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CONNECTING MODELS OF CONFIGURATION SPACES: FROM DOUBLE LOOPS TO STRINGS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Ralph Kaufmann
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Approved by Major Professor(s): Ralph Kaufmann

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CONNECTING MODELS OF CONFIGURATION SPACES:
FROM DOUBLE LOOPS TO STRINGS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Jason M. Lucas

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

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West Lafayette, Indiana

I would like to dedicate this work to all of the friends, family, colleagues, and mentors who have supported me throughout my graduate career and life, and who are too numerous to name here.

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I would like to thank my advisor, Prof. Ralph Kaufmann. Without his support and guidance, this thesis and my career would not have been possible.

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I would also like to acknowledge Prof. Ben McReynolds, who contributed the idea of indexing cells by both black-rooted and white-rooted trees.

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ABSTRACT

Lucas, Jason M. PhD, Purdue University, August 2016. Connecting Models of Configuration Spaces: From Double Loops to Strings. Major Professor: Ralph Kaufmann.

Foundational to the subject of operad theory is the notion of an E_n operad, that is, an operad that is quasi-isomorphic to the operad of little n-cubes $\mathcal{C}_n$. They are central to the study of iterated loop spaces, and the specific case of $n = 2$ is key in the solution of the Deligne Conjecture. In this paper we examine the connection between two E_2 operads, namely the little 2-cubes operad $\mathcal{C}_2$ itself and the operad of spineless cacti. To this end, we construct a new suboperad of $\mathcal{C}_2$, which we name the operad of tethered 2-cubes. Much of our analysis involves examining trees labeled by elements of the operad of little intervals, $\mathcal{C}_1$. In the final chapter, we generalize this idea of graphs decorated by elements of an operad to the notion of a decorated Feynman category, building off of the work of Kaufmann and Ward. As an immediate application, we will give a simple definition of non-Σ modular operads.

1. Introduction

The little n-cubes operads $\mathcal{C}_n$ were introduced by Boardman and Vogt in order to study homotopy invariant structures on topological spaces [Boardman and Vogt, 1968]. Very quickly May [May, 1972] used the equivalent little n-discs operads $\mathcal{D}_n$ to establish the Recognition Principle, which states that every connected $\mathcal{D}_n$-space has the homotopy type of an n-fold loop space. Soon after Boardman and Vogt themselves [Boardman and Vogt, 1973] established the same result with $\mathcal{C}_n$.

Since their creation, little n-cubes have been a key tool in the study of loop spaces. In 1988, Dunn [Dunn, 1988] related the n-fold loop structure of an n-fold loop space to its various 1-fold loop structures via the operad of little intervals $\mathcal{C}_1$ and the tensor product of operads. This was accomplished by establishing a local Σ-equivalence $\alpha : \mathcal{C}_1^{\boxtimes n} \to \mathcal{C}_n$, where $\mathcal{C}_1^{\boxtimes n}$ is the n-fold tensor product of $\mathcal{C}_1$ with itself. Later Brinkmeier [Brinkmeier, 2000] expanded on Dunn's result, establishing a local Σ-equivalence $\mathcal{C}_{n_1} \boxtimes \cdots \boxtimes \mathcal{C}_{n_l} \to \mathcal{C}_n$ where $n_1, \ldots, n_l \in \mathbb{N}$ is any collection such that $n_1 + \cdots + n_l = n$.

The tensor product construction for operads can be traced back to the tensor product for algebraic theories, defined by Boardman and Vogt and specialized by the same authors to PROPs [Boardman and Vogt, 1973]. Given two operads $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \boxtimes \mathcal{B}$ codifies the notion of an interchange relation between $\mathcal{A}$ and $\mathcal{B}$, in the sense that two operad maps $f : \mathcal{A} \to \mathcal{C}$, $g : \mathcal{B} \to \mathcal{C}$ interchange if and only if there exists a map $h : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{C}$ such that $f = h \circ i_{\mathcal{A}}$ and $g = h \circ i_{\mathcal{B}}$, where $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \boxtimes \mathcal{B}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$ are standard maps.

In 2001, Voronov defined the cacti operad to capture the BV-algebra structure on the homology of the free loop space on a compact oriented manifold discovered by

Chas and Sullivan [Voronov, 2005]. The points in the cacti operad consist of treelike arrangements of copies of S^1. In 2005, Kaufmann [Kaufmann, 2005] showed that the operad $Cact$ of spineless cacti is quasi-isomorphic to the operad $\mathcal{C}_2$ of little 2-cubes, i.e. that it is an E_2 operad (an E_n operad is any operad that is quasi-isomorphic to $\mathcal{C}_n$). This was accomplished via the Fiedorowicz Recognition Principle, which states that an operad $\mathcal{A}$ is E_2 if and only if the component spaces $\mathcal{A}(n)$ are connected and the collection of universal covers $\{\widetilde{\mathcal{A}}(k)\}$ forms a B_∞ operad.

E_2 operads hold a certain significance in the theory, in part because of their connection with the Deligne Conjecture. The conjecture, since proven, was stated by Deligne in a letter in 1993. It theorized that the Hochschild cochain complex of an associative algebra would have the structure of an algebra over the little 2-discs operad $\mathcal{D}_2$, or some equivalent operad (of which $\mathcal{C}_2$ is an example). Proofs of this and similar statements have since been provided by McClure and Smith McClure and Smith [2002], Kontsevich and Soibelman Kontsevich and Soibelman [2000], Tamarkin Tamarkin [1999], and Kaufmann and Schwell Kaufmann and Schwell [2010].

The broader theory of operads, and the related notions of cyclic operads, modular operads, PROPs, etc., can be formalized in terms of Feynman categories [Kaufmann and Ward]. Although they have a more general definition, specific Feynman categories provide a framework for studying operadic theories by capitalizing on the graph theoretic structures that exist within those theories. In this new setting, simple constructions from category theory furnish much of the tools that are familiar from these subjects.

The first portion of this paper will deal with examining further the connections between the little 2-cubes operad $\mathcal{C}_2$ and the operad of spineless cacti. To this end, we construct a new sub-operad of $\mathcal{C}_2$, the so-called operad of tethered 2-cubes. We examine a tree structure inherent to this new operad as well, and show how it indexes a corresponding CW-complex. The later portion of the paper will look at a generalization of Feynman categories to the notion of a decorated Feynman category. We will see that decorated Feynman categories make rigorous the practice long prevalent in operadic theories of decorating the vertices of a graph with elements of an operad. This idea will also lead to a simple construction of non-Σ modular operads.

1.1 Operads

We start by recalling the definition of an operad. Let Σ be the symmetric groupoid, i.e. the category whose objects are the sets $[n] = \{1, 2, \dots, n\}$ and whose Hom sets are

$$\mathrm{Hom}_\Sigma([m],[n]) = \begin{cases} \Sigma_n & m = n \\ \varnothing & m \neq n \end{cases}$$

Let $(C, \otimes, I)$ be any symmetric monoidal category. By an ***operad***, we mean a functor $\mathcal{P} : \Sigma^{\mathrm{op}} \to C$ equipped with structure maps

$$\gamma_{k;n_1,\dots,n_k} : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots n_k)$$

satisfying the following axioms [Markl et al., 2007]:

1. **Associativity**: For natural numbers m_i, $1 \leq i \leq n$, and $l_{i,j}$, $1 \leq j \leq m_i$, let $m := \sum_{1 \leq i \leq n} m_i$, $l_i = \sum_{1 \leq j \leq m_i} l_{i,j}$, and $l = \sum_{1 \leq i \leq n} l_i$. Let

$$\mathcal{P}[\mathbf{m}] := \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n),\ \mathcal{P}[\mathbf{l}_i] := \mathcal{P}(l_{i,1}) \otimes \cdots \otimes \mathcal{P}(l_{i,m_i}),$$
$$\mathcal{P}[\mathbf{l}] := \mathcal{P}(l_{1,1}) \otimes \cdots \otimes \mathcal{P}(l_{n,m_n}),\ \mathcal{P}[\mathbf{l}'] := \mathcal{P}(l_1) \otimes \cdots \otimes \mathcal{P}(l_n).$$

   Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(n) \otimes \mathcal{P}[\mathbf{m}] \otimes \mathcal{P}[\mathbf{l}] \arrow[r, "\tau"] \arrow[dd, "\gamma_{n;\mathbf{m}} \otimes 1"] & \mathcal{P}(n) \otimes (\mathcal{P}(m_1) \otimes \mathcal{P}[\mathbf{l}_1]) \otimes \cdots \otimes (\mathcal{P}(m_n) \otimes \mathcal{P}[\mathbf{l}_n]) \arrow[d, "1 \otimes \bigotimes_{i=1}^{n} \gamma_{m_i;\mathbf{l}_i}"] \\
 & \mathcal{P}(n) \otimes \mathcal{P}[\mathbf{l}'] \arrow[d, "\gamma_{n;\mathbf{l}'}"] \\
\mathcal{P}(m) \otimes \mathcal{P}[\mathbf{l}] \arrow[r, "\gamma_{m;\mathbf{l}}"] & \mathcal{P}(l)
\end{tikzcd}$$

   Here τ is the symmetry in C.

2. **Equivariance**: For $\sigma \in \Sigma_n$ and $\mathbf{m} = (m_1, \ldots, m_n)$, let $\sigma\mathbf{m} := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})$. Let $\bar{\sigma} : \mathcal{P}[\mathbf{m}] \to \mathcal{P}[\sigma\mathbf{m}]$ be permutation of the factors via the symmetry of C, and let $\sigma_{\mathbf{m}}$ be the block permutation determined by σ. Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(n) \times \mathcal{P}[\mathbf{m}] \arrow[r, "1\otimes\bar{\sigma}"] \arrow[dd, "\sigma\otimes 1"'] & \mathcal{P}(n) \otimes \mathcal{P}[\sigma\mathbf{m}] \arrow[d, "\gamma_{n;\sigma\mathbf{m}}"] \\
 & \mathcal{P}(m) \arrow[d, "\sigma_{\mathbf{m}}"] \\
\mathcal{P}(n) \otimes \mathcal{P}[\mathbf{m}] \arrow[r, "\gamma_{n;\mathbf{m}}"] & \mathcal{P}(m)
\end{tikzcd}$$

3. **Unit**: Let I be the unit object of C. Then there is a morphism $\eta : I \to \mathcal{P}(1)$ such that the composite morphisms

$$\mathcal{P}(n) \otimes I^{\otimes n} \xrightarrow{1\otimes\eta^{\otimes n}} \mathcal{P}(n) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \xrightarrow{\gamma_{n;1,\ldots,1}} \mathcal{P}(n)$$

and

$$I \otimes \mathcal{P}(m) \xrightarrow{\eta\otimes 1} \mathcal{P}(1) \otimes \mathcal{P}(m) \xrightarrow{\gamma_{1;m}} \mathcal{P}(m)$$

are the iterated right unit morphism and left unit morphism for C respectively.

Note that each object $\mathcal{P}(n)$ of C has a right Σ_n-action, since $\mathcal{P}$ is a functor out of Σ^{op}.

We have another way of defining composition for operads. Following [Markl et al., 2007], a ***pseudo-operad*** is a functor $\mathcal{P} : \Sigma^{\mathrm{op}} \to C$ along with structure maps

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), 1 \leq i \leq n$$

satisfying the following axioms:

1. **Associativity**: For the iterated compositions of $\mathcal{P}(m) \otimes \mathcal{P}(n) \otimes \mathcal{P}(p)$, we have

$$\circ_i(\circ_j \otimes 1) = \begin{cases} \circ_{j+p-1}(\circ_i \otimes 1)(1 \otimes \tau) & \text{for } 1 \leq i \leq j-1, \\ \circ_j(1 \otimes \circ_{i-j+1}) & \text{for } j \leq i \leq j+n-1, \text{ and} \\ \circ_j(\circ_{i-n+1} \otimes 1)(1 \otimes \tau) & \text{for } j+n \leq i. \end{cases}$$

Again τ is the symmetry map in C.

2. **Equivariance**: For $\sigma \in \Sigma_m$, $\rho \in \Sigma_n$, define $\sigma \circ_i \rho \in \Sigma_{m+n-1}$ by

$$\sigma \circ_i \rho := \sigma_{1,\ldots,1,n,1,\ldots,1} \circ (1 \times \ldots \times 1 \times \rho \times 1 \times \ldots \times 1)$$

($\sigma_{1,\ldots,1,n,1,\ldots,1}$ is the appropriate block permutation). Then the $\circ_i$ maps are equivariant in the sense that

$$\circ_i(\sigma \otimes \rho) = (\sigma \circ_i \rho)\circ_{\sigma(i)}$$

as maps from $\mathcal{P}(m) \otimes \mathcal{P}(n)$ to $\mathcal{P}(m+n-1)$.

As long as $\mathcal{P}(1)$ contains an identity for this composition, it is equivalent to the γ composition defined above. We think of the γ structure maps as simultaneous composition while the $\circ_i$ maps are individual composition. A map of operads $f : \mathcal{P} \to \mathcal{Q}$ is a collection of maps $f_n : \mathcal{P}(n) \to \mathcal{Q}(n)$ that commute with the composition maps.

Next we define the tensor product of operads. We follow Brinkmeier's construction [Brinkmeier, 2000].

Definition 1.1.1 *Given two maps of operads $f : \mathcal{P} \to \mathcal{R}$, $g : \mathcal{Q} \to \mathcal{R}$, we say that f and g* ***interchange*** *if the following diagram commutes for all $i, j \in \mathbb{N}$:*

$$\begin{tikzcd}
\mathcal{P}(j) \times \mathcal{Q}(k) \arrow[r, "id\times\Delta"] \arrow[d, "\simeq"'] & \mathcal{P}(j) \times \mathcal{Q}(k)^j \arrow[r, "f_j \times g_k^j"] & \mathcal{R}(j) \times \mathcal{R}(k)^j \arrow[dd, "\gamma_{j:k,\ldots,k}"] \\
\mathcal{Q}(k) \times \mathcal{P}(j) \arrow[d, "id\times\Delta"'] & & \\
\mathcal{Q}(k) \times \mathcal{P}(j)^k \arrow[r, "g_k \times f_j^k"] & \mathcal{R}(k) \times \mathcal{R}(j)^k \arrow[r, "\gamma_{k;j,\ldots,j}"] & \mathcal{R}(jk)
\end{tikzcd}$$

Here Δ is the diagonal map.

For two operads $\mathcal{P}$ and $\mathcal{Q}$, we can consider the collection of *bi-colored trees*, that is, trees whose vertices are either white or black. If we let $BiTree(j)$ be the collection of all bi-colored trees with j leaves, then $BiTree = \{BiTree(j)\}_{j\geq 1}$ forms an operad with composition given by grafting trees. For a bicolored tree T, we let

$$(\mathcal{P}, \mathcal{Q})_T = \prod_{\substack{v \in Vert(T) \\ v \text{ white}}} \mathcal{P}(|v|) \times \prod_{\substack{v \in Vert(T) \\ v \text{ black}}} \mathcal{Q}(|v|).$$

Here $|v|$ is the arity of the vertex v. We think of $(\mathcal{P}, \mathcal{Q})_T$ as labeling the white vertices of the bi-colored tree T by elements of $\mathcal{P}$ and the black verties by elements of $\mathcal{Q}$, where the arity of the label equals the arity of the vertex. We now let $F(\mathcal{P} \coprod_\Sigma \mathcal{Q})$ be given by the spaces $\coprod_{T \in BiTree(j)} (\mathcal{P}, \mathcal{Q})_T$ modulo the usual equivariance relation on labeled trees.

Define $\mathcal{P} \coprod \mathcal{Q}$ to be the quotient of $F(\mathcal{P} \coprod_\Sigma \mathcal{Q})$ by the relations (i) monochrome edges can be contracted and their labels composed and (ii) the identities of $\mathcal{P}$ and $\mathcal{Q}$ are identified with the trivial tree.

Lemma 1.1.2 *[Brinkmeier, 2000] $\mathcal{P} \coprod \mathcal{Q}$ is the direct sum of operads.*

The tensor product $\mathcal{P} \boxtimes \mathcal{Q}$ is formed by taking the quotient of $\mathcal{P} \coprod \mathcal{Q}$ with respect to one more relation, this one capturing the interchange property described in Definition 1.1.1. Namely, if a white vertex labeled by α has all incoming edges connecting to black vertices with the same label β, then the black vertices can be replaced by white vertices labeled with α and the white vertex can be replaced with a black vertex labeled with β, with the appropriate reshuffling. The similar property must hold when the starting vertex is black and the incoming vertices are white. See [Brinkmeier, 2000] for details.

1.2 The Operad of Little n-Cubes

Let $I^n = [0,1]^n$ be the unit n-cube. We define a *little n-cube* to be a linear embedding $c : I^n \to I^n$ such that the image has edges parallel to the edges of I^n. We define $\mathcal{C}_n(k)$ to be the set of all mappings $c = (c_1, \ldots, c_k) : \coprod_{i=1}^k I^n \to I^n$ such that each c_i is a little n-cube and $c_i(\mathring{I}^n) \cap c_j(\mathring{I}^n) = \varnothing$ for $i \neq j$ (that is, the various n-cubes have disjoint interiors). We topologize the set $\mathcal{C}_n(k)$ via the compact-open topology. We picture an element of $\mathcal{C}_n(k)$ as a collection of k n-dimensional rectangles contained in the unit n-cube, all with disjoint interiors.

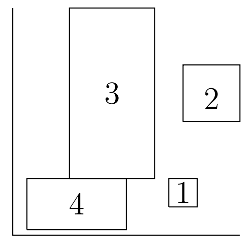


Figure 1.1.: An element of $\mathcal{C}_2(4)$

Note that while the interiors must be disjoint, the boundaries of the component rectangles may intersect.

To put an operad structure on this collection $\{\mathcal{C}_n(k)\}_{k\geq 1}$ of topological spaces, we label the component rectangles from 1 to k as in Figure 1.1. The right Σ_k-action on $\mathcal{C}_n(k)$ is given by permuting the labels of the rectangles accordingly. The maps

$$\circ_i : \mathcal{C}_n(k) \times \mathcal{C}_n(l) \to \mathcal{C}_n(k+l-1), \quad \circ_i(c_1, c_2) = c_1 \circ_i c_2$$

are defined by rescaling c_2 so that its unit cube has the same dimensions as the i-labeled rectangle in c_1 and then replacing the i-labeled rectangle with the rescaled c_2.

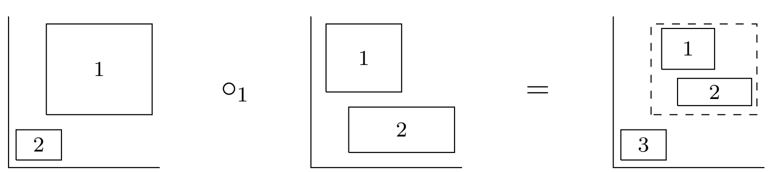


Figure 1.2.: little n-cube composition

Here the dotted rectangle is not part of the element. It is just drawn in to illustrate the composition. It is easy to check that with these maps, $\mathcal{C}_n$ is an operad. This is Boardman and Vogt's ***little*** n***-cubes*** operad.

Example 1.2.1 *Consider the operad $\mathcal{C}_1 \boxtimes \mathcal{C}_1$, where $\mathcal{C}_1$ is the operad of little intervals. We have maps $i_H, i_V : \mathcal{C}_1 \to \mathcal{C}_2$ given by extending vertically or horiztonally from the unit interval to the unit square.*

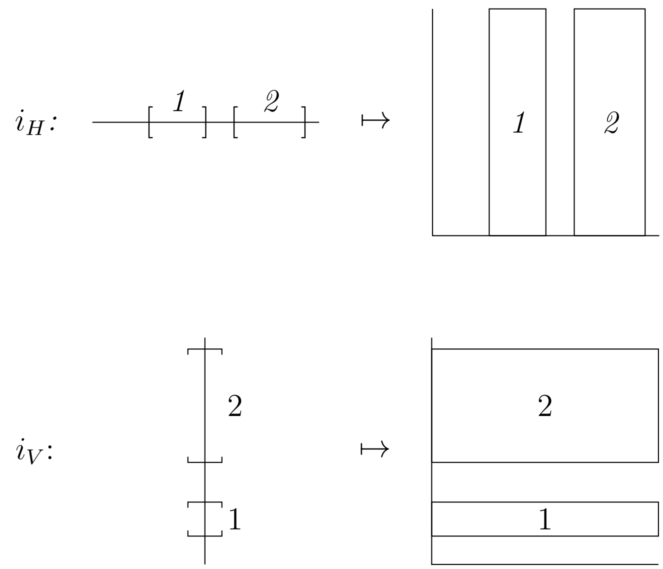


Figure 1.3.

Note that these two maps interchange, that is if we have $a, b \in \mathcal{C}_1$, then $i_H \circ (i_V(a), i_V(b))$ equals $i_V \circ (i_H(a), i_H(b))$ up to a reordering of labels. Thus we have an induced map $\mathcal{C}_1 \boxtimes \mathcal{C}_1 \to \mathcal{C}_2$. Let $\mathcal{C}_1|\mathcal{C}_1$ be the image of this map, and let $\varphi : \mathcal{C}_1 \boxtimes \mathcal{C}_1 \to \mathcal{C}_1|\mathcal{C}_1$ be the induced map. Thus we can picture the elements of $\mathcal{C}_1 \boxtimes \mathcal{C}_1$ inside $\mathcal{C}_2$ as rectangles lined up horizontally and vertically in a grid.

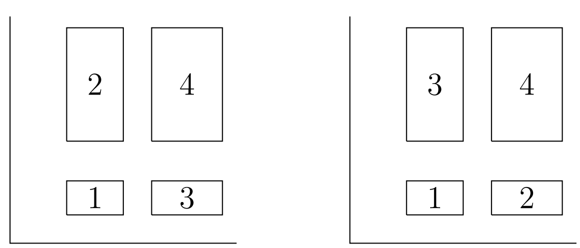


Figure 1.4.: The images of $i_H \circ (i_V, i_V)$ and $i_V \circ (i_H, i_H)$

Next we introduce a key suboperad of $\mathcal{C}_n$, namely the operad of *decomposable n-cubes*, which we denote by $\widehat{\mathcal{C}_n}$. Decomposable n-cubes were first defined by Dunn in [Dunn, 1988], and we use that definition. Here $c(r,s) \in \mathcal{C}_1(2)$ denotes the little 1-cube consisting of the intervals $[0, \frac{r}{r+s}]$ and $[\frac{r}{r+s}, 1]$.

Definition 1.2.2 *[Dunn, 1988] Let $n \geq 2$, $c \in \mathcal{C}_n(j)$ and $1 \leq i \leq n$. Call c* ***i-decomposable*** *if $j = 0, 1$ or if $j \geq 2$ and $c = \gamma(c(r,s); c_1, c_2)$ for some $r, s > 0$ and $c_k \in \mathcal{C}_n(j_k)$ with $j_k > 0$, $k = 1, 2$. We also write $c = c_1 \cup_t c_2$, where $t = \frac{r}{r+s}$.*
If there is a sequence of i-decompositions for various i, say $i_1, \ldots, i_{j-1}$, such that $c = c_1 \cup_t c_2$ is an i_1-decomposition, c_1 or c_2 is i_2-decomposable, and so on to i_{j-1}, then c is called ***decomposable****; otherwise it is* ***indecomposable****.*

Visually, an n-cube $c \in \mathcal{C}_n(j)$ is decomposable if one can insert hyperplanes that do not intersect the interiors of any of the component cubes in such a way that the j component cubes are all cordoned off. Of course not all n-cubes are decomposable. Here we see an example of a decomposable 2-cube and an indecomposable 2-cube.

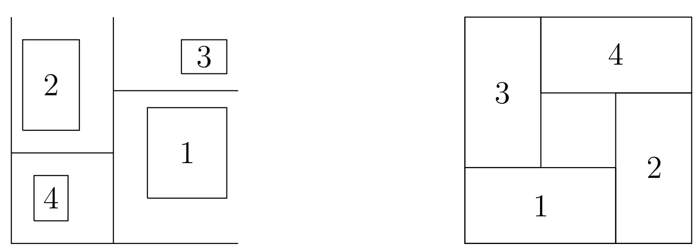


Figure 1.5.

Proposition 1.2.3 *[Dunn, 1988] There is a local Σ-equivalence $\widehat{\mathcal{C}_n} \to \mathcal{C}_n$.*

The details of the proof can be found in [Dunn, 1988]. The idea is to essentially contract the component rectangles towards their centers. At some point the arrangement produced will be decomposable.

In [Brinkmeier, 2000], Brinkmeier describes a tree structure for $\widehat{C}_n$. We will review this construction is Section 2. In this paper we will be concerned with the case $n = 2$.

1.3 The Operad of Spineless Cacti

Definition 1.3.1 *The* ***operad of spineless cacti*** *consists of component spaces* $Cact(k)$*, whose elements are connected, tree-like configurations of copies of* S^1 *in the plane. The copies of* S^1 *are called* ***lobes****. The lobes of a cactus* $c \in Cact(k)$ *are labeled from 1 to k, and the right* Σ_k*-action on such a cactus is given by permuting labels. Composition* $c_1 \circ_i c_2$ *is given by replacing the* i*-th lobe of* c_1 *with* c_2*, identifying the global marked point of* c_2 *with the marked point of the* i*-th lobe of* c_1*, where* c_2 *has been rescaled so that the total length of its lobes is equal to the length of the* i*-th lobe of* c_1*. See [Kaufmann, 2005] for full details.*

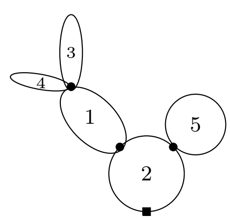


Figure 1.6.: An element of $Cact(5)$

In our pictures, the global marked point of a cactus is denoted by a black square, while the marked points of lobes are denoted by black circles. Necessarily, the global marked point of the cactus will also be the marked point of one or more lobes. We will still denote it by a square, but it is to be understood that it is the marked point of those lobes as well.

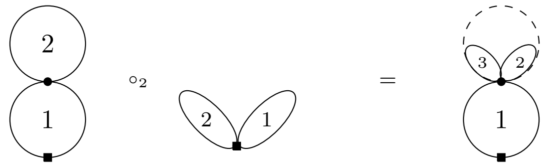


Figure 1.7.: Composition in $Cact$

An important subobject of the operad $Cact$ is $Cact^1$, or what we call ***normalized spineless cacti*** [Kaufmann, 2005]. The set $Cact^1(k)$ is defined in the same way as $Cact(k)$, with the added restriction that the lobes of an element of $Cact^1(k)$ must all have length 1. We must also alter composition in order for it to be well-defined. Given $c_1 \in Cact^1(k)$ and $c_2 \in Cact^1(l)$, $c_1 \circ_i c_2 \in Cact^1(k+l-1)$ is given by increasing the i-th lobe of c_1 so that it's length is equal to the total length of c_2 and then inserting c_2.

Remark 1.3.2 *A key consequence of this altered composition is that it is no longer strictly associative. Thus $Cact^1$ is not an operad. However, the composition is associative up to homotopy. This makes $Cact^1$ a* ***quasi-operad*** *(see [Kaufmann, 2005] for details).*

A basic fact about normalized spineless cacti that we will make use of is that the sets $Cact^1(k)$ form CW-complexes. The details are contained in [Kaufmann, 2005]. We sketch the idea here. Each lobe of an element $c \in Cact^1(k)$ has some number of arcs, say $l \geq 1$, determined by the marked points of the other lobes. The lengths of these arcs are all non-negative and add up to 1, and so they correspond to the coordinates of a point in the standard $(l-1)$-simplex Δ^{l-1}. Thus for each cactus c we have a point in a product of simplices. Two cacti who have the same shape (i.e. the same dual black and white tree, which we will define) correspond to two different points in the same product of simplices. These trees then index the cells of a CW-complex.

Given a cactus $c \in Cact^1(k)$, we can associate to it a bipartite tree as follows: For each lobe, we have a white vertex. For each marked point, we have a black vertex. There is an edge between a black vertex and a white vertex if the corresponding marked point is on the corresponding lobe. The global marked point becomes the root of the tree, and we label the white vertices with the corresponding lobe labels. Here we have an example:

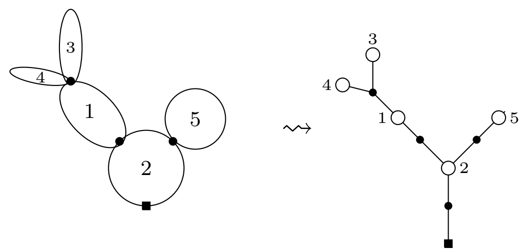


Figure 1.8.

Example 1.3.3 *$Cact^1(2)$ has a CW decomposition as copy of S^1 with two 0-cells and two 1-cells. The 0-cells correspond to the two cacti whose marked points coincide with the global marked point. The points of the 1-cells are those cacti in which one marked point differs from the global marked point.*

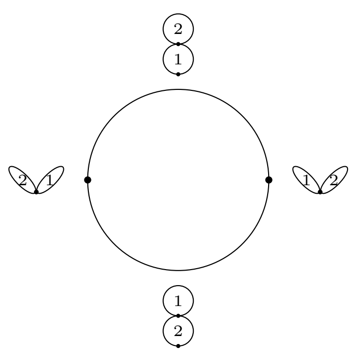


Figure 1.9.: The space $Cact^1(2)$

Lemma 1.3.4 *[Kaufmann, 2005]* $Cact(k) = Cact^1(k) \times \mathbb{R}^k_{>0}$.

We use the CW structure on $Cact^1(k)$ and this lemma to topologize $Cact(k)$.

2. The Operad of Tethered 2-Cubes

In this chapter we construct what will be the main object of study throughout Chapters 3, 4, and 5.

Definition 2.0.5 *$\mathcal{A}_2$ is the operad of* ***tethered 2-cubes*** *defined below. It is a suboperad of $\widehat{\mathcal{C}}_2$.*

Definition 2.0.6 *By an* ***arrangement of 2-cubes*** *(sometimes called just an* ***arrangement****), we mean any element of $\mathcal{C}_2$, $\widehat{\mathcal{C}}_2$, or $\mathcal{A}_2$.*

Definition 2.0.7 *We call the rectangles that make up an arrangement of 2-cubes* ***realized rectangles****.*

First we define a set of distinguished elements of $\mathcal{A}_2(n)$. An element of this set is any arrangement of realized squares in the unit square I^2 such that the side lengths of all the squares add up to 1 and such that the centers of all the squares fall on the line $y = 1 - x$.

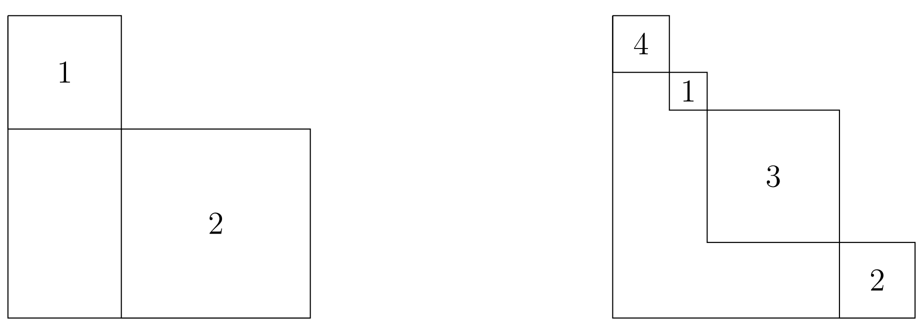


Figure 2.1.: Distinguished elements in $\mathcal{A}_2$

The other elements of $\mathcal{A}_2(n)$ are then obtained from applying certain moves to one of these distinguished elements. The arity 2 case is the simplest to describe: Take a distinguished element in $\mathcal{A}_2(2)$, pictured here:

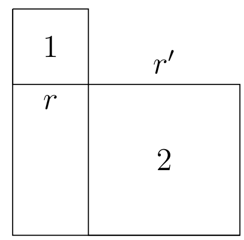


Figure 2.2.

Here $r' = 1 - r$. Square 1 has center $(\frac{r}{2}, 1 - \frac{r}{2})$ and square 2 has center $(1 - \frac{r'}{2}, \frac{r'}{2})$. We visualize the allowed movements as sliding square 1 horizontally or vertically around the boundaries of I^2 and at the same time sliding square 2 in the opposite direction, so that at any given time square 2 has gone the same proportion of the trip from one side of I^2 to the other as square 1 has. For example, if square 1 has center at $(\frac{r}{2}(1-t) + (1 - \frac{r}{2})t, 1 - \frac{r}{2})$, then square 2 has its center at $((1 - \frac{r'}{2})(1-t) + \frac{r'}{2}t, \frac{r'}{2})$ for $0 \leq t \leq 1$. Here are some sample elements of $\mathcal{A}_2(2)$ obtained from the above 2-cube.

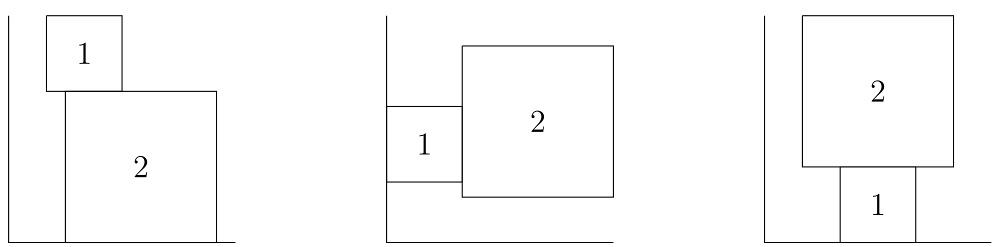


Figure 2.3.

All the elements that can be reached this way make up $\mathcal{A}_2(2)$.

To define $\mathcal{A}_2(n)$ for $n \geq 3$, we must introduce the idea of a “framing rectangle”.

Definition 2.0.8 *Given an element $c \in C_2(n)$, a **meta-rectangle** (or **meta-square**) is any rectangle (or square) that may be drawn inside I^2, with sides parallel to those of I^2. It is not a realized rectangle, it does not receive any numerical label, and it may intersect or even contain other meta-rectangles or the realized rectangles that make up c.*

We now give the somewhat recursive definition of a framing rectangle.

Definition 2.0.9 *A **framing rectangle** (or **framing square**) is a meta-rectangle (or meta-square), not equal to I^2, that contains either two realized rectangles, a realized rectangle and a smaller framing rectangle, or two smaller framing rectangles such that the positioning of those contained rectangles with respect to the framing rectangle gives an element of $\mathcal{A}_2(2)$. Also, a framing rectangle must contain any realized or framing rectangle that it intersects, and a minimal framing rectangle allowed in a given arrangement must contain two realized rectangles from the arrangement.*

Remark 2.0.10 *A framing square pairs two squares together, be they both realized, both framing, or one of each. We call such a pair of squares a **complementary pair** and each square in the pair the **complementary square** of the other. The final condition in our definition states that a minimal framing square in an arrangement must pair two realized squares together. Thus such a framing square is minimal in the sense that it contains no framing squares. Of course an arrangement may have several minimal framing squares.*

Remark 2.0.11 *We also require that every framing square be the complementary square of some other (realized or framing) square in the arrangement. The largest framing square in an arrangement and its complementary square must have side lengths totaling 1. By a **full choice** of framing squares for an arrangement of $\mathcal{A}_2(n)$, we mean a choice of framing squares such that all squares in the arrangement, realized or framing, are part of a complementary pair. We also call a full choice of framing squares an **arrangement** of framing squares.*

Every complementary pair of squares in an arrangement corresponds to some element of $\mathcal{A}_2(2)$. If a complementary pair consists of the largest framing square in the arrangement and its complementary square, then we simply replace the framing squares in this complementary pair with realized squares of the same size in the same position. Otherwise, the corresponding element of $\mathcal{A}_2(2)$ is found by rescaling the framing square creating the complementary pair and the squares it contains up to I^2 and replacing the complementary squares with realized squares of the same size and in the same position. We will use this correspondence often.

We also refer the the realized squares in an arrangement in $\mathcal{A}_2(2)$ as a complementary pair. Note that an arrangement in $\mathcal{A}_2(2)$ is never framed.

Choosing an arrangement of framing squares for a distinguished element of $\mathcal{A}_2(n)$ is equivalent to choosing a full bracketing of n letters. Below we give an enumeration of all the choices of framing squares for a distinguished element of $\mathcal{A}_2(3)$ and $\mathcal{A}_2(4)$ (up to permutations of labels):

$\mathcal{A}_2(3)$:

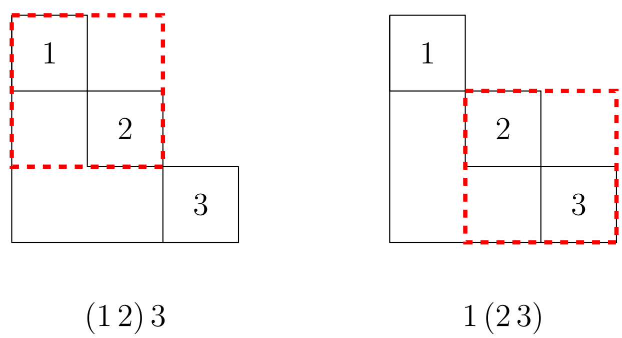


Figure 2.4.: All framings of a distinguished element in $\mathcal{A}_2(3)$

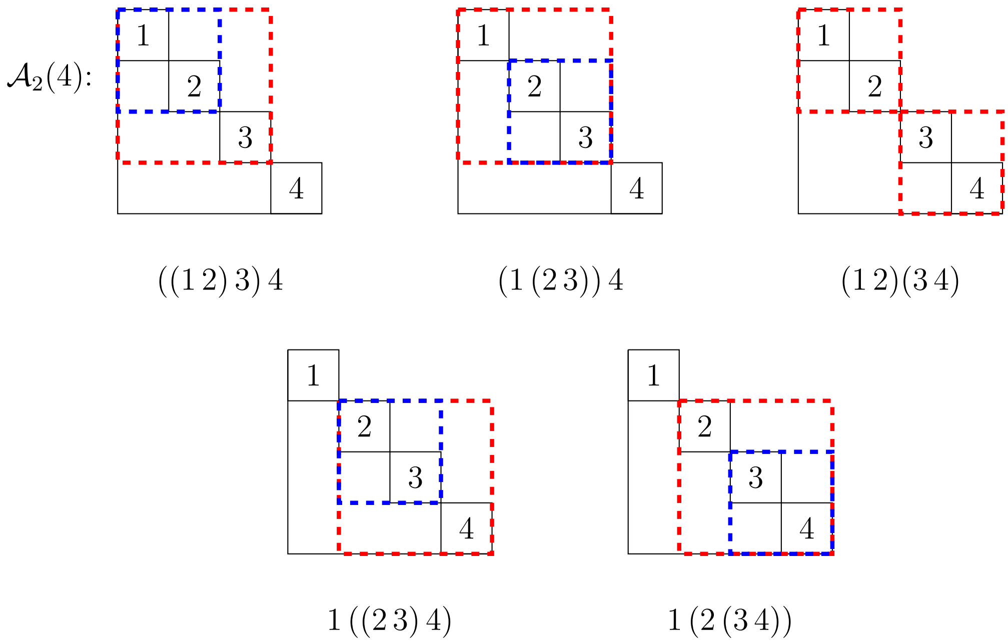


Figure 2.5.: All framings of a distinguished element in $\mathcal{A}_2(4)$

Visually we will always represent framing rectangles with dashed lines. We give some examples now.

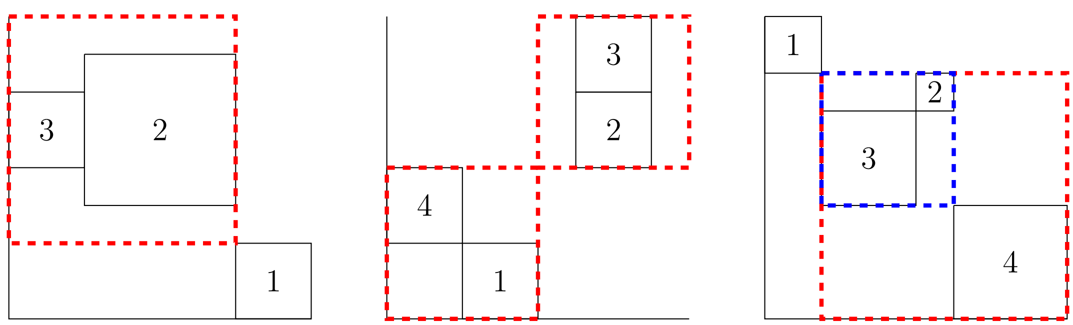


Figure 2.6.: Elements of $\mathcal{A}_2(3)$ and $\mathcal{A}_2(4)$ with framings

We may now describe $\mathcal{A}_2(n)$. Pick any distinguished element of $\mathcal{A}_2(n)$ and any arrangement of framing squares of this element. The contained squares of any framing square give us a corresponding element of $\mathcal{A}_2(2)$. We may then move these two squares around each other as we can with elements of $\mathcal{A}_2(2)$. We can do the same with the largest framing square and its complementary square. If we may switch to a different collection of framing squares for any element achieved this way, we may also do this and then perform the same movements. $\mathcal{A}_2(n)$ consists of all elements that can be formed this way. In the above picture, all three arrangements are elements of $\mathcal{A}_2$, the first being in $\mathcal{A}_2(3)$ and the second and third being in $\mathcal{A}_2(4)$.

Proposition 2.0.12 *With the composition and symmetric action induced from $\mathcal{C}_2$, $\mathcal{A}_2$ is an operad. In fact it is a suboperad of $\widehat{\mathcal{C}}_2$.*

Proof Clearly every element of $\mathcal{A}_2(n)$ is an element of $\mathcal{C}_2(n)$. Since the composition in $\mathcal{A}_2$ is induced by composition in $\mathcal{C}_2$, satisfies the axioms of an operadic composition. We must check that it is well defined.

To see that the composition of two elements from $\mathcal{A}_2$ lands in $\mathcal{A}_2$, take $c \in \mathcal{A}_2(n)$, $d \in \mathcal{A}_2(m)$. Both c and d come from sliding squares in distinguished elements of $\mathcal{A}_2$, and as such, we can undo those slides to move them back to distinguished elements. Clearly the composition of two distinguished elements produces a distinguished element. If we compose these distinguished elements and then perform the same slides that got us to c and d (with the slides giving d occuring in what was the i^{th} square of c), we will arrive at $c \circ_i d$. Thus $c \circ_i d \in \mathcal{A}_2(n+m-1)$.

It follows that $\mathcal{A}_2$ is a suboperad of $\mathcal{C}_2$. To see that it is in fact a suboperad of $\widehat{\mathcal{C}}_2$, note that an arrangement of framing squares for an element $c \in \mathcal{A}_2(n)$ gives a decomposition of c in the sense of Definition 1.2.2. A line can always be drawn, either vertically or horizontally, separating two complementary squares in an element of $\mathcal{A}_2(2)$. Thus we may separate the largest framing square of c from its complementary square. On the two sides of this line, we either have a realized square and so have cordoned it off, or we have a framing square containing a complementary pair. In the case of a framing square we repeat the process. When drawing in the line now it may extend outside of the framing square, but it will not intersect any realized squares, since there can be no realized squares not contained in this framing square on this side of the previous line. This process eventually terminates since we have only finitely many squares. Thus every element of $\mathcal{A}_2(n)$ is decomposable, and so $\mathcal{A}_2$ is a suboperad of $\widehat{\mathcal{C}}_2$. ■

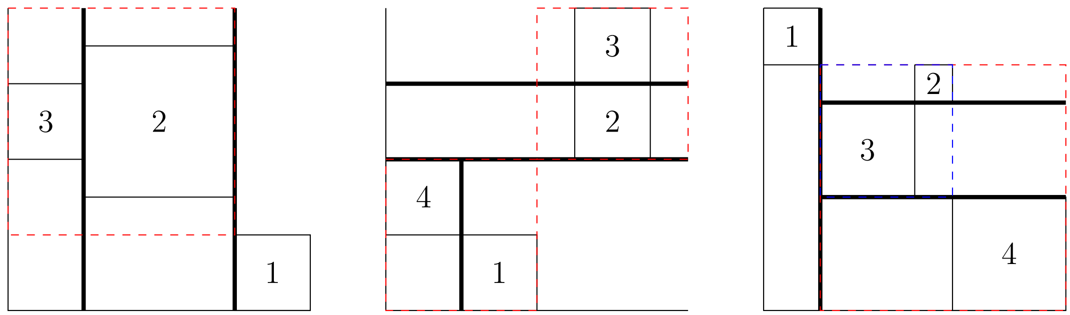


Figure 2.7.: The elements from Figure 2.6, decomposed

3. A Tree Structure for Decomposable 2-Cubes

3.1 Constructing trees

Following Brinkmeier, we describe a tree structure for decomposable 2-cubes $\widehat{\mathcal{C}}_2$. Given an element of $\widehat{\mathcal{C}}_2(n)$, we essentially alternate projecting vertically and horizontally, which will give us elements of $\mathcal{C}_1$. These will then become the labels of a planar, rooted, bipartite tree.

Lemma 3.1.1 *Given any collection of closed intervals in $I = [0,1]$, there exists a collection of disjoint closed intervals in I such that these two collections are equal as subsets of I.*

Proof Take a finite collection of closed, not necessarily disjoint intervals $\{[a_i, b_i]\}_{i=1}^n$ each contained in $I = [0,1]$. We can form a new collection of closed intervals in I with disjoint interiors. If (a_i, b_i) is disjoint from (a_j, b_j) for all $j \neq i$, leave $[a_i, b_i]$ alone. If any of the original intervals have intersecting interiors, replace them with the closed interval formed by taking their union. This gives us a collection of closed intervals contained in I with disjoint interiors, i.e. an element of $\mathcal{C}_1(k)$ for some $k \leq n$. ■

We will use Brinkmeier's notation for a rectangle in I^2 [Brinkmeier, 2000]. Thus we represent a rectangle with bottom left corner $a = (a^1, a^2)$ and top right corner $b = (b^1, b^2)$ by the pair $[a, b]$ viewing a and b as vectors. Take an element $\tau \in \widehat{\mathcal{C}}_2(n)$ consisting of rectangles $[a_i, b_i]$. If we project vertically onto the x-axis, we obtain a collection of closed, not necessarily disjoint intervals $\{[a_i^1, b_i^1]\}_{i=1}^n$ contained in $[0,1]$. By Lemma 3.1.1, these then form an element of $\mathcal{C}_1(k)$ for some $k \leq n$, up to a choice of labeling the intervals. We take the labeling given by the order of I. Similarly we can project horizontally onto the y-axis to form a collection of closed, not necessarily disjoint intervals $\{[a_i^2, b_i^2]\}_{i=1}^n$ contained in $[0,1]$ and from these form an element of $\mathcal{C}_1(l)$ for some $l \leq n$. Again we label the intervals in increasing order.

We will have two basic procedures for constructing our trees. We call the first procedure **vertical projection**: Given an element $\tau \in \widehat{\mathcal{C}}_2(n)$, we perform a vertical projection by first projecting vertically onto the x-axis and forming an element $\mu \in \mathcal{C}_1(k), k \leq n$, as described above. We then take μ, view it as living on the x-axis in I^2, and project it back up into I^2. Doing so forms k rectangles in I^2, each having height 1 and the i^{th} having width equal to the width of the i^{th} interval in μ, which partition the rectangles of τ in the sense that each such rectangle contains at least one rectangle from τ and no two such rectangles contain the same rectangle from τ. This is really just applying Brinkmeier's inclusion map i_H (Figure 1.3) and overlaying that image onto τ. Here we give a pictorial example.

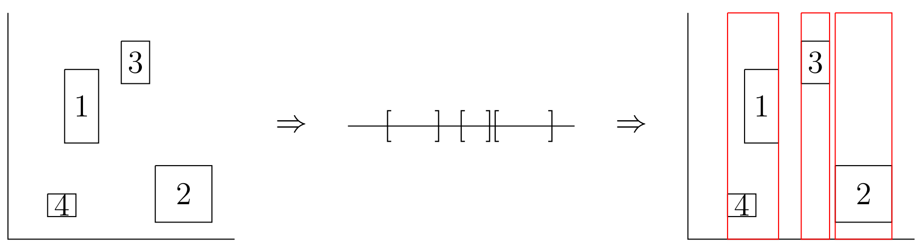


Figure 3.1.

Performing a vertical projection gives us a white vertex in what will be a bipartite tree representing τ. This vertex has k incoming edges and 1 outgoing edge, and we label it with $\mu \in \mathcal{C}_1(k)$, the element formed by Lemma 3.1.1. We can either think of μ as labeling the vertex or of the individual intervals that make up μ as labeling the incoming edges of the vertex. Both viewpoints are useful. Here we list the vertex formed from the above example.

Our second procedure is **horizontal projection**: The idea is the same, but instead of projecting onto the x-axis, we project onto the y-axis. Just as with vertical pro-

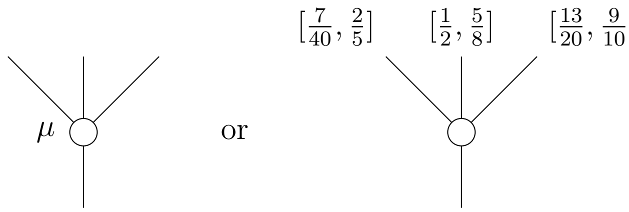


Figure 3.2.

jection, we get an element $\rho \in \mathcal{C}_1(l), l \leq n$, which we then project back onto I^2 via Brinkmeier's inclusion i_V and lay over the original 2-cube.

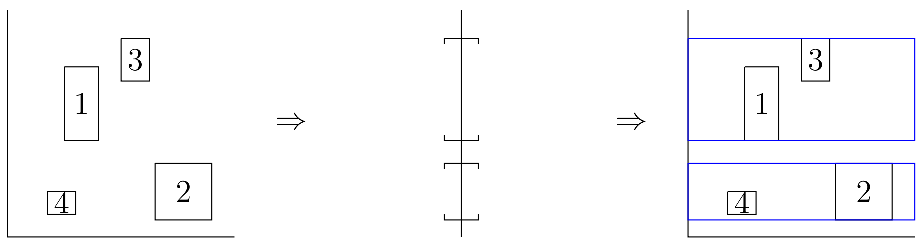


Figure 3.3.

Performing a horizontal projection gives us a black vertex. This vertex has l incoming edges and 1 outgoing edge, and we label it with $\rho \in \mathcal{C}_1(l)$. Again, we can think either of ρ as labeling the vertex or the intervals that make up ρ as labeling the incoming edges. Here is the vertex formed by our example.

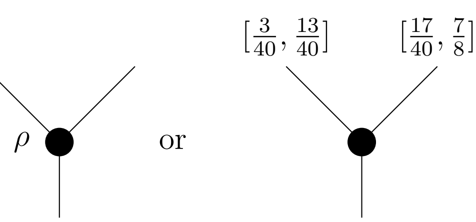


Figure 3.4.

Construction 3.1.2 *We now construct the tree associated to a given element $\tau \in \widehat{\mathcal{C}}_2(n)$. In fact we have two trees associated to τ, a white-rooted tree and a black-rooted tree. First we describe the construction of the white-rooted tree. Take τ and perform a vertical projection. This gives us a white vertex labeled by some $\mu \in \mathcal{C}_1(k)$ and k rectangles of height 1 which partition the rectangles of τ. Take the i^{th} such rectangle and scale it to the size of the unit cube, scaling the k_i rectangles of τ that it contains accordingly. Inside this unit cube, we now perform a horizontal projection. This gives us a black vertex labeled by some $\rho \in \mathcal{C}_1(l), l \leq k_i$. We then identify the outgoing edge of this black vertex with the i^{th} incoming edge of the white vertex. We take the l rectangles of length 1 created this way (which partition the k_i rectangles from τ), resize each one to the size of the unit cube (along with the rectangles it contains), and then perform a vertical projection. We attach the outgoing edges of the white vertices obtained this way to the incoming edges of the black vertex. We continue in this way, alternating between vertical and horizontal projections, building up the branches of our tree. We stop a branch when we reach a white vertex with one incoming edge such that the black vertex that would be attached to it would be have only one incoming edge labeled by the identity element of $\mathcal{C}_1(1)$ (i.e. we do not add this black vertex). Thus our white-rooted trees will always have white leaves. One exception to our stopping point is that we insist each tree has at least three levels. Even if the vertex created at level two would have one incoming edge labeled by the identity, we push on with the construction until we get a tree with three levels (and so white leaves).*

This process will eventually cordon off the rectangles in τ, with one branch for each rectangle. We label the incoming edge of each branch with the number which labels the rectangle corresponding to that branch.

The process of constructing the black-rooted tree associated to τ is analogous. The difference is that we start with a horizontal projection (instead of a vertical projection), and we stop a branch at a black vertex with one incoming edge such that the next white vertex would have only one incoming edge and this edge would be labeled by the identity of $\mathcal{C}_1(1)$. Similar to the white-rooted case, we insist that each tree have at least 3 levels. As an example, let's create the white- and black-rooted trees for the above element of $\widehat{\mathcal{C}}_2(4)$ (we leave out the labels on the vertices).

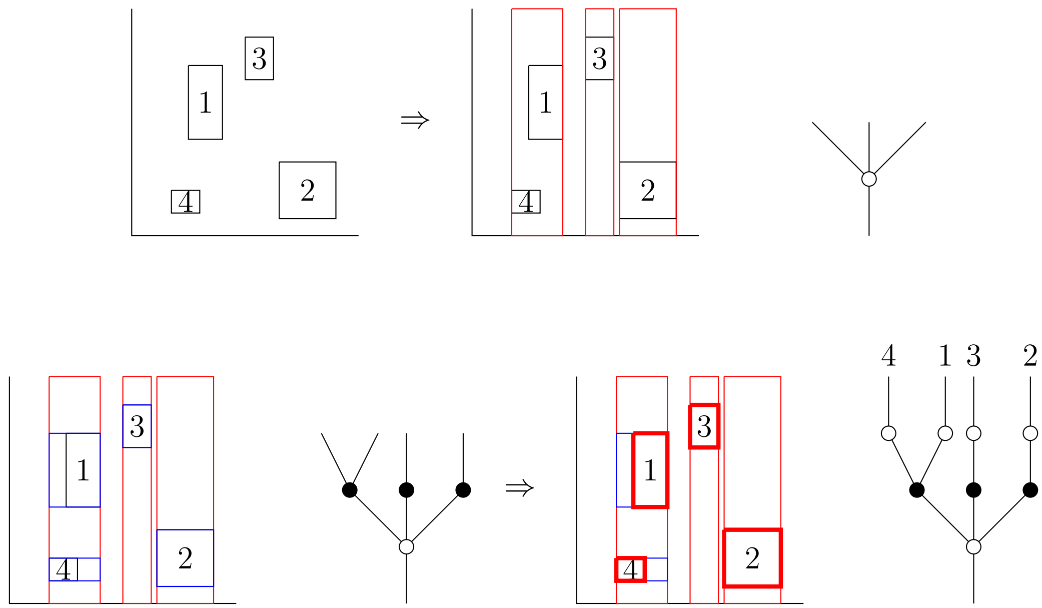


Figure 3.5.

On each branch, the next level would consist of a black vertex with one incoming edge which would be labeled by the identity. Thus we stop here. Here is the black-rooted tree.

Definition 3.1.3 *We call the trees created by the above process (either white- or black-rooted)* ***Brinkmeier trees****.*

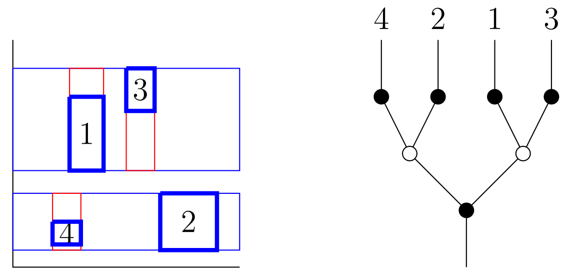


Figure 3.6.

3.2 Arity 2 trees

There are four trees that will be essential in the following. They are the trees associated with the elements of $\mathcal{A}_2(2)$. Consider the following element of $\mathcal{A}_2(2)$, along with its white-rooted and black-rooted trees:

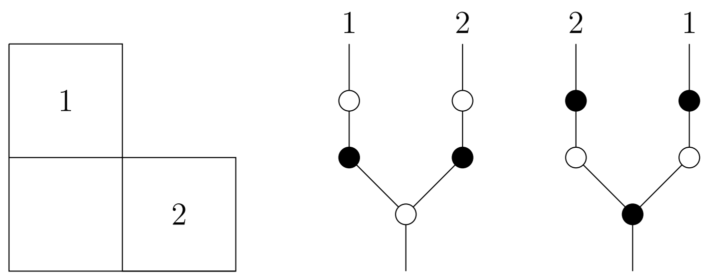


Figure 3.7.

Here both squares are of side length $\frac{1}{2}$. Recall that the other elements of $\mathcal{A}_2(2)$ where the squares have side length $\frac{1}{2}$ are formed by sliding the two squares around each other at equal speeds. Let's slide square 1 down and square 2 up. Halfway through the trip, we have this arrangement:
The shape of the white-rooted trees is the same as at the start, but of course the labels on all the trees are changing. Eventually we reach this arrangement:

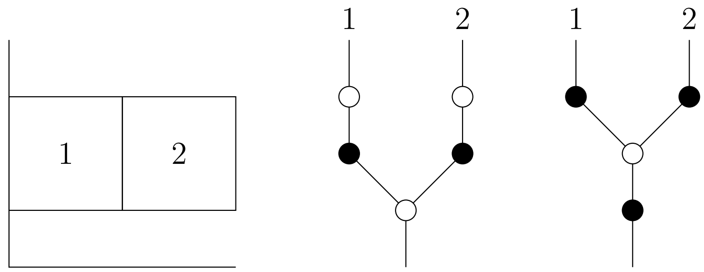


Figure 3.8.

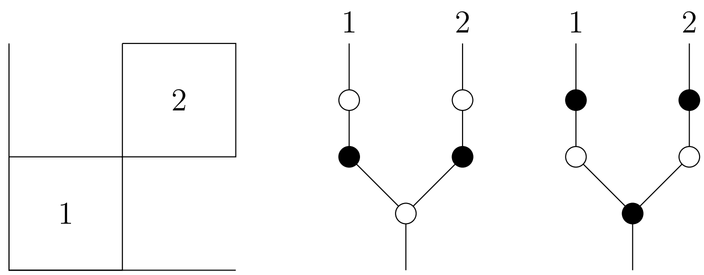
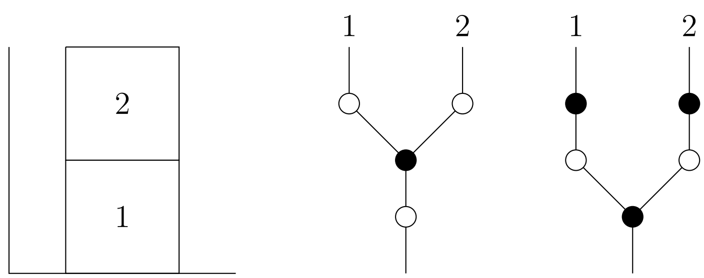
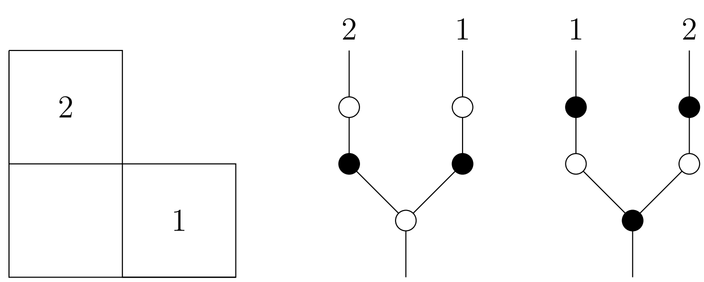


Figure 3.9.

Let’s display some elements as we continue around to the arrangement we started at.





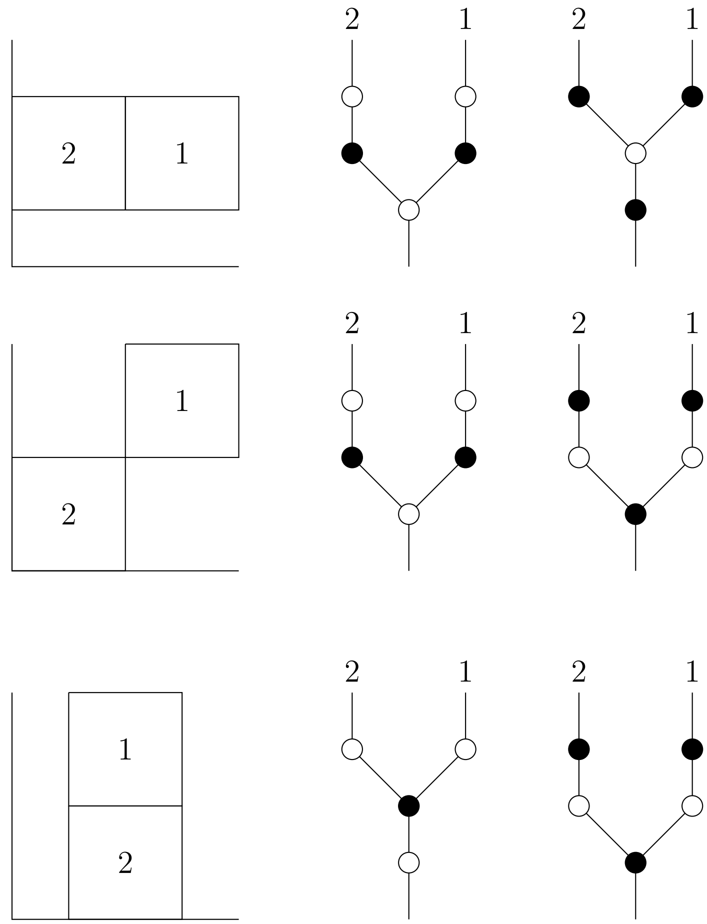


Figure 3.10.

The shape of the trees will be the same for any size of squares. This will only change the labels on the vertices. From this we can see that for fixed square sizes, the possible arrangements in $\mathcal{A}_2(2)$ form a CW complex, specifically a circle with four 0 cells and four 1 cells.

As we can see, these pairs of trees index the cells of this CW complex. Note that if we took only white-rooted or only-black rooted trees, this would not be the case. We state this result as a proposition:

Proposition 3.2.1 *Let $\mathcal{A}_2(2)_r$ be the subset of $\mathcal{A}_2(2)$ formed by all arrangements where the 1-labeled square has side length $0 < r < 1$. Then this set is a CW-complex*

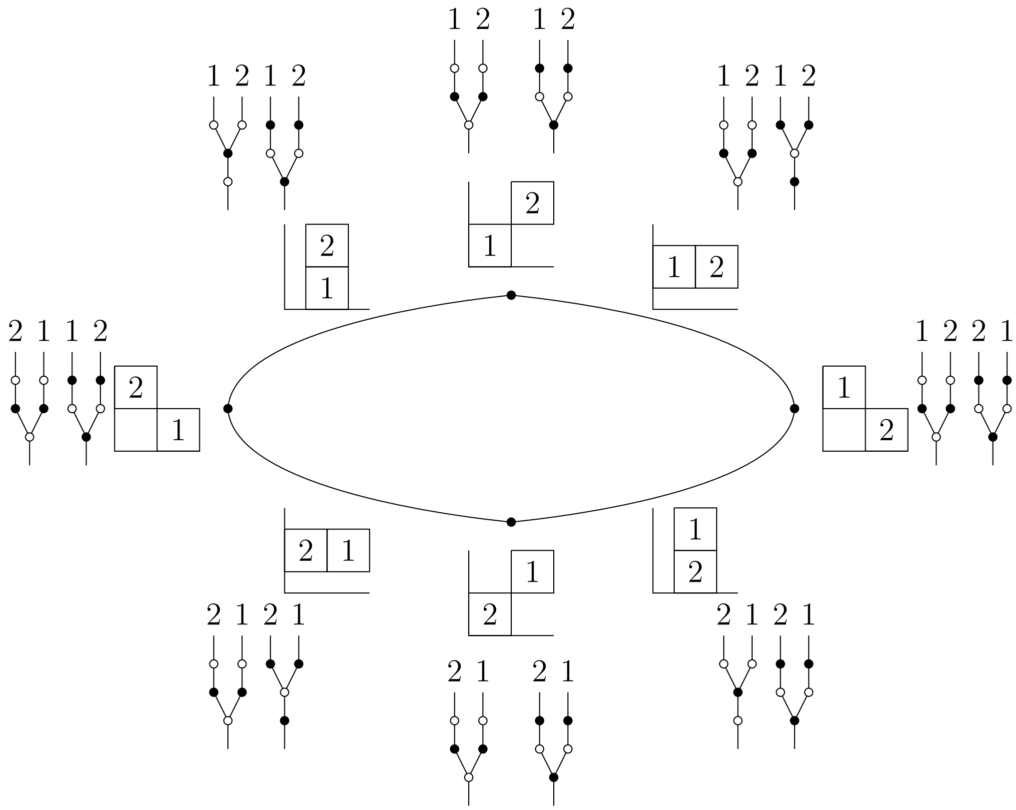


Figure 3.11.

homeomorphic to S^1. It consists of four 0-cells and four 1-cells, and these cells are indexed by pairs of Brinkmeier trees, each pair consisting of a white-rooted and black-rooted tree.

3.3 A new structure for higher arity elements

We will use the four basic trees from the arity two case to define a new tree structure on higher arity elements of $\mathcal{A}_2$. In doing so, we will create a correspondence between pairs of trees and elements of $\mathcal{A}_2$ with an arrangement of framing squares (i.e. the pair of trees will depend on both the element of $\mathcal{A}_2$ and the arrangement of framing squares for this element).

Construction 3.3.1 *Take an element $\tau \in \mathcal{A}_2(n), n \geq 3$, and choose a collection of framing squares for this element. Consider the largest framing square of τ and its complementary square. Consider the associated element of $\mathcal{A}_2(2)$. This has both a white-rooted and a black-rooted tree associated with it. We will describe how to build off of the white-rooted tree. The black-rooted tree is built similarly. Each branch of the white-rooted tree now corresponds to one of these two squares. If the branch corresponds to a realized square of τ, it stops here. If it corresponds to a framing square of τ, then we go into this framing square and repeat the process. Thus we treat the framing square as if it were the unit square, pick the largest framing square contained in it (along with its complementary square), and form the arity 2 white-rooted tree corresponding to these two squares. We now attach this tree to our original white-rooted tree by identifying the outgoing edge of the root vertex of our new tree with the incoming edge of the leaf vertex of our old tree. We do NOT contract this edge and compose the labels on the two white vertices. Notice that this means our new tree, unlike the trees we constructed previously, will not be bipartite. We now repeat the process again for each branch of this new tree, terminating if the branch corresponds to a realized square of τ and continuing if it corresponds to a framing square. Eventually we will reach a point where all branches correspond to realized squares, and then the tree will be built.*

The black rooted tree corresponding to τ and an arrangement of framing squares is built similarly. We always take black-rooted trees for the elements of $\mathcal{A}_2(2)$ and attach them in the analogous way as in the white-rooted case.

Example 3.3.2 *Consider the following element of $\mathcal{A}_2(3)$:*

Looking at the largest framing square and its complementary square gives this arrangement:

To this arrangement, we associate the following trees:

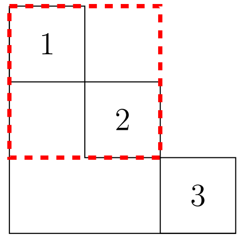


Figure 3.12.

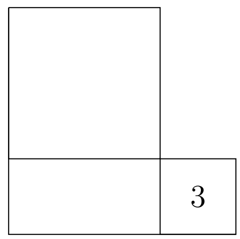


Figure 3.13.

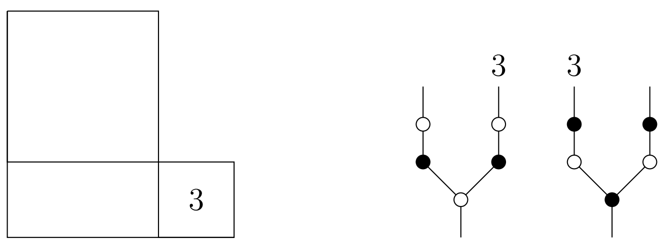


Figure 3.14.

Now we take the framing square and resize it (and the squares it contains) so that it is the unit square. This gives the following:

This arrangement has the following pair of trees associated with it.

We attach the white-rooted tree above to the previous white-rooted tree at the unlabeled leaf edge and the black-rooted tree above to the previous black-rooted tree at the unlabeled leaf edge. This gives the following two trees:

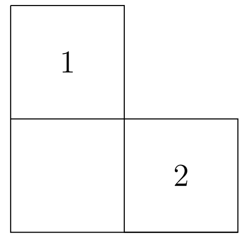


Figure 3.15.

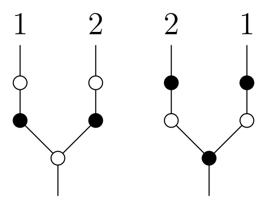


Figure 3.16.

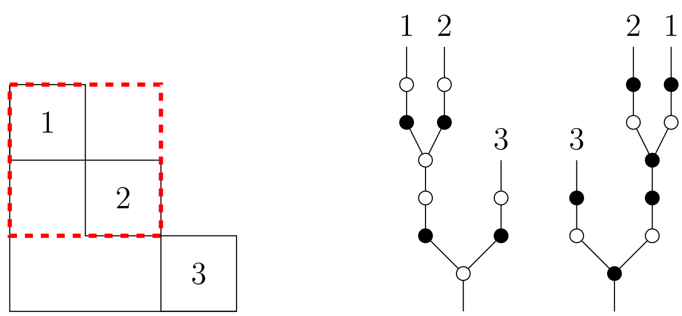


Figure 3.17.

Example 3.3.3 *This is slightly more complicated than the previous example. We just give the arrangement and the trees rather than going through the construction.*

Definition 3.3.4 *We call trees constructed in the above manner **tethered trees**.*

Remark 3.3.5 *To any decomposable 2-cube, we have two associated Brinkmeier trees, one white-rooted and one black-rooted. To any tethered 2-cube we have two*

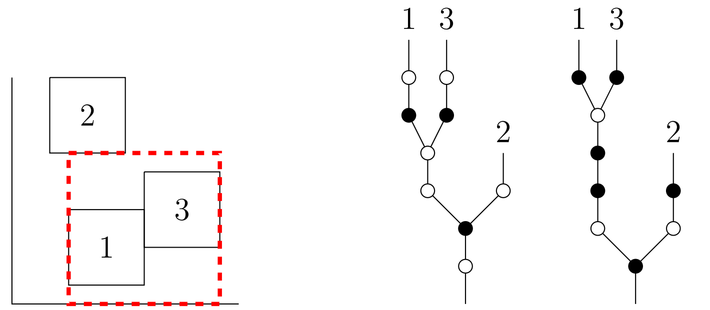


Figure 3.18.

associated tethered trees, again one white-rooted and one black-rooted. Since tethered cubes are decomposable, a tethered cube has Brinkmeier trees and tethered trees associated to it. We discuss the connection between these two tree structures in Section 5. For now we note that for an element $\tau \in \mathcal{A}_2(2)$*, the Brinkmeier and tethered trees of the same color root are equal.*

3.4 Dimension and trees

We can put an idea of dimension on tethered trees. The following two arity 2 trees will be considered to be of dimension 0.

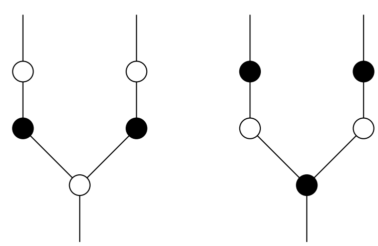


Figure 3.19.

There are really four such trees if we include the labels on the leaf edges (and infinitely many if we include the vertex labels). They are all taken to be of dimension 0. The white rooted tree corresponds to arrangements where the component squares are moving up and down. Thus they are either laying over each other horizontally or they are in the corners. The black rooted tree corresponds to arrangements where the component squares are moving left and right. Thus they are either laying over each other vertically or they are in the corners. The other two (four taking leaf labels, really infinitely many) arity 2 trees will be considered to be of dimension 1:

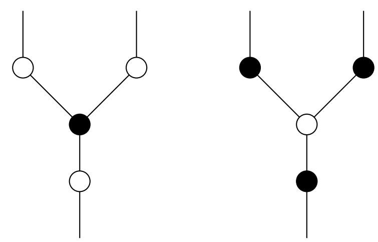


Figure 3.20.

The white rooted tree corresponds to arrangements where the component squares are moving left and right and are NOT in the corners. The black rooted tree corresponds to arrangements where the component squares are moving up and down and are NOT in the corners.

When we attach a tree to the leaf of another tree, we add their dimensions. The trees in Figure 3.17 were formed by attaching dimension 0 trees to dimension 0 trees. Thus both of these trees are of dimension 0. In Figure 3.18, we formed the white-rooted tree by attaching a dimension 0 tree to a dimension 1 tree. Thus this white-rooted tree is of dimension 1. The black-rooted tree was formed by attaching a dimension 1 tree to a dimension 0 tree. Thus this black-rooted tree is of dimension 1.

Remark 3.4.1 *By Proposition 3.2.1, pairs of the arity two trees index the cells of the CW complex that gives the structure of $\mathcal{A}_2(2)_r$. If we take the dimension of a pair* *of trees to be the sum of the dimensions of the trees, then we see that the pairs of trees have the same dimension as the cells that they are indexing.*

4. The Topological Structure of $\mathcal{A}_2$

We showed earlier that for fixed side lengths, the arrangements in $\mathcal{A}_2(2)$ form a CW complex whose cells are indexed by the pairs of trees associated with those arrangements. This CW complex is a circle consisting of four 0-cells and four 1-cells. Of course, this is not the entirety of the space $\mathcal{A}_2(2)$, since we are fixing the lengths of the squares. To get all of $\mathcal{A}_2(2)$, we must allow the side lengths to vary, namely one square can have side length r for any $r \in (0, 1)$ and the other square must have side length $1 - r$. Thus $\mathcal{A}_2(2) \simeq S^1 \times (0, 1)$. This means that while $\mathcal{A}_2(2)$ is not a CW complex, it is a CW complex crossed with a contractible space. Therefore it has the homotopy groups of the CW complex.

4.1 Framed Tethered Cubes

In order to give a thorough description of the topological structure of $\mathcal{A}_2$, we define a new operad, $\mathcal{A}_2^f$. We call this the operad of ***framed tethered cubes***. At arity 2, this operad is identical to $\mathcal{A}_2(2)$. For higher arities, $\mathcal{A}_2^f$ is defined in the same way as $\mathcal{A}_2$, with the exception that we now consider framing squares to be part of the information of the object. For example, consider the arrangements pictured below.

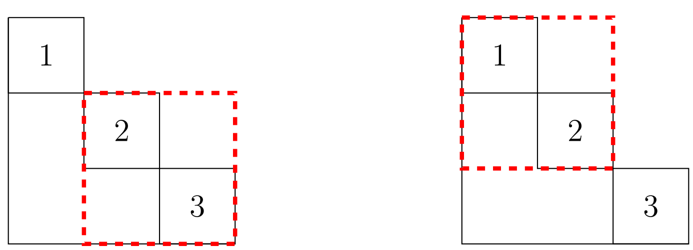


Figure 4.1.

These are the same element in $\mathcal{A}_2(3)$ (and in $\mathcal{C}_2(3)$), but they are different in $\mathcal{A}_2^f(3)$ since they have a different arrangement of framing squares. We will see that $\mathcal{A}_2^f(n)$ carries the same type of structure as $\mathcal{A}_2(2)$, that is, it is a CW complex crossed with a contractible space. It will turn out that $\mathcal{A}_2(n)$ is a quotient of the space $\mathcal{A}_2^f(n)$.

For now, let us fix the side lengths of squares in our arrangements at $\frac{1}{n}$. Again, the elements of $\mathcal{A}_2^f(n)$ are just the elements of $\mathcal{A}_2(n)$ with the framing squares included as part of the information. As a result, $\mathcal{A}_2^f(n)$ has its own "distinguished elements", namely the distinguished elements of $\mathcal{A}_2(n)$ with all possible choices of framing squares. We call these "framed distinguished elements".

As stated in Chapter 2, a choice of framing squares for a distinguished element of $\mathcal{A}_2(n)$ corresponds to a full bracketing of n letters (see Figure 2.4 and Figure 2.5), which in turn corresponds to a binary tree with n leaves. The number of binary trees with n leaves is given by the $(n-1)^{\text{st}}$ Catalan number. Let C_n denote the n^{th} Catalan number. It is defined as $C_n = \frac{(2n)!}{(n+1)!n!}$. Since there are $n!$ distinguished elements in $\mathcal{A}_2(n)$ and each distinguished element admits C_{n-1} choices of framing squares, there are a total of $n!C_{n-1} = \frac{(2n-2)!}{(n-1)!}$ framed distinguished elements in $\mathcal{A}_2^f(n)$.

The choice of framing squares ($n-2$ of them) for a distinguished element creates $n-1$ complementary pairs of squares. One pair consists of the largest framing square and its complementary square (which may be a framing square or a realized square). The other $n-2$ pairs are those squares which are contained in the $n-2$ framing squares. For each of the $n-1$ pairs, the squares can move around each other as in $\mathcal{A}_2(2)$ (all of this is the same as in $\mathcal{A}_2(n)$). As a result, each pair creates a copy of S^1 with a CW complex structure consisting of four 0-cells and four 1-cells. From this we see that starting with a framed distinguished element, the allowed movements create a copy of $T^{n-1} = S^1 \times \ldots \times S^1$ with the associated product CW structure.

So we see that there is associated to each framed distinguished element a copy of T^{n-1}, for a total of $\frac{(2n-2)!}{(n-1)!}$ copies. However, certain framed distinguished elements give the same copy of T^{n-1}, for example the two pictured below.

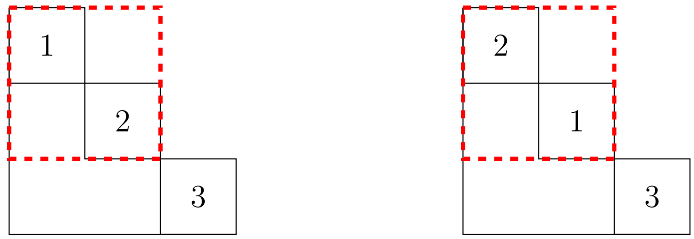


Figure 4.2.

Since each of the two arrangements can be reached from the other, they create the exact same copy of T^{n-1}. Hence our previous total of $\frac{(2n-2)!}{(n-1)!}$ is over-counting. It's clear that two framed distinguished elements will create the same copy of T^{n-1} if and only if one can be formed from the other by moving one or more pairs of squares around each other, as in Figure 4.2. If we use the binary tree representation for framed distinguished elements, this corresponds to swapping the two incoming edges of some number of internal vertices. A binary tree with n leaves has $n-1$ internal vertices, and for each such vertex we have two arrangements of incoming edges. Therefore, to get the total number of distinct copies of T^{n-1} we must divide our old total by $2(n-1)$ (in other words, we are considering non-planar binary trees with n leaves). We state these results as a proposition.

Proposition 4.1.1 *Let $n \geq 2$, let S^1 be given as a CW complex with four 0-cells and four 1-cells, and let T^{n-1} be the $(n-1)$-torus with the associated product cell structure. Let r_i denote the side length of the i-labeled square in an element of $\mathcal{A}_2^f(n)$, and let $\mathcal{A}_2^f(n)_{r_1,\ldots,r_n}$ denote the subset of $\mathcal{A}_2^f(n)$ consisting of all arrangements where the i-labeled square has side length r_i. Then $\mathcal{A}_2^f(n)_{r_1,\ldots,r_n} = \coprod_{i=1}^{N} T^{n-1}$, where $N = \frac{(2n-3)!}{(n-1)!}$.*

The fact that it is a disjoint union comes from the observation that moving squares does not allow us to shift framing squares, i.e there is no movement that can send the first arrangement in the figure below to the second arrangement.

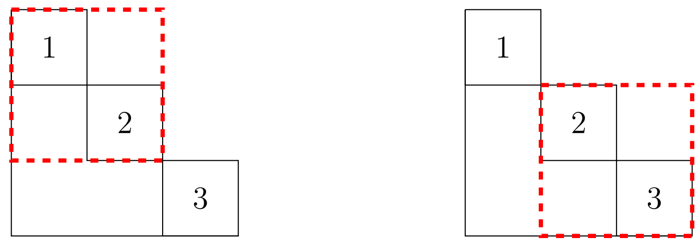


Figure 4.3.

To obtain the full space, we must allow the side lengths to vary. Let r_i be the side length of the square labeled by i. Then $0 < r_i < 1$ for all i, and $\sum r_i = 1$. The set of all possible side lengths is then the interior of the 0-face of the standard n-simplex. Denote it by F_n.

Corollary 4.1.2 *For $n \geq 2$, $\mathcal{A}_2^f(n) = \left(\coprod_{i=1}^N T^{n-1}\right) \times F_n$, with $\coprod_{i=1}^N T^{n-1}$ as in Proposition 4.1.1.*

Now that we have given a full description of $\mathcal{A}_2^f$, we may describe $\mathcal{A}_2$. We have a collection of maps $e_n : \mathcal{A}_2^f(n) \to \mathcal{A}_2(n)$ which forget the framing squares. These are quotient maps, and so $\mathcal{A}_2(n)$ is the quotient space formed from the disjoint copies of $T^{n-1} \times F_n$. Points in $\mathcal{A}_2^f(n)$ are identified if the corresponding unframed arrangement in $\mathcal{A}_2(n)$ can be given a full collection of framing squares in multiple ways. Some arrangements can be given many possible framings, for example distinguished elements. Others, like the arrangement pictured below, have only one possible framing.

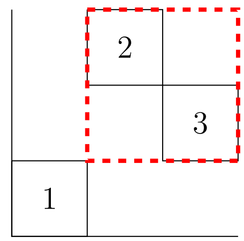


Figure 4.4.

We would like a condition that would tell us when a given arrangement can have more than one framing. Fortunately, we have such a condition. We state it as a lemma for the arity 3 case, and then we will see that the higher arity cases can be reduced to this.

Lemma 4.1.3 *An element $\tau \in \mathcal{A}_2^f(3)$ can be given a new framing if and only if the complementary square of the framing square touches some square that is contained in the framing square.*

We leave the proof of this lemma to the appendix. We reduce a general element of $\mathcal{A}_2^f(n)$ to the $\mathcal{A}_2^f(3)$ case as follows. Take any pair of complementary squares such that at least one of the squares is a framing square. This framing square must contain a pair of complementary squares. We can now take our first pair, one of which is a framing square, and the pair that the framing square contains, and treat these squares as an element of $\mathcal{A}_2^f(3)$. Then we see whether or not we can shift the framing square in this set up. If we can, then we can shift it in this sub-arrangement of the original element of $\mathcal{A}_2^f(n)$. Here is an example:

Here we take the pair of framing squares as our first pair of complementary squares. Even though both of these are framing squares, we will treat one as a framing square and one as a realized square. Let's consider the framing square containing squares 1 and 2 as the framing square and the framing square containing squares 3 and 4 as the realized square (and so we will ignore the squares inside it).

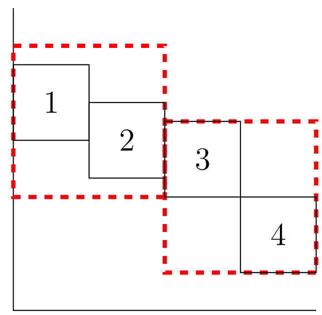


Figure 4.5.

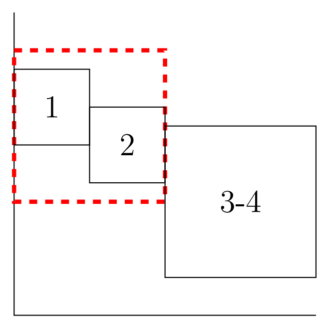


Figure 4.6.

As an element of $\mathcal{A}_2^f(3)$, we can see that we can shift the framing square here. We get the following arrangement.

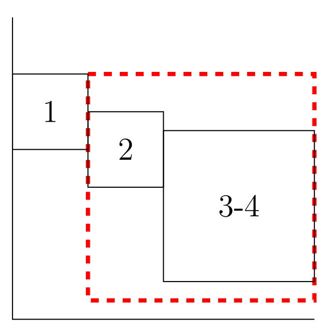


Figure 4.7.

Then the alternate framing of our original arrangement is given by

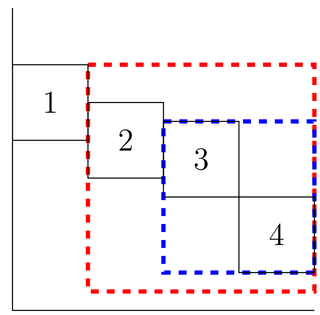


Figure 4.8.

Of course we could have chosen the framing square containing squares 3 and 4 in the original arrangement as our framing square for this procedure. This would have led to yet another collection of framing squares. And after doing either of these shifts, with this arrangement it will be possible to do yet another shift. Here we list all five possible framings of the original arrangement.

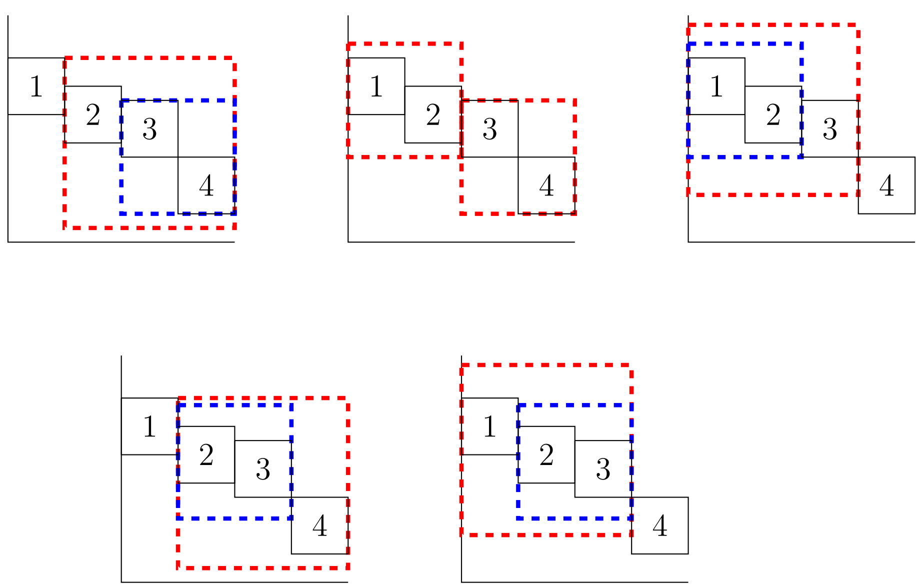


Figure 4.9.

In $\mathcal{A}_2(4)$, these five elements are identified.

The last thing we will discuss here is how this shifting of framing squares affects the corresponding trees. Consider the following arrangement in $\mathcal{A}_2^f(3)$ and its corresponding white-rooted tethered tree.

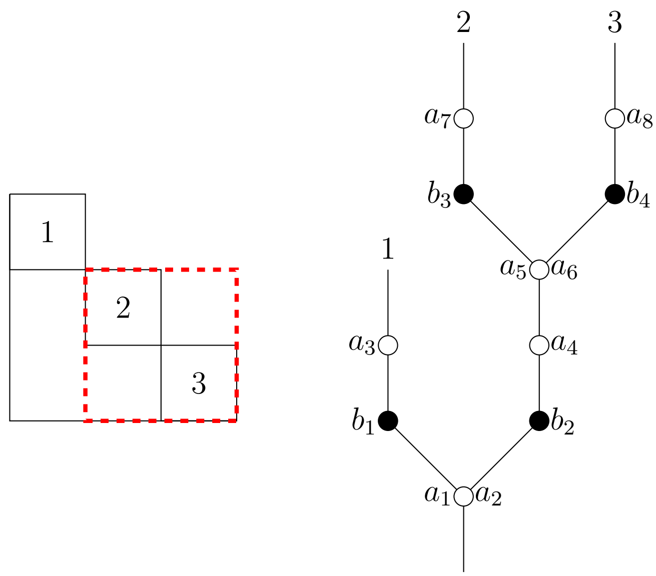


Figure 4.10.

Here $a_i = [r_{2i-1}, r_{2i}]$ and $b_i = [s_{2i-1}, s_{2i}]$ are the subintervals of $[0, 1]$ which give the pictured arrangement. As square 1 moves down and squares 2 and 3 move down and up, the shape of this tree doesn't change, although the labels do. Thus this tree represents any arrangement in that motion. We want to show how to draw the tree that results from shifting the framing square.

In the new framing, squares 1 and 2 will be paired together, so they need to share an immediate root. Similarly, the framing square 1-2 and the square 3 are paired and so must share an immediate root. Thus we have the following tree:

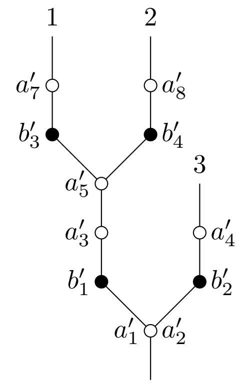


Figure 4.11.

The new labels on the vertices can be determined from the old ones by using the formulas laid out in the proof of the lemma (in the appendix). One just has to set x_i (the side length of the i-labeled square) equal to the appropriate amount determined by the labels in the original tree.

When we do this shifting of framing squares for higher arity arrangements, the affect on the corresponding tree is essentially the same. Say we start with an arrangement as in Figure 4.5, where we have two framing squares. We proceeded by only considering one of these squares to be a framing square and treating the other as a realized square, ignoring what was inside it for the time being. In the tree picture, this corresponds to cutting off the branch of the tree above where this framing square is given. We then proceed as above, and once we are done we reattach this severed branch in the appropriate place.

5. Connections Between $\mathcal{C}_2$, $\mathcal{A}_2$, and *Cact*

In Section 5.1 we deliver on our promise from Remark 3.3.5 and make explicit the connection between Brinkmeier trees and tethered trees. Recall that Brinkmeier trees were defined in Definition 3.1.3 and tethered trees in Definition 3.3.4. In fact we work more broadly and show how any black and white tree whose vertices are labeled by elements of the little 1-cubes operad $\mathcal{C}_1$ (tethered trees are examples) can be reduced to a Brinkmeier tree. In this sense Brinkmeier trees are the minimal black and white, $\mathcal{C}_1$-labeled trees corresponding to decomposable 2-cubes.

In Section 5.2 we examine the relations on $\mathcal{A}_2$ and compare them to those of spineless cacti.

5.1 Brinkmeier Trees

In Chapter 3, we described how to form a black and white (in fact bipartite), $\mathcal{C}_1$-labeled tree from a given decomposable 2-cube. We defined Brinkmeier trees to be those trees formed in this way. Conversely, given any black and white, $\mathcal{C}_1$-labeled tree (not necessarily Brinkmeier), there is a corresponding decomposable 2-cube. It is formed by starting at the root, moving up the tree, and projecting the labels on the vertices either vertically or horizontally into I^2 (vertically for white vertices, horizontally for black). In this section we characterize Brinkmeier trees entirely in terms of tree structure and vertex labels and in doing so develop a procedure to reduce a general black and white, $\mathcal{C}_1$-labeled tree to a Brinkmeier tree.

Definition 5.1.1 *A collection of closed subintervals of* $I = [0, 1]$ *form a* ***good cover*** *of* $[0, 1]$ *if the union of these subintervals cover* $[0, 1]$ *and the union of their interiors cover* $(0, 1)$.

Consider a black and white bipartite tree of height ≥ 3, either white-rooted or black-rooted, such that each leaf vertex is the same color as the root vertex. For each vertex, suppose it is labeled by an element of $\mathcal{C}_1(k)$ where $|v| = k$. We only allow the identity to be a label on the root vertex, leaf vertices, and vertices that are one level below a leaf vertex such that the vertex below them has more than one incoming edge (There is one exception, shown in Figure 5.1). Further, suppose the leaf edges are labeled by $1, 2, \ldots, n$, where n is the total number of leaf edges. Partition the vertices at height ≥ 3 of the tree so that two vertices are grouped together if their outgoing edges are incident to the same vertex. From now on, we call this partition the **vertex partition** of the given tree.

Proposition 5.1.2 *A tree is a Brinkmeier tree if and only if it is of the above form and the labels on the vertices in each group of the vertex partition form a good cover of* $[0, 1]$. *Any labeling is allowed on the vertices at height 1 and 2.*

Proof Let τ be a white-rooted Brinkmeier tree, and let $\tau_c \in \widehat{\mathcal{C}}_2(n)$ be the decomposable 2-cube that gives τ. It is clear that τ is a black and white, bipartite tree of height ≥ 3 whose leaf edges are labeled by $1, 2, \ldots, n$ and whose vertices v are labeled by elements of $\mathcal{C}_1(|v|)$, with the identity appearing only on possibly the root vertex, the leaf vertices, and the vertices one level below the leaves. The root label of τ is formed by taking the realized rectangles in τ_c, projecting to the x-axis, and taking the union of any of the closed intervals whose interiors intersect. The rectangles formed by projecting these new closed intervals back into I^2 can be thought of as being represented by the incoming edges of the root vertex, and they partition the realized rectangles of τ_c. These projected rectangles also give the vertex partition of the height 3 vertices of τ, in the sense that vertices grouped together in the vertex partition have their outgoing edges incident to the black vertex whose outgoing edge is the incoming edge of the root vertex represented by this rectangle. The widths of the realized rectangles contained in such a projected rectangle necessarily form a good cover of the width of this rectangle, since the realized rectangles fill up the width of the projected rectangle. The labels on the height 3 vertices grouped together by the vertex partition are formed by scaling this rectangle up to I^2 (and scaling the realized rectangles it contains appropriately) and projecting the widths of the contained realized rectangles down. Since the widths of the realized rectangles formed a good cover of the width of the projected rectangle before the rescaling, and since the projected rectangle has been rescaled to I^2, the widths of the rescaled realized rectangles form a good cover of $[0, 1]$. Thus the labels on the height 3 vertices grouped together by the vertex partition form a good cover, as required.

The same argument holds for the vertices at any height which are grouped together by the vertex partition. We form labels by rescaling the projected rectangle to I^2 and projecting the contained realized rectangles, which then form a good cover of $[0, 1]$ for the same reason described above.

Now suppose τ is a white-rooted, white-leafed black and white, bipartite tree of height ≥ 3 whose vertices are appropriately labeled by elements of $\mathcal{C}_1$ and whose leaves are labeled by $1, 2, \ldots, n$. Further suppose that each group of vertices from the vertex partition of τ is labeled by a good cover of $[0, 1]$. To form the 2-cube associated with τ, we take the root label of τ and vertically project those intervals into I^2. Then, treating each projected rectangle as it if were I^2, we horizontally project the labels of the black vertices at height 2 into the projected rectangle representing their outgoing edge. We vertically project the labels of the height 3 edges into the rectangles formed by their outgoing edges, and so on until we reach the leaf edges. The final rectangles that we end up with form the realized rectangles of this 2-cube, and we label them by the corresponding leaf edge label.

Let us now form the white-rooted Brinkmeier tree associated with this 2-cube. Since the groups in the vertex partition were labeled by good covers throughout τ, at every level the realized rectangles filled out the projected rectangles in the appropriate dimension (width or height). Thus when we project the realized rectangles (either vertically or horizontally), we end up with the same labels as those of τ. Therefore τ comes from a 2-cube and so is a Brinkmeier tree. ■

Remark 5.1.3 *We have seen that every black and white tree whose vertices are labeled by elements of $\mathcal{C}_1$ and whose leaf edges are labeled by $1, 2, \ldots, n$ gives an element of $\widehat{\mathcal{C}}_2(n)$. This gives us an obvious equivalence relation on the set of such trees: Two trees with the same root color are equivalent if and only if they give the same element of $\widehat{\mathcal{C}}_2(n)$. By definition, every decomposable 2-cube has exactly one Brinkmeier tree associated with it. Therefore every tree as described above is equivalent to exactly one Brinkmeier tree (and so no two distinct Brinkmeier trees are equivalent to each other). We can think of the Brinkmeier trees as being the "prefered representatives" of the equivalence classes of our trees.*

As stated in the remark, every labeled black and white tree is equivalent to exactly one Brinkmeier tree. Our goal now is to give a procedure for obtaining from any $\mathcal{C}_1$-labeled black and white tree its equivalent Brinkmeier tree.

Let τ be a white-rooted, black and white, $\mathcal{C}_1$-labeled tree. We wish to construct a Brinkmeier tree τ_b which is equivalent to τ. First, if τ consists of just a root vertex with one incoming edge labeled by $\mu \in \mathcal{C}_1(1)$, τ_b will be the following tree:

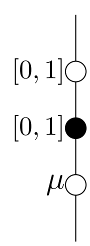


Figure 5.1.

Suppose τ is not just a root vertex with one incoming edge. We will perform a series of alterations to τ, transforming it into new trees, each of which is equivalent to τ. When we finish, we will have τ_b. Since τ has a white root, so will all of these equivalent trees.

Step 1: The leaf vertices of τ_b must be white vertices with one incoming edge. If a leaf vertex of τ is black, we may attach a white vertex labeled by the identity to each leaf edge on this vertex, and we may put the label from the leaf edge on the outgoing edge of the leaf vertex that was attached to it. If a leaf of τ is white with two or more incoming edges, we may attach the following tree to each incoming edge of this vertex:

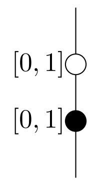


Figure 5.2.

Note that this also guarantees that the tree has height at least 3. Since all the vertices we are adding are labeled by the identity, we are not changing the associated cube picture, and so the resulting tree is equivalent to τ.

Step 2: If any vertices of τ other than the root vertex, the leaf vertices, or the vertices one level below the leaf vertices are labeled by the identity, delete those vertices and attach the outgoing edge of the vertex above them to the vertex below them.

Step 3: τ_b must be bipartite. If τ is not bipartite, then it contains at least one edge connecting two black vertices or two white vertices. Suppose such an edge is the i^{th} incoming edge of a vertex labeled by $\mu \in \mathcal{C}_1(n)$ and the outgoing edge of a vertex labeled by $\nu \in \mathcal{C}_1(m)$. We can contract this edge and replace the two vertices by a new vertex of the same color labeled by $\mu \circ_i \nu \in \mathcal{C}_1(n+m-1)$. If we do this with all such edges in τ, the resulting tree will be bipartite. It is also clear that none of the alterations performed on τ so far effect the corresponding cube picture, i.e. the tree we have at this point is still equivalent to τ.

Step 4: The groups of the vertex partition of τ_b must be labeled by good covers of $[0,1]$. To finish transforming τ into τ_b, we must make sure that this property holds for our altered tree. We start at the highest level of the tree and ensure this property for each group of vertices there. Then we move down to the next level, and so on until we've done this for all the groups in the partition (recall that the root vertex and the vertices at height 2 don't need to have this property).

Suppose we are at a group of the vertex partition located at height $h \geq 3$ of the tree. If the labels of this group form a good cover of $[0,1]$, we do nothing and move on. Let's assume they do not form a good cover of $[0,1]$. Suppose there are k vertices in this partition, and let the j^{th} vertex (reading from left to right in the planar structure) have v_j incoming edges. Thus $\sum_{j=1}^{k} v_j = v$ is the total number of closed intervals making up the labels of this group. Let $[a^i_{v_j}, b^i_{v_j}]$ be the label on the i^{th} incoming edge of the j^{th} vertex in this group. First we take these closed intervals and form a new collection of closed intervals via Lemma 3.1.1. Let these new closed intervals be given by $[e_1, e_2], \ldots, [e_{2p-1}, e_{2p}]$. Of course these give us an element $\rho \in \mathcal{C}_1(p)$.

The k vertices in this group all have outgoing edges incident to the same vertex at height $h-1$. This vertex necessarily has k incoming edges, and its outgoing edge is incident to some vertex at height $h-2$. Suppose this vertex at height $h-2$ is labeled by $\mu \in \mathcal{C}_1(q)$ and that its r^{th} incoming edge is the outgoing edge of our vertex at height $h-1$. We replace this vertex at height $h-2$ with a vertex of the same color labeled by $\mu \circ_r \rho$, and we replace our vertex at height $h-1$ by p vertices of the same color, the i^{th} such vertex having as its outgoing edge the incoming edge of the vertex at $h-2$ represented by resized interval $[e_{2i-1}, e_{2i}]$. The incoming edges on these p new vertices will be discussed momentarily.

We will replace the k vertices in our original group with new vertices of the same color. Recall that v was the number of incoming edges in the original group. The number of vertices in the new group will be

$$v - \sum_{j=1}^{k} \sum_{i=1}^{p} \max\{n^j_i - 1, 0\}$$

where n_i^j is the number of intervals on vertex j that are contained in $[e_{2i-1}, e_{2i}]$. The incoming edges of these vertices are arranged in the following manner: If two incoming edges were originally on the same vertex and their labels were grouped with the same $[e_{2i-1}, e_{2i}]$, they are still on the same vertex. If they were on different vertices and their labels were grouped with the same $[e_{2i-1}, e_{2i}]$, they are still on different vertices, but the outgoing edges of those vertices are incident to the same vertex at height $h-1$. If their labels were not grouped with the same $[e_{2i-1}, e_{2i}]$, they are on different vertices whose outgoing edges are not incident to the same vertex at height $h-1$. Thus each new vertex can only contain labels that were grouped with the same $[e_{2i-1}, e_{2i}]$, and the outgoing edge of a vertex containing a label that was grouped with $[e_{2i-1}, e_{2i}]$ is incident to the vertex at height $h-1$ whose outgoing edge is labeled by the resized interval $[e_{2i-1}, e_{2i}]$.

Since the pertinent labels at height $h-2$ are the intervals $[e_{2i-1}, e_{2i}]$ rescaled, we must rescale the labels at height h appropriately in order to not change the associated 2-cube. Suppose an incoming edge at height h was originally labeled by $[a^l_{v_j}, b^l_{v_j}]$ and that in the rearrangment this edge is now incident to a vertex at height h whose outgoing edge is the incoming edge of the vertex at height $h-1$ whose outgoing edge is labeled by $[e_{2i-1}, e_{2i}]$. Then the new label on the edge at height h is $[a'_{jl}, b'_{jl}]$, where

$$a'_{jl} = \frac{a^l_{v_j} - e_{2i-1}}{e_{2i} - e_{2i-1}}, \qquad b'_{jl} = \frac{b^l_{v_j} - e_{2i-1}}{e_{2i} - e_{2i-1}}.$$

The labels of the incoming edges at height $h-1$ are determined as follows: Take a vertex at height $h-1$ and go to one of the vertices at height h incident to it. Look at a label of one of the incoming edges of this vertex at height h. Suppose it is $[a'_{jl}, b'_{jl}]$. Then the original label on this edge was $[a^l_{v_j}, b^l_{v_j}]$. The label on the outgoing edge of the vertex that contained the label $[a^l_{v_j}, b^l_{v_j}]$ before rearranging, i.e. the vertex v_j, will be the label on the incoming edge of our vertex at height $h-1$. This is well defined, in the sense that we get the same result if we choose a different incoming edge of vertex at height h connected to our vertex at height $h-1$. This is because in order for two edges to end up on the same vertex at height h after rearranging, they had to have been on the same vertex before rearranging.

After doing all of this, we now have a good cover of $[0,1]$ for this group of the vertex partition. This is because we have created new (smaller) groups in the partition, one for each interval $[e_{2i-1}, e_{2i}]$. We then rearranged the incoming edges at height h so that their labels formed a good cover of $[e_{2i-1}, e_{2i}]$ and rescaled these labels so that they formed a good cover of $[0,1]$. Due to the rescaling of labels, the associated 2-cube is unchanged.

Step 5: Recall that we start at the top of the tree and work our way down, so that by the time we are at height h, we can assume all the groups of the vertex partition at height $\geq h+1$ are labeled by good covers. However, in performing Step 4, it is possible that we may have ruined good covers for the groups of the vertex partition at height $h+1$. This is because when we rearrange the incoming edges at height h, we are rearranging the vertex partition at height $h+1$. For example, a vertex at height h may have had three incoming edges, and so there was a group of three vertices in the vertex partition at height $h+1$. The labels on the incoming edges of these three vertices formed a good cover of $[0,1]$. If we had to move one or two of the incoming edges at height h, we would have moved some of these labels around, and the labels that are left may no longer form a good cover.

We can handle this apparent defect. Suppose we have finished Step 4 for the groups of the partition at height h. We then simply go back to height $h+1$ and perform Step 4 again, making sure we have good covers everywhere. If this destroys a good cover at height $h+2$, we move there and perform Step 4 again, and so on. One may wonder if this procedure could go on forever, but we can see that it won't. Recall how we rearrange the incoming edges at height h. Two edges only end up on the same vertex after rearranging if they started on the same vertex before rearranging, and so this rearranging only makes the groups of the vertex partitions at height $h+1$ smaller (or leaves them unaffected). If a group of the partition at height $h+1$ was unaffected, its labels will still form a good cover of $[0,1]$. If it gets smaller, it will eventually stabilize at some size ≥ 1 with a good cover. If it does get down to size one (i.e. there is only one vertex in this group of the vertex partition), then once we form a good partition for this single vertex, we can never break it again.

Now that our relabeling procedure is described, we have a way of going from any element in an equivalence class of trees to our preferred representative. This is the connection between tethered trees and Brinkmeier trees, namely, for any tethered tree representing some element of $\mathcal{A}_2(n)$, we may perform the above procedure and arrive at the Brinkmeier tree for this same element, now viewed as living in $\widehat{\mathcal{C}}_2(n)$. Over the next several pages we describe a situation where this is very useful.

Example 5.1.4 *Consider the following two 2-cubes and their corresponding Brinkmeier trees:*

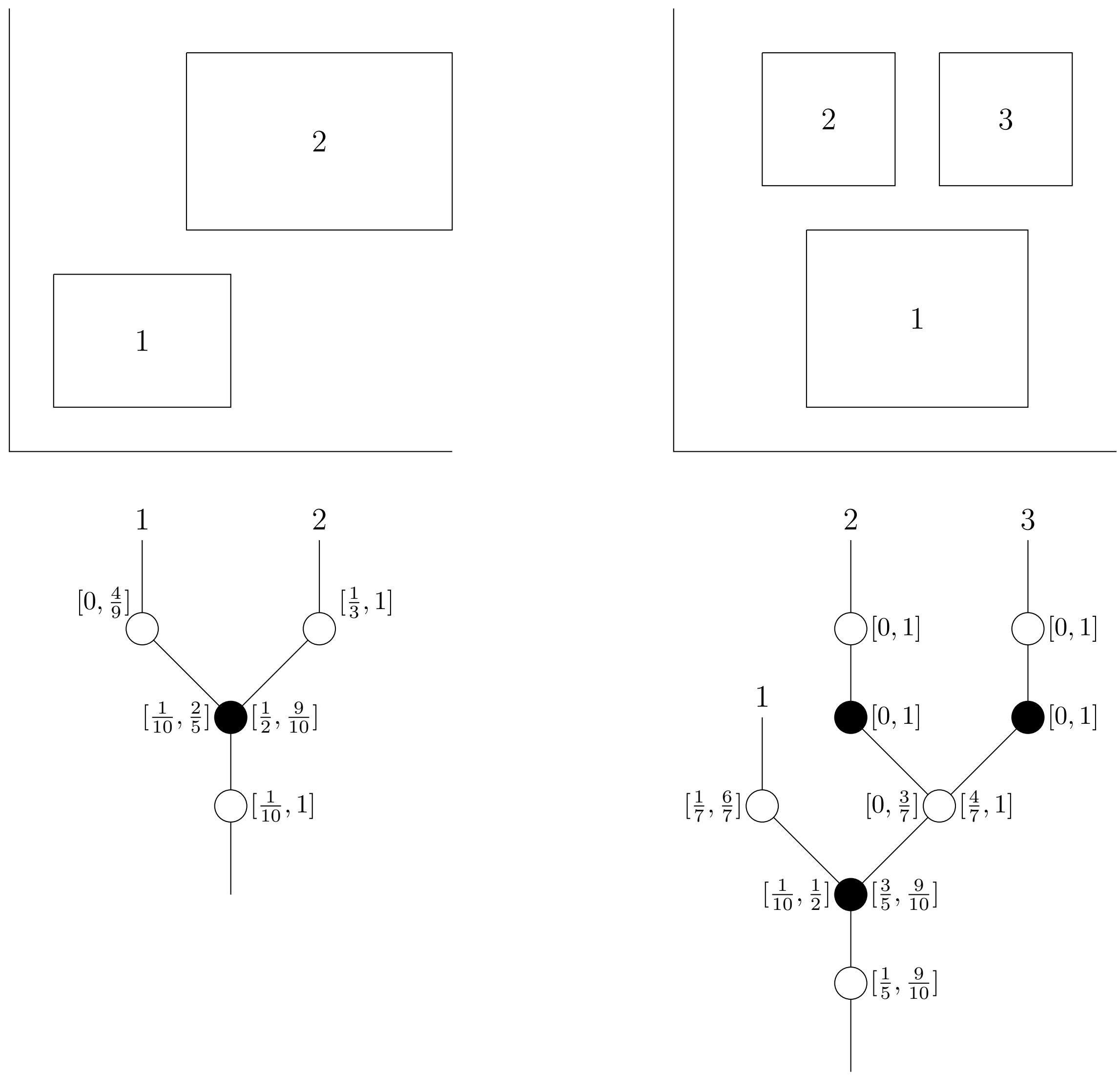


Figure 5.3.

Suppose we compose these 2-cubes via the $\circ_2$ operation. We can also compose the trees to get a tree corresponding to the new 2-cube. We do this by identifying the root edge of the right tree with the incoming edge of the left tree labeled by 2, contracting this edge, and composing the labels to create a new white vertex as follows:

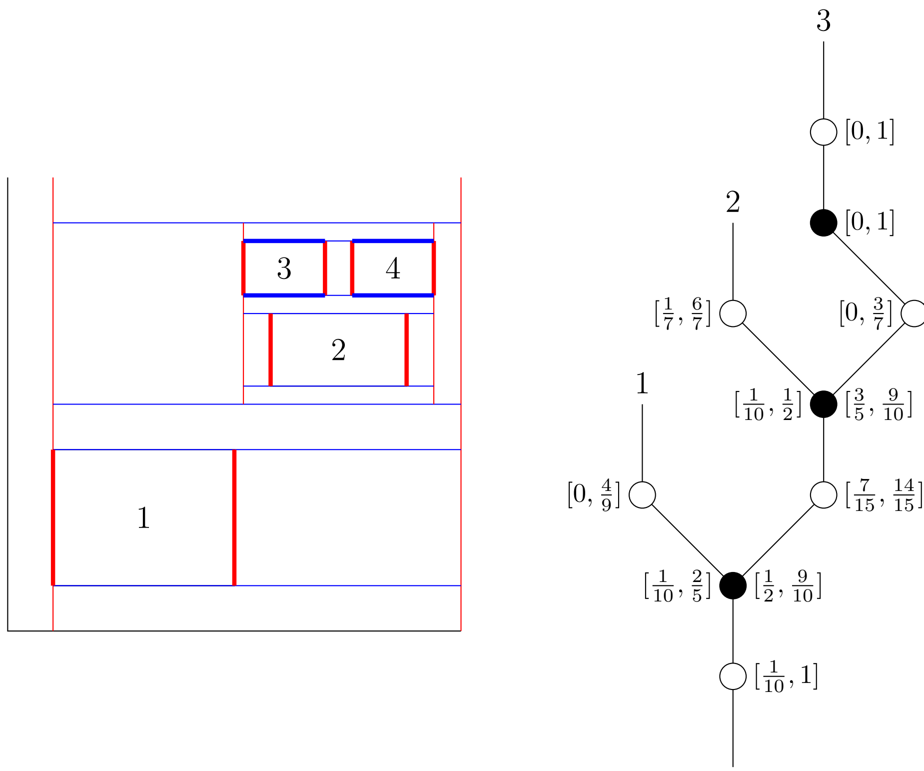


Figure 5.4.

Note that this tree is not a Brinkmeier tree. The labels on the vertex groups at height 3 and 4 do not form a good cover of $[0,1]$. *We must perform Step 4 of our relabeling procedure on the vertex group at height 4. From now on, we will only list vertex labels if they change from step to step.*

First we project the labels on the vertex group at height 4. This will not change them at all, since there is only one vertex in the partition. Since we have two intervals here, we form the element $\rho \in \mathcal{C}_1(2)$ *consisting of* $[\frac{1}{10}, \frac{1}{2}]$, $[\frac{3}{5}, \frac{9}{10}]$. *We add two white vertices at height three and compose* ρ *with the label on the height 2 vertex.*

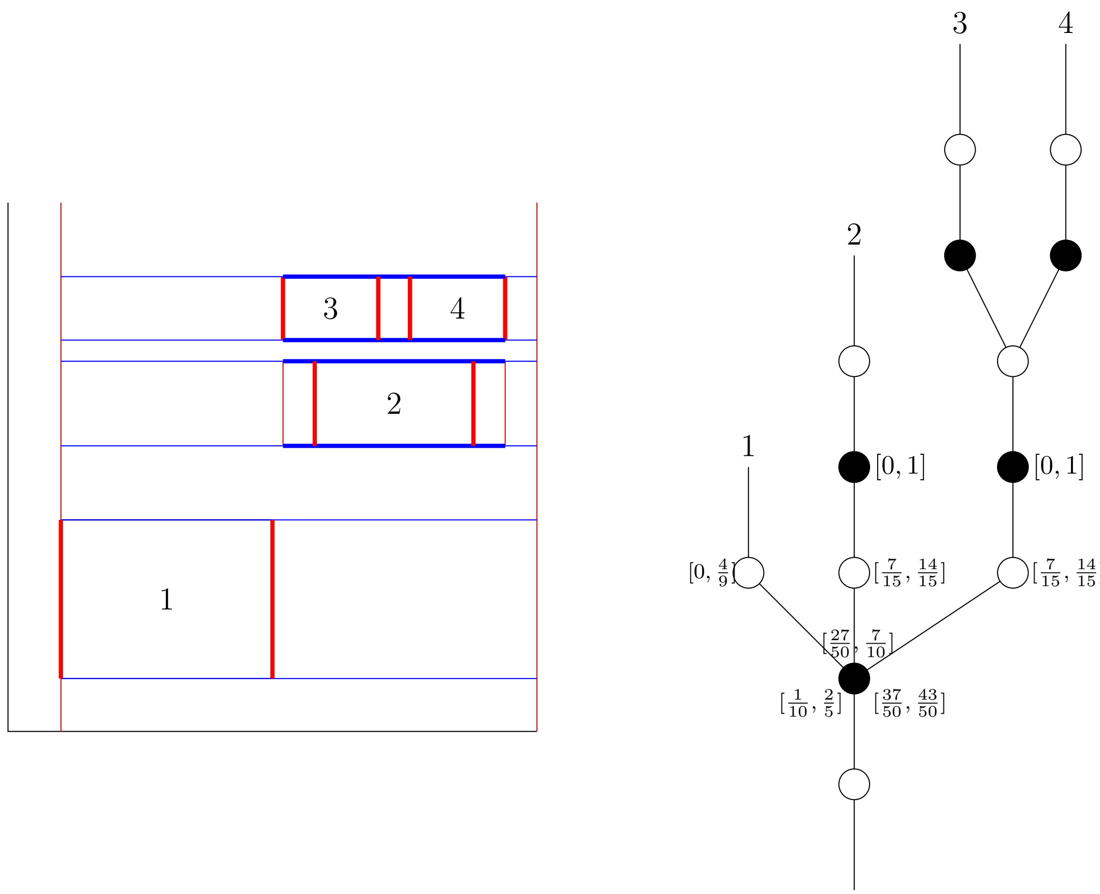


Figure 5.5.

We now have unnecessary identity labels on the vertices at height 4, so we will delete these vertices and contract the edges, composing the white vertices at height 5 with those at height 3 (remember that the labels at height 5 for the previous tree have not yet changed).

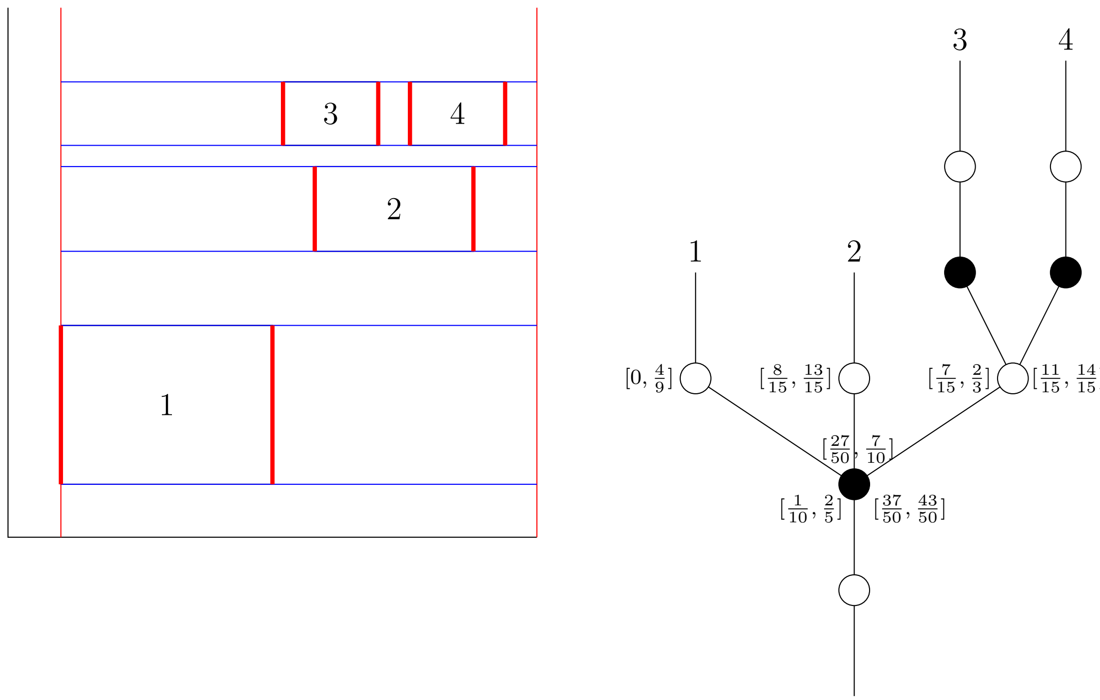


Figure 5.6.

The labels at height 3 do not form a good cover of $[0,1]$*, so we must perform Step 4 again. This time when we project, our original four intervals give us only two intervals,* $[0,\frac{4}{9}]$ *and* $[\frac{7}{15},\frac{14}{15}]$*. We take the corresponding element of* $\mathcal{C}_1(2)$ *and compose it with the label of the root vertex. As a result the new root vertex will have two incoming edges.*

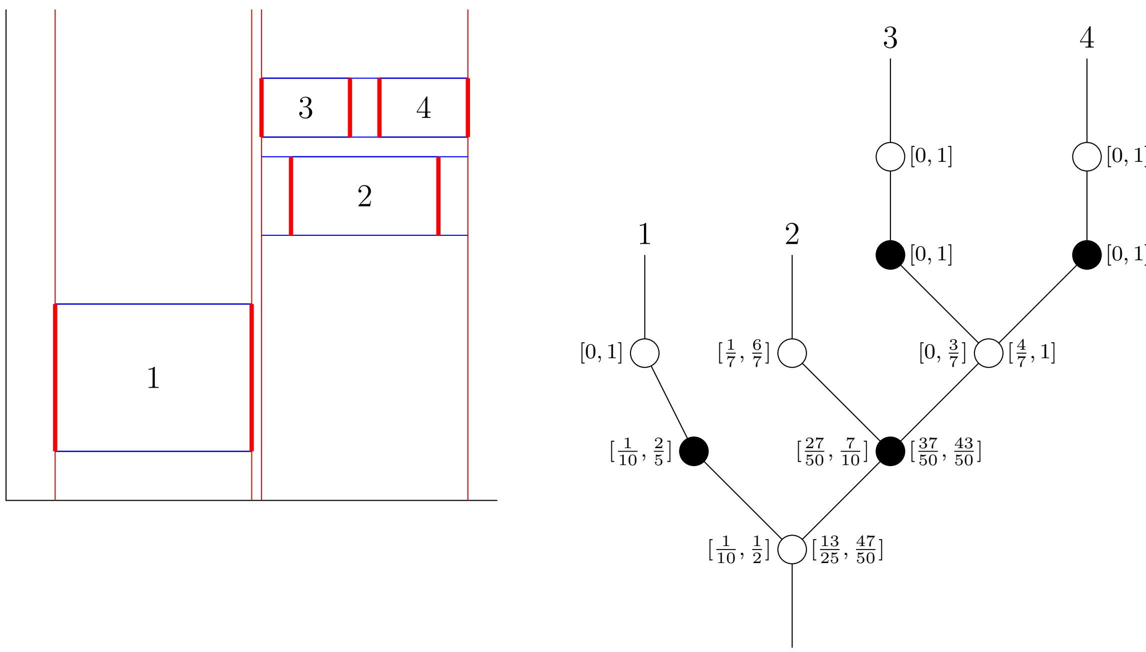


Figure 5.7.

This final tree is the Brinkmeier tree associated with the 2-cube.

Among other things, our relabeling procedure, illustrated by this example, shows that we have a well-defined operadic composition of Brinkmeier trees. Composing two Brinkmeier trees with the same color root using the usual composition will never produce a Brinkmeier tree, but the procedure will always produce a Brinkmeier tree equivalent to the result. It is clear that this composition is both associative and equivariant (with respect to the usual symmetric group action) and that the tree of Figure 5.1 with $\mu = [0, 1]$ is the identity for white-rooted trees (and similarly for black-rooted trees). So we have

Theorem 5.1.5 *The collection $BT_w(n)$ of white-rooted Brinkmeier trees with n leaves forms the arity n component of the operad BT_w of white-rooted Brinkmeier trees. Similarly there is the operad BT_b of black-rooted Brinkmeier trees. Further, we have*

$$BT_w \cong \widehat{\mathcal{C}}_2 \cong BT_b$$

where $\cong$ denotes isomorphism of operads.

The isomorphisms above from cubes to trees are given by Construction 3.1.2 while those from trees to cubes are the procedure describe in the beginning of Section 5.1. This relabeling procedure also makes explicit the connection between tethered trees and Brinkmeier trees.

5.2 Relations on $\mathcal{A}_2$

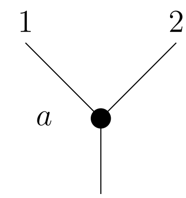
Let $FA_2 := \mathcal{F}ree(\mathcal{A}_2(2))$, where $\mathcal{F}ree(\mathcal{A}_2(2))$ is the free operad on the collection $\{\mathcal{A}_2(2)\}$ (see [Gizburg and Kapranov, 1994] for details). Thus FA_2 is the operad of non-planar, binary trees whose vertices are labeled by elements of $\mathcal{A}_2(2)$. In this section, we prove the following proposition:

Proposition 5.2.1 *$\mathcal{A}_2^f \cong FA_2$, where $\cong$ again denotes isomorphism of operads.*

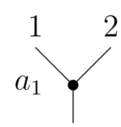
First we point out that both $\mathcal{A}_2$ and $\mathcal{A}_2^f$ are generated by their arity 2 components. A decomposition of an element in $\mathcal{A}_2(n)$ or $\mathcal{A}_2^f(n)$ is given in the obvious way by an arrangement of framing squares placed on that element. This is because a full choice of framing squares creates complementary pairs, each of which correspond to an element in $\mathcal{A}_2(2) = \mathcal{A}_2^f(2)$. Of course there may be multiple arrangements that can be placed on an element of $\mathcal{A}_2(n)$ while there is only one arrangement (the given arrangement) for an element of $\mathcal{A}_2^f(n)$. This implies that there may be several decompositions of an element in $\mathcal{A}_2(n)$ while there is only one decomposition of an element in $\mathcal{A}_2^f(n)$.

Proof Recall that $\mathcal{A}_2(2) = \mathcal{A}_2^f(2)$. Let $\Psi : \mathcal{A}_2^f(2) \to FA_2(2)$ be given by

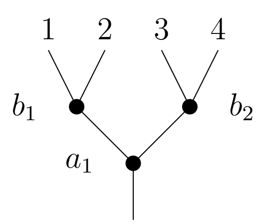
$\Psi(a) \qquad = $



Take $a \in \mathcal{A}_2^f(n)$. Define $\Psi(a)$ as follows: Take the complementary pair of squares in a that contains the largest framing square. Consider the corresponding element $a_1 \in \mathcal{A}_2^f(2)$. The tree $\Psi(a)$ has base subtree



If either of the squares in the complementary pair is a framing square, take the largest complementary pair contained in that framing square. Take the corresponding element $b_i \in \mathcal{A}_2^f(2)$ $(i = 1, 2)$. Attach $\Psi(b_i)$ to $\Psi(a_1)$ on leaf i:



Continue up each branch until a realized square is reached. Label the leaf with the label of this realized square. The process is done when this happens on all branches. The resulting tree is $\Psi(a)$. This of course amounts to taking the unique decomposition of an element in $\mathcal{A}_2^f(n)$ into elements of $\mathcal{A}_2^f(2)$, applying Ψ to these terms, and then composing again. Note that this implies that Ψ is operadic.

Also, this is an invertible process. Starting from the top, cut off a binary subtree with 2 leaves. Its vertex is labeled by the element of $\mathcal{A}_2^f(2)$ which corresponds to the complementary pair of realized squares. Do the same for all top level subtrees with 2 leaves. Next cut off the binary, 2-leafed trees below these and take the labels on their vertex. Compose these elements in $\mathcal{A}_2^f$. Continue down to the base subtree. This gives the original $a \in \mathcal{A}_2^f(n)$. Thus Ψ is injective. It is clear that Ψ is surjective. All elements of $FA_2(n)$ are binary trees with vertices labelled by elements of $\mathcal{A}_2^f(2) = \mathcal{A}_2(2)$. Perform the cutting process described above to get an element of $\mathcal{A}_2^f(n)$. Since this process was inverse to Ψ, we have that Ψ is surjective. ■

We now have the following diagram:

$$\begin{tikzcd}
\mathcal{A}_2^f \arrow[rr, "\cong", "\Psi"'] \arrow[dr, "e"'] & & \mathcal{F}ree(\mathcal{A}_2(2)) \arrow[dl, "\circ"] \\
& \mathcal{A}_2 &
\end{tikzcd}$$

Figure 5.8.

Here e is the map of Chapter 4 which erases framing squares to give an element in $\mathcal{A}_2$, and $\circ$ is the standard map from the free operad to $\mathcal{A}_2$ given by composing. It is clear that this diagram commutes.

Let $\mathbb{P}Cact$ be the operad of spineless projective cacti (see [Kaufmann, 2005]). We claim $\mathcal{F}ree(\mathcal{A}_2(2)) = \mathcal{F}ree(\mathbb{P}Cact(2))$. Actually, we have $\mathcal{A}_2(2) = \mathbb{P}Cact(2)$. Both are CW complexes homeomorphic to S^1 crossed with the open interval $(0, 1)$ (the scaling factor). The identification then gives the claim. Thus we can expand our diagram to the following:

$$\begin{tikzcd}
\mathcal{A}_2^f \arrow[r, "\cong", "\Psi"'] \arrow[dr, "e"'] & \mathcal{F}ree(\mathcal{A}_2(2)) \arrow[r, equal] \arrow[dl, "\circ_A"] & \mathcal{F}ree(\mathbb{P}Cact(2)) \arrow[d, "\circ_C"] \\
& \mathcal{A}_2 & \mathbb{P}Cact
\end{tikzcd}$$

Figure 5.9.

This diagram expresses the connection between tethered 2-cubes and spineless cacti. Further, this appears to be the extent of their connection. While they are both generated by their arity 2 components and these arity 2 components are in fact equal, the full operads are not quasi-isomorphic (and so $\mathcal{A}_2$ is not quasi-isomorphic to $\mathcal{C}_2$). This is due to the fact that tethered 2-cubes have many more relations than spineless cacti. For suppose we had a quasi-isomorphism $h : \mathcal{A}_2 \to \mathbb{P}Cact$ filling in the above diagram as follows:

$$\begin{tikzcd}
\mathcal{A}_2^f \arrow[r, "\cong", "\Psi"'] \arrow[dr, "e"'] & \mathcal{F}ree(\mathcal{A}_2(2)) \arrow[r, equal] \arrow[dl, "\circ_A"] & \mathcal{F}ree(\mathbb{P}Cact(2)) \arrow[d, "\circ_C"] \\
& \mathcal{A}_2 \arrow[r, "h"] & \mathbb{P}Cact
\end{tikzcd}$$

Figure 5.10.

There is little choice as to how such a map could be defined on $\mathcal{A}_2(2)$, since the image space is identical. Suppose we have

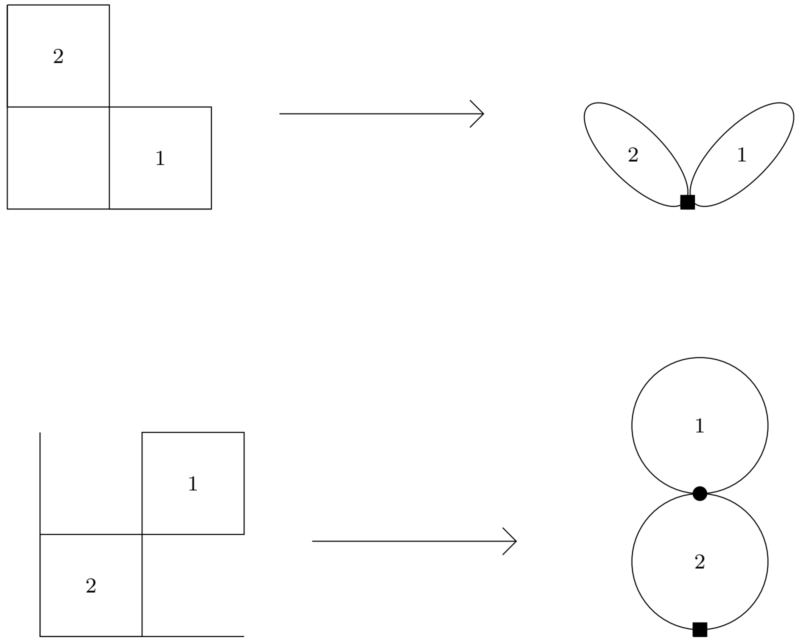


Figure 5.11.

with the rest of h defined similarly. That is, as square 1 moves up and square 2 moves down, lobe 1 moves over lobe 2 at the rate shown, and then we continue on at this same rate. With the map defined as such, we run into the following situation. Consider the element of $\mathcal{A}_2(3)$ pictured below:

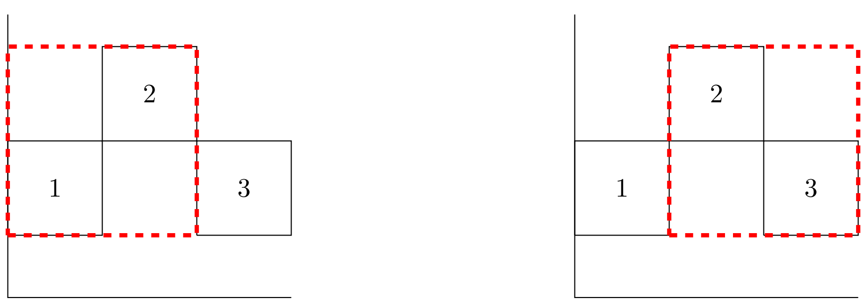


Figure 5.12.

Recall that the framing squares tell us how to decompose the elements of $\mathcal{A}_2$. What we see here is one element of $\mathcal{A}_2(3)$ with two possible framings. If we take the first element, decompose it according to the framing, map over to cacti, and then recompose, we get the following element of $\mathbb{P}Cact(3)$:

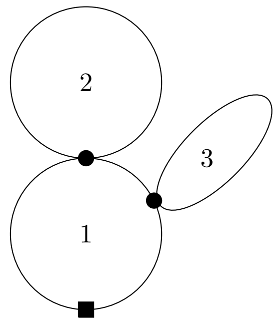


Figure 5.13.

Doing the same with the second element produces

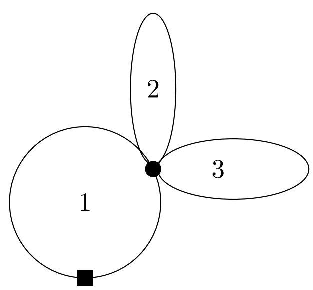


Figure 5.14.

If h was operadic, these would have to be the same element in $\mathbb{P}Cact(3)$. Obviously they are not.

Any variation on this very natural map will have the same issue. Remember that $\mathcal{A}_2(2)$ and $\mathbb{P}Cact(2)$ are topologically the same space (a cylinder). The map h as defined is the identity map. In order for h to be a quasi-isomorphism, there is not much variation in how we can define it. It may include a rotation or some sort of stretching or contracting of the cylinder, but in the end this same problem will always arise. We must conclude that $\mathcal{A}_2$ has many more relations than $\mathbb{P}Cact$.

6. Decorated Feynman Categories

Feynman categories were first defined by Kaufmann and Ward in 2014 [Kaufmann and Ward]. While they are defined more generally, one of their main applications is to provide a categorical framework for studying general operadic theories. The prototypical Feynman category $\mathfrak{G}$ (described below) achieves this by capitalizing on the graph structures that appear in these theories, and this is made precise via Manin's graph formalism.

Cast in this new light, the tools of category theory are readily available in our general analysis of these theories of composition. Common constructions in these theories will appear in the guise of basic categorical tools, and previously arduous arguments will be simplified greatly.

This chapter will consist of three sections. The first will recount the necessary details of Feynman categories. For a full treatment, the interested reader may consult [Kaufmann and Ward]. The second section will consist of original work, defining and utilizing the notion of a decorated Feynman category. We will establish the basic results and constructions. In the closing section, we will use this new setting to present a simple definition of non-Σ modular operads.

6.1 Feynman Categories

For a category $\mathcal{C}$, let $\mathrm{Iso}(\mathcal{C})$ be the wide subcategory whose morphisms consist of only the isomorphisms of $\mathcal{C}$. For two categories $\mathcal{C}$ and $\mathcal{D}$, $(\mathcal{C} \downarrow \mathcal{D})$ denotes the associated comma category. Recall that a category is said to be **essentially small** if it is equivalent to a small category, that is a category whose collections of objects and arrows are sets.

Let $\mathcal{F}$ be a symmetric monoidal category, and let $\mathcal{V}$ be a groupoid with $\mathcal{V}^{\otimes}$ the free symmetric monoidal category generated by $\mathcal{V}$. Let $\iota : \mathcal{V} \to \mathcal{F}$ be a functor with $\iota^{\otimes} : \mathcal{V}^{\otimes} \to \mathcal{F}$ the induced symmetric monoidal functor.

Definition 6.1.1 *[Kaufmann and Ward] The triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ is called a **Feynman category** if it satisfies the following three conditions:*

(i) ***Isomorphism condition**: The symmetric monoidal functor $\iota^{\otimes}$ induces an equivalence of symmetric monoidal categories $\iota^{\otimes} : \mathcal{V}^{\otimes} \to Iso(\mathcal{F})$.*

(ii) ***Hereditary condition**: The symmetric monoidal functor $\iota^{\otimes}$ induces an equivalence of symmetric monoidal categories $\iota^{\otimes} : Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes} \to Iso(\mathcal{F} \downarrow \mathcal{F})$.*

(iii) ***Size condition**: For any object $* \in Ob(\mathcal{V})$, the comma category $(F \downarrow *)$ is essentially small.*

Remark 6.1.2 *We make some comments on the above definition.*

(i) The isomorphism condition provides us with a factorization of objects in $\mathcal{F}$. By fixing a quasi-inverse functor $j : Iso(\mathcal{F}) \to \mathcal{V}^{\otimes}$ to $\iota^{\otimes}$, for any object $X \in Ob(\mathcal{F})$ (recall that the objects of $Iso(\mathcal{F})$ are the same as those of $\mathcal{F}$) we have $X \cong \iota^{\otimes} jX$ where

$$\iota^{\otimes} jX = \iota^{\otimes}(\otimes_{v \in I} *_v) = \otimes_{v \in I} \iota(*_v)$$

with I a finite indexing set.

(ii) Similarly, the hereditary condition provides us with a factorization of arrows in $\mathcal{F}$. For any morphism $\phi : X \to X'$, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[d, "\cong"'] & & X' \arrow[d, "\cong"] \\
\otimes_{v \in I} X_v \arrow[rr, "\otimes_{v \in I} \phi_v"] & & \otimes_{v \in I} \iota(*_v)
\end{tikzcd}$$

where $*_v \in Ob(\mathcal{V})$, $X_v \in Ob(\mathcal{F})$, and $\phi_v \in Hom(X_v, \iota(*_v))$.

(iii) Often it is the case that $\mathcal{V}$ is a subcategory of $\mathcal{F}$ and $\iota : \mathcal{V} \to \mathcal{F}$ is the inclusion functor.

Given two Feynman categories $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ and $\mathfrak{F}' = (\mathcal{V}', \mathcal{F}', \iota')$, a morphism from $\mathfrak{F}$ to $\mathfrak{F}'$ is a pair of functors (v, f) with $v : \mathcal{V} \to \mathcal{V}'$ and monoidal $f : \mathcal{F} \to \mathcal{F}'$. We also require that all necessary structure is preserved, namely, that the functors commute with ι and ι', that $v^{\otimes} : \mathcal{V}^{\otimes} \to \mathcal{V}'^{\otimes}$ is compatible with f, and that the decompositions of objects and arrows described above are preserved. Much of this is summed up in the following diagram:

$$\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d, "v"'] \arrow[rr, bend left, "\iota"] & \mathcal{V}^{\otimes} \arrow[r, "\iota^{\otimes}"] \arrow[d, "v^{\otimes}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{V}' \arrow[r] \arrow[rr, bend right, "\iota'"'] & \mathcal{V}'^{\otimes} \arrow[r, "\iota'^{\otimes}"] & \mathcal{F}'
\end{tikzcd}$$

For a Feynman category $\mathcal{B}$, we say that $\mathfrak{F}$ is **indexed over** $\mathcal{B}$ if there exists a morphism $B : \mathfrak{F} \to \mathcal{B}$ that is surjective on objects.

Fix a symmetric monoidal category $\mathcal{C}$ and a Feynman category $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$. Then we define $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := \mathrm{Hom}_{\otimes}(\mathcal{F}, \mathcal{C})$, the category of strong symmetric monoidal functors from $\mathcal{F}$ to $\mathcal{C}$. We call an object of this category an $\mathcal{F}\text{-}\mathcal{O}p$. These are the generalization of operads to this setting. Similarly we define $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := \mathrm{Hom}(\mathcal{V}, \mathcal{C})$, the category of functors from $\mathcal{V}$ to $\mathcal{C}$. Such a functor is called a $\mathcal{V}$-module, and these are the generalizations of $\mathbb{S}$-modules from operad theory.

For a morphism (v, f) from a Feynman category $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ to another Feynman category $\mathfrak{F}' = (\mathcal{V}', \mathcal{F}', \iota')$, the pull-back of f is the functor $f^* : \mathcal{F}'\text{-}\mathcal{O}ps_{\mathcal{C}} \to \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ given by $f^*(\mathcal{O}) = \mathcal{O} \circ f$. The pull-back v^* of v is defined similarly.

Suppose that $\mathcal{C}$ is cocomplete, so that we can construct the left Kan extension along a functor $f : \mathcal{F} \to \mathcal{F}'$. Also suppose that the monoidal product of $\mathcal{C}$ preserves colimits in both variables. Then the push-forward $f_* : \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} \to \mathcal{F}'\text{-}\mathcal{O}ps_{\mathcal{C}}$ of f is given on objects by

$$f_*(\mathcal{O})(X') = \mathrm{colim}_{(f \downarrow X')} \mathcal{O} \circ P$$

where $X' \in \mathrm{Ob}(\mathcal{F}')$ and P is the projection functor $P(Y, \phi : f(Y) \to X) = Y$ of the comma category. The push-forward v_* is again defined similarly.

Theorem 6.1.3 (Kaufmann-Ward) *[Kaufmann and Ward] The functor $f_*(\mathcal{O})$ is monoidal. Moreover, f_* and f^* form an adjunction of symmetric monoidal functors between symmetric monoidal categories.*

Proof See [Kaufmann and Ward]. ■

The quintessential Feynman category is $\mathfrak{G}$, the Feynman category of graphs. Using Manin's graph formalism, we take a graph to consist of a set of vertices, a set of flags (i.e. half-edges), a function from flags to vertices (that we interpret as assigning a flag to the vertex it is attached to), and an involution on flags. If this involution maps a flag to itself, we call that flag a tail. Otherwise a flag and it's image are called an edge. Other structure can be added, such as directing or ordering edges. See Appendix A of [Kaufmann and Ward] for a detailed account.

By a corolla, we mean a graph with one vertex, and by an aggregate of corollae, we mean a finite disjoint union of corollae. Let Crl be the groupoid of corollae. Here a morphism acts by rearranging flags on a given corolla. Let Agg be the symmetric monoidal category of aggregates of corollae. Its objects are finite disjoint unions of corollae, and its morphisms are graphs. That is, a morphism between two aggregates of corollae act by identifying certain flags to form edges and then contracting those edges to return to a corolla. The figure gives an example.

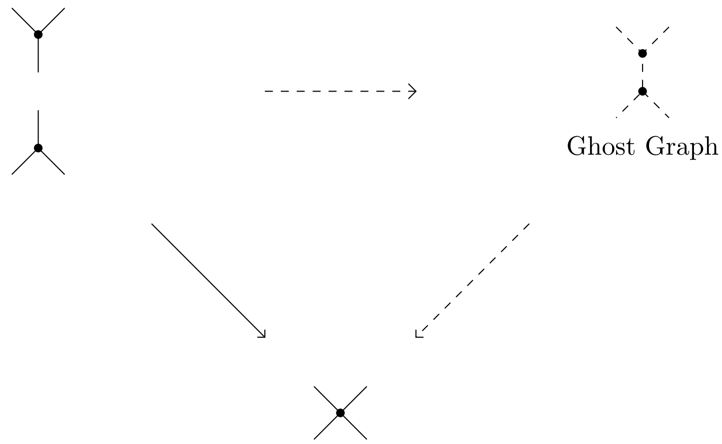


Figure 6.1.

Here the morphism is given by the solid arrow. We sometimes call the graph associated with a morphism the ghost graph. Notice that the graph is a key part of the information of the morphism. Without it, we don't know which flags have been identified, and so we do not know the full morphism. It is also key to note that the ghost graph in the above figure is NOT an object in our category. It is part of the data of a morphism.

Letting $\iota : Crl \to Agg$ be the inclusion functor, we have the Feynman category $\mathfrak{G} = (Crl, Agg, \iota)$ of graphs. Other Feynman categories can now be formed by restricting the types of morphisms we consider (and so the types of graphs we consider) or adding conditions to the corollae (such as making them directed). We will return to specific Feynman categories in Section 6.3

6.2 Decorated Feynman Categories

Decorated Feynman categories formalize the notion of decorating the vertices of a graph with elements of an operad, though, like Feynman categories, their definition is more general than that. In this section we introduce the definition of a decorated Feynman category and prove some basic results.

First we construct the components of a decorated Feynman category. Recall that a concrete category is one whose objects are sets with structure and whose morphisms are functions that preserve that structure. Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$ be a Feynman category, let $\mathcal{C}$ be a fixed concrete symmetric monoidal category, and take $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. First we define a symmetric monoidal category $\mathcal{F}_{dec\mathcal{O}}$. Take an object $X \in \mathrm{Ob}(\mathcal{F})$, and note that since $\mathcal{C}$ is concrete, $\mathcal{O}(X)$ is some set with structure. We take as the objects of $\mathcal{F}_{dec\mathcal{O}}$ the pairs (X, a_X) where $X \in \mathrm{Ob}(\mathcal{F})$ and $a_X \in \mathcal{O}(X)$. The only restriction we place on a_X is that it be an element of $\mathcal{O}(X)$. We define morphisms in $\mathcal{F}_{dec\mathcal{O}}$ as follows: Take two objects (X, a_X) and (Y, a_Y). A morphism from (X, a_X) to (Y, a_Y) consists of a morphism $\phi : X \to Y$ such that $\mathcal{O}(\phi)(a_X) = a_Y$. Note that $\mathcal{O}(\phi)$ is a map of sets. We denote such a morphism by $\phi : (X, a_X) \to (Y, a_Y)$.

Note that we have identity morphisms $id_X : (X, a_X) \to (X, a_X)$ since $\mathcal{O}(id_X)(a_X) = a_X$. We also have the obvious composition rule: Given $\phi : (X, a_X) \to (Y, a_Y)$ and $\psi : (Y, a_Y) \to (Z, a_Z)$, we have that $\mathcal{O}(\psi \circ \phi)(a_X) = \mathcal{O}(\psi) \circ \mathcal{O}(\phi)(a_X) = \mathcal{O}(\psi)(a_Y) = a_Z$. Therefore $\psi \circ \phi$ is a well-defined morphisms from (X, a_X) to (Z, a_Z). Associativity of this composition follows from that of composition in $\mathcal{F}$ and of set maps.

Since $\mathcal{O}$ is a strong symmetric monoidal functor, we have natural isomorphisms $\tau_{X,Y} : \mathcal{O}(X) \otimes_{\mathcal{C}} \mathcal{O}(Y) \to \mathcal{O}(X \otimes_{\mathcal{F}} Y)$. We define $\otimes_{\mathcal{F}_{dec\mathcal{O}}}$ so that

$$(X, a_X) \otimes_{\mathcal{F}_{dec\mathcal{O}}} (Y, a_Y) = (X \otimes_{\mathcal{F}} Y, a_{X \otimes Y})$$

where $a_{X \otimes Y} = \tau(a_X \otimes_{\mathcal{C}} a_Y)$. The definition on morphisms is obvious, and with this definition, $\otimes_{\mathcal{F}_{dec\mathcal{O}}}$ is a functor from $\mathcal{F}_{dec\mathcal{O}} \times \mathcal{F}_{dec\mathcal{O}}$ to $\mathcal{F}_{dec\mathcal{O}}$.

For the identity object, we take the pair $(I_{\mathcal{F}}, e)$, where $I_{\mathcal{F}}$ is the identity object of $\mathcal{F}$ and e is the distinguished element of $\mathcal{O}(I_{\mathcal{F}})$ (the single element if $\mathcal{O}(I_{\mathcal{F}})$ is a one-point set, the identity element if $\mathcal{O}(I_{\mathcal{F}})$ is a ground field k, etc.). With this definition, we have

$$(X, a_X) \otimes (I, e) = (X \otimes I, \tau(a_X \otimes e)) \cong (X, a_X)$$

where the last isomorphism comes from the corresponding isomorphism in the symmetric monoidal category $\mathcal{F}$. Let's check this definition. We must have

$$(X, a_X) \otimes (I, e) = (X \otimes I, a_{X \otimes I}) \cong (X, a_X)$$

where $a_{X\otimes I} = \tau(a_X \otimes e)$. We have the natural isomorphism $\lambda_X^{\mathcal{F}} : X \otimes I \to X$ in $\mathcal{F}$. Applying $\mathcal{O}$ gives the natural isomorphism $\mathcal{O}(\lambda_X^{\mathcal{F}}) : \mathcal{O}(X \otimes I) \to \mathcal{O}(X)$. Since $\mathcal{O}$ is strong symmetric monoidal, it must preserve the left identity isomorphism, that is $\mathcal{O}(\lambda_X^{\mathcal{F}}) = \lambda_{\mathcal{O}(X)}^{\mathcal{C}} \circ \tau_{X,I}^{-1} : \mathcal{O}(X) \otimes \mathcal{O}(I) \to \mathcal{O}(X)$. Under this map, we have

$$\lambda_{\mathcal{O}(X)}^{\mathcal{C}} \circ \tau_{X,I}^{-1}(a_{X\otimes I}) = \lambda_{\mathcal{O}(X)}^{\mathcal{C}}(a_X \otimes e) = a_X$$

Thus $\lambda_X^{\mathcal{F}} : (X, a_X)\otimes(I, e) \to (X, a_X)$ is the left identity isomorphism in $\mathcal{F}_{dec\mathcal{O}}$. Similar arguments give the isomorphisms for the right identity and associativity. Therefore $(\mathcal{F}_{dec\mathcal{O}}, \otimes_{\mathcal{F}_{dec\mathcal{O}}}, (I_{\mathcal{F}}, e), \alpha^{\mathcal{F}}, \lambda^{\mathcal{F}}, \rho^{\mathcal{F}})$ is a symmetric monoidal category.

Next we define a groupoid $\mathcal{V}_{dec\mathcal{O}}$. We take as the objects of $\mathcal{V}_{dec\mathcal{O}}$ pairs $(*, a_*)$ with $* \in \mathrm{Ob}(\mathcal{V})$ and $a_* \in \mathcal{O}(\iota(*))$. Similar to the above, we define a morphism $\phi : (*_v, a_{*_v}) \to (*_w, a_{*_w})$ to be a morphism $\phi : *_v \to *_w$ in $\mathcal{V}$ such that $\mathcal{O}(\iota(\phi))(a_{*_v}) = a_{*_w}$. Identity morphisms and compositions for this category are defined as in $\mathcal{F}_{dec\mathcal{O}}$. Since $\phi : *_v \to *_w$ in $\mathcal{V}$ is an isomorphism and $\mathcal{O}(\iota)$ is a functor, $\mathcal{O}(\iota(\phi))$ is an isomorphism. Thus $\phi : (*_v, a_{*_v}) \to (*_w, a_{*_w})$ is an isomorphism, and so $\mathcal{V}_{dec\mathcal{O}}$ is a groupoid.

Finally we define a functor $\iota_{dec\mathcal{O}} : \mathcal{V}_{dec\mathcal{O}} \to \mathcal{F}_{dec\mathcal{O}}$ by $\iota_{dec\mathcal{O}}(*, a_*) = (\iota(*), a_*)$. For a morphism $\phi : (*_v, a_{*_v}) \to (*_w, a_{*_w})$, we take $\iota_{dec\mathcal{O}}(\phi) = \iota(\phi) : (\iota(*_v), a_{*_v}) \to (\iota(*_w), a_{*_w})$. It is clear that $\iota_{dec\mathcal{O}}$ is a functor, since ι is.

Theorem 6.2.1 $\mathfrak{F}_{dec\mathcal{O}} = (\mathcal{F}_{dec\mathcal{O}}, \mathcal{V}_{dec\mathcal{O}}, \iota_{dec\mathcal{O}})$ *is a Feynman category.*

Proof To prove this, we must check the isomorphism condition, the hereditary condition, and the size condition.

Isomorphism Condition: Let $j : \mathcal{F} \to \mathcal{V}^{\otimes}$ be a quasi-inverse of $\iota^{\otimes}$. Take $\otimes_{v\in I}(*_v, a_{*_v}) \in \mathcal{V}^{\otimes}_{dec\mathcal{O}}$ where I is a finite indexing set. We have $\iota^{\otimes}_{dec\mathcal{O}}(\otimes_{v\in I}(*_v, a_{*_v})) = \otimes_{v\in I}(\iota(*_v), a_{*_v}) \in \mathcal{F}_{dec\mathcal{O}}$. Take $X \in \mathrm{Ob}(\mathcal{F})$. Since j is fixed, X has a decomposition $X \cong \iota^{\otimes} j(X) = \otimes_{v\in I}\iota(*_v)$. This gives a natural isomorphism between the identity functor on $\mathcal{F}$ and $\iota^{\otimes} j$. Call the components of this transformation $\xi_X : X \to \otimes_{v\in I}\iota(*_v)$. Then we have the set map $\mathcal{O}(\xi_X) : \mathcal{O}(X) \to \mathcal{O}(\otimes_{v\in I}\iota(*_v))$. Let $a_{\otimes\iota(*_v)} := \mathcal{O}(\xi_X)(a_X)$ and $\otimes_{v\in I} a_{\iota(*_v)} := \tau^{-1}(a_{\otimes\iota(*_v)})$. We define $j_{dec\mathcal{O}} : \mathcal{F}_{dec\mathcal{O}} \to \mathcal{V}^{\otimes}_{dec\mathcal{O}}$ by $j_{dec\mathcal{O}}(X, a_X) = \otimes_{v\in I}(*_v, a_{*_v})$. Then we have

$$\iota^{\otimes}_{dec\mathcal{O}} j_{dec\mathcal{O}}(X, a_X) = \iota^{\otimes}_{dec\mathcal{O}}(\otimes_{v\in I}(*_v, a_{*_v})) = \otimes_{v\in I}(\iota(*_v), a_{*_v}) = (\otimes_{v\in I}\iota(*_v), a_{\otimes\iota(*_v)}),$$

and $\xi_{(X,a_X)} : (X, a_X) \to \otimes_{v\in I}(\iota(*_v), a_{*_v})$ is a component of a natural isomorphism from the identity on $\mathcal{F}_{dec\mathcal{O}}$ to $\iota^{\otimes}_{dec\mathcal{O}} j_{dec\mathcal{O}}$.

Hereditary Condition: An object in $\mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{V}_{dec\mathcal{O}})^{\otimes}$ is an arrow

$$\otimes_{v\in I}\phi_v : \otimes_{v\in I}(X_v, a_{X_v}) \to \otimes_{v\in I}(\iota(*_v), a_{\iota(*_v)}) \qquad (\dagger)$$

such that each $\phi_v : (X_v, a_{X_v}) \to (\iota(*_v), a_{\iota(*_v)})$ is an isomorphism. Define $K : \mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{V}_{dec\mathcal{O}})^{\otimes} \to \mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{F}_{dec\mathcal{O}})$ on objects so that K maps ($\dagger$) to

$$\phi = \otimes_{v\in I}\phi_v : (\otimes_{v\in I} X_v, a_{\otimes X_v}) \to (\otimes\iota(*_v), a_{\otimes\iota(*_v)})$$

where $a_{\otimes X_v} = \tau(\otimes_{v\in I} a_{X_v})$ and similarly for $a_{\otimes\iota(*_v)}$. To be clear, this is well-defined, since

$$\begin{aligned}\mathcal{O}(\phi)(a_{\otimes X_v}) = \mathcal{O}(\otimes\phi_v)(\tau(\otimes a_{X_v})) &= \tau(\otimes\mathcal{O}(\phi_v)(\otimes a_{X_v})) \\ &= \tau(\otimes\mathcal{O}(\phi_v)(a_{X_v})) \\ &= \tau(\otimes a_{\iota(*_v)}) \\ &= a_{\otimes\iota(*_v)}\end{aligned}$$

where the second equality follows by the naturality of τ. Notice that K is essentially just $\otimes_{\mathcal{F}_{dec\mathcal{O}}}$. Thus we take K equal to $\otimes_{\mathcal{F}_{dec\mathcal{O}}}$. We define K so that it is the identity on morphisms of $\mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{V}_{dec\mathcal{O}})^{\otimes}$.

Recall from Remark 6.1.2 that any morphism $\phi : X \to Y$ in $\mathcal{F}$ can be factored in a specific way. We denote this factorization $\hat{\phi} : X \to Y$. We define $L : \mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{F}_{dec\mathcal{O}}) \to \mathrm{Iso}(\mathcal{F}_{dec\mathcal{O}} \downarrow \mathcal{V}_{dec\mathcal{O}})^{\otimes}$ on objects so that it maps $\phi : (X, a_X) \to (Y, a_Y)$ to

$$(\otimes_{v\in I}\iota(*_v), \tau(\otimes_{v\in I}a_{\iota(*_v)})) \xrightarrow[\xi_X^{-1}]{\cong} (X, a_X) \xrightarrow{\hat{\phi}} (Y, a_Y) \xrightarrow[\xi_Y]{\cong} (\otimes_{w\in J}\iota(*_w), \tau(\otimes_{w\in J}a_{\iota(*_w)}))\ .$$

Here ξ is as in our proof of the isomorphism condition. Denote this composition by $\tilde{\phi} : (\otimes\iota(*_v), a_{\otimes\iota(*_v)}) \to (\otimes\iota(*_w), a_{\otimes\iota(*_w)})$. On morphisms, L sends the left diagram to the right diagram below:

$$\begin{tikzcd}
(X, a_X) \arrow[r, "f"] \arrow[d, "\phi"'] & (X', a_{X'}) \arrow[d, "\psi"] \\
(Y, a_Y) \arrow[r, "g"] & (Y', a_{Y'})
\end{tikzcd}
\qquad
\begin{tikzcd}
(\otimes\iota(*_v), a_{\otimes\iota(*_v)}) \arrow[r, dashed, "\hat{f}"] \arrow[d, "\cong"] \arrow[ddd, bend right, "\tilde{\phi}"'] & (\otimes\iota(*_u), a_{\otimes\iota(*_u)}) \arrow[d, "\cong"] \arrow[ddd, bend left, "\tilde{\psi}"] \\
(X, a_X) \arrow[r, "f"] \arrow[d, "\hat{\phi}"] & (X', a_{X'}) \arrow[d, "\hat{\psi}"] \\
(Y, a_y) \arrow[r, "g"] \arrow[d, "\cong"] & (Y', a_{Y'}) \arrow[d, "\cong"] \\
(\otimes\iota(*_w), a_{\otimes\iota(*_w)}) \arrow[r, dashed, "\hat{g}"] & (\otimes\iota(*_t), a_{\otimes\iota(*_t)})
\end{tikzcd}$$

Here $\hat{f}$ and $\hat{g}$ are the morphisms that make the top and bottom squares commute. The right hand diagram condenses to

$$\begin{tikzcd}
(\otimes\iota(*_v), a_{\otimes\iota(*_v)}) \arrow[r, "\hat{f}"] \arrow[d, "\tilde{\phi}"'] & (\otimes\iota(*_u), a_{\otimes\iota(*_u)}) \arrow[d, "\tilde{\psi}"] \\
(\otimes\iota(*_w), a_{\otimes\iota(*_w)}) \arrow[r, "\hat{g}"] & (\otimes\iota(*_t), a_{\otimes\iota(*_t)})
\end{tikzcd}$$

We must show that K and L are quasi-inverse to each other. Applying KL to $\phi : (X, a_X) \to (Y, a_Y)$ gives $\tilde{\phi} : (\otimes\iota(*_v), a_{\otimes\iota(*_v)}) \to (\otimes\iota(*_w), a_{\otimes\iota(*_w)})$, and these arrows are isomorphic via the pair (ξ_X, ξ_Y). Now applying LK to $\otimes\phi_v : \otimes(X_v, a_{X_v}) \to \otimes(\iota(*_v), a_{\iota(*_v)})$ yields $\tilde{\phi} : (\otimes\iota(*_v), a_{\otimes\iota(*_v)}) \to (\otimes\iota(*_w), a_{\otimes\iota(*_w)})$, and these are isomorphic via ξ as well.

Size Condition: Fix $(*, a_{\iota(*)}) \in \mathcal{V}_{dec\mathcal{O}}$. We must show that $(\mathcal{F}_{dec\mathcal{O}} \downarrow (*, a_{\iota(*)}))$ is essentially small. Let $\mathcal{A}$ be a small category that is equivalent to $(\mathcal{F} \downarrow *)$ with equivalences $\Theta : \mathcal{A} \to (\mathcal{F} \downarrow *)$ and $\Sigma : (\mathcal{F} \downarrow *) \to \mathcal{A}$. An arrow in $(\mathcal{F}_{dec\mathcal{O}} \downarrow (*, a_{\iota(*)}))$ is of the form

$$\begin{tikzcd}
(X, a_X) \arrow[rr, "f"] \arrow[dr, "\phi"'] & & (Y, a_Y) \arrow[dl, "\psi"] \\
& (\iota(*), a_{\iota(*)}) &
\end{tikzcd}$$

and from this we obtain an arrow in $(\mathcal{F} \downarrow *)$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\
& \iota(*) &
\end{tikzcd}$$

Define a category $\tilde{\mathcal{A}}$ as follows: Let

$$\mathrm{Ob}(\tilde{\mathcal{A}}) = \times_{\Sigma(\phi) \in \Sigma(\mathrm{Ob}(\mathcal{F} \downarrow *))} \mathcal{O}(\phi)^{-1}(\mathcal{O}(\iota(*))),$$

that is, pairs $(\Sigma(\phi), a_X)$ where $\mathcal{O}(\phi)(a_X) = a_{\iota(*)}$. For the morphisms of $\tilde{\mathcal{A}}$, between two pairs $(\Sigma(\phi), a_X)$ and $(\Sigma(\psi), a_Y)$, we have all morphisms $\Sigma(f)$ where $f : X \to Y$ with $\mathcal{O}(f)(a_X) = a_Y$. Notice that with these definitions, the collections of objects and morphisms of $\tilde{\mathcal{A}}$ are sets, so that $\tilde{\mathcal{A}}$ is small. We can now define a functor $\tilde{\Sigma}$: $(\mathcal{F}_{dec\mathcal{O}} \downarrow a_{\iota(*)}) \to \tilde{\mathcal{A}}$ by $\tilde{\Sigma}(\phi) = (\Sigma(\phi), a_X)$ for objects where $\phi : (X, a_X) \to (\iota(*), a_{\iota(*)})$ and $\tilde{\Sigma}(f) = \Sigma(f)$ for morphisms. The quasi-inverse $\tilde{\Theta} : \tilde{\mathcal{A}} \to (\mathcal{F}_{dec\mathcal{O}} \downarrow a_{\iota(*)})$ sends a pair $(\Sigma(\phi), a_X)$ to the morphism $\phi : (X, a_X) \to (\iota(*), a_{\iota(*)})$. It sends a morphisms $\Sigma(f)$ to f. We these definitions, it is clear that $\tilde{\Sigma}$ and $\tilde{\Theta}$ are quasi-inverse. Hence $(\mathcal{F}_{dec\mathcal{O}} \downarrow \iota(*))$ is essentially small. ■

6.2.1 Push-Forwards

Theorem 6.2.2 *Take two Feynman categories $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$, $\mathfrak{F}' = (\mathcal{V}', \mathcal{F}', \iota')$, a morphism $(v, f) : \mathfrak{F} \to \mathfrak{F}'$, and $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$. Then there exists a functor $f^{\mathcal{O}} : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_*(\mathcal{O})}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "f^{\mathcal{O}}"] \arrow[d, "forget"'] & \mathfrak{F}'^{f_*(\mathcal{O})}_{dec} \arrow[d, "forget"] \\
\mathfrak{F} \arrow[r, "f"] & \mathfrak{F}'
\end{tikzcd}$$

Here the vertical arrows are the forgetful functors which forget the decorations:

$$f(forget(X, a_X)) = f(X) = forget(f(X), \mu_X(a_X)) = forget(f^{\mathcal{O}}(X, a_X))$$

This functor is natural in $\mathcal{O}$, that is for any morphism $\sigma : \mathcal{O} \to \mathcal{P}$ in $\mathfrak{F}$-$\mathcal{O}ps$ there is a diagram

$$\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "\sigma_{dec}"] \arrow[d, "f^{\mathcal{O}}"'] & \mathfrak{F}_{dec\mathcal{P}} \arrow[d, "f^{\mathcal{P}}"] \\
\mathfrak{F}'_{decf_*(\mathcal{O})} \arrow[r, "\sigma'_{dec}"] & \mathfrak{F}'_{decf_*(\mathcal{P})}
\end{tikzcd}$$

This functor $f^{\mathcal{O}}$ will be called the push-forward of f to the decorated Feynman category $\mathfrak{F}_{dec\mathcal{O}}$.

Proof Recall that $f_*(\mathcal{O})$ is defined by

$$f_*(\mathcal{O})(X') = colim_{(f \downarrow X')} \mathcal{O} \circ P$$

where $(f \downarrow X')$ is the comma category of diagrams

$$\begin{tikzcd}
f(X) \arrow[rr, "f(\xi)"] \arrow[dr, "\varphi"'] & & f(Y) \arrow[dl, "\psi"] \\
& X' &
\end{tikzcd}$$

and P is the projection functor $P : (f \downarrow X') \to \mathcal{F}$ given by

$$P(X, \varphi : f(X) \to X') = X$$

Define $f^{\mathcal{O}}$ on objects as follows:

$$(X, a_X) \in \mathcal{F}_{dec\mathcal{O}} \mapsto (f(X), \mu_X(a_X)) \in \mathcal{F}'_{decf_*(\mathcal{O})}$$

where μ is the natural transformation that is paired with the colimit object $f_*(\mathcal{O})(f(X))$, and μ_X is the specific arrow

$$\mu_X : \mathcal{O} \circ P(X, id : f(X) \to f(X)) \to f_*(\mathcal{O})(f(X))$$

Notice then that $\mu_X(a_X) \in f_*(\mathcal{O})(f(X))$, and so $(f(X), \mu_X(a_X))$ is in fact an object of $\mathcal{F}'_{decf_*(\mathcal{O})}$. For morphisms, we define $f^{\mathcal{O}}$ by

$$\begin{tikzcd}
(X, a_X) \arrow[d, "\varphi"'] & \mapsto & (f(X), \mu_X(a_X)) \arrow[d, "f(\varphi)"'] \\
(Y, a_Y) & & (f(Y), \nu_Y(a_Y))
\end{tikzcd}$$

Here ν is the transformation associated to $f_*(\mathcal{O})(f(Y))$. To see that this is a viable definition, we must check that the right hand side is in fact an arrow in $\mathcal{F}'_{decf_*(\mathcal{O})}$, that is, that $f(\varphi)$ is an arrow from $f(X)$ to $f(Y)$ and that $f_*(\mathcal{O})(f(\varphi))(\mu_X(a_X)) = \nu_Y(a_Y)$. Clearly $f(\varphi)$ is an arrow from $f(X)$ to $f(Y)$. Now we have the colimit diagram

$$\begin{tikzcd}
\mathcal{O} \circ P(X, f(\varphi) : f(X) \to f(Y)) \arrow[rr, "\mathcal{O}(\varphi)"] \arrow[dr, "\nu_X"'] \arrow[dd, equal] & & \mathcal{O} \circ P(Y, id : f(Y) \to f(Y)) \arrow[dl, "\nu_Y"] \\
& f_*(\mathcal{O})(f(Y)) & \\
& f_*(\mathcal{O})(f(X)) \arrow[u, dotted] & \\
\mathcal{O} \circ P(X, id : f(X) \to f(X)) \arrow[ur, "\mu_X"] & &
\end{tikzcd}$$

Here the dotted arrow is $f_*(\mathcal{O})(f(\varphi))$, induced by $\nu_X : \mathcal{O}(X) \to f_*(\mathcal{O})(f(Y))$. Take $a_X \in \mathcal{O}(X)$ at the bottom of the diagram. Following the arrows up gives

$f_*(\mathcal{O})(f(\varphi))(\mu_X(a_X))$. Following the equality, going right across the top, and then going down gives $\nu_Y(\mathcal{O}(\varphi)(a_X)) = \nu_Y(a_Y)$. Thus $f^{\mathcal{O}}$ is well defined on arrows. That $f^{\mathcal{O}}$ respects compositions and identities (and is therefore a functor) follows immediately from the fact that f is a functor and the nature of composition and identities in decorated Feynman categories. That $f^{\mathcal{O}}$ is monoidal again follows from the same being true for f and the monoidal structure on decorated Feynman categories. Hence we have a monoidal functor $f^{\mathcal{O}} : \mathcal{F}_{dec\mathcal{O}} \to \mathcal{F}'_{decf_*(\mathcal{O})}$.

The construction of a functor $v^{\mathcal{O}} : \mathcal{V}_{dec\mathcal{O}} \to \mathcal{V}'_{decf_*(\mathcal{O})}$ is nearly identical. We must only remember that an object in $\mathcal{V}_{dec\mathcal{O}}$ is a pair $(*_w, a_{*_w})$ where $a_{*_w} \in \mathcal{O}(\iota(*_w))$. Thus we can define $v^{\mathcal{O}}$ on objects by sending $(*_w, a_{*_w})$ to $(v(*_w), \mu_{*_w}(a_{*_w}))$ where $\mu_{*_w} = \mu_{\iota(*_w)}$ from above (recall $\iota(*_w)$ is an object of $\mathcal{F}$) and on morphisms by

$$\begin{tikzcd}
(*_w, a_{*_w}) \arrow[d, "\varphi"'] & \mapsto & (v(*_w), \mu_{*_w}(a_{*_w})) \arrow[d, "v(\varphi)"'] \\
(*_z, a_{*_z}) & & (v(*_z), \nu_{*_z}(a_{*_z}))
\end{tikzcd}$$

where again $\nu_{*_z} = \nu_{\iota(*_z)}$ from above. The proof that $v^{\mathcal{O}}$ is a well-defined functor is now identical to that of $f^{\mathcal{O}}$.

The necessary compatibilities of $v^{\mathcal{O}}$ and $f^{\mathcal{O}}$ with all relevant structure will follow readily. Thus we have a functor between Feynman categories $(v^{\mathcal{O}}, f^{\mathcal{O}}) : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_*(\mathcal{O})}$. We will usually denote this functor simply by $f^{\mathcal{O}}$.

Take $\mathcal{O}, \mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ and consider a natural transformation $\sigma : \mathcal{O} \to \mathcal{P}$. Then σ induces a functor $\sigma_{\mathcal{F}dec} : \mathcal{F}_{dec\mathcal{O}} \to \mathcal{F}_{dec\mathcal{P}}$ via

$$(X, a_X) \mapsto (X, \sigma_X(a_X))$$

(here σ_X is the component arrow $\sigma_X : \mathcal{O}(X) \to \mathcal{P}(X)$ of the natural transformation). For $\phi : (X, a_X) \to (Y, a_Y)$, we let $\sigma_{\mathcal{F}dec}(\phi) = \mathcal{P}(\phi)$. Since

$$\mathcal{P}(\phi)(\sigma_X(a_X)) = \sigma_Y(\mathcal{O}(\phi)(a_X)) = \sigma_Y(a_Y)$$

(because $\sigma : \mathcal{O} \to \mathcal{P}$ is natural), $\mathcal{P}(\phi)$ is in fact an arrow from $(X, \sigma_X(a_X)) = \sigma_{\mathcal{F}dec}(X, a_X)$ to $(Y, \sigma_Y(a_Y)) = \sigma_{\mathcal{F}dec}(Y, a_Y)$. By our usual abuse of notation, we will denote $\sigma_{\mathcal{F}dec}(\phi) = \mathcal{P}(\phi)$ by simply

$$\phi : (X, \sigma_X(a_X)) \to (Y, \sigma_Y(a_Y))$$

as an arrow in $\mathcal{F}_{dec\mathcal{P}}$. Respecting identities and composition follows from the commutativity diagrams for the naturality of σ.

The transformation σ also induces a functor $\sigma_{\mathcal{V}dec} : \mathcal{V}_{dec\mathcal{O}} \to \mathcal{V}_{dec\mathcal{P}}$. Let $\sigma_{*_v} := \sigma_{\iota(*_v)} : \mathcal{O}(\iota(*_v)) \to \mathcal{P}(\iota(*_v))$. We define $\sigma_{\mathcal{V}dec}$ on objects of $\mathcal{V}_{dec\mathcal{O}}$ by

$$\sigma_{\mathcal{V}dec}(*_v, a_{*_v}) = (*_v, \sigma_{*_v}(a_{*_v}))$$

and on arrows by

$$\sigma_{\mathcal{V}dec}(\phi : (*_v, a_{*_v}) \to (*_w, a_{*_w})) = \phi : (*_v, \sigma_{*_v}(a_{*_v})) \to (*_w, \sigma_{*_w}(a_{*_w}))$$

where on the right-hand side ϕ follows our usual abuse of notation of the arrow in $V_{dec\mathcal{P}}$ induced by $\phi : *_v \to *_w$ such that $\mathcal{P}(\iota(\phi))$ maps one decoration to the other. As above the functorial axioms follow from the diagrams for the natural transformation σ.

Now taking $\sigma_{dec} = (\sigma_{\mathcal{V}dec}, \sigma_{\mathcal{F}dec}) : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}_{dec\mathcal{P}}$ gives a functor between Feynman categories. The necessary compatibility conditions (e.g. with $\iota_{dec\mathcal{O}}$, $\iota_{dec\mathcal{P}}$) will follow readily.

We also have an induced functor $\sigma'_{dec} : \mathfrak{F}'_{decf_*(\mathcal{O})} \to \mathfrak{F}'_{decf_*(\mathcal{P})}$. Namely, the natural transformation $\sigma : \mathcal{O} \to \mathcal{P}$ induces a natural transformation $\sigma' : f_*(\mathcal{O}) \to f_*(\mathcal{P})$, and this second natural transformation induces the functor σ'_{dec} in the same manner as above. The induced transformation σ' comes from the colimit diagrams for $f_*(\mathcal{O})$ and $f_*(\mathcal{P})$. We have the diagram

$$
\begin{tikzcd}
\mathcal{O} \circ Pr(X, \phi : f(X) \to X') \arrow[rr, "\mathcal{O}\circ\mathrm{Pr}(\xi)"] \arrow[dr, "\mu_X"] \arrow[dd, "\sigma_X"'] & & \mathcal{O} \circ Pr(Y, \psi : f(Y) \to X') \arrow[dl, "\mu_Y"'] \arrow[dd, "\sigma_Y"] \\
& f_*(\mathcal{O})(X') \arrow[dd, dotted, "\sigma'_{X'}"] & \\
\mathcal{P} \circ Pr(X, \phi : f(X) \to X') \arrow[rr, "\mathcal{P}\circ\mathrm{Pr}(\xi)"] \arrow[dr, "\nu_X"] & & \mathcal{P} \circ Pr(Y, \psi : f(Y) \to X') \arrow[dl, "\nu_Y"'] \\
& f_*(\mathcal{P})(X') &
\end{tikzcd}
$$

Here $\xi : X \to Y$ is such that

$$
\begin{tikzcd}
f(X) \arrow[rr, "f(\xi)"] \arrow[dr, "\phi"'] & & f(X) \arrow[dl, "\psi"] \\
& X' &
\end{tikzcd}
$$

The dotted vertical arrow is the component arrow $\sigma'_{X'} : f_*(\mathcal{O})(X') \to f_*(\mathcal{P})(X')$ of the natural transformation σ'.

We now claim that the diagram

$$
\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "\sigma_{dec}"] \arrow[d, "f^{\mathcal{O}}"'] & \mathfrak{F}_{dec\mathcal{P}} \arrow[d, "f^{\mathcal{P}}"] \\
\mathfrak{F}'_{decf_*(\mathcal{O})} \arrow[r, "\sigma'_{dec}"] & \mathfrak{F}'_{decf_*(\mathcal{P})}
\end{tikzcd}
$$

commutes. This follows directly from the previous colimit diagram:

$$
\begin{tikzcd}
(X, a_X) \arrow[rr, "\sigma_X"] \arrow[d, "f^{\mathcal{O}}"'] & & (X, \sigma_X(a_X)) \arrow[d, "f^{\mathcal{P}}"] \\
(f(X), \mu_X(a_X)) \arrow[r, "\sigma'_{f(X)}"] & (f(X), \sigma'_{f(X)}(\mu_X(a_X))) = (f(X), \nu_X(\sigma_X(a_X))) &
\end{tikzcd}
$$

This diagram shows the commutativity for objects. Arrows are similar. $\blacksquare$

With the push-forward constructed, it is obvious that the diagram

$$\begin{tikzcd} \mathfrak{F}_{dec\mathcal{O}} \arrow[r, "f^{\mathcal{O}}"] \arrow[d, "forget"'] & \mathfrak{F}'^{f_*(\mathcal{O})}_{dec} \arrow[d, "forget"] \\ \mathfrak{F} \arrow[r, "f"] & \mathfrak{F}' \end{tikzcd}$$

commutes.

6.3 Non-Σ Modular Operads

One thing we immediately gain from decorated Feynman categories is a simple formulation of non-Σ modular operads, as follows.

Consider the Feynman category $\mathfrak{O} = (Crl^{rt}, Agg^{rt}, \iota)$ of rooted corollae whose morphisms are rooted trees. This is the Feynman category of operads, in the sense that functors out of $\mathfrak{O}$ are classical non-unital pseudo-operads. The details may be found in [Kaufmann and Ward]. We may now consider the associative operad $Assoc$ whose elements can be thought of as linear orderings. Consider the decorated Feynman category $\mathfrak{O}_{decAssoc}$. It consists of rooted corollae paired with linear orders, and we can think of these orders as being on the non-root flags of the corollae. The morphisms consists of rooted trees that respect the linear orders of the corollae in the usual operadic sense and so have linear orders on their leaves. Thus we see that $\mathfrak{O}_{decAssoc}$ is the Feynman category of non-Σ operads.

Similarly, we can take the Feynman category $\mathfrak{C} = (Crl, Cyc, \iota)$ of cyclic operads. Here Cyc is the symmetric monoidal category of aggregates of corollae whose morphisms are (undirected) trees. There is an inclusion functor $inc : \mathfrak{O} \to \mathfrak{C}$ which forgets the direction on edges, and we have

$$inc_*(Assoc) = CAssoc,$$

that is, the push forward of the associative operad under the inclusion functor is the cyclic associative operad. The elements of its arity n components can be thought of as cyclic orders of 1 through n. As in the previous paragraph, we see that $\mathfrak{C}_{decCAssoc}$ is the Feynman category of non-Σ cyclic operads. Its objects consist of unrooted corollae along with cyclic orderings of their flags.

Finally, consider $\mathfrak{M} = (Crl^{\gamma}, Agg^{\gamma,ct}, \iota)$, the Feynman category of modular operads. Here Crl^{γ} consists of corollae with genus labels γ on their vertices and $Agg^{\gamma,ct}$ is the category of aggregates of such corollae with morphisms represented by general connected graphs. We have another inclusion functor $i : \mathfrak{C} \to \mathfrak{M}$ which applies a label of 0 to all vertices, and the push forward of i, when it exists, is the modular envelope, i.e. if $\mathcal{O}$ is a $\mathfrak{C}$-$\mathcal{O}p$ and so a cyclic operad, then the $\mathfrak{M}$-$\mathcal{O}p$ $i_*(\mathcal{O})$ is the modular envelope of $\mathcal{O}$. Now

$$i_*(CAssoc) = MAssoc,$$

the modular associative operad. The elements of $MAssoc(n)$ can be thought of as polycyclic orders of 1 through n, that is, a partition of 1 through n and a cyclic order on each group of that partition. Following the previous pattern, $\mathfrak{M}_{decMAssoc}$ is then the Feynman category of non-Σ modular operads.

These results for operads and cyclic operads, while nice, are not of particular interest. Non-Σ operads and non-Σ cyclic operads are not particularly hard to define, and so an alternate formulation is perhaps not that exciting. However, this is not the case for non-Σ modular operads, which until very recently have been difficult to define. This fairly straightforward definition agrees with Markl's in Markl. As described in that paper, an attempt to push cyclic orders to the modular operad setting will fail, since in addition to the standard composition operations, modular operads also have contraction operations. These contractions do not behave well with cyclic orderings, and so the process breaks down. The solution, as Markl describes it, is to introduce polycyclic orders on finite sets. The definition presented above, while requiring the cost of understanding decorated Feynman categories, saves a great deal of technical work in defining these objects.

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APPENDIX

A. Appendix

This appendix is entirely devoted to the proof of Lemma 4.1.3, restated here.

Lemma A.0.1 *An element $\tau \in \mathcal{A}_2^f(3)$ can be given a new framing if and only if the square not contained in the framing square touches some square that is contained in the framing square.*

Note that the two squares contained in the framing square must touch in at least one point, so this statement is equivalent to saying that a framing square can be shifted if and only one square touches the other two squares.

Proof If the unframed square does not touch either of the framed squares in at least one point, then we cannot shift the framing square. If we were to frame the previously unframed square with one of the other two, the resulting framing rectangle would in fact not be a square, and so the arrangement would not be an element of $\mathcal{A}_2^f(3)$. Thus the forward direction is true by contrapositive.
For the backwards direction, we can reduce the general case to the following:

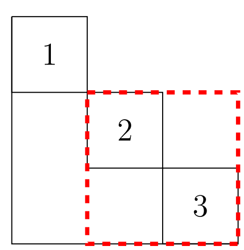

Figure A.1.

Let x_i be the side length of the square labeled by i. Then $x_1 + x_2 + x_3 = 1$, and the framing square (which we may refer to as square 2-3) has side length $x_2 + x_3$. Let

Figure A.1 correspond to the parameters $t = 0, s = 0$. As t goes to 1, we slide square 1 down and the framing square up. As s goes to 1, we slide square 2 down and square 3 up. The other possible movements we could make (sliding squares left or right) and the other framing clearly reduce to this one, i.e. if we can show that we can shift the framing square if and only if square 1 touches square 2 as the squares traverse along these paths, then we can show it in the other situations.

At time t, the centers of the squares are the following points:

- Center of square 1: $(\frac{x_1}{2}, 1 - \frac{x_1}{2} - (x_2 + x_3)t)$
- Center of framing square 2-3: $(1 - \frac{x_2+x_3}{2}, \frac{x_2+x_3}{2} + x_1 t)$
- Center of square 2: $(x_1 + \frac{x_2}{2}, \frac{x_2+x_3}{2} + x_1 t + \frac{x_3}{2} - x_3 s)$
- Center of square 3: $(1 - \frac{x_3}{2}, \frac{x_2+x_3}{2} + x_1 t - \frac{x_2}{2} + x_2 s)$

We are proving that when square 1 touches square 2, the framing square can be shifted. The condition that square 1 touches square 2 can be expressed in one of four inequalities.

(a) $1 - x_1 - (x_2 + x_3)t \leq x_2 + x_3 + x_1 t - x_3 s$ and $1 - (x_2 + x_3)t \geq x_2 + x_3 + x_1 t - x_3 s$ (this states that the bottom of square 1 is below or even with the top of square 2 and the top of square 1 is above or even with the top of square 2)

(b) $1 - x_1 - (x_2 + x_3)t \geq x_3 + x_1 t - x_3 s$ and $1 - (x_2 + x_3)t \leq x_2 + x_3 + x_1 t - x_3 s$ (the bottom of square 1 is above or even with the bottom of square 2 and the top of square 1 is below or even with the top of square 2)

(c) $1 - x_1 - (x_2 + x_3)t \leq x_3 + x_1 t - x_3 s$ and $1 - (x_2 + x_3)t \geq x_3 + x_1 t - x_3 s$ (the bottom of square 1 is below or even with the bottom of square 2 and the top of square 1 is above or even with the bottom of square 2)

(d) $1 - x_1 - (x_2 + x_3)t \leq x_3 + x_1 t - x_3 s$ and $1 - (x_2 + x_3)t \geq x_2 + x_3 + x_1 t - x_3 s$ (the bottom of square 1 is below or even with the bottom of square 2 and the top of square 1 is above or even with the top of square 2)

These conditions can be restated as follows:

(a') $0 \leq t - x_3 s \leq x_1$

(b') $x_1 \leq t - x_3 s \leq x_2$

(c') $x_2 \leq t - x_3 s \leq x_1 + x_2$

(d') $x_2 \leq t - x_3 s \leq x_1$

Let us consider for a moment what the arrangements with square 3 as the unframed square look like.

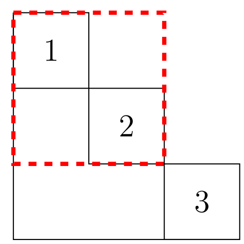

Figure A.2.

Here we use parameters r and q where Figure A.2 displays the arrangement when $r = q = 0$. As r goes to 1, square 3 rises and framing square 1-2 falls. As q goes to 1, square 2 rises and square 1 falls. In this situation, the centers of the square are given by

- Center of square 3: $(1 - \frac{x_3}{2}, \frac{x_3}{2} + (x_1 + x_2)r)$
- Center of framing square 1-2: $(\frac{x_1+x_2}{2}, 1 - \frac{x_1+x_2}{2} - x_3 r)$
- Center of square 1: $(\frac{x_1}{2}, 1 - \frac{x_1+x_2}{2} - x_3 r + \frac{x_2}{2} - x_2 q)$
- Center of square 2: $(x_1 + \frac{x_2}{2}, 1 - \frac{x_1+x_2}{2} - x_3 r - \frac{x_1}{2} + x_1 q)$

The remainder of the proof will proceed as follows. First, assuming that we can shift the framing square (so that the actual squares are in the same positions with either framing square), we will solve for r and q in terms of t and s. Next, we will assume one of the conditions a',...,d'. From the range of values for t and s that this creates, we will get a range of values for r and q. If all the values obtained in this way give an arrangement that is contained in $\mathcal{A}_2^f(3)$, i.e. if $0 \le r, q \le 1$, we will be done. Equating the center of square 3 in the two arrangements gives

$$\begin{aligned}
&\frac{x_2+x_3}{2} + x_1 t - \frac{x_2}{2} + x_2 s = \frac{x_3}{2} + (x_1+x_2)r \\
&\Rightarrow x_1 t + x_2 s = (x_1+x_2)r \\
&\Rightarrow r = \frac{x_1 t + x_2 s}{x_1+x_2}.
\end{aligned}$$

Equating the center of square 1 gives

$$\begin{aligned}
&1 - \frac{x_1}{2} - (x_2+x_3)t = 1 - \frac{x_1+x_2}{2} - x_3 r + \frac{x_2}{2} - x_2 q \\
&\Rightarrow -(x_2+x_3)t = -x_3 r - x_2 q \\
&\Rightarrow q = \frac{x_2+x_3}{x_2} t - \frac{x_3}{x_2} r.
\end{aligned}$$

Further simplification gives $q = \frac{t - x_3 s}{x_1+x_2}$.

Now assume condition a', that $0 \le t - x_3 s \le x_1$. Then $x_3 s \le t \le x_1 + x_3 s$, which implies

$$0 \le \tfrac{x_1 x_3 s + x_2 s}{x_1+x_2} \le r \le \tfrac{x_1^2 + x_1 x_3 s + x_2 s}{x_1+x_2} < 1$$

and

$$0 = \tfrac{x_3 s - x_3 s}{x_1+x_2} \le q \le \tfrac{x_1}{x_1+x_2} < 1.$$

To see that $\frac{x_1^2 + x_1 x_3 s + x_2 s}{x_1+x_2} < 1$, note the following string of equivalent statements:

$$\begin{aligned}
\frac{x_1^2 + x_1 x_3 s + x_2 s}{x_1+x_2} < 1 &\iff x_1^2 + x_1 x_3 s + x_2 s < x_1 + x_2 \\
&\iff x_1^2 + x_1 x_3 s - x_1 + x_2 s - x_2 < 0 \\
&\iff x_1(x_1 + x_3 s - 1) + (s-1)x_2 < 0 \\
&\iff x_1(x_3 s - x_2 - x_3) + (s-1)x_2 < 0 \\
&\iff x_1 x_3 s - x_1 x_3 - x_1 x_2 + (s-1)x_2 < 0 \\
&\iff (s-1)x_1 x_3 + (s-1)x_2 - x_1 x_2 < 0 \\
&\iff (s-1)(x_1 x_3 + x_2) - x_1 x_2 < 0
\end{aligned}$$

This final statement is true since $s-1$ is negative and $x_1x_3+x_2$ and x_1x_2 are positive. Since r and q must have values between 0 and 1, we can shift the framing square under condition a'. The arguments for conditions b',c', and d' are very similar. ∎

VITA

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