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Abstract

Suppose that $A \in \mathbb{C}^{n,n}$ is a block p-cyclic consistently ordered matrix and let B and S_ω denote, respectively, the block Jacobi and the block Symmetric Successive Overrelaxation (SSOR) iteration matrices associated with A. Neumaier and Varga found (in the $(\rho(|B|), \omega)$-plane) the exact convergence and divergence domains of the SSOR method for the class of H-matrices. Hadjidimos and Neumann applied Rouché's theorem to the functional equation connecting the eigenvalue spectra $\sigma(B)$ and $\sigma(S_\omega)$, obtained by Varga, Niethammer and Cai, and derived in the $(\rho(B), \omega)$-plane the convergence domains for the SSOR method associated with p-cyclic consistently ordered matrices, for any $p \geq 3$. In the present work it is further assumed that the eigenvalues of B^p are real of the same sign. Under this assumption the exact convergence domains in the $(\rho(B), \omega)$-plane are derived in both the nonnegative and the nonpositive cases for any $p \geq 3$.

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Running Title: SSOR convergence domains

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1 Introduction

Consider the linear system

$$Ax = b \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$ and $x, b \in \mathbb{C}^n$, and suppose that A is written in the $p \times p$ block form

$$A = D(I - L - U) \tag{1.2}$$

with D being a $p \times p$ block diagonal invertible matrix and L and U being strictly lower and strictly upper triangular matrices, respectively. Suppose also that for the solution of (1.1)–(1.2) the Symmetric Successive Overrelaxation (SSOR) iterative method (see, e.g., [13], [15], [1]) is used. The SSOR method is defined by

$$\begin{aligned} x^{(m+1/2)} &= (I - \omega L)^{-1}[(1-\omega)I + \omega U]x^{(m)} + \omega(I - \omega L)^{-1}b \\ &\text{and} \\ x^{(m+1)} &= (I - \omega U)^{-1}[(1-\omega)I + \omega L)x^{(m+1/2)} + (I - \omega U)^{-1}b, \qquad m = 1, 2, \ldots, \end{aligned} \tag{1.3}$$

where $x^{(0)} \in \mathbb{C}^n$ arbitrary and $\omega \in (0, 2)$ is the relaxation factor. The block SSOR iteration matrix, associated with A, relative to its block partitioning, is given by

$$S_\omega := (I - \omega U)^{-1}[(1-\omega)I + \omega L](I - \omega L)^{-1}[(1-\omega)I + \omega U]. \tag{1.4}$$

Let $B := L + U$ be the block Jacobi matrix associated with A. If A is block p-cyclic consistently ordered then, without loss of generality, B may be assumed to have the block form

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & B_1 \\ B_2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & B_3 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & B_{p-1} & 0 \end{bmatrix}. \tag{1.5}$$

It is well known that the sets of eigenvalues μ of B (or of B^T) and λ of S_ω satisfy the functional equation obtained by Varga, Niethammer and Cai [14]

$$[\lambda - (1-\omega)^2]^p = \lambda(\lambda + 1 - \omega)^{p-2}(2-\omega)^2\omega^p\mu^p. \tag{1.6}$$

It is noted that (1.6) generalized the corresponding relationship for $p = 2$ (see [4], [10]) and was later generalized by Chong and Cai [3] to cover the entire class of p-cyclic, not necessarily consistently ordered, matrices.

Recently, Hadjidimos and Neumann [7] have found in the (ν, ω)-plane, with $\nu = \rho(B)$ being the spectral radius of the Jacobi matrix B, the domain of convergence for the SSOR method for block

p-cyclic consistently ordered matrices A, $p \geq 3$. Later the same authors generalized their result to cover the entire class of p-cyclic matrices (see [8]). In the analyses in [7] and [8] the application of Rouché's theorem (see, e.g., [9], [12]) led to the determination of the aforementioned convergence domains. The main result of [7] is presented in Theorem 1.1 below and a typical SSOR convergence domain is illustrated in Figure 1.

Theorem 1.1: Let A be a nonsingular block p-cyclic consistently ordered, $p \geq 3$, matrix. Let B and S_ω be the block Jacobi and the block SSOR iteration matrices associated with A and given in (1.5) and (1.4), respectively. Suppose that $\rho(B) = \nu$. Then

$$\rho(S_\omega) < 1$$

provided that $(\nu, \omega) \in R(p)$, where $R(p)$ is the region in the (ν, ω)-plane defined by

$$R(p) := \begin{cases} 0 < \omega \leq 1, \quad 0 \leq \nu < 1 =: \nu_1(\omega) \\ 1 \leq \omega \leq \hat{\omega}, \quad 0 \leq \nu < \frac{1+(1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}} =: \nu_2(\omega) \\ \hat{\omega} \leq \omega < 2, \quad 0 \leq \nu < \frac{[1+(1-\omega)^4 - 2(1-\omega)^2\varphi]^{1/2}}{\omega(2-\omega)^{2/p}[1+(1-\omega)^2+2(1-\omega)\varphi]^{1/2-1/p}} =: \nu_3(\omega) \end{cases} \tag{1.7}$$

where

$$\hat{\omega} := \frac{2(-\hat{y}+2)^{1/2}}{(-\hat{y}+2)^{1/2} + (-\hat{y}-2)^{1/2}}, \; \hat{y} = -\frac{p + (9p^2 - 16p)^{1/2}}{2(p-2)} \tag{1.8}$$

and

$$\begin{aligned} \varphi &:= \varphi(\omega) := \tfrac{1}{4}[-(p-2)y^2 - py + 2(p-2)], \\ y &:= y(\omega) = 1 - \omega + \tfrac{1}{1-\omega}. \end{aligned} \tag{1.9}$$

Note: It is worth pointing out that on the right boundary of $R(p)$ given by the union of the three arcs $\nu_1(\omega)$, $\nu_2(\omega)$, and $\nu_3(\omega)$ of (1.7) the following hold: i) When $|\mu| = 1 \equiv \nu_1(\omega)$, a necessary and sufficient condition for λ, $|\lambda| = 1$, to be an eigenvalue of S_ω is that $\lambda = 1$ and $\mu^p = 1$. This property extends to all $\omega \in (0,2)$ provided that the domain of $\nu_1(\omega)$ is extended accordingly. ii) When $|\mu| = \nu_2(\omega)$, a necessary and sufficient condition for λ, $|\lambda| = 1$, to be an eigenvalue of S_ω is that $\lambda = -1$ and $\mu^p = -\frac{[1+(1-\omega)^2]^p}{(2-\omega)^2\omega^{2p-2}}$, a property that can be extended to cover all $\omega \in (0,2)$ provided the domain of $\nu_2(\omega)$ is also extended.

It is interesting to note that as $p \to \infty$, then, from (1.8), $\hat{y} \to -2^-$, $\hat{\omega} \to 2^-$ and the right boundary of $R(p)$ in (1.7) tends to $\nu(\omega) = \frac{1+(1-\omega)^2}{\omega^2}$ (or $\omega = \frac{2}{1+(2\nu-1)^{1/2}}$, $\frac{1}{2} < \nu \leq 1$). In this limiting case, however, $R(p)$ describes the domain of convergence of the point SSOR method for the entire class of H-matrices A found by Neumaier and Varga [11] (see Figure 2). An open question in [11] regarding convergence on the upper part of the right boundary of the aforementioned region was

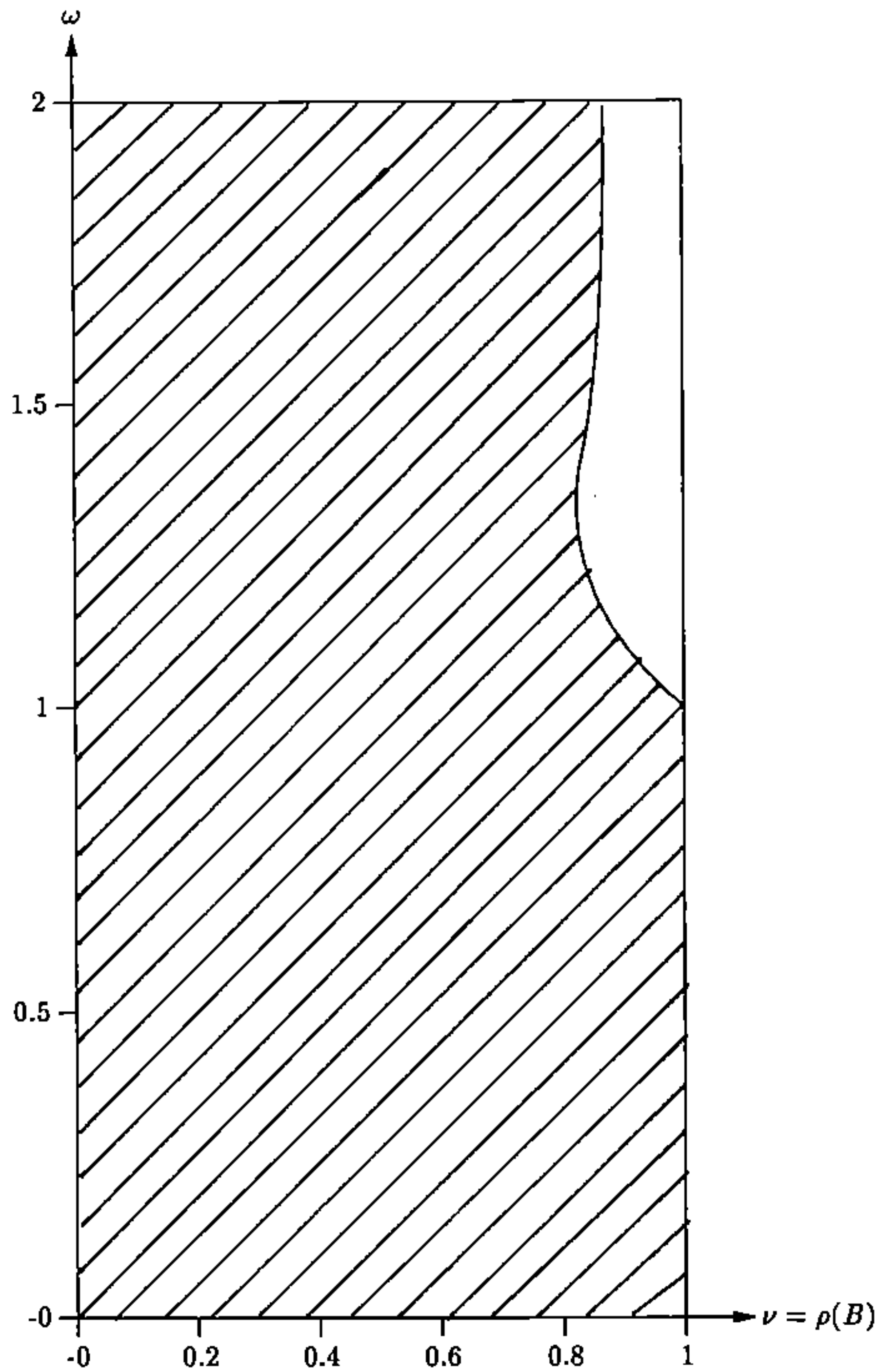

Figure 1: Convergence domain of SSOR for p-cyclic matrices ($p = 5$).

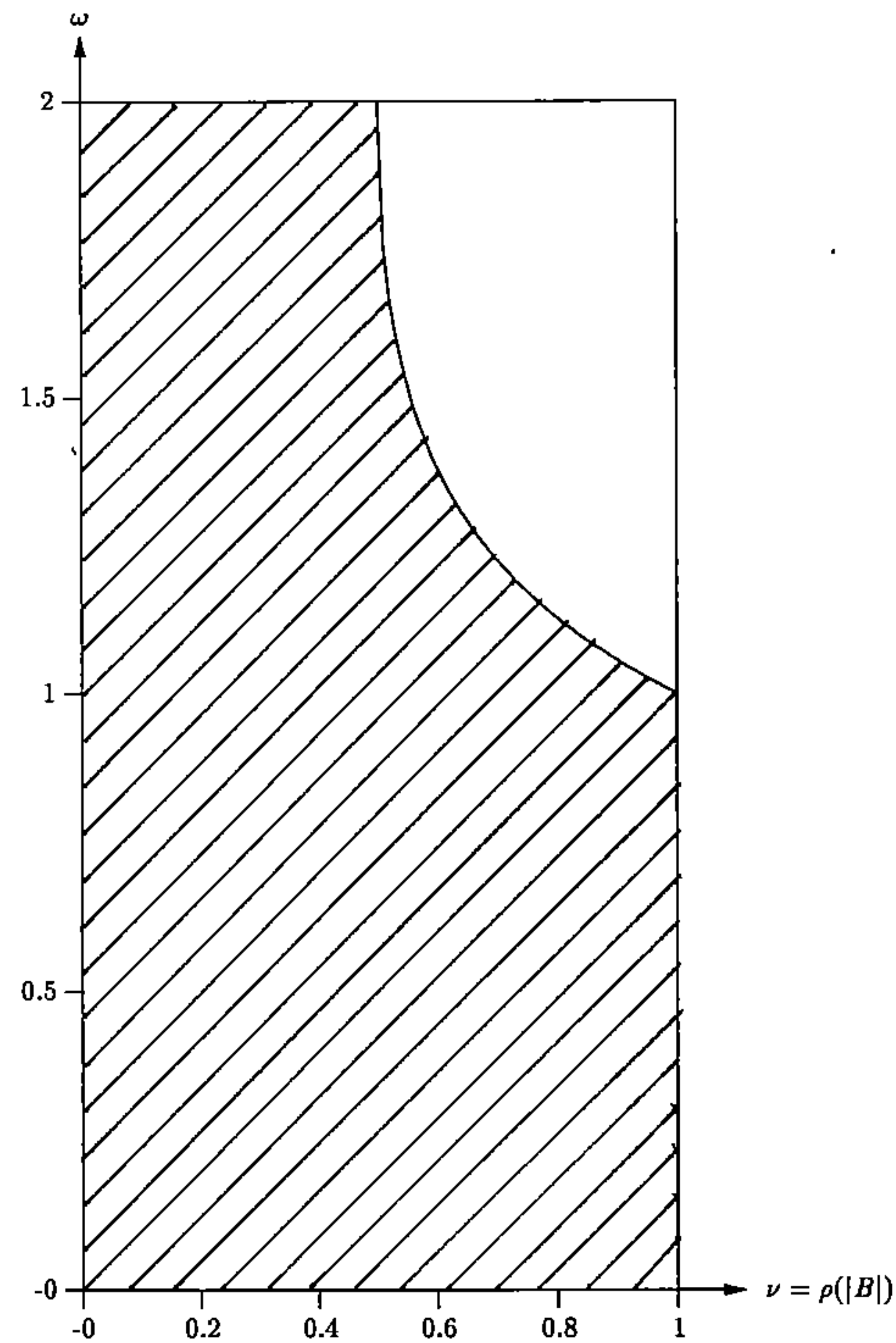

Figure 2: Exact convergence domain of SSOR for H-matrices.

settled later in [6]. We also note here that ν in [11] and [6] denotes $\nu = \rho(|B|)$ and **not** $\nu = \rho(B)$ as it is considered in [7], [8] and which is also considered in the present work.

In this manuscript we obtain, in the (ν,ω)-plane, the **exact** convergence domains of the SSOR method for (block) p-cyclic consistently ordered matrices whose spectrum of the pth power of the Jacobi matrix B, $\sigma(B^p)$, is: i) nonnegative, $\mu_i^p \geq 0$, and ii) nonpositive, $\mu_i^p \leq 0$. In either case $\mu_i \in \sigma(B)$, $i = 1(1)n$. However, having in mind the previous results presented in Theorem 1.1 and its Note we notice that what actually are to be found are the following: i) In the nonnegative case, the right boundary of the domain in question for $1 < \omega < 2$, since for $0 < \omega \leq 1$ it is $\nu_1(\omega) = 1$. Obviously, this boundary must lie strictly to the right of $\nu(\omega) = \frac{1+(1-\omega)^2}{\omega^2}$ and to the left of $\nu_1(\omega) = 1$ and ii) In the nonpositive case, the corresponding right boundary for $0 < \omega < 1$ and $\hat{\omega} < \omega < 2$, since for $1 \leq \omega \leq \hat{\omega}$ it is $\nu_2(\omega) = \frac{1+(1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}}$. This boundary must lie strictly to the right of $\nu_1(\omega) = 1$ and to the left of $\nu_2(\omega)$, for $0 < \omega < 1$, while for $\hat{\omega} < \omega < 2$ strictly to the right of $\nu(\omega) = \frac{1+(1-\omega)^2}{\omega^2}$ and to the left of $\nu_2(\omega)$.

To derive the parts of the desired right boundaries our study will have as a starting point the functional equation (1.6) which, except for some trivial cases, can be rewritten as

$$\mu^p = \frac{[\lambda - (1-\omega)^2]^p}{(2-\omega)^2\omega^p\lambda(\lambda+1-\omega)^{p-2}}. \tag{1.10}$$

The basic idea is to use (1.10) and find, for either nonnegative or nonpositive spectra $\sigma(B^p)$, all possible pairs (μ^p,ω) (or equivalently (ν,ω), with $\nu = |\mu|$), where μ^p belongs to a real interval having as one of its endpoints the point 0, such that $|\lambda| < 1$. For this we set

$$|\lambda| = 1 \Longleftrightarrow \lambda = e^{i\theta} \tag{1.11}$$

and replace λ in (1.10) by the expression in (1.11) to obtain

$$F := F(\omega,\theta) = \frac{[e^{i\theta} - (1-\omega)^2]^p}{(2-\omega)^2\omega^p e^{i\theta}[e^{i\theta}+1-\omega]^{p-2}}. \tag{1.12}$$

In Section 2, after we identify our problem, a complete study of the function F for each fixed $\omega \in (0,2)$ and for all $\theta \in [0,\pi]$ is made. In Sections 3 and 4 the application of the results obtained in Section 2 allows us to determine the **exact** domains of convergence of the SSOR method in the nonnegative and nonpositive case, respectively. Finally, in Section 5 some remarks are made and some particular cases treated in the previous sections are further investigated.

2 Study of the Function F in (1.12)

2.1 Introduction

Before we begin with the study of the function $F(\omega,\theta)$ we shall try to identify the problem we are to solve as well as possible ways of attacking it.

For this we consider first the following simple transformations which will facilitate the subsequent analysis

$$x := x(\omega) := 1-\omega,\ \ \omega \in (0,2),\ \ y := y(\omega) := x + \frac{1}{x},\ \ \omega \in (0,2)\setminus\{1\}. \tag{2.1}$$

The function $F(\omega,\theta)$ can be written explicitly as

$$F(\omega,\theta) = ReF + iImF \tag{2.2}$$

where

$$\begin{aligned} ReF = \tfrac{1}{D}\Bigg\{&[(1+x^4)\cos\theta - 2x^2]\left[E^{p-2} - \binom{p-2}{2}E^{p-4}G^2 + \dots\right] \\ &-(1-x^4)\sin\theta\left[\binom{p-2}{1}E^{p-3}G - \binom{p-2}{3}E^{p-5}G^3 + \dots\right]\Bigg\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} ImF = \tfrac{1}{D}\Bigg\{&[(1-x^4)\sin\theta\left[E^{p-2} - \binom{p-2}{2}E^{p-4}G^2 + \dots\right] \\ &+[(1+x^4)\cos\theta - 2x^2]\left[\binom{p-2}{1}E^{p-3}G - \binom{p-2}{3}E^{p-5}G^3 + \dots\right]\Bigg\} \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} D &= (1+x)^2(1-x)^p(1+x^2+2x\cos\theta)^{p-2}, \\ E &= (1-x^3) + x(1-x)\cos\theta, \qquad G = x(1+x)\sin\theta. \end{aligned} \tag{2.5}$$

Obviously, for $\theta = 0$ or π, $ImF = 0$. More specifically, for $\theta = 0$

$$ReF(\omega,0) = 1 > 0, \tag{2.6}$$

while for $\theta = \pi$

$$ReF(\omega,\pi) = -\frac{(1+x^2)^p}{(1+x)^2(1-x)^{2p-2}} = -\frac{y^p}{(y+2)(y-2)^{p-1}} < 0. \tag{2.7}$$

To find $\theta \in (0,\pi)$, if any, such that $ImF = 0$, we have to find the zeros of the polynomial equation, with respect to (wrt) $z := \cos\theta$,

$$H(z) := H(\cos\theta) := \frac{ImF}{\sin\theta} = 0, \tag{2.8}$$

of degree $p-2$. Note that the coefficients of $H(z)$ in (2.8) are functions of both ω and p. It is obvious that from all possible real roots of (2.8) we are only interested in those belonging to the interval $(-1,1)$. Let θ^+ be the set of all $\theta \in [0,\pi)$ such that

$$ImF(\omega,\theta) = 0, \qquad ReF(\omega,\theta) \geq 0. \tag{2.9}$$

Let also θ^- be the set of all $\theta \in (0,\pi]$ such that

$$ImF(\omega,\theta) = 0, \qquad ReF(\omega,\theta) \leq 0. \tag{2.10}$$

Evidently, from (2.6) and (2.7), we have that $0 \in \theta^+$ and $\pi \in \theta^-$. So, our problem is twofold. Specifically: For the nonnegative case; determine $\theta \in \theta^+$ such that

$$ReF(\omega,\theta) \quad \text{is a minimum.} \tag{2.11}$$

For the nonpositive case; determine $\theta \in \theta^-$ such that

$$ReF(\omega,\theta) \quad \text{is a maximum.} \tag{2.12}$$

In general for $p-2 \geq 5$ ($p \geq 7$) equation (2.8) cannot be solved. Even for $p-2 = 3,4$ ($p = 5,6$) its zeros are very complicated functions of ω. So, what we do in the subsequent analysis is the following. For each fixed $\omega \in (0,2)\backslash\{1\}$ (or, equivalently, for each fixed x or each fixed y, considered in their appropriate intervals) we find all p such that besides the obvious solution $\theta = 0$, for problem (2.9) (resp. $\theta = \pi$ for problem (2.10)), there exists at least one $(0 \neq)\theta \in \theta^+$ (resp. $(\pi \neq)\theta \in \theta^-$) that solves problem (2.11) (resp. problem (2.12)).

2.2 Study of $F(\omega,\theta)$

Our analysis is greatly facilitated if we rewrite the function $F(\omega,\theta)$ in (1.12) in the form below

$$F = F_1F_2^{p-2} \tag{2.13}$$

where

$$F_1 := F_1(\omega,\theta) := \frac{[e^{i\theta} - (1-\omega)^2]^2}{(2-\omega)^2\omega^2e^{i\theta}} \tag{2.14}$$

and

$$F_2 := F_2(\omega,\theta) := \frac{e^{i\theta} - (1-\omega)^2}{\omega(e^{i\theta} + 1 - \omega)} \tag{2.15}$$

Next, we introduce the following functions

$$\begin{gathered} a_1 := a_1(\omega,\theta) := arg(F_1), \quad a_2 := a_2(\omega,\theta) := arg(F_2), \\ a := a(\omega,\theta) := arg(F) = a_1 + (p-2)a_2, \\ r_1 := r_1(\omega,\theta) := |F_1|, \quad r_2 := r_2(\omega,\theta) := |F_2|, \\ r := r(\omega,\theta) = r_1r_2^{p-2}, \end{gathered} \tag{2.16}$$

and then we distinguish the two cases $\omega \in (0,1)$ and $\omega \in (1,2)$ which are studied separately.

2.2.1 Case $\omega \in (0,1)$

From the expressions (2.13)-(2.16) and in view of (2.1) it can be readily obtained that

$$\sin a_1 = \frac{y(y^2-4)^{1/2}\sin\theta}{y^2-2-2\cos\theta}, \qquad \cos a_1 = \frac{(y^2-2)\cos\theta - 2}{y^2-2-2\cos\theta}, \tag{2.17}$$

$$\sin a_2 = \frac{(y+2)^{1/2}\sin\theta}{(y^2-2-2\cos\theta)^{1/2}(y+2\cos\theta)^{1/2}}, \quad \cos a_2 = \frac{(y-2)^{1/2}(y+1+\cos\theta)}{(y^2-2-2\cos\theta)^{1/2}(y+2\cos\theta)^{1/2}} \tag{2.18}$$

and

$$r_1 = \frac{y^2-2-2\cos\theta}{(y+2)(y-2)}, \qquad r_2 = \left(\frac{y^2-2-2\cos\theta}{(y-2)(y+2\cos\theta)}\right)^{1/2}, \tag{2.19}$$

$$r = \frac{(y^2-2-2\cos\theta)^{p/2}}{(y-2)^{p/2}(y+2)(y+2\cos\theta)^{p/2-1}}. \tag{2.20}$$

Based on the functions introduced a number of statements can be proved. Most of them are given as Lemmas and the most important ones as Theorems. To simplify various relationships we shall use the "new" relationship "$A \sim B$" to denote that the expressions A and B are of the same sign.

Lemma 2.1: For a fixed $\omega \in (0,1)$, the function a_1 in (2.16) as a function of θ strictly increases in $[0,\pi]$, with $a_1(\omega,0) = 0$ and $a_1(\omega,\pi) = \pi$.

Proof: Differentiating $\cos a_1$ *wrt* θ and using the expression for $\sin a_1$, from (2.17), one obtains

$$\frac{\partial a_1}{\partial\theta} = \frac{y(y^2-4)^{1/2}}{y^2-2-2\cos\theta} > 0, \tag{2.21}$$

since $y > 2$.□

Lemma 2.2: For a fixed $\omega \in (0,1)$, the function a_2 in (2.16) as a function of θ strictly increases in $[0,\underline{\theta}]$ and strictly decreases in $[\underline{\theta},\pi]$, with $a_2(\omega,0) = a_2(\omega,\pi) = 0$, where

$$\underline{\theta} = arc\cos\left(-\frac{1}{y+1}\right) \in (0,\pi). \tag{2.22}$$

Proof: From (2.18) we can obtain in an analogous way that

$$\frac{\partial a_2}{\partial\theta} = \frac{(y^2-4)^{1/2}[(y+1)\cos\theta+1]}{(y^2-2-2\cos\theta)(y+2\cos\theta)}. \tag{2.23}$$

Obviously, $\frac{\partial a_2}{\partial\theta} > 0$ for $\theta \in [0,\underline{\theta})$ and $\frac{\partial a_2}{\partial\theta} < 0$ for $\theta \in (\underline{\theta},\pi]$ which proves our assertions. □

Theorem 2.3: For a fixed $\omega \in (0,1)$, a of (2.16) strictly increases with $\theta \in [0,\pi]$ if $\omega \in (\omega^{**},1)$. On the other hand, if $\omega \in (0,\omega^{**})$ then a strictly increases with $\theta \in [0,\theta_0]$ and strictly decreases with $\theta \in [\theta_0,\pi]$. Moreover, it is $a(\omega,0)=0$, $a(\omega,\pi)=\pi$, while

$$\omega^{**} = \frac{2(p-2)^{1/2}}{(p+2)^{1/2}+(p-2)^{1/2}} \tag{2.24}$$

and

$$\theta_0 = arc\cos\left(-\frac{y^2+p-2}{py+p-2}\right) \in (0,\pi). \tag{2.25}$$

Proof: Differentiating a of (2.16) wrt $\theta \in [0,\pi]$ and by virtue of Lemmas 2.1 and 2.2 we obtain

$$\frac{\partial a}{\partial \theta} = \frac{(y^2-4)^{1/2}\left[(py+p-2)\cos\theta + y^2+p-2\right]}{(y^2-2-2\cos\theta)(y+2\cos\theta)} \tag{2.26}$$

Obviously,

$$\frac{\partial a}{\partial \theta} \sim (py+p-2)\cos\theta + y^2+p-2 \tag{2.27}$$

which gives

$$\frac{\partial a}{\partial \theta}|_{\theta=0} \sim y^2+py+2(p-2) > 0, \tag{2.28}$$

while

$$\frac{\partial a}{\partial \theta}|_{\theta=\pi} \sim y(y-p). \tag{2.29}$$

From (2.26) – (2.29) it becomes clear that for $y > y^{**} = p$, $\frac{\partial a}{\partial \theta}$ can not vanish in $(0,\pi]$, while for $y \le y^{**}$, $\frac{\partial a}{\partial \theta}$ does vanish for $\theta = \theta_0$ given by (2.25). Furthermore, from (2.1) it is found that $y^{**}=p$ corresponds to ω^{**} given by (2.24). Considering now the variation of the sign of $\frac{\partial a}{\partial \theta}$, in the various cases, the assertions of the present theorem are readily verified. □

Theorem 2.4: For a fixed $\omega \in (0,1)$, the function r as a function of θ strictly increases with $\theta \in [0,\pi]$. Moreover

$$r(\omega,0)=1, \qquad r(\omega,\pi) = \frac{y^p}{(y+2)(y-2)^{p-1}}. \tag{2.30}$$

Proof: Differentiating (2.19) wrt θ it is easily found out that both r_1 and r_2 strictly increase with $\theta \in [0,\pi]$. Consequently, so does r. The values in (2.30) are directly obtained from (2.20), which completes the proof. □

2.2.2 Case $\omega \in [1,2)$

For $\omega = 1$ we already know that the point $(\nu, \omega) = (1,1)$ is a point of the right boundary of the convergence domain for both the nonnegative and the nonpositive cases. Thus, we concentrate on $\omega \in (1,2)$.

This time, in view of (2.1), we have that $x \in (-1,0)$ while $y \in (-\infty, -2)$. Working in exactly the same way as in §2.2.1 we obtain almost identical expressions to those in (2.17)–(2.20) except that when an expression comes from another like $(z^2)^{1/2} = |z|$, $z \in I\!R$, and one gets ride of the absolute value one has to be careful as to the sign one has to use. Below, in (2.31)–(2.34) we give a complete list of the corresponding expressions. Here they are

$$\sin a_1 = -\frac{y(y^2-4)^{1/2}\sin\theta}{y^2-2-2\cos\theta}, \quad \cos a_1 = \frac{(y^2-2)\cos\theta-2}{y^2-2-2\cos\theta}, \tag{2.31}$$

$$\sin a_2 = -\frac{(-y-2)^{1/2}\sin\theta}{(y^2-2-2\cos\theta)^{1/2}(-y-2\cos\theta)^{1/2}}, \quad \cos a_2 = -\frac{(-y+2)^{1/2}(y+1+\cos\theta)}{(y^2-2-2\cos\theta)^{1/2}(-y-2\cos\theta)^{1/2}}, \tag{2.32}$$

$$r_1 = \frac{y^2-2-2\cos\theta}{(y+2)(y-2)}, \quad r_2 = \left(\frac{y^2-2-2\cos\theta}{(y+2)(y-2)}\right)^{1/2}, \tag{2.33}$$

$$r = \frac{(y^2-2-2\cos\theta)^{p/2}}{(-y+2)^{p/2}(-y-2)(-y-2\cos\theta)^{p/2-1}}. \tag{2.34}$$

Again, statements corresponding to those in §2.2.1 can be stated and proved. The conclusions, this time, differ from the previous ones. More specifically:

Lemma 2.5: For a fixed $\omega \in (1,2)$, the function a_1 in (2.16) as a function of θ strictly increases with $\theta \in [0,\pi]$. Moreover, $a_1(\omega, 0) = 0$ and $a_1(\omega,\pi) = \pi$.

Proof: Same as the one in Lemma 2.1. This time (2.31) are used from which we have

$$\frac{\partial a_1}{\partial \theta} = \frac{-y(y^2-4)^{1/2}}{y^2-2-2\cos\theta} > 0. \quad \Box$$

Lemma 2.6: For a fixed $\omega \in (1,2)$, the function a_2 in (2.16) as a function of θ strictly decreases in $[0, \underline{\theta}]$ and strictly increases in $[\underline{\theta}, \pi]$, with $a_2(\omega, 0) = a_2(\omega, \pi) = 0$. $\underline{\theta}$ is given by (2.22).

Proof: It is readily obtained that

$$\frac{\partial a_2}{\partial \theta} = \frac{(y^2-4)^{1/2}[(y+1)\cos\theta + 1]}{(y^2-2-2\cos\theta)(-y-2\cos\theta)}$$

and our assertions follow immediately. $\Box$

Theorem 2.7: Suppose $\omega \in (1,2)$ is fixed. Then for $p = 3,4$ the function a in (2.16) strictly increases with $\theta \in [0,\pi]$. For $p > 5$, a strictly increases with $\theta \in [0,\pi]$ for any $\omega \in (1, \omega^*]$; while if $\omega \in [\omega^*, 2)$ then a strictly decreases for $\theta \in [0, \theta_0]$ and strictly increases for $\theta \in [\theta_0, \pi]$. It is $a(\omega, 0) = 0$ and $a(\omega, \pi) = \pi$. The value of θ_0 is given again by (2.25) while

$$\omega^* = \frac{2p^{1/2}}{p^{1/2} + (p-4)^{1/2}}. \tag{2.35}$$

Proof: We work in an analogous way as in the proof of Theorem 2.3. Thus, relationships (2.27), (2.28) and (2.29) are obtained. This time, however, due to the fact that $y < -2$, the expression in (2.28) changes sign at $y^* = -(p-2)$ provided $p \geq 5$. For $p = 3, 4$, the expression in (2.28) is positive implying that the function a strictly increases with $\theta \in [0, \pi]$. For $p \geq 5$, it is $\frac{\partial a}{\partial \theta}|_{\theta=0} > 0$, for any $y < y^*$. Hence, a strictly increases with $\theta \in [0, \pi]$. On the other hand, $\frac{\partial a}{\partial \theta}|_{\theta=0} < 0$, for any $y > y^*$, and $\frac{\partial a}{\partial \theta} = 0$ has a unique root θ_0 given by (2.25). Obviously, the monotonicity of the function a in the two subintervals of $\omega, (1, \omega^*]$ and $[\omega^*, 2)$ claimed in the statement of the theorem directly follows. The value of ω^* in (2.35) is obtained from (2.1) for $y = y^* = -(p-2)$. □

Theorem 2.8: Suppose $\omega \in (1, 2)$ is fixed. Then, the function r in (2.16) as a function of $\theta \in [0, \pi]$ strictly decreases if $\theta \in (1, \hat{\omega}]$. If $\omega \in [\hat{\omega}, 2)$ r strictly decreases for $\theta \in [0, \theta_1]$ and strictly increases for $\theta \in [\theta_1, \pi]$. The value of $\hat{\omega}$ is given by

$$\hat{\omega} = \frac{2(-\hat{y}+2)^{1/2}}{(-\hat{y}+2)^{1/2} + (-\hat{y}-2)^{1/2}}, \qquad \hat{y} = -\frac{p + (9p^2 - 16p)^{1/2}}{2(p-2)} \tag{2.36}$$

while that of θ_1 by

$$\theta_1 = arc\cos\left(-\frac{(p-2)y^2 + py - 2(p-2)}{4}\right). \tag{2.37}$$

Moreover $\hat{y} > y^*$. (Note: It is worth nothing that the values in (2.36) are the ones in (1.8) obtained in [7].)

Proof: Differentiating r in (2.34) we obtain

$$\frac{\partial r}{\partial \theta} \sim -4\cos\theta - (p-2)(y^2 - 2) - py. \tag{2.38}$$

The expression in (2.38) is positive if and only if $\theta \in (\theta_1, \pi)$, with θ_1 being given by (2.37). Since it can be readily checked that

$$\lim_{y \to -2^-} \left(-\frac{(p-2)y^2 + py - 2(p-2)}{4}\right) = 1,$$

the existence of a unique $\theta_1 \in (0, \pi)$ is guaranteed if and only if

$$-\frac{(p-2)y^2 + py - 2(p-2)}{4} > -1,$$

which, in turn, holds if and only if $y > \hat{y}$, where $\hat{y}$ is given by (2.36). The monotonicity of the function r in the intervals stated in the theorem are direct consequences of the equivalences of the inequality relationships shown above. Finally, it can be checked that $\hat{y} > y^*$, which concludes the proof of the theorem. □

3 The Nonnegative Case

As has been made clear from the analysis in Sections 1 and 2.1 to derive the right boundary of the convergence domain we are interested in one has to solve the problem (2.9), (2.11) for any fixed $\omega \in (0,2)$ (and any fixed $p \geq 3$). Since $\theta = 0$ satisfies (2.9) it is the possible solution. As is known [7], for $\omega \in (0,1]$, $\theta = 0$ is the only element of θ^+. So, in this case our problem has been solved and the corresponding arc of the right boundary is given by

$$\nu_1(\omega) = 1, \qquad \omega \in (0,1]. \tag{3.1}$$

We concentrate then on $\omega \in (1,2)$.

From Theorem 2.7 we have that for $p = 3, 4$, $\theta = 0$ is the only $\theta \in \theta^+$ satisfying (2.11). Hence the right boundary in (3.1) is also the right boundary of the convergence domain for all $\omega \in (1,2)$. So, the convergence domain $R^+(p)$, $p = 3, 4$, is the whole rectangle with vertices (0,0), (1,0), (1,2) and (2,0), illustrated in Figure 3, except its bottom, right and top sides. (Note: It is noted that the result for $p = 3$ was known [5], [2].)

For $p \geq 5$, from Theorem 2.7 we also have that for a fixed $\omega \in (1, \omega^*)$ the only solution to (2.9), (2.11) is $\theta = 0$. Consequently, the arc of the right boundary is given by (3.1). From the same Theorem we have that for a fixed $\omega \in (\omega^*, 2)$ the function $a(\omega,\theta)$ strictly decreases in $[0, \theta_0]$ and strictly increases in $[\theta_0, \pi]$, with $\alpha(\omega, 0) = 0$, $\alpha(\omega, \pi) = \pi$. This implies that there is at least one value of $\theta \in (\theta_0, \pi)$, such that $\theta \in \theta^+ \setminus \{0\}$. The question that arises is the following. Among all $\theta \in \theta^+ \setminus \{0\}$ is there one that satisfies (2.11)?

For $\omega \in (\omega^*, \hat{\omega}]$ the answer can be given immediately by appealing to Theorem 2.8. This is because $r = |F(\omega, \theta)|$ strictly decreases for $\theta \in [0, \pi]$. Therefore among all $\theta \in \theta^+ \setminus \{0\}$ there will be one that will satisfy (2.11).

To proceed with our analysis in the case of $\omega \in (\hat{\omega}, 2)$ we prove three lemmas which are useful in the sequel.

Lemma 3.1: There exists a value of $y = \overline{y} \in (\hat{y}, -2)$ such that for all $y \in (\overline{y}, -2)$ the existence of a $\theta_2 \in (\theta_1, \pi)$ satisfying

$$\cos\theta_2 = \cos\theta_1 - \frac{(p-2)(y^2+y-2)}{4} = \frac{-(p-2)y^2 - (p-1)y + 2(p-2)}{2} \tag{3.2}$$

is guaranteed.

Proof: From (3.2) we have that $\cos\theta_2$ strictly increases with $y \in (\hat{y}, -2)$. Since $\cos\theta_2|_{y=-2} = 1$, the existence of θ_2 is guaranteed if and only if the rightmost expression in (3.2) is greater than -1 or, equivalently, if and only if

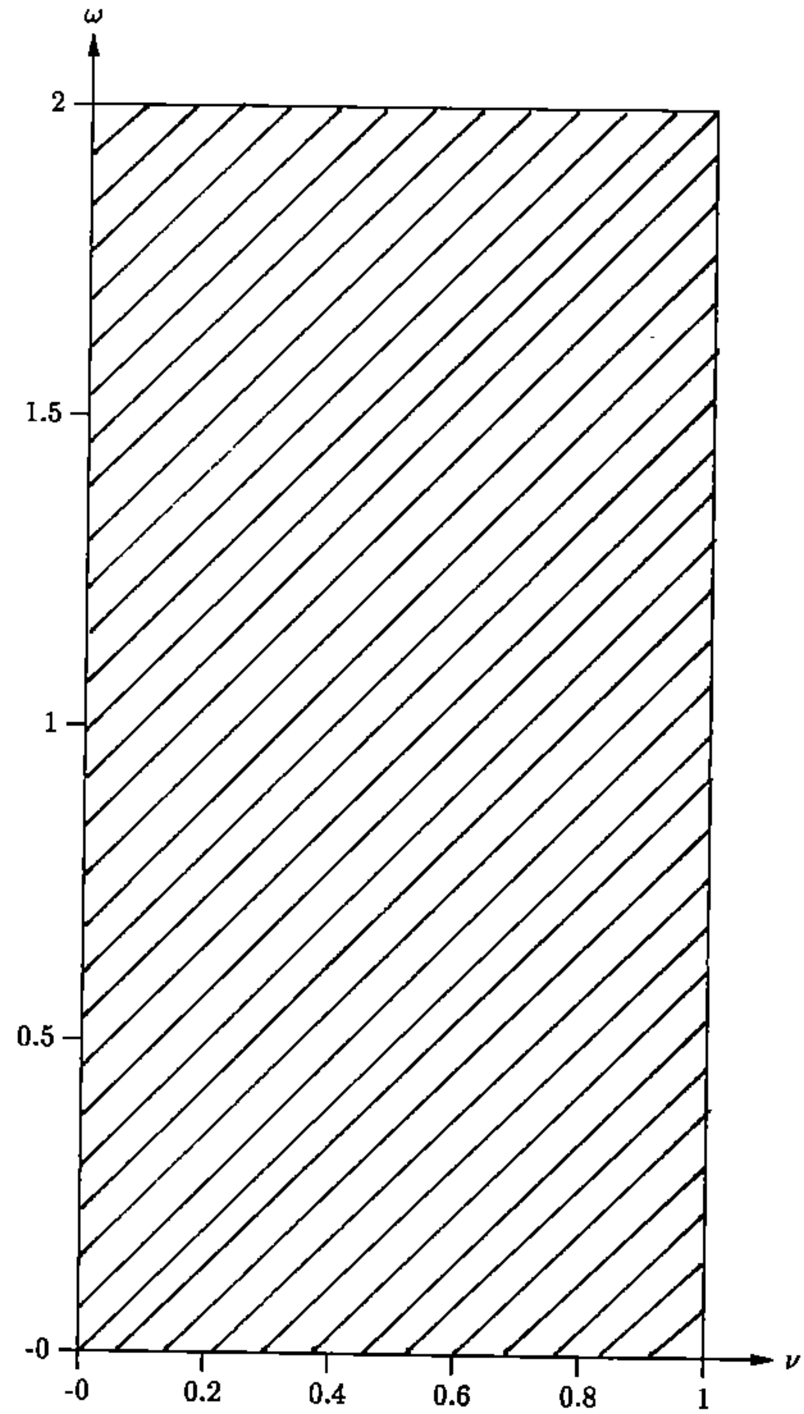

Figure 3: Nonnegative case ($p = 3, 4$).

$$y > \overline{y} := \frac{-(p-1)-(9p^2-28p+17)^{1/2}}{2(p-2)}. \tag{3.3}$$

It can be readily checked that $\overline{y} \in (\hat{y}, 2)$ and that $\theta_2 \in (\theta_1, \pi)$ which completes the proof. □

Lemma 3.2: For $5 \le p \le 24$, $a(\omega, \theta_2) > 0$ for all $y \in (\overline{y}, -2)$.

Proof: By using (3.2) in (2.31), (2.32) we obtain

$$\cos a_1|_{\theta=\theta_2} = \frac{(2-y)^{1/2}[(p-2)y-(p-1)]}{2(p-1)^{1/2}(p-2)^{1/2}(y-1)} \tag{3.4}$$

and

$$\cos a_2|_{\theta=\theta_2} = \frac{-(p-2)y^3+(p-3)y^2+2(p-1)y-2(p-1)}{2(p-1)(y-1)}, \tag{3.5}$$

respectively. Differentiating (3.4), (3.5) *wrt* to y we have

$$\frac{\partial}{\partial y}(\cos a_1|_{\theta=\theta_2}) \sim y[-2(p-2)y^2+(4p-9)y-2(p-3)] > 0 \tag{3.6}$$

and

$$\frac{\partial}{\partial y}(\cos a_2|_{\theta=\theta_2}) \sim -(p-2)y^2+(2p-5)y-(p-5) < 0, \tag{3.7}$$

with the inequalities being valid for all $p \ge 5$. The inequalities in (3.6), (3.7) together with $\cos a_1|_{\theta=\theta_2} > 0$ and $\cos a_2|_{\theta=\theta_2} < 0$ imply that both $a_1(\omega, \theta_2)$ and $a_2(\omega, \theta_2)$ are strictly decreasing functions of y. So is $a(\omega, \theta_2)$. It can be checked that for $p \ge 5$ the largest value of p giving the smallest positive value for $a(\omega, \theta_2)$, and corresponding to $y = -2$, which is $a(2, \theta_2) \approx 0.0206$, is $p = 24$. Hence the present statement is proved. □

Lemma 3.3: The function $r(\omega, \theta_2)$ is given by

$$r(\omega, \theta_2) = -\frac{(p-1)^{p/2}(y-1)}{(p-2)^{p/2-1}(2-y)^{p/2}} \tag{3.8}$$

and is a strictly increasing function of $y \in (\overline{y}, -2)$.

Proof: (3.8) is readily obtained if (3.2) is used in (2.34). The monotonic behavior of $r(\omega, \theta_2)$ is easily proved since

$$\frac{\partial r(\omega, \theta_2)}{\partial y} \sim -(p-2)y+(p-4) > 0. \quad \square \tag{3.9}$$

Lemma 3.4: The function $F(\omega, \pi)$ is given by

$$F(\omega,\pi) = -\frac{(-y)^p}{(2-y)^{p-1}(-y-2)} \tag{3.10}$$

and strictly decreases for all $y \in [\hat{y}, -2)$ with $\lim_{y\to -2^-} F(\omega,\pi) = -\infty$.

Proof: (3.10) is directly derived from (1.12) and (2.1). Its monotonicity in the aforementioned interval is proved by differentiation and by observing that $\hat{y} > -\frac{2p}{p-2}$. Its limiting value is then readily derived. □

From Theorem 2.8 we have that for a fixed $\omega \in (\hat{\omega}, 2)$, $r(\omega,\theta)$, as a function of θ, strictly decreases in $[0,\theta_1]$ and strictly increases in $[\theta_1,\pi]$. Its maximum value is then attained at one of the endpoints of $[0,\pi]$. So, if $r(\omega,\pi) < 1 (= r(\omega,0))$, then $r(\omega,\theta) < 1$, $\theta \in (0,\pi]$. Since by virtue of Lemma 3.4, $r(\omega,\pi)$ $(= -F(\omega,\pi))$ strictly increases with $y \in (\hat{y},-2]$ then $r(\omega,\pi)$ will be less than 1 for all $y \in (\hat{y},\overline{y}]$, if $r(\overline{\omega},\pi) < 1$, where $\overline{\omega}$ is the value of $\omega \in (1,2)$ that gives $\overline{y}$. As can be checked $r(\overline{\omega},\pi) < 1$ for all $5 \le p \le 24$. This implies that there is a value of $\theta \in \theta^+ \setminus \{0\}$ that satisfies (2.9) and (2.11) for all $y \in (\hat{y},\overline{y}]$, or for all $\omega \in (\hat{\omega},\overline{\omega})$.

For $y \in (\overline{y},-2)$, from Lemma 3.2, the real positive value of $F(\omega,\theta)$ corresponds to a $\theta \in (0,\theta_2]$. Thus, as in the previous case, if $r(\omega,\theta_2) < 1$ then $r(\omega,\theta) < 1$ for all $\theta \in (0,\theta_2]$. Since, from Lemma 3.3, $r(\omega,\theta_2)$ increases *wrt* y then $r(2,\theta_2) < 1$ will imply $r(\omega,\theta_2) < 1$ for all $y \in (\overline{y},-2)$. By direct computation it can be verified that the values $r(\overline{\omega},\pi)$ and $r(2,\theta_2)$ are indeed less than 1 for all $5 \le p \le 24$.

The analysis so far effectively shows that for any $5 \le p \le 24$ and for each $\omega \in (\omega^*,2)$ there exists a value of $\theta \in \theta^+ \setminus \{0\}$ that satisfies (2.11). For this value of θ, $F(\omega,\theta) < 1$. Consequently, the right boundary of the convergence domain will be given by an expression of the form

$$\nu_1' = (F(\omega,\theta))^{1/p}, \qquad \omega \in (\omega^*,2). \tag{3.11}$$

A typical convergence domain for $5 \le p \le 24$ is illustrated in Figure 4.

For $p \ge 25$ we can study the sequences of values $a_1(\hat{\omega},\hat{\theta}_0)$, $a_2(\hat{\omega},\hat{\theta}_0)$ and $a(\hat{\omega},\hat{\theta}_0)$, where $\hat{\omega}$ and $\hat{\theta}_0$ are given by (2.36) and (2.25) with $y = \hat{y}$, as functions of p only. It can be found that $a(\hat{\omega},\hat{\theta}_0)|_{p=25} \approx -7.4578 < -2\pi$. This means that there are more than one real nonnegative values of $F(\omega,\theta)$ for $\theta \in (0,\pi)$, with at least one of them being less than 1. This is because $r(\omega,\theta)$ strictly decreases in $(0,\theta_1)$ and $\theta_1 > \theta_0$. So, for $26 \le p \le 30$ we arrive at exactly the same conclusion as before, since $a(\hat{\omega},\hat{\theta}_0)$ strictly decreases as a function of p. For $p = 31$, we can find that $(p-2)a_2(\hat{\omega},\hat{\theta}_0)|_{p=31} \approx -9.5058 < -3\pi$ and since $0 < a_1(\hat{\omega},\hat{\theta}_0)|_{p=31} < \pi$, it is $a(\hat{\omega},\hat{\theta}_0)|_{p=31} = a_1(\hat{\omega},\hat{\theta}_0)|_{p=31} + (p-2)a_2(\hat{\omega},\hat{\theta}_0)|_{p=31} < -2\pi$. Therefore the same conclusion as before is true. For $p > 31$, we simply, note that $a_2(\hat{\omega},\hat{\theta}_0)$ strictly decreases as a function of p and so, again, the same conclusion follows. Consequently, the right boundary for $\omega \in (\omega^*,2)$ is given by (3.11).

We summarize the analysis in this section in the following statement.

Theorem 3.5: For $p = 3, 4$, the right boundary of the convergence domain $R^+(p)$ is given by

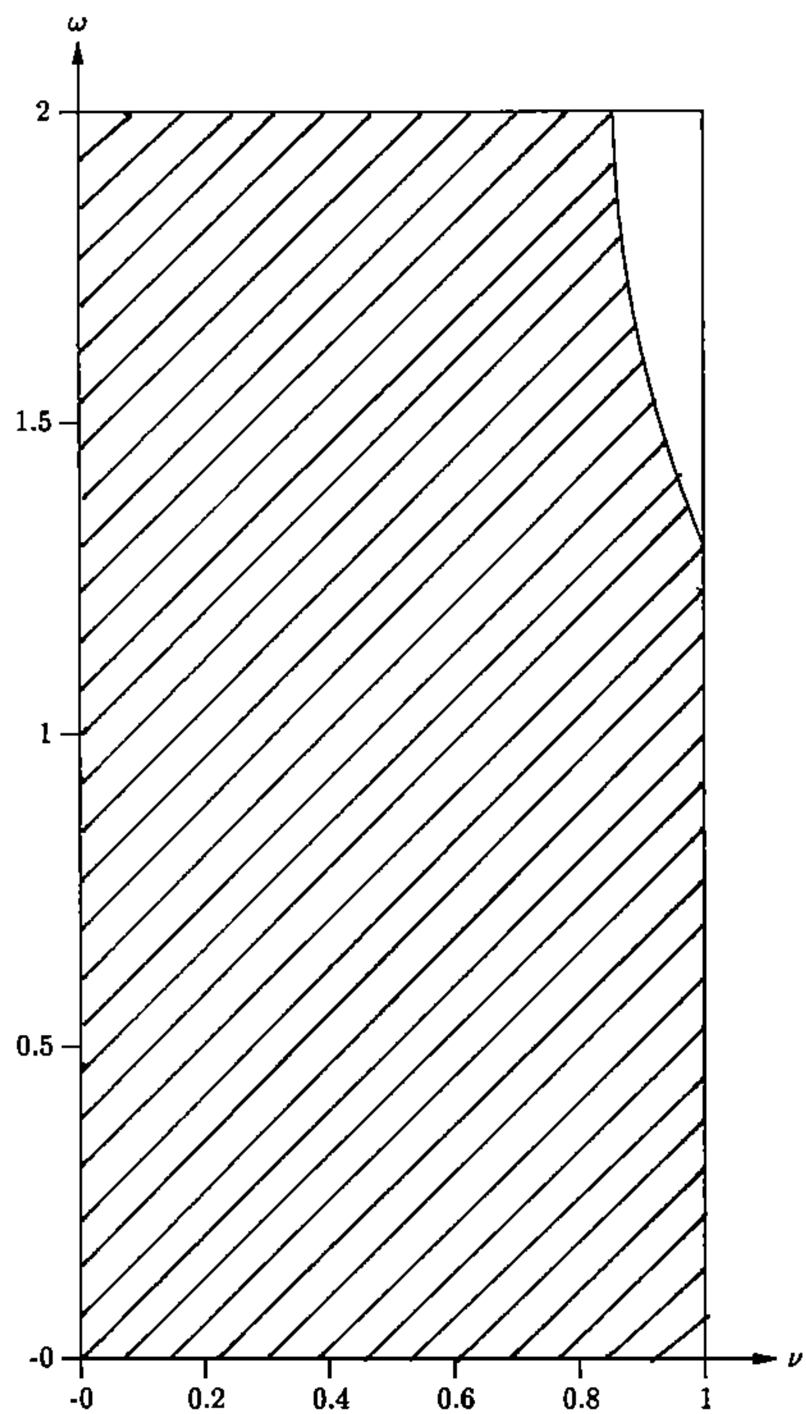

Figure 4: Nonnegative case ($p \geq 5$).

$$\nu_1 := \nu_1(\omega) = 1, \qquad \omega \in (0,2). \tag{3.12}$$

For $p \geq 5$, it is given by the union of the two arcs ν_1 and ν_1', where

$$\nu_1 := \nu_1(\omega) = 1, \qquad \omega \in (0,\omega^*] \tag{3.13}$$

and

$$\nu_1' := \nu_1'(\omega) = (F(\omega,\theta))^{1/p}, \qquad \omega \in (\omega^*,2) \tag{3.14}$$

with $\theta \in \theta^+ \setminus \{0\}$ being the solution to (2.11).

4 The Nonpositive Case

In an analogous way to that in Section 3 we shall try to find out for each $\omega \in (0,2) \setminus \{1\}$ whether an element $\theta \in \theta^- \setminus \{\pi\}$ exists that satisfies (2.12). It is reminded that from the analysis in [7], for any $\omega \in [1,\hat{\omega}]$, $\theta = \pi$ is the only element of θ^- and thus gives the solution to problem (2.12). So, the right boundary of the convergence domain in this case is given by

$$\nu_2(\omega) := -\frac{y}{(-y-2)^{1/p}(-y+2)^{1-1/p}} = \frac{1+(1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}}, \qquad \omega \in [1,\hat{\omega}]. \tag{4.1}$$

Also from Theorem 2.3 we have that for a fixed $\omega \in (\omega^{**},1)$ there is not any other real nonpositive value of $F(\omega,\theta)$ except that corresponding to $\theta = \pi$. This is obvious because $a(\omega,\theta)$ strictly increases. Therefore the right boundary of the convergence domain will be given by

$$\nu_2(\omega) := \frac{y}{(y+2)^{1/p}(y-2)^{1-1/p}} = \frac{1+(1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}}, \qquad \omega \in [\omega^{**},1]. \tag{4.2}$$

To proceed with our analysis for a fixed $\omega \in (0,\omega^{**})$ we recall from Theorem 2.3 that there exists a $\theta_0 \in [0,\pi]$ corresponding to the maximum value of $a(\omega,\theta)$. This value of $a(\omega,\theta)$ is greater than π. So, there will exist a $\theta \in (0,\theta_0]$ which will satisfy (2.10), (2,12). Since, by Theorem 2.4, $r(\omega,\theta)$ strictly increases with θ it will be $F(\omega,\pi) < F(\omega,\theta) < 0$. In case there are more than one $\theta \in \theta^- \setminus \{\pi\}$ satisfying (2.12) the smallest one, let it be θ_m, will give the right boundary of the convergence domain. In other words it will be

$$\nu_2''(\omega) := (-F(\omega,\theta_m))^{1/p}, \qquad \omega \in (0,\omega^{**}). \tag{4.3}$$

For $\omega \geq 1$ only the case $\omega \in (\hat{\omega},2)$ remains to be studied. The following two lemmas will facilitate the analysis.

Lemma 4.1: For all $11 \leq p \leq 30$, the function $a_1(\omega,\theta_0)$ is a strictly decreasing function of $y \in [\hat{y},-2)$, where θ_0 and $\hat{y}$ are given by (2.25) and (2.36), respectively.

Proof: From (2.31) we obtain

$$tana_1 = -\frac{y(y^2-4)^{1/2}\sin\theta}{(y^2-2)\cos\theta - 2} \tag{4.4}$$

and therefore

$$tana_1(\omega,\theta_0) = \frac{[y(y-2)(p-y)(y+p-2)]^{1/2}}{y^2-2y+p}. \tag{4.5}$$

Differentiating the square of the expression in (4.5) *wrt* y it can be obtained that

$$\frac{\partial}{\partial y}\left(tana_1(\omega,\theta_0)\right)^2 \sim K(y,p) := -(p^2+12)y^5$$

$$+(5p^2-12)y^4 - (15p^2-8p-2)y^3 - 2p(p^2-5p-4)y^2 \tag{4.6}$$

$$+p^2(p^2-2p-4)y - p^3(p-2).$$

By Descartes' rule of sings it is checked that $K(y,p)$ has one negative zero for all $p \geq 5$. It can be found that $K(-2,p) < 0$ and also, computationally, that $K(\hat{y},p) < 0$ for all $11 \leq p \leq 30$. This implies that $tana_1(\omega,\theta_0)$ strictly decreases. So does $a_1(\omega,\theta_0)$, with its smallest (limiting) value being given by

$$\lim_{y\to -2^-} tana_1(\omega,\theta_0) = \frac{2\sqrt{2}(p+2)^{1/2}(p-4)^{1/2}}{p+8}. \quad \square \tag{4.7}$$

Lemma 4.2: For all $p \geq 3$, the function $a_2(\omega,\theta_0)$ is a strictly decreasing function of y for all $y < -2$.

Proof: Working as in the proof of the previous lemma we can obtain

$$\frac{\partial}{\partial y}\left(tana_2(\omega,\theta_0)\right)^2 \sim -y+1 > 0. \tag{4.8}$$

Since $a_2(\omega,\theta) \in (-\frac{\pi}{2},0)$ it is implied that $a_2(\omega,\theta_0)$ strictly decreases *wrt* y with its smallest (limiting) value being given by

$$\lim_{y\to -2^-} (tana_2(\omega,\theta_0)) = -\frac{\sqrt{2}(p+2)^{1/2}(p-4)^{1/2}}{4(p-1)}. \quad \square \tag{4.9}$$

One of our main results is given in the following statement.

Theorem 4.3: i) For any $3 \leq p \leq 14$ and a fixed $\omega \in (\hat{\omega},2)$ there exists a unique real negative value of $F(\omega,\theta)$ satisfying (2.10) and corresponding to $\theta = \pi$. ii) For $p \geq 15$, there exists a $\tilde{y}$ such that for any fixed $y \in [\tilde{y},-2)$ there is at least one real negative value of $F(\omega,\theta)$ other than that corresponding to $F(\omega,\pi)$ and such that $F(\omega,\theta) > F(\omega,\pi)$.

Proof: i) For $3 \leq p \leq 11$, by virtue of Lemma 4.2, it is

$$a_2(\omega,\theta_0) > \lim_{y\to -2^-} a_2(\omega,\theta_0) = arc\ tan\left(-\frac{\sqrt{2}(p+2)^{1/2}(p-4)^{1/2}}{4(p-1)}\right). \tag{4.10}$$

By direct computation and for the aforementioned values of p it can be obtained that $(p-2)\lim_{y\to -2^-} a_2(\omega,\theta_0) > -\pi$. Since $a_1(\omega,\theta_0) > 0$, it is $a(\omega,\theta_0) > -\pi$ implying that there is no value of θ other than $\theta = \pi$ for which (2.10) holds true. For $p = 12, 13, 14$, using the results of Lemmas 4.1 and 4.2 it can be obtained, computationally, that

$$\begin{aligned}\min a(\omega,\theta_0) \quad &= \quad arc\ tan\left(\tfrac{2\sqrt{2}(p+2)^{1/2}(p-4)^{1/2}}{p+8}\right) \\ &+ \quad (p-2)arc\ tan\left(-\tfrac{\sqrt{2}(p+2)^{1/2}(p-4)^{1/2}}{4(p-1)}\right) > -\pi.\end{aligned}$$

In other words the same conclusion as before holds.

ii) As in the analysis of the nonnegative case we study the sequences of values $a_1(\hat{\omega},\hat{\theta}_0)$, $a_2(\hat{\omega},\hat{\theta}_0)$, and $a(\hat{\omega},\hat{\theta}_0)$ corresponding to $\hat{y}$, given by (2.36), as functions of p. From Lemmas 4.1 and 4.2, $a_1(\hat{\omega},\hat{\theta}_0)$ is a strictly decreasing function of p, for $11 \le p \le 30$, while $a_2(\hat{\omega},\hat{\theta}_0)$ is a strictly decreasing function for all p. This is because, $\hat{y}$ strictly increases with p and $\lim_{p\to\infty}\hat{y} = -2$. Therefore $a(\hat{\omega},\hat{\theta}_0)$, as a function of p, strictly decreases for $11 \le p \le 30$. Computationally it can be found out that

$$a(\hat{\omega},\hat{\theta}_0)|_{p=15} \approx -2.985 > -\pi > a(\hat{\omega},\hat{\theta}_0)|_{p=16} \approx -3.311. \tag{4.11}$$

Result (4.11) implies that for all $16 \le p \le 30$ and for all $y \in [\hat{y},-2)$ it will hold $a(\omega,\theta_0) < -\pi$. Hence, there exists $\bar{y} \in (y^*,\hat{y}]$ such that (2.10) will be satisfied for more than one $\theta \in \theta^-$ (one of them is that corresponding to $\theta = \pi$), for any fixed $y \in [\bar{y},-2)$ or, equivalently, for any fixed $\omega \in [\bar{\omega},2)$. On the other hand it is $a_1(\hat{\omega},\hat{\theta}_0)|_{p\ge3} \in (0,\pi)$ while $(p-2)a_2(\hat{\omega},\hat{\theta}_0)|_{p=21} \approx -6.090 > -2\pi > (p-2)a_2(\hat{\omega},\hat{\theta}_0)|_{p=22} \approx -6.432$. Therefore $a(\hat{\omega},\hat{\theta}_0)|_{p\ge22} < -\pi$. Consequently the same conclusion as before holds for any $p \ge 30$. For $p = 15$, it can be checked that $\min\ a(\omega,\theta_0) < -\pi$, meaning that there exists $\bar{y} \in (\hat{y},-2)$ such that there are more than one $\theta \in \theta^-$ for any fixed $y \in [\bar{y},-2)$. This completes our proof. □

From the first part of Theorem 4.3 it is concluded that the right boundary of the convergence domain for $3 \le p \le 14$ and for all $\omega \in (1,2)$ will be given by the formula (4.1). A typical region of convergence in this case is illustrated in Figure 5.

For $p \ge 16$ and for a fixed $y \in [\bar{y},\hat{y}]$, Theorem 2.8 states that the largest real negative value of $F(\omega,\theta)$ is $F(\omega,\pi) = -r(\omega,\pi)$. From (2.34) this value is given by

$$F(\omega,\pi) = -\frac{(-y)^p}{(2-y)^{p-1}(-y-2)}. \tag{4.12}$$

Differentiating the above expression *wrt* y it can be proved that it is a strictly decreasing function for all $y \ge -\frac{2p}{p-2}$. Since $\hat{y} > -\frac{2p}{p-2}$ it is concluded that $F(\omega,\pi)$ strictly decreases for $y \in [\hat{y},-2)$, with $\lim_{y\to -2^-} F(\omega,\pi) = -\infty$. Based on continuity arguments we can say that the above value, $F(\omega,\pi)$, must be the largest one in an interval of y whose right endpoint is strictly to the right of $\hat{y}$. Let

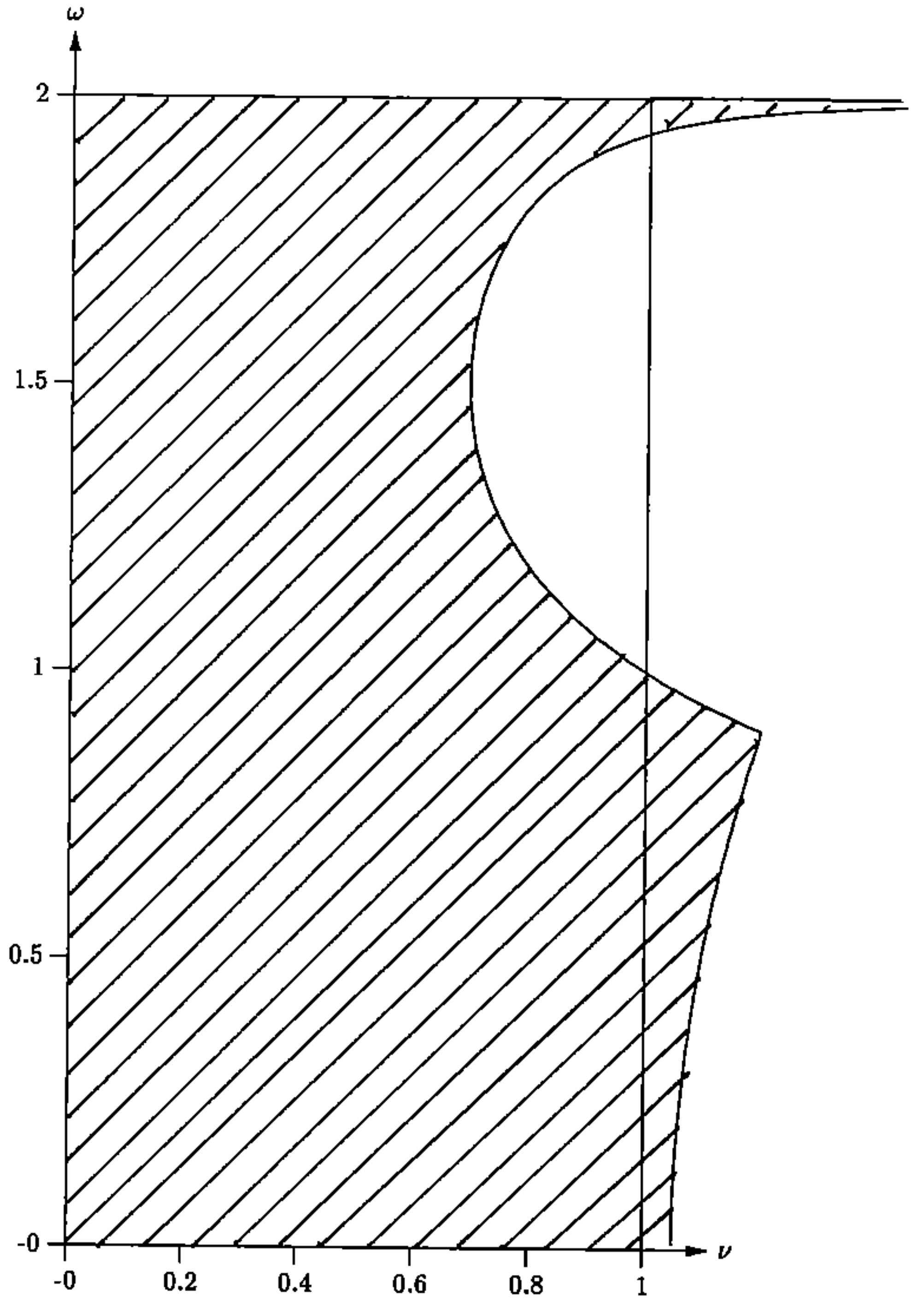

Figure 5: Nonpositive case ($5 \leq p \leq 14$).

y' be the aforementioned right endpoint. Then, it is readily concluded that for $y \in (y', -2)$ the largest real negative value satisfying (2.12) will become greater than -1.

Summarizing the conclusions so far we have that for $y \leq y'$ (or equivalently $\omega \in (1, \omega']$, with ω' corresponding to y') the right boundary of the convergence domain will be given by $\nu_2(\omega)$ of (4.1) while for $y > y'$ (or $\omega \in (\omega', 2)$) there will exist a right boundary, other than $\nu_2(\omega)$, corresponding to the solution of (2.10), (2.12).

In the previous analysis the case $p = 15$ was not included. The following lemma fills this gap.

Lemma 4.4: The function $r(\omega, \theta_0)$, which is given by

$$r(\omega, \theta_0) = \frac{p^{p/2}}{(p-2)^{p/2-1}} \left(\frac{-y}{2-y}\right)^{p/2} (1-y), \tag{4.13}$$

is a strictly decreasing function *wrt* $y \in (y^*, -2)$.

Proof: A direct substitution of (2.25) in (2.34) yields (4.13). Since both $\frac{-y}{2-y}$ and $1-y$ are positive and strictly decreasing functions of y, so is $r(\omega, \theta_0)$. □

Now, for $p = 15$, it can be found computationally that: For $y_1 = -2.0959$ and $y_2 = -2.0949$ there will be

$$a(\omega_1, \theta_0) = -3.1406 > -\pi > a(\omega_2, \theta_0) = -3.1421.$$

On the other hand we can find out that

$$\begin{aligned} r(\omega_1, \theta_0) &= 0.65519, \quad r(\omega_2, \theta_0) = 0.65508, \\ r(\omega_1, \pi) &= 0.431965, \quad r(\omega_2, \pi) = 0.432354. \end{aligned} \tag{4.14}$$

Since $r(\omega, \theta_0)$ strictly decreases while $r(\omega, \pi)$ strictly increases with y, it is implied from (4.14) that there will be a $\tilde{y} \in (-2.0959, -2.0949)$ such that $F(\tilde{\omega}, \theta_0) \in (-0.65519, -0.65508)$ and $F(\tilde{\omega}, \pi) \in (-0.431965, -0.432354)$. Consequently, $F(\tilde{\omega}, \theta_0) < F(\tilde{\omega}, \pi)$. The rest of the argumentation is that of the case $p \geq 16$, implying that for $p = 15$ exactly the same conclusions hold.

Therefore for all $p \geq 15$ and for any $\omega \in (\omega', 2)$ the right boundary will be given by an expression of the form

$$\nu_2''(\omega) := (-F(\omega, \theta))^{1/p}, \tag{4.15}$$

with $\theta \in \theta^- \setminus \{\pi\}$ being the solution to (2.12).

A typical convergence domain for any $p \geq 15$ is illustrated in Figure 6.

5 Final Remarks and Particular Cases

The analysis in the previous sections allowed us to determine the exact convergence domains for the block SSOR iterative method when the corresponding block Jacobi matrix B (or its transpose) is weakly cyclic of index $p \geq 3$. This was done in the two cases where $\sigma(B^p)$ is nonnegative or

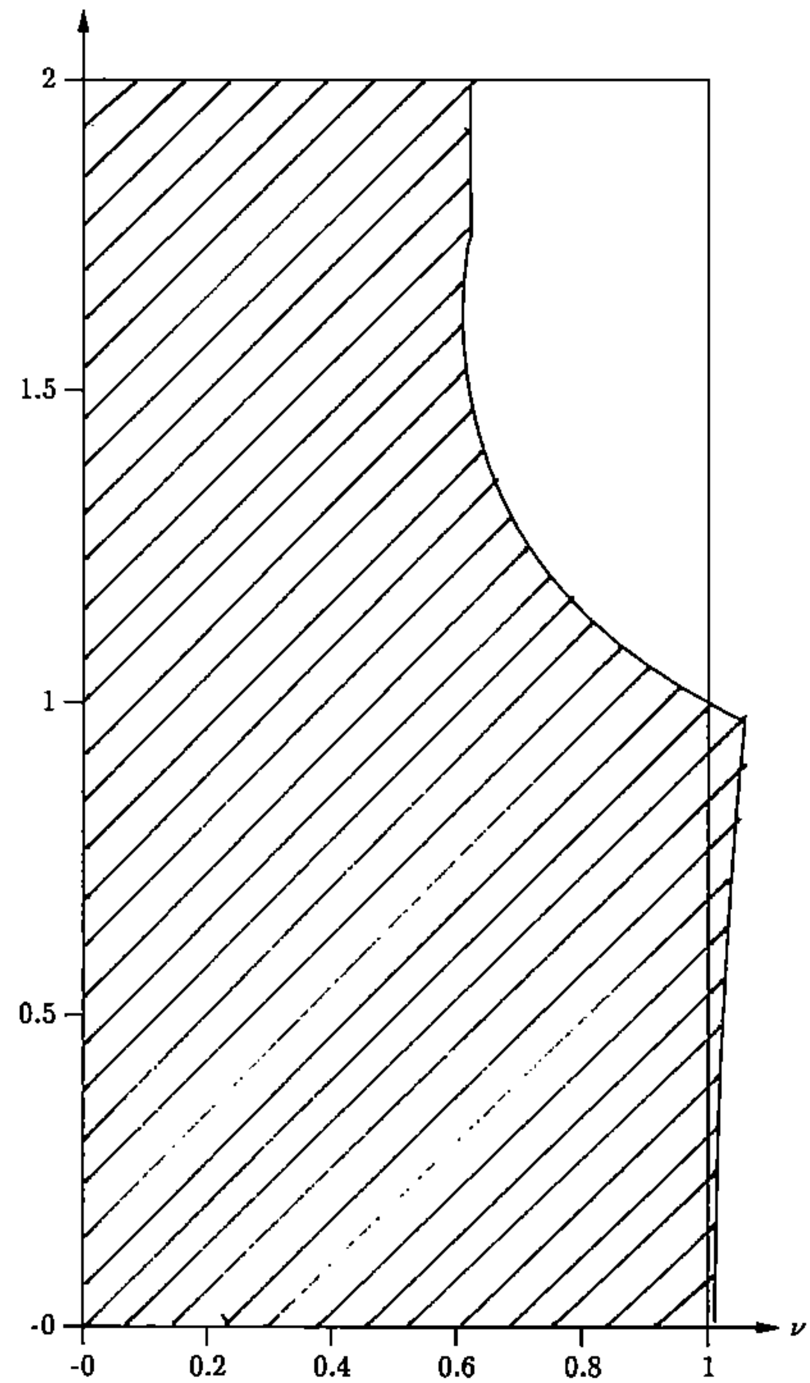

Figure 6: Nonpositive case ($p \geq 15$).

nonpositive. It is reminded that except for those parts of the arcs of the right boundaries of the convergence domains that were known (see [7]) or are extensions of the known ones, the remaining parts of the right boundaries can be determined through (2.8) and (2.9), (2.11) (resp. (2.10), (2.12)). It is also reminded that from (2.8) analytic expressions for $\cos\theta$ can only be found for $p = 3, 4, 5$ and 6. In all other nontrivial cases, for each $p \geq 7$ and each ω, $\cos\theta$ has to be found computationally. Consequently, the same holds true for the corresponding parts of the right boundaries.

In what follows we work out the cases $p = 3$ and 4 for $\sigma(B^p)$ nonpositive, since the corresponding nonnegative cases have already been examined in Section 3.

p = 3: From (2.8) we take

$$\cos\theta = -\frac{(1 - 3x^3 - 3x^4 + x^7)}{2x(1 + x^5)}, \qquad \omega \in (0, \omega_3^{**}) \tag{5.1}$$

or using (2.1)

$$\cos\theta = -\frac{y^3 - y^2 - 2y - 2}{2(y^2 - y - 1)} \qquad \omega \in (0, \omega_3^{**}) \tag{5.2}$$

with

$$\omega_3^{**} = \frac{-1 + \sqrt{5}}{2} \qquad \text{(the golden section number).} \tag{5.3}$$

So, using (5.1) in (2.2) we have

$$F(\omega, \theta) = -\frac{y^3}{y^2 - y - 1}, \qquad \omega \in (0, \omega_3^{**})$$

or

$$\nu_2''(\omega) := \frac{[(1 - \omega)^2 + 1](2 - \omega)^{1/3}}{(1 - \omega)^{1/3}[(1 - \omega)^5 + 1]^{1/3}}. \tag{5.4}$$

It is interesting to point out that

$$\lim_{\omega \to 0+} \nu_2''(\omega) = 2. \tag{5.5}$$

The convergence domain $R^-(3)$ is illustrated in Figure 7.

p = 4: This time it can be found that

$$\cos\theta = \frac{-y^2 + 2y + 2}{2(y - 1)}, \qquad \omega \in (0, \omega_4^{**}) \tag{5.6}$$

with

$$\omega_4^{**} = -1 + \sqrt{3}. \tag{5.7}$$

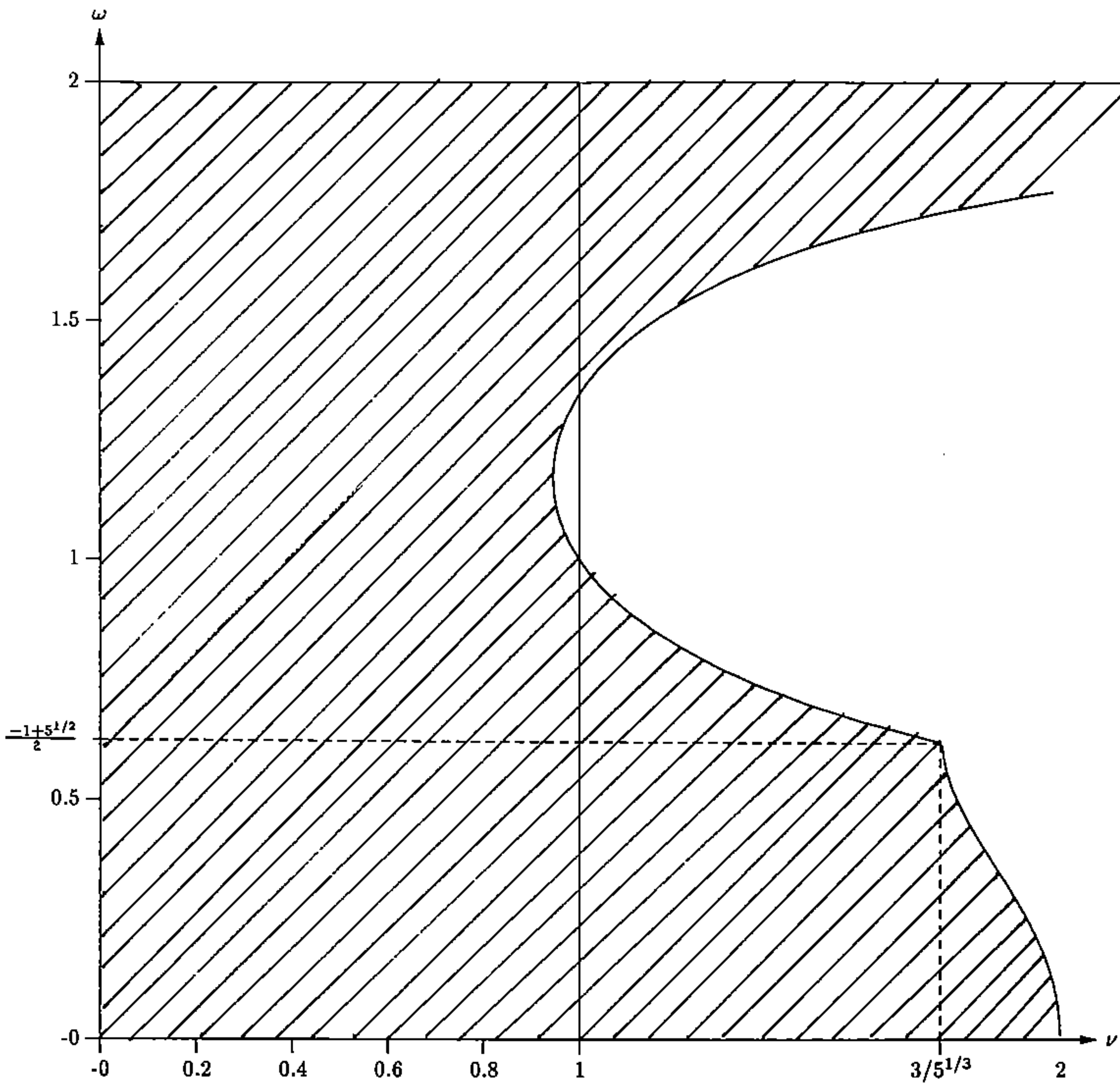

Figure 7: Nonpositive case ($p = 3$).

From (5.6) we have

$$F(\omega,\theta) = -\frac{y^2}{y-1}, \qquad \omega \in (0, \omega_4^{**}) \tag{5.8}$$

or

$$\nu_2''(\omega) := \frac{[(1-\omega)^2+1]^{1/2}}{(1-\omega)^{1/4}[(1-\omega)^2-(1-\omega)+1]^{1/4}}, \qquad \omega \in (0, \omega_4^{**}). \tag{5.9}$$

On the other hand we have

$$\lim_{\omega \to 0^+} \nu_2''(\omega) = \sqrt{2} \tag{5.10}$$

and $R^-(4)$ is illustrated in Figure 8.

Finally, we would like to report that we have worked out the case $p = 5$, computationally, by using Sturm sequences [9]. The computational results simply confirm the theoretical ones obtained in Section 4.

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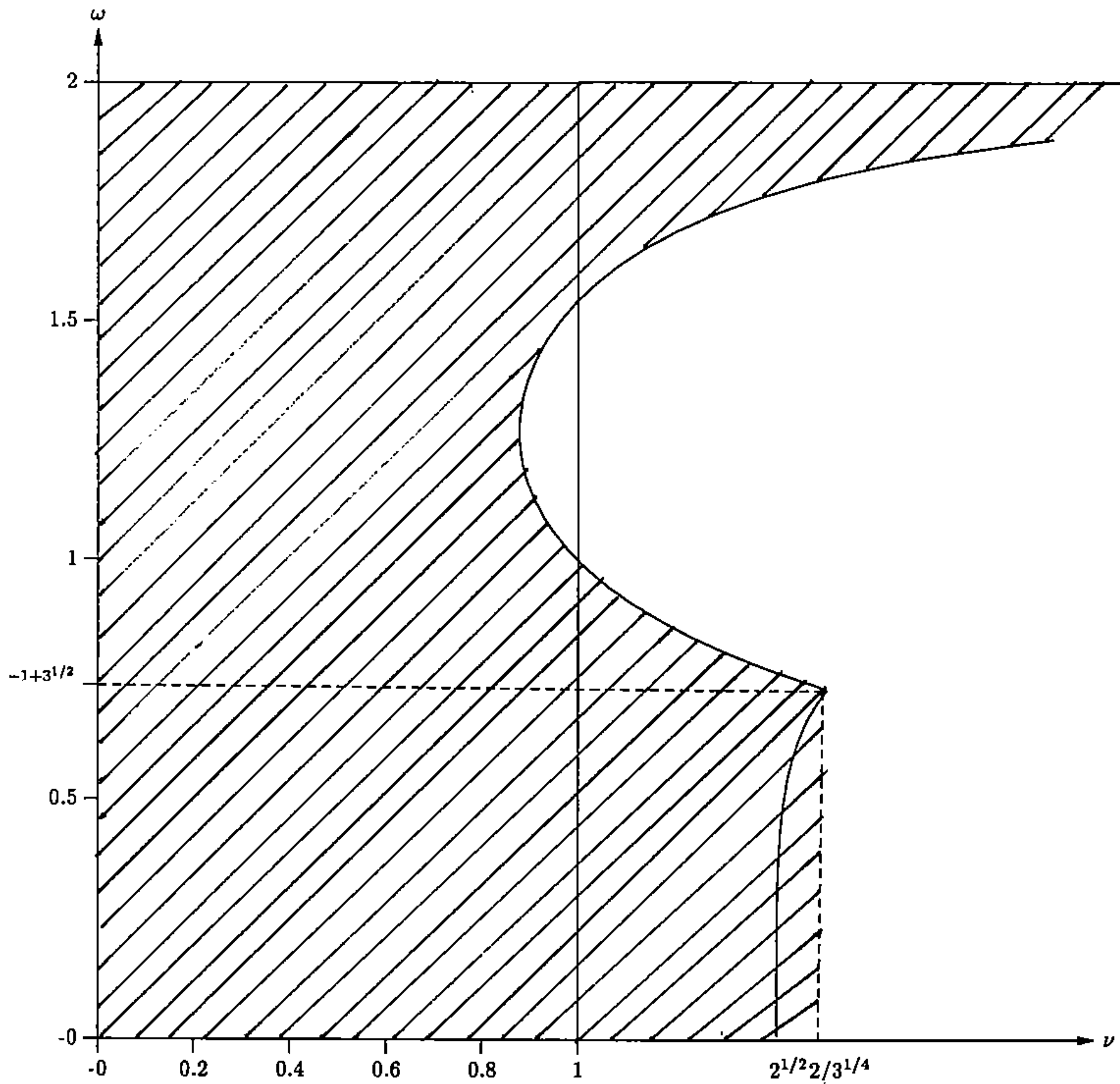

Figure 8: Nonpositive case ($p = 4$).

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