } \binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\prod_{i=s}^m \alpha_i!}.$$

Here $s = 0$ in the context of the BB form and $s = 1$ for the power form. Thus the power form of a polynomial $p_{m,d}$ of degree d with the multiindex $\beta := (\beta_1, \dots, \beta_m)$ is

$$p_{m,d}(x) = \sum_{|\beta| \leq d} c(\beta) x^\beta,$$

where $c(\beta)$ is the coefficient corresponding to the multiindex β. To define the m-variate BB form (cf. [B87] [F88]), we need $m+1$ linear polynomials ξ_v, the barycentric coordinate functions defined by

$$\sum_{v \in V} \xi_v p(v) = p \quad \text{for all polynomials } p \text{ with } \deg\ p \leq 1.$$

Here $V := [v_0, \dots, v_m]$ is the domain m-simplex formed by the vertices $v_i \in \mathbb{R}^m$. Thus $\xi_v(x)$ is the barycentric coordinate of x corresponding to $v \in V$. The BB form of total degree d in m variables and with coefficients $c(\alpha)$, $\alpha := (\alpha_0, \dots, \alpha_m)$, is

$$b_{m,d}[V] := \sum_{|\alpha|=d} \binom{d}{\alpha} c(\alpha) \xi^\alpha,$$

where $\xi := [\xi_{v_0}, \dots, \xi_{v_m}]$. For example, denoting the j^{th} unit vector by e_j,

$$b_{2,3}[0, e_1, e_2] = \sum_{\alpha_0+\alpha_1+\alpha_2=3} \frac{3}{\alpha_0!\alpha_1!\alpha_2!} c(\alpha_0, \alpha_1, \alpha_2)(1 - x - y)^{\alpha_0} x^{\alpha_1} y^{\alpha_2}$$

is a bivariate cubic on the standard 2-simplex. Latin letters are used for superscripts, e.g.

$$V^i := [v_0^i, \dots, v_m^i]$$

is the ith simplex, while greek letters denote exponents.

In the pseudo code an algorithm has a list of input and output parameters separated by ';'. If $b_{m,d}[V]$ is an argument to a pseudocode function, then m, d, V and the coefficients $c(\alpha)$ are passed. If $p_{m,d}$ is an argument, then m, d and the coefficients $c(\beta)$ are passed. To avoid specializations of subroutines, a dummy return argument $*$ is used that fits any returned object and indicates that the object is not needed for further computation. The scope of a **for**-loop is indicated by indentation. A statement like **for** $|\alpha| < d$ can be implemented as a sequence of nested loops, one for each of α_0 to α_m, if m is fixed or recursively if m is variable.

Following [SH82 p.331], the theoretical stability of an evaluation process can be measured in terms of the basic operations and the level of indirection. Extrapolation and differencing are considered less stable than averaging since cancelation of leading terms is unlikely for averaging on the coefficients of a sufficiently smooth function. The level of indirection, l, indicates how many intermediate operations separate the output from the original coefficients of the polynomial. For example, if each f_i is a basic operation, c represents the original coefficients and a point is generated by $f_k \circ \ldots \circ f_1(c)$, then $l = k$. One can distinguish a worst case and an average case level of indirection, l_{worst} and l_{average}. A large l indicates potential loss of stability through round-off.

DeCasteljau [C59]. Applied one point at a time, de Casteljau's algorithm serves as a standard for accuracy, since the BB form is naturally defined in terms of the algorithm. The reader familiar with the evaluation triangle in the univariate case (see e.g. [BFK84, p.8]), will recognize that the variable l below counts the levels of an $(m+1)$-dimensional simplex, filled layer by layer first with the original $\binom{m+d}{m}$ coefficients, then with barycentric combinations of coefficients in the lower layer. For arbitrary m and d, the following algorithm computes $b_{m,d}[V]$ at x.

```
**DeCasteljau** $(b_{m,d}[V], x; c(0), b_{m,d}[V^0], \ldots, b_{m,d}[V^m])$

$\beta_v = \xi_v(x)$ for all $v \in V$        [determine the barycentric coordinates]
**for** $l = 1..d$        [levels of the $(m+1)$-dimensional subdivision simplex]
    **for** $|\alpha| = d - l$
        $c(\alpha) = \sum_{v \in V} \beta_v c(\alpha + e_v)$
```

Here e_v is the vector that is 1 in the jth slot if $v = v_j$ and 0 otherwise. We note that de Casteljau's algorithm not only generates the point $c(0) = b_{m,d}[V](x)$, but in the process also computes the coefficients of the $m+1$ subpolynomials

$$b_{m,d}[V^i] := \sum_{|\alpha^i|=d} (\eta^i)^{\alpha^i} \binom{d}{\alpha^i} c(\alpha^i), \quad \text{where } V^i := [v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_m],$$

η^i is the vector of barycentric functions corresponding to V^i and $\alpha^i := (\alpha_0, \ldots, \alpha_{i-1}, l, \alpha_{i+1}, \ldots, \alpha_m)$.

Each $b_{m,d}[V^i]$ represents $b_{m,d}[V]$ over a subsimplex of the original domain simplex and takes its coefficients from a facet of the $m+1$ dimensional subdivison simplex (cf. [Go83], [B87]).

(2.1) Example We consider the case $d = 2$ and $m = 1$. This is the simplest case of interest since Section 3 will reduce multivariate evaluation and subdivision to the univariate case. If $b_{1,2}[0,1] = 18(1-x)x + 18x^2$, then the coefficients are $c(20) = 0$, $c(11) = 9$, and $c(02) = 18$. A call to DeCasteljau($b_{1,2}[0,1], \frac{1}{3}; \ldots$) generates

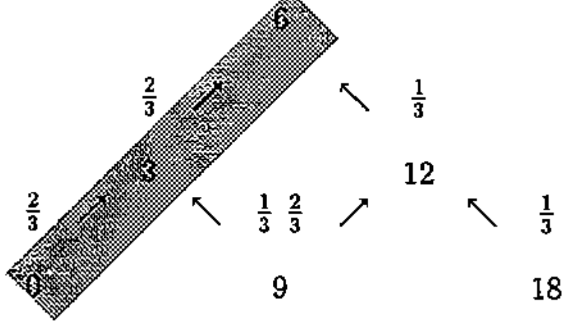
$$c(10) = \frac{2}{3}c(20) + \frac{1}{3}c(11) = 3, \quad c(01) = \frac{2}{3}c(11) + \frac{1}{3}c(02) = 12,$$

$b_{1,2}[0,1](\frac{1}{3}) = c(00) = 6$ and the subpolynomials

$$b_{1,2}[V^0] = b_{1,2}[\frac{1}{3}, 1] \text{ with coefficients } c_{[\frac{1}{3},1]}(20) = 6, c_{[\frac{1}{3},1]}(11) = 12, c_{[\frac{1}{3},1]}(02) = 18$$

and

$$b_{1,2}[V^1] = b_{1,2}[0, \frac{1}{3}] \text{ with coefficients } c_{[0,\frac{1}{3}]}(20) = 0, c_{[0,\frac{1}{3}]}(11) = 3, c_{[0,\frac{1}{3}]}(02) = 6:$$



■

The space efficiency of the algorithm can be improved by overwriting lower levels of the evaluation $(m+1)$-simplex. Thus the computation can be done in the input array, an m-simplex with $\binom{d+m}{m}$ coefficients ([B87], [F88]). On the downside, now only the coefficients of the subpatch $b_{m,d}[V^0]$ are available.

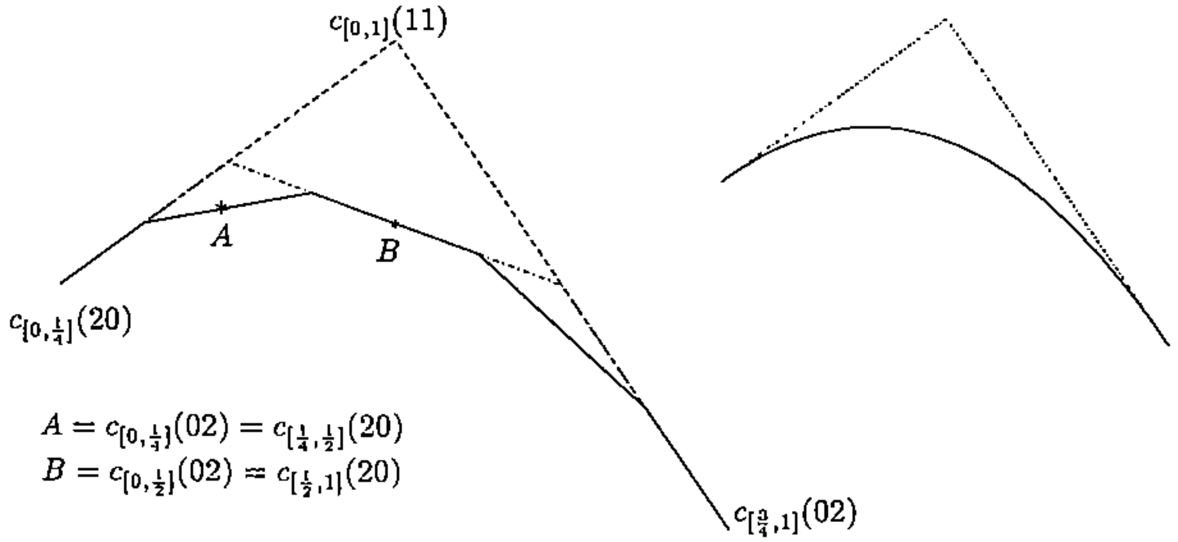
```
DeCasteljau_2 ($b_{m,d}[V], x; c(0), b_{m,d}[V^0]$)

$\beta_v = \xi_v(x)$ for all $v \in V$
for $l = 1..d$
  for $|\alpha'| \le d - l$,   where $\alpha' = (\alpha_1, \ldots, \alpha_m)$
    $c(\alpha') = \beta_{v_0} c(\alpha') + \sum_{v \in V \setminus v_0} \beta_v c(\alpha' + e_v)$
```

Each evaluation by DeCasteljau or DeCasteljau_2 costs $(m+1)$ additions and the same number of multiplications for each of $\binom{d-1+m+1}{m+1}$ intermediate values yielding a total complexity of $2(m+1)\binom{d+m}{m+1}$. As Section 3 shows, this is unnecessarily slow for evaluation over a lattice and even for evaluation at a single point since the number of operations is larger by a factor of d than the number of coefficients. However, the level of indirection is minimal: $l_{\text{worst}} = l_{\text{average}} = d$ and, for evaluation over the simplex, the basic operation is a convex combination of the coefficients, which is stable for densely sampled polynomials.

Approximate evaluation by subdivision is motivated by the following theorem.

(2.2) Theorem. *[D86, Thm 4.1] A particular piecewise linear interpolant to the coefficients of the polynomial, the so-called BB-net, converges quadratically in the size of the domain simplex and linearly in the size of the second derivative to the polynomial.*

The theorem is a direct consequence of the facts that any BB-net as defined in [D86] reproduces linear functions and that the BB form is a stable basis. If, for example, the diameter of the domain simplex is halved at each subdivision step, then 10 subdivision steps reduce the distance between the polynomial and BB-net to $(1/2^{10})^2 \approx 10^{-6}$ times the initial distance. The convergence is speeded up as pieces of maximal curvature are confined to subsimplices and thus the maximal curvature for the other polynomial pieces decreases. Consequently, after a number of subdivisions depending on the desired accuracy, all coefficients generated by the subdivision process can be accepted as good approximations to points on the surface.


$c_{[0,1]}(11)$
A
B
$c_{[0,\frac{1}{4}]}(20)$
$A = c_{[0,\frac{1}{4}]}(02) = c_{[\frac{1}{4},\frac{1}{2}]}(20)$
$B = c_{[0,\frac{1}{2}]}(02) = c_{[\frac{1}{2},1]}(20)$
$c_{[\frac{3}{4},1]}(02)$

(2.3) Figure: (Left) The solid polygon is the result of two steps of subdivision applied to $b_{1,2}[01]$ with coefficients $c = \begin{bmatrix} 0 & 9 & 18 \\ 9 & 18 & 0 \end{bmatrix}$ at $t = 1/2$.
(Right) An approximation generated by 4 subdivision steps with $t = 1/2$.

3. Two algorithms for generating points on a regularly partitioned simplex

Since an m-variate polynomial of total degree d has $\binom{d+m}{d}$ coefficients, Schumaker and Volk's variant of nested multiplication [SV86] and its generalization (cf. SV-NestMult in the Appendix) are, up to a constant, optimal when a single point is to be generated. However, for a large number of evaluations, $n \gg d$, two simple ideas lead to lower cost per evaluation. The first is to nest computations both with respect to the degree and the dimension of the domain. The second is to use a specialized version of de Casteljau's algorithm that evaluates on an edge of the parameter domain. The low cost specialization works as follows. Given an input polynomial (and hence m and d), $x \in [0,1] \subset \mathbb{R}$, and a point $w := (1-x)v_i + xv_j$ on an edge of the domain simplex V, the routine EdgeDeCasteljau returns the subpolynomials that represent the input polynomial over the two simplices $V^i := [v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_{m+1}]$ and $V^j := [v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_{m+1}]$ respectively.

```
**EdgeDeCasteljau** $(b_{m,d}[V], i, j, x; b_{m,d}[V^i], b_{m,d}[V^j])$

for $|\alpha_{\setminus ij}| < d$     where $\alpha_{\setminus ij} := (\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_m)$
    $d' = d - |\alpha_{\setminus ij}|$
    $b_{1,d'}[0,1] = \sum_{\alpha_i+\alpha_j=d'} \xi_1^{\alpha_i}\xi_2^{\alpha_j}\frac{d'!}{\alpha_i!\alpha_j!}c(\alpha)$
    DeCasteljau$(b_{1,d'}[0,1], x; *, b_{1,d'}[x,1], b_{1,d'}[0,x])$
```

Here ξ_1 and ξ_2 are the barycentric coordinate functions ξ_i and ξ_j restricted to the 1-dimensional simplex $[v_i, v_j]$. A key point is that $\sum \xi_1^{\alpha_i}\xi_2^{\alpha_j}\frac{d'!}{\alpha_i!\alpha_j!}c(\alpha)$ is treated as a univariate polynomial. This allows applying the *univariate* DeCasteljau algorithm to the coefficients $c(\alpha)$ where α_l is fixed for $l = 0..m, i \neq l \neq j$.

(3.1) Example. A call to EdgeDeCasteljau$(b_{2,3}[0, e_1, e_2], 1, 2, \frac{1}{2}; \dots)$ results in

```
for $\alpha_0 = 0..2$        $[\alpha_{\setminus 12} = \alpha_0$ since $m = 2.]$
    $d' = d - \alpha_0$
    $b_{1,d'}[0,1] = \sum_{\alpha_1+\alpha_2=d'} (1-s)^{\alpha_1}s^{\alpha_2}\binom{d'}{\alpha_1}c(\alpha_0,\alpha_1,\alpha_2)$
    DeCasteljau$(b_{1,d'}[0,1], \frac{1}{2}; *, b_{1,d'}[\frac{1}{2},1], b_{1,d'}[0,\frac{1}{2}])$
```

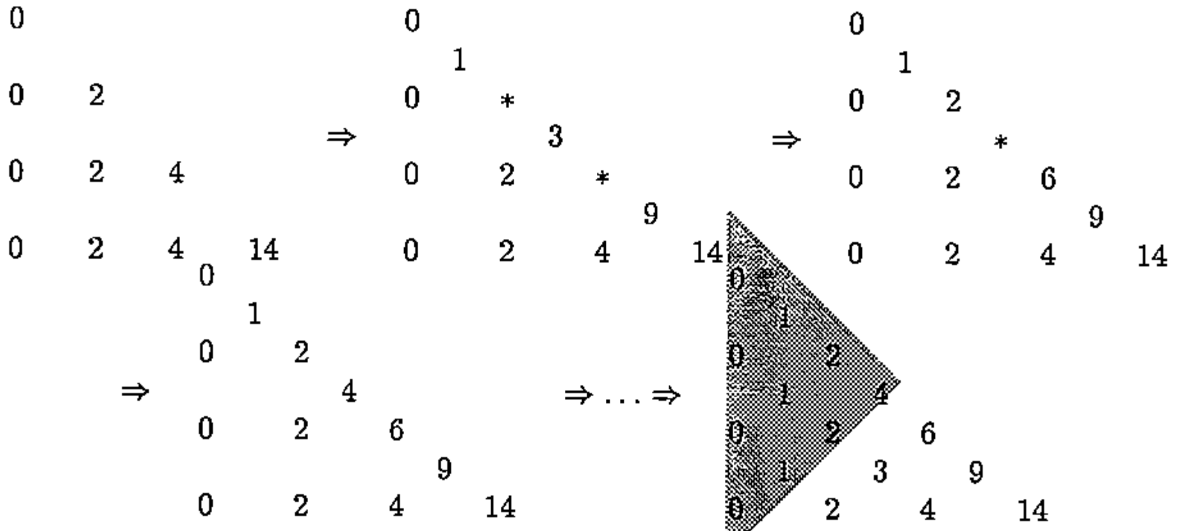
The assignment to $b_{1,d'}[0,1]$ interprets the right hand side as a univariate polynomial of degree $d - \alpha_0$ and with coefficients $c'(\alpha_1,\alpha_2) := c(\alpha_0,\alpha_1,\alpha_2)$ over $[0,1]$. This is possible, because α_0 is fixed. If $b_{2,3}$ has the coefficients

$$c(300) = c(201) = c(102) = c(003) = 0, \quad c(210) = c(111) = c(012) = 2,$$
$$c(120) = c(021) = 4, \quad c(030) = 14,$$

then $\alpha_0 = 0$ calls DeCasteljau$(b_{1,3}[0,1], \frac{1}{2}; \dots)$ where $b_{1,3}[0,1]$ has the coefficients

$$c'(30) = 0, \quad c'(21) = 2, \quad c'(12) = 4, \quad c'(03) = 14.$$

Next $\alpha_0 = 1$ works on the coefficients $c'(20) = 0, c'(11) = 2, c'(02) = 4$, and finally the linear polynomial with coefficients $c'(10) = 0, c'(01) = 2$ is evaluated. The restriction $|\alpha_{\setminus ij}| < d$ avoids an evaluation of the constant polynomial. Associating the coefficients in a canonical way with the domain, the algorithm generates the following sequence of coefficients.



One reads off the coefficients of the subpolynomial $b_{2,3}[V^1]$

$$c(300) = c(201) = c(102) = c(003) = 0, \quad c(210) = c(111) = c(012) = 1,$$
$$c(120) = c(021) = 2, \quad c(030) = 4,$$

and of the subpolynomial $b_{2,3}[V^2]$

$$c(300) = 0, c(201) = 1, c(102) = 2, c(003) = 4, c(210) = 2, c(111) = 3, c(012) = 6,$$
$$c(120) = 4, c(021) = 9, c(030) = 14.$$

■

(3.2) Lemma. *The complexity of EdgeDeCasteljau is $3\binom{d+m}{m+1}$ operations. It is $\binom{d+m}{m+1}$ add-and-shifts if $x = \frac{1}{2}$.*

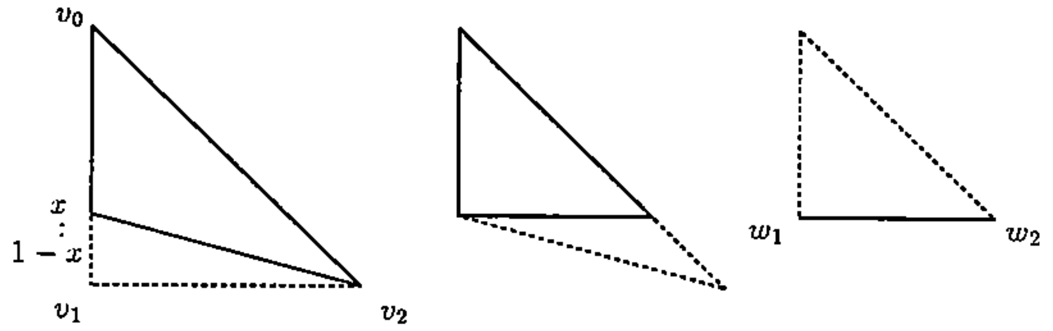
Proof. Only two barycentric weights, ξ_{v_i} and ξ_{v_j}, are non-zero. Without loss of generality, the edge v_i, v_j is parametrized by $x \in [0,1]$. Then the sum in DeCasteljau simplifies to

$$\textbf{for } |\alpha| = d - l \qquad c(\alpha) = (1-x)c(\alpha + e_i) + xc(\alpha + e_j) \tag{3.3}$$

implying that only coefficients on the same mesh line parallel to the boundary edge v_i, v_j are averaged to obtain V^i and V^j. Coefficients on a mesh line parallel to v_i, v_j have an index α such that $\alpha_{\setminus ij} := (\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m)$ is fixed and $d' := \alpha_i + \alpha_j = d - |\alpha_{\setminus ij}|$. It follows that there are exactly $\binom{d+m-1}{d}$ mesh lines parallel to v_i, v_j in the simplex and that each mesh line with $\alpha_i + \alpha_j = d'$ has $d' + 1$ coefficients. The multivariate version of DeCasteljau performs the same operation (3.3) on the coefficients as does univariate DeCasteljau. Since the complexity of the univariate algorithm is $d(d+1)/2$ additions and $d(d+1)$ multiplications, the total complexity is $3\binom{d+m}{m+1}$. If $x = \frac{1}{2}$, then (3.3) becomes a single add-and-shift operation. ■

3.1 Isoparametric Evaluation

The following algorithm shows how to extract a univariate polynomial for each choice of $(m-1)$ fixed parameters. Simply fixing $(m-1)$ barycentric coordinates still leaves the main task of aggregating the terms as coefficients of the univariate polynomial. The routine IsoParamEval is a systematic and stable way of performing this aggregation for polynomials in the BB form. With the help of EdgeDeCasteljau, the routine Slice extracts the $(m-1)$-variate polynomial $b_{m-1,d}[W]$ from $b_{m,d}[V]$. The domain vertices w_j correspond to the intersection of the hyperplane $\xi_0(u)=(1-x)$ with V.



(3.4) Figure: Slice($b_{2,d}[v_0,v_1,v_2],x;b_{1,d}[w_1,w_2]$).

```
Slice($b_{m,d}[V^0],x;b_{m-1,d}[W]$)

for $j = 1..m$
    EdgeDeCasteljau($b_{m,d}[V^{j-1}],0,j,x;*,b_{m,d}[V^j]$)
        where $V^j=[v_0^{j-1},\dots,v_{j-1}^{j-1},w_j,v_{j+1}^{j-1},\dots,v_m^{j-1}]$ and $w_j=(1-x)v_0+xv_j$
$b_{m-1,d}[W]=b_{m-1,d}[v_1^m,\dots,v_m^m]$
```

The last assignment in Slice is based on the well-known fact that the restriction of the BB form to a facet is entirely defined by the coefficients associated with that facet. By dropping v_0, only the polynomial of $(m-1)$ variables associated with W is retained. Slice can be simplified if $x=0$ or $x=1$.

(3.5) Example. A call to Slice($b_{2,3}[2e_2,0,2e_1],\frac{1}{2};\dots$) results in

```
for $j = 1..2$
    EdgeDeCasteljau($b_{2,3}[V^{j-1}],0,j,\frac{1}{2};*,b_{2,3}[V^j]$)
$b_{1,3}[W]=b[e_2,e_1+e_2]$
```

Here $V^1=[2e_2,e_2,2e_1]$ and $V^2=[2e_2,e_2,e_1+e_2]$. With the coefficients associated with the domain triangle $[2e_2,0,2e_1]$ in the canonical way as in Example 3.1, the coefficient matrix transforms as follows.

$$\begin{array}{cccc} 0 & & & \\ & & & \\ 0 & 2 & & \\ & & & \\ 0 & 2 & 4 & \\ & & & \\ 0 & 2 & 4 & 14 \end{array} \quad \Rightarrow .. \Rightarrow \quad \begin{array}{cccc} 0 & & & \\ 0 & & & \\ 0 & 2 & & \\ 0 & 2 & & \\ 0 & 2 & 4 & \\ 0 & 2 & 4 & \\ 0 & 2 & 4 & 14 \end{array} \quad \Rightarrow .. \Rightarrow \quad \begin{array}{ccccccc} 0 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 2 & & & & \\ 0 & 1 & 2 & 4 & & & \\ & & 2 & 3 & 6 & & \\ & & & & 4 & 9 & \\ & & & & & & 14 \end{array}$$

From this one reads off the coefficients of $b_{1,3}[e_2, e_1 + e_2]$ as

$$c'(30) = 0, \quad c'(21) = 1, \quad c'(12) = 2, \quad c'(03) = 4.$$

That is the boundary coefficients of $b_{2,3}[2e_2, e_2, e_1 + e_2]$ determine the polynomial over the boundary $[e_2, e_1 + e_2]$ of the domain of $b_{2,3}$. ∎

The routine Slice is the basic building block of the algorithm IsoParamEval. In the pseudocode below, $n \gg d$ is the total number of evaluations per edge of the parameter simplex. If m is variable, then the nested loops are implemented by recursion.

```
**IsoParamEval** ($b_{m,d}[V^0], n$)

output $c(d, 0, .., 0)$ of $b_{m,d}[V^0]$
for $i_1 = 1..n$
  Slice($b_{m,d}[V^0], \frac{i_1}{n}; b_{m-1,d}[V^1]$)
  output $c(d, 0, .., 0)$ of $b_{m-1,d}[V^1]$
    $\ddots$
      for $i_l = 1..i_{l-1}$
        Slice($b_{m+1-l,d}[V^{l-1}], \frac{i_l}{n}; b_{m-l,d}[V^l]$)
        output $c(d, 0, .., 0)$ of $b_{m-l,d}[V^l]$
          $\ddots$
            for $i_{m-1} = 1..i_{m-2}$
              Slice($b_{2,d}[V^{m-2}], \frac{i_{m-1}}{n}; b_{1,d}[V^{m-1}]$)
              output $c(d, 0)$ of $b_{1,d}[V^{m-1}]$
              UnivariateEvaluateBB($b_{1,d}[v_0^{m-1}, v_1^{m-1}], i_{m-1}$)
```

Here

$$V^l = [(1 - x_l)v_0^{l-1} + x_l v_1^{l-1}, \ldots, (1 - x_l)v_0^{l-1} + x_l v_{m+1-l}^{l-1}] \text{ and } x_l = \frac{i_l}{n},$$

and UnivariateEvaluateBB($b_{1,d}, k$) is any routine that outputs the value of the univariate polynomial $b_{1,d}$ in BB-form at j/k, $j = 1..k$. To apply IsoParamEval to tensor-product polynomials only the length of the for-loops has to be changed from i_j to n.

Stability. The level of indirection in IsoParamEval increases with m. Since the calls to the univariate de Casteljau routine in EdgeDeCasteljau are independent and $m + (m-1) +$

$\ldots + 2$ applications of EdgeDeCasteljau are necessary, $l_{\text{worst}} = ((m+1)m/2 - 1)(d-1)$ and $l_{\text{average}} = ((m+1)m/2 - 1)d'$, where $\frac{d'(d'+1)}{2} = \frac{d(d-1)}{4}$. The basic operation in EdgeDeCasteljau is a convex combination of the form $C = (1-v)A + vB$ for a $v \in [0..1]$ and adjacent BB-coefficients A and B. That is, the extraction is more stable than for example the initialization of ForDiff. The only other numerical task in IsoParamEval is the univariate evaluation. In the case of SV-NestMult the basic operations are of the form $C = uA + B$.

Theoretical Time Complexity. EdgeDeCasteljau applied to a polynomial $b_{m,d}$ forms $\binom{m+1+d-1}{d-1}$ convex combinations at a cost of 2 multiplications and one addition each. The cost of slicing $b_{m,d}$ to obtain $b_{m-1,d}$ is therefore $3m\binom{m+d}{d-1}$. Due to the nesting, this cost is distributed over $\binom{m+n}{n}$ points, where $n \gg d$ so that its contribution to the cost per point is small. Away from the apex the cost per point is therefore essentially independent of the parameter dimension and equal to the cost of the univariate evaluation routine. The extra cost for evaluation close to the apex in the bivariate case is measured by e in Section 4.

Storage Complexity. EdgeDeCasteljau needs one array of size $\binom{m+d}{m}$. This array can be reused by Slice. For time efficiency it is reasonable to allocate $\binom{j+d}{j}$ space for the coefficients of each $b[V^j]$. If the univariate evaluation is not by subdivision, then the final points can be output directly and need not be stored.

Other considerations. Evaluation along isoparametric lines does not readily yield multidirectional information, like normal fields or curvature fields. Two possible fixes are: (1) running the algorithm with different choices of fixed parameters or (2) storing and connecting the surface points to analyze the piecewise linear approximant.

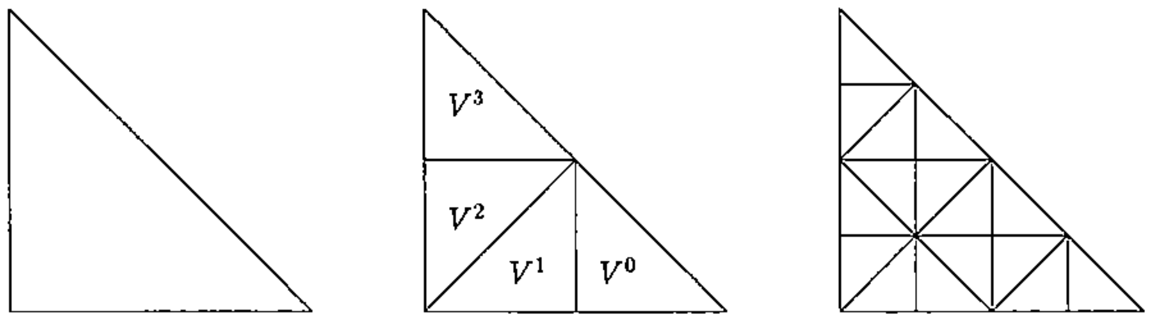
3.2 Congruent subdivision

The algorithm CongruentSub is motivated by Theorem 2.2 which states that a piecewise linear interpolant to the coefficients, the BB net, converges quadratically in the diameter of the domain simplex to the polynomial. If a polynomial surface is to be displayed as a suitably connected mesh of points, the connecting lines between the points do in general not lie on the surface and hence there is little gain in evaluating exactly. Convergence of the mesh to the surface is guaranteed by applying the subroutine CongruentStep below since it halves the domain diameter.

```
**CongruentStep**($b_{m,d}[V]; b[V^0], \dots, b[V^{2^m-1}]$)

$V^0 = V$
**for** $i = 0..(m-1)$        [a subdivision sub-step]
  **for** $j = 2^i - 1..0$       [for each subsimplex]
    EdgeDeCasteljau($b[V^j], i, m, \frac{1}{2}; b[V^{2j}], b[V^{2j+1}]$)
      where $w_i = (v_i^j + v_m^j)/2$        [$v_m^j \neq v_m$ in general]
        $V^{2j} = [v_0^j, \dots, v_{i-1}^j, w_i, v_{i+1}^j, \dots, v_m^j]$
        $V^{2j+1} = [v_0^j, \dots, v_{i-1}^j, w_i, v_i^j, \dots, v_{m-1}^j]$
```

(3.6) Example. The subdivision of the domain simplex by CongruentStep($b_{2,3}[V]; \dots$) is illustrated in Figure 3.4 for $V := [2e_2, 0, 2e_1] = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$. The domain simplices are

$$i = 0, j = 0: \quad w_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, V^0 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}, V^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$
$$i = 1, j = 1: \quad w_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, V^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, V^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$
$$i = 1, j = 0: \quad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, V^0 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, V^1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$



(3.7) Figure: Subtriangles generated by 2 steps of CongruentSub ($m = 2$).

Starting with the coefficients of Example 3.1, the sequence of coefficient matrices is

$$\begin{array}{ccccccc} 0 &&&&&& \\ &&&&&& \\ 0 && 2 &&&& \\ &&&&&& \\ 0 && 2 && 4 && \\ &&&&&& \\ 0 && 2 && 4 && 14 \end{array} \Rightarrow .. \Rightarrow \begin{array}{ccccccc} 0 &&&&&& \\ & 1 &&&&& \\ 0 && 2 &&&& \\ & 1 && 4 &&& \\ 0 && 2 && 6 && \\ & 1 && 3 && 9 & \\ 0 && 2 && 4 && 14 \end{array} \Rightarrow .. \Rightarrow \begin{array}{ccccccc} 0 &&&&&& \\ 0 & 1 &&&&& \\ 0 & 1 & 2 &&&& \\ 0 & 1 & 2 & 4 &&& \\ 0 & 1 & 2 & 4 & 6 && \\ 0 & 1 & 2 & 4 & 6 & 9 & \\ 0 & 1 & 2 & 4 & 6 & 9 & 14 \end{array}$$

From this we read off for example the coefficients of $b[V^2] = b\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$:

$$c(030) = c(021) = c(012) = c(003) = 0, \quad c(120) = c(111) = c(102) = 1,$$
$$c(201) = c(210) = 2, \quad c(300) = 4.$$

■

To complete the algorithm, CongruentStep has to be called for each subsimplex at each step of the subdivision. Below σ is the number of subdivisions necessary to generate $n = d2^\sigma$ points per edge of the parameter simplex.

```
**CongruentSub**($b_{m,d}[V], \sigma; b[V^0], \ldots, b[V^{2^{m\sigma}}]$)

for $s = 1..\sigma$        [steps of the subdivision]
  $l = 2^{m(s-1)} - 1$
  for $i = l..0$        [for each subsimplex]
    $j = i2^m$
    CongruentStep($b[V^i]; b[V^{j+2^m-1}], \ldots, b[V^j]$)
```

The name of the routine is motivated by the observation that for the canonical simplex, $V = [0, e_1, \ldots, e_m]$, the resulting subsimplices are congruent to one another.

Stability. The only operation is of the form $A = (B + C)/2$, i.e. add-and-shift. The level of indirection is

$$l_{\text{worst}} = \sigma m(d-1), \quad l_{\text{average}} = \sigma m \frac{\sum_{i=0}^{d} \binom{d+m-2+i}{d} i}{\binom{d+m-1}{d}}.$$

Theoretical Time Complexity. The asymptotic time complexity decreases with m for fixed d. Let σ be the number of subdivisions necessary to generate n coefficients per edge of the domain simplex, i.e. $n/d =: 2^\sigma$. Each traversal of the inner loop of CongruentSub generates $(2^m)^s$ simplices from $(2^m)^{s-1}$ since each call to CongruentStep replaces one simplex by 2^m simplices. The ith step of CongruentStep requires 2^i calls to EdgeDeCasteljau and each call to EdgeDeCasteljau consists of $\binom{d+m}{m+1}$ add-and-shifts according to Lemma 3.2. This yields the total work count

prev. subd's.	simplices per subd.	eval.'s per simplex	add-shift per eval.
$\sum_{s=0}^{\sigma-1}$	$(2^m)^s$	$\left(\sum_{i=0}^{m-1} 2^i\right)$	$\binom{d+m}{m+1}$

$$= (2^{m\sigma} - 1)\binom{d+m}{m+1}$$

Since the number of generated points is

$$\binom{n+m}{m} \quad \text{where } n := 2^\sigma d,$$

we obtain the following bound on the number of operations per coefficient.

(3.8) Theorem. *The number of add-and-shifts per point generated by CongruentSub is*

$$w(m,d) := \frac{d}{m+1}\frac{2^{m\sigma}(d+m)\cdots(d+1)}{(2^\sigma d+m)\cdots(2^\sigma d+1)} < \frac{2^m}{2^m-1}\frac{d}{m+1}\frac{(d+m)\cdots(d+1)}{d^m}.$$

In particular,

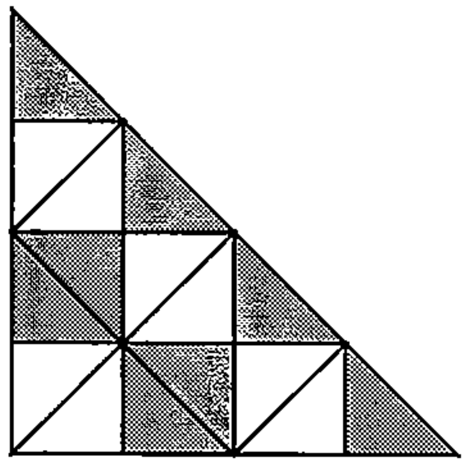
$$w(2,d) < \frac{4}{9}d + \frac{4}{3} + \frac{8}{9d},$$
$$w(3,d) < \frac{2}{7}d + \frac{12}{7} + \frac{22}{7d} + \frac{12}{7d^2}$$
$$w(4,d) < \frac{16}{75}d + \frac{160}{75} + const * d^{-1}.$$

The cost can be further decreased by storing the coefficients of all subpolynomials in a common m-dimensional simplicial array as in Example 3.6. That is, the coefficients of the subpolynomials are not returned but rather each polynomial piece is represented by a sector of the array. This storage allocation avoids redundant evaluation at facets shared by 2 or more subsimplices.

(3.9) Example: Figure 3.7(middle) illustrates Example 3.6 after the first subdivision step. In the second step, the edge from $[1,1]$ to $[0,0]$ is independently subdivided at the midpoint both for

$$b\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix} \quad \text{and} \quad b\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

This can be avoided by placing the subpolynomials into a common 2-dimensional array and making the algorithm works on pairs of triangles. That is, edge-adjacent triangles of the same color in Figure 3.10 are subdivided in one sweep (cf. [P90]). ■



(3.10) Figure: Pairs of subtriangles that are subdivided in one sweep.

For additional efficiency, the coefficients of adjacent C^0-connected triangles can be put into the same array to make use of a standard (tensor) m-dimensional array and share the computational effort on the common face. Surface constructions based on splitting, e.g. [F83], [P90a], naturally pair up polynomials in the bivariate case.

Storage Complexity. As explained above, all computations can be performed in the simplicial array of size $\binom{n+m}{m}$ that stores the result. If more than one triangular patch is to be evaluated, storage efficiency can be improved by storing two adjacent polynomial pieces in one square array.

Remarks. (1) Multidirectional information such as directional derivatives and therefore normal and curvature fields can be obtained at all subdivision points by differencing of adjacent coefficients (cf. [B87]). (2) CongruentSub, as stated above, can be modified to allow for a biased distribution of parameter values, namely by choosing the evaluation parameter in EdgeDeCasteljau to be $\neq \frac{1}{2}$. (3) If the *in situ* construction is not used, subdivision need not proceed through all edges. For example, the edge with the longest associated BB polygon or highest curvature can be subdivided at each stage. With this strategy the same edge may be subdivided repeatedly in order to get more uniform coverage of the image of a parametric map. By selecting the subdivision edge globally, this adaptively subdivides adjacent polynomials in the same time step and creates a subdivision surface without gaps.

4. A comparison of evaluation and approximate evaluation methods for the bivariate BB-form

This section compares implementations of algorithms for bivariate polynomials in BB-form defined over a triangle. Figure 4.1 shows time per evaluation for a range of $d = 2..14$. The time axis is linear and in the *msec* range. Table 4.2 below lists the time complexity, the code length and miscellaneous observations on stability and storage complexity of the algorithms. Time complexity counts additions plus multiplications per point. For the run time comparison, $\binom{2^4 d+2}{2}$ points were generated to take account of the binary distribution of the parameter values for the subdivision algorithms. The observations on stability and storage requirements are encoded as follows.

- A A conversion from the BB form to power form is necessary; the additional lines of C-language code for the conversion are listed in parentheses.
- B Multidirectional information such as normal and curvature fields are not easily generated together with the points.
- C The quantities γ used in the refinement grow rapidly with d and the reduction step. In the implementation, following [V88] and [V90], this leads to overflow.
- D(x) Instability due to round-off for $d \geq x$. Both D.I.M. and ForDiff were started with an accurate difference table.
- E All intermediate subdivision values have to be stored.
- F The parameters are distributed non-uniformly (see Figure 6.1).
- G The number of coefficients generated is not arbitrary, but a power of 2. Hence the spacing over a parameter interval of length l is $l/(d2^\sigma)$.

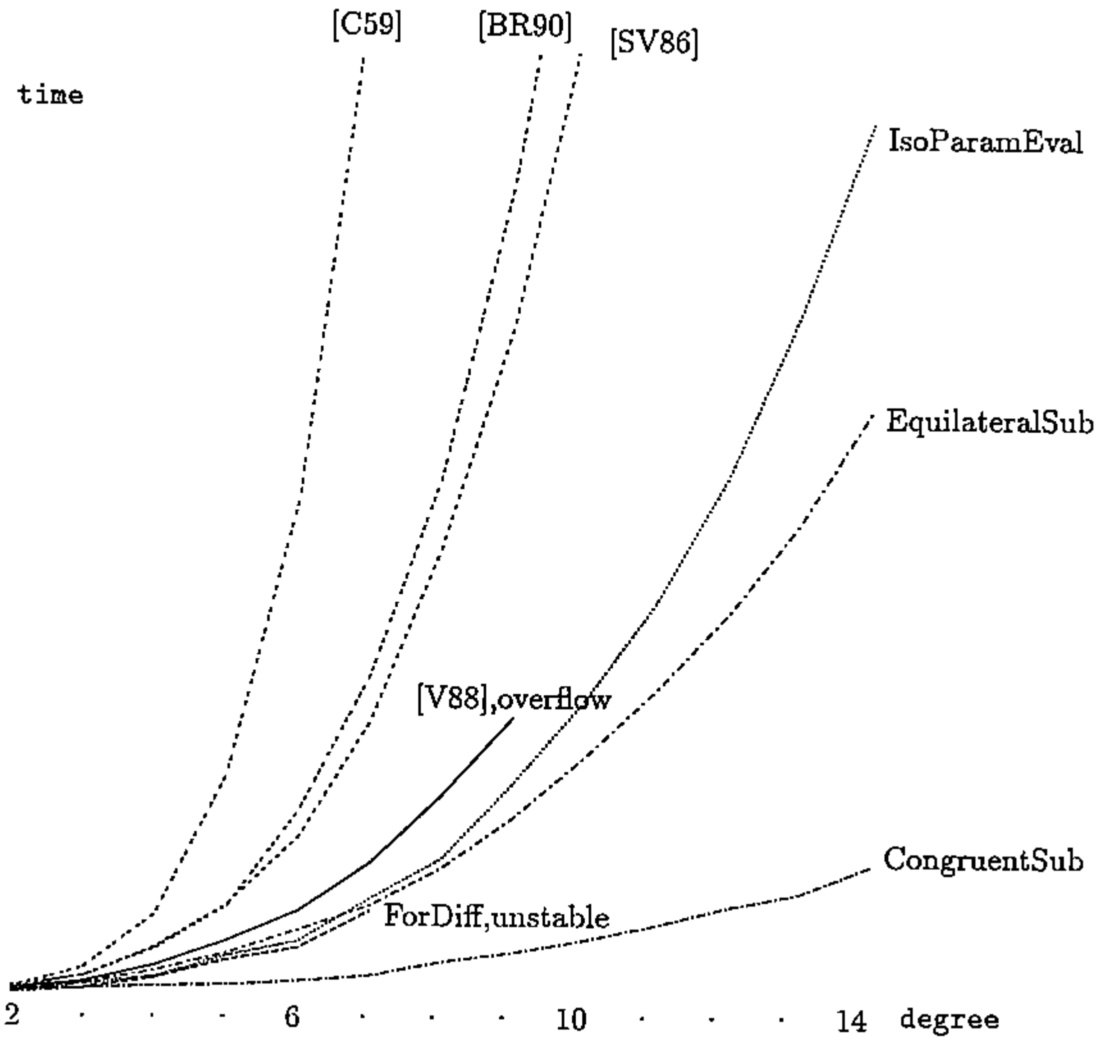
algorithm	time complexity	stability storage	≈ lines of code	reference
DeCasteljau_2	$d(d+1)(d+2)$		30	[C59]
NestMult	$2d(d+1)$	A	20 (+20)	[BR90]
SV-NestMult	$(d^2+5d+4)/2$		25	[SV86]
D.I.M.	$1.5d+e+f$	B,C	200	[V88]
IsoParamEval	$2d+e$	B	60	§3.1
ForDiff	$d+e+f$	B,D(5),	50	eg[B78, p15]
EquilateralSub	$1.4d+7/3$	E,G	180	[Go83],§6
CongruentSub	$\frac{4}{9}(d+3)$	E,G	50	§3.2

(4.2) Table: Evaluation of bivariate (triangular) patches in BB form.

Stability of the algorithms was checked against the numbers generated by DeCasteljau_2 in double precision. Given the finite number of evaluations, the time complexity must account for the linear overhead when generating a quadratic number of points. This overhead is measured by e and f.

$e := \frac{12}{n-1}\binom{d+2}{3}$ is the distributed cost of n calls to EdgeDeCasteljau to extract a univariate polynomial, distributed over $n(n-1)/2$ points.

$f := \frac{2}{n-1}\binom{d+1}{2}$ is n times the cost of building the finite difference table distributed over $n(n-1)/2$ points.



(4.1) Figure: Time per point for $\binom{2^4 d+2}{2}$ points corresponding to the bivariate BB-form. The time scale is linear.

CongruentSub has a distinct time advantage over its competitors already for polynomials of low degree. The table below show the time per point for the evaluation of low degree polynomials.

degree	CongruentSub	EquilateralSub	ForDiff	IsoParamEval	[V88]
2	0.03	0.03	0.04	0.03	0.06
3	0.05	0.09	0.08	0.10	0.10
4	0.05	0.16	0.14	0.14	0.22
5	0.10	0.28	0.22	0.32	0.36

If low storage is required, then ForDiff is advantageous for polynomials of low degree, while IsoParamEval coupled with the univariate version of SV-NestMult is preferable for polynomials of high degree.

5. Conclusion

The two algorithms presented in this paper have a linear time complexity and low actual run time for generating a mesh of points corresponding to an m-variate total-degree polynomial in Bernstein-Bézier form. Both algorithms are based on a low cost specialization of de Casteljau's algorithm that amounts to repeated univariate averaging, and on nesting computations both with respect to the domain and the degree. While the isoparametric decomposition of Section 3.1 requires little storage and offers flexibility in the number and location of evaluation points, the subdivision method in Section 3.2 has a better run time and yields derivatives as a simple extension of the evaluation. Implementation in the bivariate case and a general complexity analysis show that congruent subdivision and isoparametric evaluation are compete well with standard algorithms in the literature.

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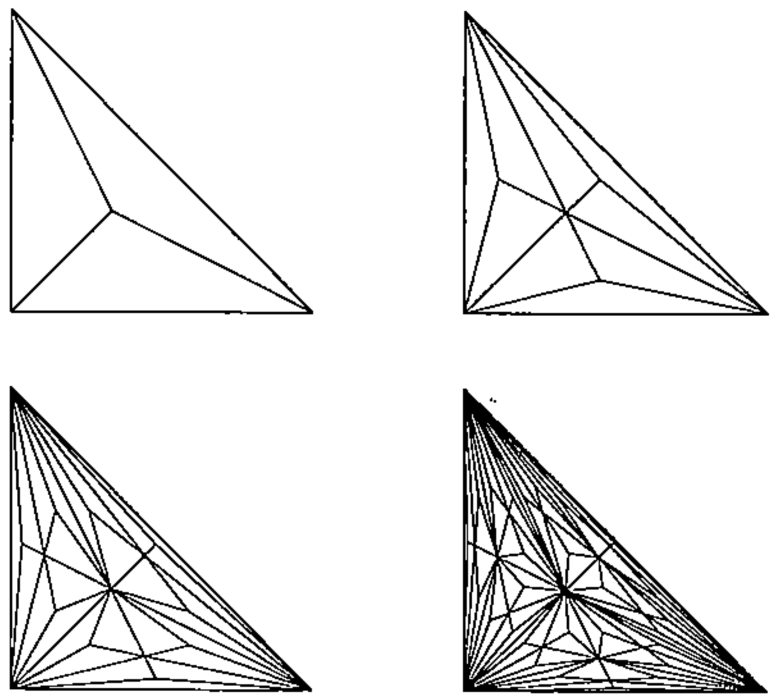
6. Appendix: evaluation methods for polynomials in the BB-form

This section reviews algorithms in the literature referred to in Section 4: recursive use of de Casteljau's algorithm [Go83],[Boe83], two forms of nested multiplication [BR90],[SV86], generic forward differencing and the related difference interpolation method [V88]. The time and space complexity and stability of each algorithm are stated as a remark rather than as a formal lemma since they are either well-known or are easily derived from the pseudocode.



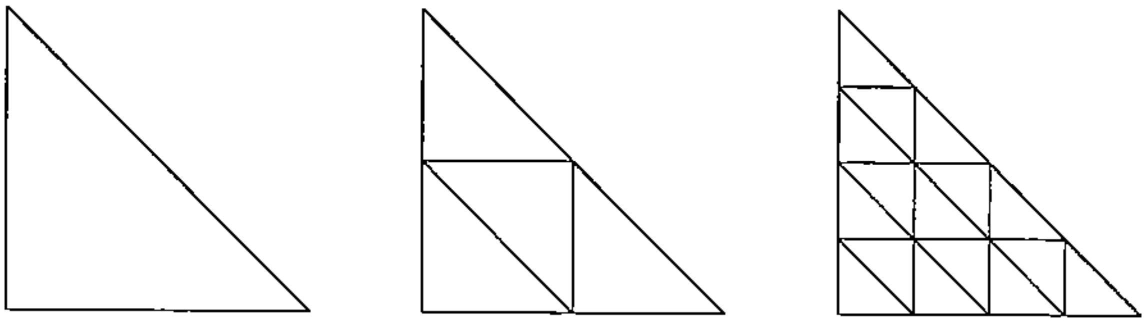
(6.1) Figure: Subtriangles generated by splitting at the centroid ($m = 2$).

As Figure 6.1 shows, recursive subdivision at the centroid is not a good generalization of the univariate subdivision at the midpoint. In particular, Theorem 2.2 does not guarantee convergence of the piecewise linear interpolant.

EquilateralSub covers the domain uniformly by splitting, for $m = 2$, any equilateral domain triangle into four equilateral triangles. It can also be viewed as a natural generalization of the univariate subdivision at midpoints and was suggested in [Go83, Fig. 9]. The subdivision of one triangle into four is achieved by invoking de Casteljau's algorithm repeatedly at the midpoints of the triangle edges. An efficient sequencing of these intermediate evaluations is given in [Boe83, Fig11]. However, in this sequencing, half the basic operations are of the form $A = B + C - D$, i.e. extrapolate, the symmetric subtriangles are treated unequally and $l_{\text{worst}} = \sigma 4(d-1)$, where σ is the number of subdivision levels. To improve stability, the implementation compared in this paper uses a different sequencing that reduces the share of extrapolations to one seventh of the overall operations and l_{worst} to $\sigma 3(d-1)$. C-language code is given in [P90]. Goodman [Goo87] shows that extrapolation can be avoided in exchange for a higher operation count.

```
**EquilateralSub** $(b[V], \sigma; b[V^0], \ldots, b[V^{4^\sigma}])$        [$m = 2$ only!]

for $s = 1..\sigma$        [each subdivision]
$\ell = 4^{s-1} - 1$
  for $i = \ell..0$    [for each subtriangle; overwrites]
    for $j = 0,1,2$        [each vertex in $V^i$]
      $b[V^{4i+j}] = b[V^i]$
      for $l = 0,1,2,\ l \neq j$        [each edge emanating from the vertex]
        EdgeDeCasteljau($b[V^{4i+j}], j, l, \frac{1}{2}; b[V^{4i+j}], *$)
    Extrapolate($b[V^{4i}], b[V^{4i+1}], b[V^{4i+2}]; b[V^{4i+3}]$)
      [extrapolate the interior triangle from the 3 surrounding ones]
```


(6.2) Figure: Subtriangles generated if EquilateralSub is applied to a nonequilateral triangle.

The next method, Horner's Scheme and its multivariate generalization, NestMult, are tailored to polynomials in power form, i.e. of the form $p_{m,d}(x) = \sum_{|\beta| \leq d} c(\beta) x^\beta$, where $\beta \in \mathbb{N}^m$. This requires conversion to power form (see SV-NestMult below).

NestMult [BR90]. Also known as Horner's Scheme, the classical univariate version of nested multiplication (see e.g. [CB80 p.33]) computes the value $p(x)$ of the polynomial p with coefficients $c(0), \ldots, c(d)$ by overwriting $c(0)$ with $p(x)$:

$$c(k) = c(k+1) * x + c(k) \quad \text{for } k = (d-1)..0.$$

We observe that $c(k) = \frac{D^k p_d}{k!}(0)$. With $D^\beta := D_1^{\beta_1} \cdots D_m^{\beta_m}$ and D_i the derivative with respect to the ith variable, [BR90] generalizes the algorithm to the multivariate power form in the following natural way. Note the similarity to DeCasteljau.

```
**NestMult** $(p_{m,d}, x; p_{m,d}(x))$

$c(\beta) = D^\beta p_d(0)/|\beta|!$    for $|\beta| = d$.
**for** $|\beta| = (d-1)..0$
    $c(\beta) = D^\beta p_d(0)/|\beta|! + \sum_{i=1}^m x_i c(\beta + e_i)$
$p_{m,d}(x) = c(0)$.
```

(6.3) Example. If $p_{2,3} = y^3 + 4x^2 + 2xy + 3x + 1$, then NestMult$(p_{2,3}, 2; 39)$. First the algorithm computes $c(\beta) = \begin{cases} 3!/3! & \text{if } \beta = (03) \\ 0 & \text{else} \end{cases}$, then

$$c(20) = 2c(30) + 2c(21) + \frac{1}{2}D_1^2 p(0) = 0 + 0 + 4$$
$$c(11) = 2c(21) + 2c(12) + \frac{1}{2}D_1 D_2 p(0) = 0 + 0 + 1$$
$$c(02) = 2c(12) + 2c(03) + \frac{1}{2}D_2^2 p(0) = 0 + 2 + 0$$

and finally

$$\begin{array}{lll} c(20) = 4 & & \\ & c(10) = 8 + 2 + 3 & \\ c(11) = 1 & & c(00) = 26 + 12 + 1 = 39 \\ & c(01) = 4 + 2 + 0 & \\ c(02) = 2 & & \end{array}$$

■

The algorithm is easy to implement and generates after conversion to power form one point in $2m\binom{d-1+m}{m}$ operations. Since the algorithm overwrites the input polynomial, only $\binom{d+m}{m}$ space is necessary. In fact, by initializing $c(\alpha)$ only with $D^\alpha p_d(0)/|\alpha|!$ for $|\alpha| = d$ and computing $D^\alpha p_d(0)/|\alpha|!$ for $|\alpha| < d$ while iterating, the space requirement for $c(\alpha)$ can be reduced to $\binom{d+m-1}{m-1}$. The level of indirection is $l = d$.

SV-NestMult [SV86]. This approach converts the BB-form to a *modified power form*. The modified power form for $m = 1$ is obtained (see [Schö59, p.252]) by observing that

$$\sum_{\alpha_0+\alpha_1=d} (1-x)^{\alpha_0} x^{\alpha_1} \frac{d!}{\alpha_0!\alpha_1!} c(\alpha_0, \alpha_1)$$
$$= x^d \sum_{\alpha_0=0}^{d} \left(\frac{1-x}{x}\right)^{\alpha_0} \frac{d!}{\alpha_0!(d-\alpha_0)!} c(\alpha_0, d-\alpha_0)$$
$$= x^d \sum_{\beta=0}^{d} \eta^\beta c'(\beta),$$

where $\beta \in \mathbb{Z}$, $\eta := (1-x)/x$ and $c'(\beta) := \binom{d}{\beta}c(\beta, d-\beta) = \binom{d}{\beta}c(\alpha)$. In general,

$$\sum_{|\alpha|=d} \xi^\alpha \binom{d}{\alpha} c(\alpha) = \xi_i^d \sum_{|\beta|\le d} \eta^\beta c'(\beta) \tag{6.4}$$

for $\beta := (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{m+1})$ and $\eta := (\frac{\xi_1}{\xi_i}, \dots, \frac{\xi_{i-1}}{\xi_i}, \frac{\xi_{i+1}}{\xi_i}, \dots, \frac{\xi_{m+1}}{\xi_i})$. That is, ξ_i is pulled outside the summation and the coefficients of the modified power form are $\binom{d}{\beta}c(\alpha)$. By choosing i to be the index corresponding to the largest barycentric coordinate, the transformation is stable and the stability of the evaluation is the same as for nested multiplication; [SV86] explains the bivariate case. In view of Section 3.1, we note that extraction of an isoparametric line from the transformed polynomial by keeping all variables fixed except for η_j amounts to tracing the patch along rays that emanate from the j^{th} vertex. This is different from tracing along isoparametric lines and does not cover the domain uniformly.

ForDiff. This extension of a difference table, is based on the fact that the value of the dth divided difference of a polynomial of degree d is independent of the parameter value. Thus it is possible and efficient to obtain points at equal parameter intervals by extrapolating the difference table as follows.

```
ForDiff ($P(0), \dots, P(d)$)    [$m = 1$]
for $l = 1..d$    [build the difference table]
  for $j = 0..d-l$
    $P(j) = P(j+1) - P(j)$;
for $i = d+1..n$    [extrapolate the table]
  for $l = 1..d$
    $P(l) = P(l) + P(l-1)$
  output $P(d)$
```

The method does not depend on a particular representation of the polynomial; it can start with just $d+1$ equally spaced points. There are numerous improvements of the above generic forward differencing scheme based on other anihilating differential operators and there are extensions of forward differencing to multivariate polynomials (see e.g. [V88a], [LSP87]). ForDiff requires little coding and only const$\binom{d+m}{m}$ space. While for a large number of evaluations in the univariate case, the cost of building the difference table is negligible, this cost has to be considered when the domain is a simplex since there are fewer and fewer evaluations per univariate isoparametric slice as the evaluation moves towards the apex. The main problem with ForDiff as stated above is *loss of stability with increasing degree* due to round-off. The start requires subtracting terms of similar magnitude and subsequent evaluations do not work with the original divided differences but with the newly computed differences. Thus $l_{\text{worst}} = n^m$, $l_{\text{average}} = (n/2)^m$, where $n = d2^\sigma$.

D.I.M. [V88]. The difference interpolation method, introduced by Volk in [V88], evaluates at many equidistant points along the real line. The idea is to improve stability of the forward difference calculation by reducing the level of indirection to $l_{\text{worst}} = (n/\lambda)^m$, where $\lambda \geq 2$. Thus forward differences Δ_h with a short step length h are computed from differences $\Delta_{\lambda h}$ by (back-)solving a $d \times d$ upper triangular system of the form ([V90, (1.5)].

$$G\Delta_h = \Delta_{\lambda h} + C.$$

The entries in G and C can be computed stably according to the reference [V90]. However, the entries in G increase rapidly with d: at least one entry is of size $\binom{d(d-2)}{d(d-2)/2}$ (cf. [V90 (2.3)]). Like ForDiff, D.I.M. does not depend on a particular representation of the polynomial, but needs only $d+1$ points to start. D.I.M. can also be extended to multivariate polynomials. Terms of similar magnitude are subtracted to start the algorithm and this may cause instability. Extensive pseudocode of D.I.M. is given in [V88].