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**A SET OF ORTHOGONAL POLYNOMIALS INDUCED
BY A GIVEN ORTHOGONAL POLYNOMIAL**

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A SET OF ORTHOGONAL POLYNOMIALS INDUCED BY A GIVEN ORTHOGONAL POLYNOMIAL*

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ABSTRACT. Given an integer $n \geq 1$, and the orthogonal polynomial $\pi_n(\cdot; d\sigma)$ of degree n relative to some positive measure $d\sigma$, the polynomial system “induced” by π_n is the system of orthogonal polynomials $\{\hat{\pi}_{k,n}\}$ corresponding to the modified measure $d\hat{\sigma}_n = \pi_n^2 d\sigma$. Our interest here is in the problem of determining the coefficients in the three-term recurrence relation for the polynomials $\hat{\pi}_{k,n}$ from the recursion coefficients of the orthogonal polynomials belonging to the measure $d\sigma$. A stable computational algorithm is proposed, which uses a sequence of *QR* steps with shifts. For all four Chebyshev measures $d\sigma$, the desired coefficients are obtained analytically in closed form, which in part reproduces (with different methods) results obtained previously by Al-Salam, Allaway and Askey via sieved orthogonal polynomials, and by Van Assche and Magnus via polynomial transformations. Interlacing properties involving the zeros of π_n and those of $\hat{\pi}_{n+1,n}$ are studied for Gegenbauer measures, as well as the orthogonality – or lack thereof – of the polynomial sequence $\{\hat{\pi}_{n,n-1}\}$.

1. INTRODUCTION

Let $d\sigma$ be a positive measure on $\mathbb{R}$ (with infinitely many points of increase) having finite moments of all orders, and let $\{\pi_m\}$ be the corresponding (monic) orthogonal polynomials,

$$\pi_m(t) = \pi_m(t; d\sigma), \quad m = 0, 1, 2, \dots \quad (1.1)$$

They are known to satisfy a three-term recurrence relation

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$$\begin{aligned}\pi_{k+1}(t) &= (t - \alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, 2, \dots, \\ \pi_0(t) &= 1, \quad \pi_{-1}(t) = 0,\end{aligned}\tag{1.2}$$

where $\alpha_k = \alpha_k(d\sigma) \in \mathbb{R}$, $\beta_k = \beta_k(d\sigma) > 0$, and by convention, $\beta_0 = \beta_0(d\sigma) = \int_{\mathbb{R}} d\sigma(t)$.

For a fixed $n \geq 1$, we define

$$d\hat{\sigma} = d\hat{\sigma}_n := [\pi_n(t; d\sigma)]^2 d\sigma(t).\tag{1.3}$$

The set of orthogonal polynomials we wish to study is

$$\hat{\pi}_k(t) = \hat{\pi}_{k,n}(t) = \pi_k(t; d\hat{\sigma}_n), \quad k = 0, 1, 2, \dots$$

The existence of $\{\hat{\pi}_k\}$ is assured, since $d\hat{\sigma}$ is a positive measure satisfying the same assumptions as those made for $d\sigma$. We consider the following problem:

Given the recursion coefficients $\alpha_k(d\sigma), \beta_k(d\sigma)$ for $d\sigma$, determine the recursion coefficients $\hat{\alpha}_{k,n} = \alpha_k(d\hat{\sigma}_n)$, $\hat{\beta}_{k,n} = \beta_k(d\hat{\sigma}_n)$ for $d\hat{\sigma}_n$.

Modification of a measure $d\sigma$ by multiplying it with the square of a polynomial occurs, for example, in constrained polynomial least squares approximation, where the approximant is required to agree at certain given points exactly with the function to be approximated (cf. [4, §8]). Other modifications, involving more general multipliers, are discussed in [11], where the class of Bernstein-Szegő weights is significantly generalized, and in [12], [10] in connection with the asymptotic theory – an extension of Szegő's theory – of orthogonal polynomials. The particular modification in (1.3) has been proposed by Bellen [2] to provide a source of additional interpolation points – the zeros of $\hat{\pi}_{n+1,n}$ – in the process of extending Lagrange interpolation at the zeros of π_n . When $d\sigma$ is a Chebyshev measure of the first or second kind, the polynomials $\hat{\pi}_k$ also arise in the work of Al-Salam, Allaway and Askey [1] on sieved ultraspherical polynomials, and in the related work of Van Assche and Magnus [13] on new orthogonal polynomials generated by a polynomial mapping (see also [8]). The problem stated above, in these special cases, has been completely solved in [13]. For polynomials π_n other than those of Chebyshev, the measure $d\hat{\sigma}$ in (1.3) appears to define a new

set of orthogonal polynomials, which we term “induced orthogonal polynomials”, or more precisely, “orthogonal polynomials induced by the (fixed) polynomial π_n ”.

In §2, we give an algorithm based on successive QR decompositions with shifts, which, for any given $N \geq 1$, constructs $\hat{\alpha}_{j,n}, \hat{\beta}_{j,n}, j = 0, 1, \dots, N-1$. In §3 we treat the Chebyshev measures $d\sigma$ of all four kinds, and for those of the first and second kind provide new (elementary) derivations of the results in [13]. They are here phrased in terms of monic, rather than normalized, polynomials. The zeros of $\pi_n(\cdot) = \pi_n(\cdot; d\sigma)$ and those of $\hat{\pi}_{n+1,n}(\cdot) = \pi_{n+1}(\cdot; d\hat{\sigma}_n)$, in particular their interlacing properties, will be the topic of §4. In §5 we will show that, for any Chebyshev measure, $\{\hat{\pi}_{n,n-1}\}$ is a sequence of polynomials orthogonal relative to a measure of Bernstein-Szegő type, but give reasons to believe that for other Gegenbauer or Jacobi measures, they do not form an orthogonal system.

2. COMPUTATIONAL METHOD

Let $J = J(d\sigma)$ be the Jacobi matrix for the measure $d\sigma$, i.e., the infinite symmetric tridiagonal matrix

$$J = J(d\sigma) = \text{tri}(\alpha_0, \alpha_1, \alpha_2, \dots, \sqrt{\beta_1}, \sqrt{\beta_2}, \dots) \quad (2.1)$$

with the recursion coefficients $\alpha_k = \alpha_k(d\sigma)$ (cf. (1.2)) down the main diagonal, and the $\sqrt{\beta_k} = \sqrt{\beta_k(d\sigma)}$ down the two side diagonals. Similarly, let

$$\hat{J} = J(d\hat{\sigma}) = \text{tri}(\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2, \dots, \sqrt{\hat{\beta}_1}, \sqrt{\hat{\beta}_2}, \dots) \quad (2.2)$$

be the Jacobi matrix for the measure $d\hat{\sigma}$ (cf. (1.3)). By $J_m = J_m(d\sigma)$ and $\hat{J}_m = J_m(d\hat{\sigma})$ we denote the leading principal minor matrices of order m of $J(d\sigma)$ and $J(d\hat{\sigma})$, respectively.

Our computational method is based on the following observation, due to Kautsky and Golub [9]. Given the Jacobi matrix $J_{m+1}(d\sigma)$ and a number $\tau \in \mathbb{R}$, one can obtain the Jacobi matrix of order m , $J_m((\cdot - \tau)^2 d\sigma)$, for the measure $(t - \tau)^2 d\sigma(t)$ by one step of the QR algorithm with shift τ :

Let Q (orthogonal) and R (upper triangular with $\text{diag } R \geq 0$) be the matrices of order $m+1$ such that

$$J_{m+1}(d\sigma) - \tau I_{m+1} = QR; \quad (2.3)$$

then

$$J_m((\cdot - \tau)^2 d\sigma) = (RQ + \tau I_{m+1})_{m \times m}, \quad (2.4)$$

where, as indicated by the subscript, the last row and last column of the transformed matrix are discarded.

We define

$$d\hat{\sigma}(t; j) = \prod_{\nu=1}^j (t - \tau_\nu)^2 d\sigma(t), \quad j = 0, 1, 2, \dots, n, \quad (2.5)$$

where τ_ν are the zeros of $\pi_n(\cdot; d\sigma)$ and the empty product is defined to be 1. It follows immediately from (2.5) that

$$d\hat{\sigma}(t; j) = (t - \tau_j)^2 d\hat{\sigma}(t; j-1), \quad j = 1, 2, \dots, n; \quad d\hat{\sigma}(\cdot; 0) = d\sigma, \quad d\hat{\sigma}(\cdot; n) = d\hat{\sigma}, \quad (2.6)$$

and hence, from (2.3), (2.4), that $\hat{J}_N = J_N(d\hat{\sigma})$ can be obtained by the following algorithm:

Algorithm

```

For j = 1, 2, ..., n do
  let
    Jold = Jn+N-j+1(d $\hat{\sigma}(\cdot; j-1)$ )
  do QR step with shift  $\tau_j$ 
    Jold -  $\tau_j I$  = QR
    Jnew = (RQ +  $\tau_j I$ )(n+N-j) \times (n+N-j)
  end
end

```

Upon exit, J_{new} will be $\hat{J}_N$. For the individual QR steps, one can use the algorithm in [14, p. 567].

Comparison of double-precision with single-precision results suggests that the algorithm is remarkably stable, numerically. This is also confirmed by comparing the computed coefficients with the exact ones, given below in Theorems 3.2, 3.3, 3.5 and 3.7, for the three Chebyshev measures $d\sigma^{[i]}$, $i = 1, 2, 3$ (cf. (3.1)). Indeed, if $E_{n,\alpha}^{[i]} = \max_{0 \leq k \leq N-1} |\hat{\alpha}_{k,n}^{[i]} - \hat{\alpha}_{k,n}^{[i]*}|$, where the starred quantities are those computed (in single precision on the Cyber 205; machine precision $\approx 7 \times 10^{-15}$) and $\hat{\alpha}_{k,n}^{[i]} = \alpha_k(d\hat{\sigma}_n^{[i]})$ (cf. (3.2)), and if $E_{n,\beta}^{[i]}$

similarly denotes the maximum error in the first N of the β -coefficients, then, for $N = 320$, we have observed the results given in Table 2.1.

n	$E_{n,\beta}^{[1]}$	$E_{n,\beta}^{[2]}$	$E_{n,\alpha}^{[3]}$	$E_{n,\beta}^{[3]}$
1	3.6(-15)	3.6(-15)	1.2(-11)	5.6(-11)
2	1.7(-12)	1.1(-12)	5.3(-11)	2.7(-11)
5	5.4(-12)	1.1(-11)	8.3(-11)	4.2(-11)
10	2.8(-11)	2.4(-11)	1.2(-10)	5.4(-11)
20	2.6(-11)	4.8(-11)	1.5(-10)	9.3(-11)
40	1.8(-10)	9.6(-11)	4.3(-10)	3.7(-10)
80	7.6(-10)	1.5(-10)	1.1(-8)	8.6(-9)
160	4.3(-9)	8.2(-10)	1.9(-8)	9.7(-9)
320	5.2(-9)	8.4(-10)	5.7(-9)	2.1(-9)

TABLE 2.1. *Accuracy of Chebyshev recursion coefficients*

3. RECURRENCE COEFFICIENTS FOR CHEBYSHEV MEASURES

In general, there are no analytic expressions known for the coefficients $\hat{\alpha}_j$, $\hat{\beta}_j$, and one has to resort to the algorithm of §2 to compute them numerically. As so often is the case in the constructive theory of orthogonal polynomials, the four Chebyshev measures on $(-1,1)$,

$$\begin{aligned}
 d\sigma^{[1]}(t) &= (1-t^2)^{-1/2}dt, & d\sigma^{[2]}(t) &= (1-t^2)^{1/2}dt, \\
 d\sigma^{[3]}(t) &= (1-t)^{-1/2}(1+t)^{1/2}dt, & d\sigma^{[4]}(t) &= (1-t)^{1/2}(1+t)^{-1/2}dt,
 \end{aligned}
 \tag{3.1}$$

are exceptional in that they permit closed-form expressions for the desired coefficients. In the case of $d\sigma^{[2]}$, they have already been obtained in [1] from the theory of sieved ultraspherical polynomials, and for both $d\sigma^{[1]}$ and $d\sigma^{[2]}$ in [13] via a theory of polynomial transformation of measures and orthogonal polynomials (cf. [8]). Here we give a derivation from first principles, and also obtain the result for $d\sigma^{[3]}$ (and hence for $d\sigma^{[4]}$).

We shall denote the n th modified measures by

$$d\hat{\sigma}_n^{[i]}(t) = [\pi_n^{[i]}(t)]^2 d\sigma^{[i]}(t), \quad i = 1, 2, 3, 4, \quad (3.2)$$

where $\pi_n^{[i]}$ is the n th-degree monic Chebyshev polynomial of the i th kind. We will suppress the superscript i when there is no danger of confusion.

One of the reasons for the existence of analytic formulae is the fact that the polynomials $\pi_k(\cdot; d\hat{\sigma}_n^{[i]})$, $i = 1, 2, 3, 4$, of degree $k \leq n$ ($k \leq n+1$ if $i = 2$) are pure Chebyshev polynomials of the first kind (cf. Theorems 3.1, 3.4 and 3.6). Another is the special algebraic structure enjoyed by Chebyshev polynomials (for example, (3.3)).

3.1. Chebyshev measure of the first kind. Here, and throughout this subsection, $d\sigma(t) = d\sigma^{[1]}(t)$ and $\pi_n(t) = 2^{-(n-1)}T_n(t) = \hat{T}_n(t)$, the monic Chebyshev polynomial of the first kind. Heavy use will be made of the well-known identity

$$T_p(t)T_q(t) = \frac{1}{2} (T_{|p-q|}(t) + T_{p+q}(t)), \quad (3.3)$$

valid for arbitrary nonnegative integers p, q .

THEOREM 3.1. *For any integer $n \geq 1$, there holds*

$$\hat{\pi}_{k,n}(t) = \hat{T}_k(t), \quad 0 \leq k \leq n, \quad (3.4)$$

where $\hat{\pi}_{k,n}(\cdot) = \pi_k(\cdot; d\hat{\sigma}_n^{[1]})$.

Proof. It suffices to prove that

$$\int_{-1}^1 T_k(t)T_r(t)T_n^2(t)d\sigma(t) = 0, \quad 0 \leq r < k \leq n. \quad (3.5)$$

We use (3.3) to write $T_n^2 = \frac{1}{2} (1 + T_{2n})$ and then use $T_k T_r \in \mathbb{P}_{2k-1} \subset \mathbb{P}_{2n-1}$ together with orthogonality to obtain for the integral on the left in (3.5)

$$\frac{1}{2} \int_{-1}^1 T_k(t)T_r(t)d\sigma(t) = 0. \quad \square$$

THEOREM 3.2. *If $n = 1$, then*

$$\hat{\pi}_{k,n}(t) = \begin{cases} \sum_{j=0}^{k/2} (-1)^j 4^{-j} \hat{T}_{k-2j}(t), & k \text{ (even)} \geq 2, \\ \sum_{j=0}^{(k-1)/2} (-1)^j 4^{-j} \frac{k-2j}{k} \hat{T}_{k-2j}(t), & k \text{ (odd)} \geq 1. \end{cases} \quad (3.6)$$

Moreover, the recurrence relation for the polynomials $\hat{\pi}_k = \hat{\pi}_{k,1}$ is given by

$$\begin{aligned} \hat{\pi}_{k+1}(t) &= t\hat{\pi}_k(t) - \hat{\beta}_k \hat{\pi}_{k-1}(t), \quad k = 0, 1, 2, \dots, \\ \hat{\pi}_0(t) &= 1, \quad \hat{\pi}_{-1}(t) = 0, \end{aligned} \quad (3.7)$$

where

$$\hat{\beta}_0 = \frac{\pi}{2}, \quad \hat{\beta}_k = \begin{cases} \frac{1}{4} \left(1 - \frac{2}{k+1}\right), & k \text{ (even)} \geq 2, \\ \frac{1}{4} \left(1 + \frac{2}{k}\right), & k \text{ (odd)} \geq 1. \end{cases} \quad (3.8)$$

Proof. For k even, it is known [13, Lemma 1] that

$$t\hat{\pi}_{k,1}(t) = \hat{T}_{k+1}(t), \quad k \text{ even}. \quad (3.9)$$

Expanding $\hat{\pi}_{k,1}$ in monic Chebyshev polynomials,

$$\hat{\pi}_{k,1}(t) = \hat{T}_k(t) + \sum_{j=1}^{k/2} a_j \hat{T}_{k-2j}(t),$$

and using (3.9) together with the recurrence relation for monic Chebyshev polynomials, one obtains the identity

$$0 \equiv \left(\frac{1}{4} + a_1\right) \hat{T}_{k-1} + \left(\frac{1}{4} a_1 + a_2\right) \hat{T}_{k-3} + \dots + \left(\frac{1}{4} a_{(k/2)-1} + a_{k/2}\right) \hat{T}_1,$$

from which $a_j = (-1)^j 4^{-j}$, $j = 1, 2, \dots, k/2$. This proves the first relation in (3.6). A similar argument works in the case k odd, where now (op. cit.)

$$t^2 \hat{\pi}_{k,1}(t) = \hat{T}_{k+2}(t) + \frac{k+2}{4k} \hat{T}_k(t), \quad k \text{ odd}, \quad (3.10)$$

and the recurrence relation for Chebyshev polynomials is to be applied twice.

The recursion coefficients (3.8) for $k \geq 1$ follow by squaring those given in [13, Eq. (2.4)] for the orthonormal polynomials, while for $k = 0$ one gets directly $\hat{\beta}_0 = \int_{-1}^1 t^2 d\sigma(t) = \int_0^\pi \cos^2 \theta d\theta = \pi/2$. $\square$

THEOREM 3.3. *For any $n \geq 2$, the polynomials $\hat{\pi}_{m,n}(\cdot) = \pi_m(\cdot; d\hat{\sigma}_n^{[1]})$, $m = 0, 1, 2, \dots$, satisfy*

$$\begin{aligned} \hat{\pi}_{k+1,n}(t) &= t \hat{\pi}_{k,n}(t) - \hat{\beta}_{k,n} \hat{\pi}_{k-1,n}(t), \quad k = 0, 1, 2, \dots, \\ \hat{\pi}_{0,n}(t) &= 1, \quad \hat{\pi}_{-1,n}(t) = 0, \end{aligned} \quad (3.11)$$

where

$$\hat{\beta}_{k,n} = \begin{cases} \frac{\pi}{2^{2n-1}} & \text{if } k = 0, \\ \frac{1}{4} \left(1 - \frac{(-1)^{k/n}}{1 + (k/n)} \right) & \text{if } k = 0 \bmod n \quad (k \neq 0), \\ \frac{1}{4} \left(1 + \frac{(-1)^{(k-1)/n}}{1 + (k-1)/n} \right) & \text{if } k = 1 \bmod n, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \quad (3.12)$$

Proof. For $k = 0$ one obtains directly

$$\hat{\beta}_{0,n} = \int_{-1}^1 d\hat{\sigma}_n = \frac{1}{2^{2n-2}} \int_{-1}^1 T_n^2(t) d\sigma(t) = \frac{1}{2^{2n-2}} \int_0^\pi \cos^2 n\theta d\theta = \frac{\pi}{2^{2n-1}}.$$

We will thus assume $k \geq 1$.

Instead of the monic polynomial $\hat{\pi}_{k+1,n}$ in (3.11), we consider

$$\hat{p}_{k+1}(t) = 2^k \hat{\pi}_{k+1,n}(t) = T_{k+1}(t) + a_{k-1}T_{k-1}(t) + a_{k-3}T_{k-3}(t) + \cdots, \quad (3.13)$$

where we have already taken care of symmetry. In view of (3.3) (with $p = q = n$), the coefficients in (3.13) must be determined so as to have

$$\int_{-1}^1 (T_{k+1}(t) + a_{k-1}T_{k-1}(t) + a_{k-3}T_{k-3}(t) + \cdots) T_r(t) (1 + T_{2n}(t)) d\sigma(t) = 0,$$

$$\text{for } 0 \leq r \leq k,$$

that is, again by (3.3),

$$\begin{aligned} \int_{-1}^1 (T_{k+1}(t) + a_{k-1}T_{k-1}(t) + a_{k-3}T_{k-3}(t) + \cdots) (T_r(t) + \frac{1}{2} T_{|r-2n|}(t) \\ + \frac{1}{2} T_{r+2n}(t)) d\sigma(t) = 0 \quad \text{for } 0 \leq r \leq k. \end{aligned} \quad (3.14)$$

We first determine a_{k-1} for $k \geq 1$.

To do this, we solve a linear system of equations involving $a_{k-1}, a_{|k-1-2n|}, \dots, a_{|k-1-2sn|}$, where s is defined by

$$s = \begin{cases} \left\lfloor \frac{k}{2n} \right\rfloor & \text{if } k \equiv 1 \pmod{2n}, \\ \frac{k}{n} - 1 & \text{if } k \equiv 0 \pmod{n}, \\ \left\lfloor \frac{k}{n} \right\rfloor & \text{otherwise.} \end{cases} \quad (3.15)$$

The system is obtained by taking $r = k-1, |k-1-2n|, \dots, |k-1-2sn|$ in (3.14). It is easy to check that these values of r are strictly less than k .

Assume for the time being that $k > 2n+1$ (hence $s \geq 2$).

We first let $r = k-1$ in (3.14) and observe that $k-1+2n > k+1$ (since $n \geq 2$). Then, by the orthogonality of the $\{T_m\}$, we obtain

$$2a_{k-1} + a_{|k-1-2n|} = 0 \quad (3.16_1)$$

(where the absolute value signs in the second index could be dropped under the present assumption on k).

Next, letting $r = |k - 1 - 2(j - 1)n|$, $j = 2, 3, \dots, s$, in (3.14), we will show that the following additional equations result (the first if $s \geq 3$):

$$a_{|k-1-2(j-2)n|} + 2a_{|k-1-2(j-1)n|} + a_{|k-1-2jn|} = 0, \quad j = 2, 3, \dots, s-1, \quad (3.16_j)$$

$$a_{|k-1-2(s-2)n|} + 2a_{|k-1-2(s-1)n|} + \gamma a_{|k-1-2sn|} = 0, \quad (3.16_s)$$

where

$$\gamma = \begin{cases} 1 & \text{if } k \not\equiv 1 \pmod{2n}, \\ 2 & \text{otherwise.} \end{cases} \quad (3.17)$$

To verify (3.16_j), (3.16_s), we note, first of all, that for the values of r presently being considered, we have $r > 0$, and $r = 2n$ if and only if $k = 1 \pmod{2n}$ and $j = s$. Indeed, if we had $r = 0$, then $k = 1 + 2(j - 1)n$, that is, $k = 1 \pmod{2n}$. But then, by definition of s in (3.15), $k = 1 + 2sn$, and hence $j = s + 1$, which is presently excluded. Similarly, if $r = 2n$, then again $k = 1 + 2sn$, hence $r = (s - j + 1)2n$, which, since $j \leq s$, equals $2n$ only if $j = s$. Conversely, $k = 1 \pmod{2n}$ and $j = s$ imply respectively $k = 1 + 2sn$ and $r = |k - 1 - 2(s - 1)n| = 2n$.

Now consider in turn the contributions from the three terms in the second factor of the integrand in (3.14). The first term yields $(\pi/2)a_r$, where

$$r = |k - 1 - 2(j - 1)n|, \quad (3.18)$$

since $r > 0$. The contribution from the second term is $(\pi/4)a_{|r-2n|}$, or $(\pi/2)a_{|r-2n|}$ if $r = 2n$, where

$$|r - 2n| = \begin{cases} |k - 1 - 2(j - 2)n| & \text{if } k < 1 + 2(j - 1)n, \\ |k - 1 - 2jn| & \text{otherwise.} \end{cases} \quad (3.19)$$

The third term contributes $(\pi/4)a_{r+2n}$, where

$$r + 2n = \begin{cases} |k - 1 - 2(j - 2)n| & \text{if } k \geq 1 + 2(j - 1)n, \\ |k - 1 - 2jn| & \text{otherwise.} \end{cases} \quad (3.20)$$

Thus, we get the equations

$$\frac{\pi}{4} a_{|r-2n|} + \frac{\pi}{2} a_r + \frac{\pi}{4} a_{r+2n} = 0, \quad r \neq 2n,$$

$$\frac{\pi}{2} a_{|r-2n|} + \frac{\pi}{2} a_r + \frac{\pi}{4} a_{r+2n} = 0, \quad r = 2n,$$

with $r, |r - 2n|$ and $r + 2n$ as given in (3.18)–(3.20), where in (3.19) for $j = s$ the second relation holds, and in (3.20) the first. This proves (3.16_j), (3.16_s), (3.17).

Finally, let $j = s + 1$. In this case, we distinguish between $k \not\equiv 1 \pmod{2n}$ and $k \equiv 1 \pmod{2n}$. In the former case, the contributions from the first two terms of the second factor in (3.14) are determined in the same way as before, noting that in (3.19) the first case holds. Indeed, by definition of s , we have $k = sn + \nu$, $0 < \nu \leq n$, and $s > 0$ (since $k > 2n + 1$), so that $k \leq (s + 1)n \leq 2s \cdot n < 1 + 2sn$. For the third term, $\frac{1}{2} T_{r+2n}$, we note that $r + 2n = |k - 1 - 2s \cdot n| + 2n = 1 + 2sn - k + 2n = 1 + 2(k - \nu) - k + 2n = k + 1 + 2(n - \nu)$, which is $> k + 1$ if $k \not\equiv 0 \pmod{n}$ and $= k + 1$ otherwise. Thus, we obtain a contribution, $\pi/4$, only if $k \equiv 0 \pmod{n}$, and we get the final equation

$$a_{|k-1-2(s-1)n|} + 2a_{|k-1-2sn|} = \delta, \quad (3.16_{s+1})$$

where

$$\delta = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod{n}, \\ -1 & \text{otherwise.} \end{cases} \quad (3.21)$$

In the second case, $k \equiv 1 \pmod{2n}$ (and $j = s + 1$), we have $k = 1 + 2sn$ by definition of s , hence $r = |k - 1 - 2sn| = 0$ and $|k - 1 - 2(s - 1)n| = 2n$. Therefore, the contribution from the first term of the second factor in (3.14) is $\pi a_{|k-1-2sn|}$, and the contributions from the last two terms are both

$(\pi/4)a_{|k-1-2(s-1)n|}$. So, the final equation in this case is again (3.16_{s+1}), where $\delta = 0$ in agreement with (3.21).

With the definition $\alpha_j = a_{|k-1-2(j-1)n|}$, $j = 1, 2, \dots, s+1$, our findings may be summarized by the following system of equations,

$$\begin{aligned} 2\alpha_1 + \alpha_2 &= 0, \\ \alpha_{j-1} + 2\alpha_j + \alpha_{j+1} &= 0, \quad 2 \leq j \leq s-1, \\ \alpha_{s-1} + 2\alpha_s + \gamma\alpha_{s+1} &= 0, \\ \alpha_s + 2\alpha_{s+1} &= \delta, \end{aligned} \tag{3.22}$$

where γ and δ are given by (3.17) and (3.21), respectively. In particular, if $k \neq 0 \pmod n$, then $\delta = 0$, and (3.22) represents a tridiagonal, nonsingular homogeneous system of $s+1$ equations in the $s+1$ unknowns $\alpha_1, \alpha_2, \dots, \alpha_{s+1}$. It follows that all $\alpha_j = 0$, in particular, $\alpha_1 = 0$, that is, $a_{k-1} = 0$.

We assumed so far that $k > 2n+1$. The same conclusion, $a_{k-1} = 0$, however can also be reached in the case $1 \leq k \leq 2n+1$, $k \neq 0 \pmod n$, by considering separately the three cases $1 \leq k < n$, $n < k < 2n$, and $k = 2n+1$, letting in the first case $r = k-1$ in (3.14), and in the other two cases $r = k-1$ and $r = |k-1-2n|$ in (3.14).

It thus follows from (3.13) that

$$\hat{\pi}_{k+1} = \hat{T}_{k+1} + \{\hat{T}_{k-3}, \hat{T}_{k-5}, \dots\}, \quad k \neq 0 \pmod n, \tag{3.23}$$

where $\{x, y, z, \dots\}$ means a linear combination of $x, y, z, \dots$ or zero, if the list is empty. If, in addition, $k \neq 1 \pmod n$, then (3.23) also holds when k is replaced by $k-1$, and we obtain

$$\hat{\pi}_{k+1}(t) - t\hat{\pi}_k(t) = \hat{T}_{k+1} - t\hat{T}_k + \{\hat{T}_{k-3}, \hat{T}_{k-5}, \dots\} = -\frac{1}{4}\hat{T}_{k-1} + \{\dots\},$$

where $\{\dots\}$ is a linear combination of $\hat{T}_{k-3}, \hat{T}_{k-5}, \dots$, hence of $\hat{\pi}_{k-3}, \hat{\pi}_{k-5}, \dots$ if $k \geq 3$, or zero otherwise. Since $\hat{T}_{k-1} = \hat{\pi}_{k-1} + \{\dots\}$, there follows

$$\hat{\pi}_{k+1}(t) - t\hat{\pi}_k(t) = -\frac{1}{4}\hat{\pi}_{k-1}(t),$$

the linear combination $\{\dots\}$ left over being necessarily zero. Therefore, $\hat{\beta}_k = 1/4$, proving the last formula in (3.12).

Consider next the case $k = 0 \pmod n$ and assume first $k > 2n + 1$. Then $k = mn$ for some integer $m \geq 3$, and $s = m - 1$ in (3.15). Since $\gamma = 1$ by (3.17), and $\delta = -1$ by (3.21), the system (3.22) represents a homogeneous second-order difference equation with constant coefficients for $\{\alpha_j\}_{j=1}^m$ subject to two boundary conditions, $2\alpha_1 + \alpha_2 = 0$, $\alpha_{m-1} + 2\alpha_m = -1$. Straightforward solution gives

$$\alpha_j = \frac{(-1)^{j+m-1}}{m+1}, \quad 1 \leq j \leq m,$$

hence, in particular,

$$\alpha_1 = a_{k-1} = \frac{(-1)^{k/n}}{1 + (k/n)}, \quad k = 0 \pmod n. \quad (3.24)$$

The same result holds also for $m = 1$ and $m = 2$, as can be verified from (3.14) by putting there $k = n$, $r = k - 1 = n - 1$ in the first case, and $k = 2n$, $r = k - 1 = 2n - 1$ and $r = 1$ in the second case. Using (3.24) in (3.13) now gives

$$\begin{aligned} \hat{\pi}_{k+1} &= \hat{T}_{k+1} + \frac{1}{4} a_{k-1} \hat{T}_{k-1} + \{\hat{T}_{k-3}, \hat{T}_{k-5}, \dots\} \\ &= \hat{T}_{k+1} + \frac{1}{4} \frac{(-1)^{k/n}}{1 + (k/n)} \hat{T}_{k-1} + \{\hat{T}_{k-3}, \hat{T}_{k-5}, \dots\}, \quad k = 0 \pmod n. \end{aligned} \quad (3.25)$$

Subtracting $t\hat{\pi}_k(t)$ from both sides, and using (3.23) with k replaced by $k - 1$ ($\neq 0 \pmod n$), then yields with the same argument as above,

$$\hat{\pi}_{k+1}(t) - t\hat{\pi}_k(t) = -\frac{1}{4} \left(1 - \frac{(-1)^{k/n}}{1 + (k/n)} \right) \hat{\pi}_{k-1}(t),$$

proving the second formula in (3.12).

It remains to consider $k = 1 \pmod n$. If $k \neq 1$, we subtract $t\hat{\pi}_k(t)$ from both sides of (3.23), and use (3.25) with k replaced by $k - 1$ (since now $k - 1 = 0 \pmod n$), to get as before

$$\hat{\pi}_{k+1}(t) - t\hat{\pi}_k(t) = -\frac{1}{4} \left(1 + \frac{(-1)^{(k-1)/n}}{1 + (k-1)/n} \right) \hat{\pi}_{k-1}(t),$$

which proves the third formula in (3.12) when $k \neq 1$. Finally, the case $k = 1$ can be disposed of directly, by calculating

$$\hat{\beta}_{1,n} = \frac{1}{\hat{\beta}_{0,n}} \int_{-1}^1 \hat{\pi}_1^2 d\hat{\sigma}_n = \frac{2}{\pi} \int_{-1}^1 t^2 T_n^2(t) d\sigma(t) = \frac{2}{\pi} \int_0^\pi \cos^2 \theta \cos^2 n\theta d\theta = \frac{1}{2}.$$

This completes the proof of Theorem 3.3. $\square$

We remark that for $k \neq 0$ the result (3.12) can be written as

$$\hat{\beta}_{k,n} = \begin{cases} \hat{\beta}_{2k/n,2} & \text{if } k = 0 \bmod n, \\ \hat{\beta}_{1+2(k-1)/n,2} & \text{if } k = 1 \bmod n, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \quad (3.12')$$

It is also of some interest to note that the polynomials $\{\hat{\pi}_{k,n}\}_{k=0}^\infty$ are special symmetric random walk polynomials, since (cf. [3, §7]), as is readily verified, $\hat{\beta}_{k,n} = D_k(1 - D_{k-1})$ for $k = 1, 2, 3, \dots$, with $D_k = \frac{1}{2}(1 - (-1)^{k/n}/(1 + k/n))$ if $k = 0 \bmod n$, and $D_k = \frac{1}{2}$ otherwise.

3.2. Chebyshev measure of the second kind. We now consider $d\sigma(t) = d\sigma^{[2]}(t)$, for which $\pi_n(t) = 2^{-n}U_n(t) = \hat{U}_n(t)$ is the monic Chebyshev polynomial of the second kind. The well-known identity

$$U_n^2(t) = \frac{1}{2} \frac{1 - T_{2n+2}(t)}{1 - t^2} \quad (3.26)$$

allows us to reduce the present case essentially to the one treated in the previous section.

THEOREM 3.4. *For any integer $n \geq 1$, there holds*

$$\hat{\pi}_{k,n}(t) = \hat{T}_k(t), \quad 0 \leq k \leq n+1,$$

where $\hat{\pi}_{k,n}(\cdot) = \pi_k(\cdot; d\hat{\sigma}_n^{[2]})$ and $\hat{T}_k$ is the monic Chebyshev polynomial of the first kind.

Proof. The basic orthogonality to be shown, in view of (3.26), is

$$\int_{-1}^1 T_k(t) T_r(t) \frac{1 - T_{2n+2}(t)}{1 - t^2} d\sigma^{[2]}(t) = 0, \quad 0 \leq r < k \leq n+1,$$

which, since $d\sigma^{[2]}(t)/(1-t^2) = d\sigma^{[1]}(t)$, is equivalent to

$$\int_{-1}^1 T_k(t)T_r(t)d\sigma^{[1]}(t) = 0, \quad 0 \leq r < k \leq n+1.$$

This is clearly true. $\square$

The result in Theorem 3.4 has also been noted in [1, Eq. (2.16)].

THEOREM 3.5. *For any $n \geq 1$, the polynomials $\hat{\pi}_{m,n}(\cdot) = \pi_m(\cdot; d\hat{\sigma}_n^{[2]})$, $m = 0, 1, 2, \dots$, satisfy*

$$\begin{aligned} \hat{\pi}_{k+1,n}(t) &= t\hat{\pi}_{k,n}(t) - \hat{\beta}_{k,n}\hat{\pi}_{k-1,n}(t), \quad k = 0, 1, 2, \dots, \\ \hat{\pi}_{0,n}(t) &= 1, \quad \hat{\pi}_{-1,n}(t) = 0, \end{aligned} \tag{3.27}$$

where

$$\hat{\beta}_{k,n} = \begin{cases} \frac{\pi}{2^{2n+1}} & \text{if } k = 0, \\ \frac{1}{4} \left(1 - \frac{1}{1 + k/(n+1)} \right) & \text{if } k = 0 \bmod n+1, \quad k \neq 0, \\ \frac{1}{4} \left(1 + \frac{1}{1 + (k-1)/(n+1)} \right) & \text{if } k = 1 \bmod n+1, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{3.28}$$

Proof. For $k = 0$, one obtains directly, using (3.26),

$$\begin{aligned} \hat{\beta}_{0,n} &= \int_{-1}^1 d\hat{\sigma}_n^{[2]} = \frac{1}{2^{2n}} \int_{-1}^1 U_n^2(t)d\sigma^{[2]}(t) \\ &= \frac{1}{2^{2n+1}} \int_{-1}^1 (1 - T_{2n+2}(t))d\sigma^{[1]}(t) = \frac{\pi}{2^{2n+1}}. \end{aligned}$$

Writing $\hat{p}_{k+1}(t) = 2^k \hat{\pi}_{k+1,n}(t)$ again in the form (3.13), and using (3.26), we find that the basic orthogonality condition is identical to (3.14), where $d\sigma = d\sigma^{[1]}$, except that n is to be replaced by $n+1$ and the two plus signs in the second factor of the integrand are now minus signs. It is easily

checked that the latter change reverses the signs of all off-diagonal entries of the tridiagonal system (3.22) as well as the sign of δ . As a consequence, the formula (3.24) for a_{k-1} holds without the alternating signs in the numerator. With these two changes – replacement of n by $n+1$ and removal of the alternating signs – the proof of Theorem 3.3 remains in force and produces the result of Theorem 3.5. $\square$

We may write (3.28), for $k \neq 0$, alternatively in the form

$$\hat{\beta}_{k,n} = \begin{cases} \hat{\beta}_{2k/(n+1),1} & \text{if } k = 0 \bmod n+1, \\ \hat{\beta}_{1+2(k-1)/(n+1),1} & \text{if } k = 1 \bmod n+1, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \quad (3.28')$$

Again, $\{\hat{\pi}_{k,n}\}$ are symmetric random walk polynomials with parameters $D_k = \frac{1}{2} (1 - 1/(1 + k/(n+1)))$ if $k = 0 \bmod n+1$, $D_k = \frac{1}{2}$ otherwise, in $\hat{\beta}_{k,n} = D_k(1 - D_{k-1})$, $k \geq 1$.

3.3. Chebyshev measure of the third and fourth kind. To the measure $d\sigma^{[3]}$ (cf. (3.1)) there corresponds $\pi_n^{[3]}(t) = 2^{-n}V_n(t)$, the monic Chebyshev polynomial of the third kind, and to $d\sigma^{[4]}$ the one of the fourth kind, $\pi_n^{[4]}(t) = 2^{-n}W_n(t)$. It is well known that

$$V_n(\cos \theta) = \frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta}, \quad W_n(\cos \theta) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}. \quad (3.29)$$

Replacing θ by $\theta + \pi$ in the second formula yields $W_n(-t) = (-1)^n V_n(t)$, so that

$$\int_{-1}^1 \hat{\pi}_k(t) p(t) V_n^2(t) d\sigma^{[3]}(t) = 0, \quad p \in \mathbb{P}_{k-1},$$

upon changing $t \mapsto -t$ implies

$$\int_{-1}^1 \hat{\pi}_k(-t) q(t) W_n^2(t) d\sigma^{[4]}(t) = 0, \quad q \in \mathbb{P}_{k-1}.$$

Consequently,

$$\pi_k(t; d\hat{\sigma}_n^{[4]}) = (-1)^k \pi_k(-t; d\hat{\sigma}_n^{[3]}), \quad (3.30)$$

and it suffices to consider the case of $d\sigma^{[3]}$.

THEOREM 3.6. *For any integer $n \geq 1$, there holds*

$$\hat{\pi}_{k,n}(t) = \hat{T}_k(t), \quad 0 \leq k \leq n,$$

where $\hat{\pi}_{k,n}(\cdot) = \pi_k(\cdot; d\hat{\sigma}_n^{[3]})$ and $\hat{T}_k$ is the monic Chebyshev polynomial of the first kind.

Proof. We need to show that

$$\int_{-1}^1 T_k(t) T_r(t) V_n^2(t) d\sigma^{[3]}(t) = 0, \quad 0 \leq r < k \leq n.$$

This follows at once from the first relation in (3.29) by noting that, with $t = \cos \theta$,

$$\begin{aligned} V_n^2(t) d\sigma^{[3]}(t) &= \frac{\cos^2(n + \frac{1}{2})\theta}{\cos^2 \frac{1}{2} \theta} (1-t)^{-1/2} (1+t)^{1/2} \\ &= \frac{1 + \cos(2n+1)\theta}{1 + \cos \theta} (1-t)^{-1/2} (1+t)^{1/2} \\ &= (1 + T_{2n+1}(t)) d\sigma^{[1]}(t). \quad \square \end{aligned} \tag{3.31}$$

THEOREM 3.7. *For any $n \geq 1$, the polynomials $\hat{\pi}_{m,n}(\cdot) = \pi_m(\cdot; d\hat{\sigma}_n^{[3]})$, $m = 0, 1, 2, \dots$, satisfy*

$$\begin{aligned} \hat{\pi}_{k+1,n}(t) &= (t - \hat{\alpha}_{k,n}) \hat{\pi}_{k,n}(t) - \hat{\beta}_{k,n} \hat{\pi}_{k-1,n}(t), \quad k = 0, 1, 2, \dots, \\ \hat{\pi}_{0,n}(t) &= 1, \quad \hat{\pi}_{-1,n}(t) = 0, \end{aligned} \tag{3.32}$$

where

$$\hat{\alpha}_{k,n} = \begin{cases} \frac{1}{4} \frac{1}{1 + (k-n)/(2n+1)} & \text{if } k = n \pmod{2n+1}, \\ -\frac{1}{4} \frac{1}{1 + (k-n-1)/(2n+1)} & \text{if } k = n+1 \pmod{2n+1}, \\ 0 & \text{otherwise} \end{cases} \quad (3.33)$$

and

$$\hat{\beta}_{k,n} = \begin{cases} \frac{\pi}{2^{2n}} & \text{if } k = 0, \\ \frac{1}{4} \left(1 - \frac{1}{1 + 2k/(2n+1)} \right) & \text{if } k = 0 \pmod{2n+1}, \quad k \neq 0, \\ \frac{1}{4} \left(1 + \frac{1}{1 + 2(k-1)/(2n+1)} \right) & \text{if } k = 1 \pmod{2n+1}, \\ \frac{1}{4} \left(1 - \frac{1}{4(1 + (k-n-1)/(2n+1))^2} \right) & \text{if } k = n+1 \pmod{2n+1}, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \quad (3.34)$$

Proof. The first relation in (3.34) is a simple consequence of (3.31):

$$\hat{\beta}_{0,n} = \int_{-1}^1 d\hat{\sigma}_n^{[3]} = \frac{1}{2^{2n}} \int_{-1}^1 (1 + T_{2n+1}(t)) d\sigma^{[1]}(t) = \frac{\pi}{2^{2n}}.$$

We thus assume $k \geq 1$.

In view of (3.31) and (3.3), the polynomial

$$\hat{p}_k(t) = 2^{k-1} \hat{\pi}_{k,n}(t) = T_k(t) + a_{k-1} T_{k-1}(t) + a_{k-2} T_{k-2}(t) + \cdots + a_0 T_0(t) \quad (3.35)$$

needs to be determined such that

$$\begin{aligned} & \int_{-1}^1 (T_k(t) + a_{k-1}T_{k-1}(t) + a_{k-2}T_{k-2}(t) + \cdots + a_0T_0(t)) \\ & \times (T_r(t) + \frac{1}{2} T_{|r-2n-1|}(t) + \frac{1}{2} T_{r+2n+1}(t)) d\sigma^{[1]}(t) = 0 \quad \text{for } 0 \leq r < k. \end{aligned} \quad (3.36)$$

In contrast to the proof of Theorem 3.3, it will be necessary now to determine both a_{k-1} and a_{k-2} (if $k \geq 2$). This will be done by setting up appropriate systems of linear equations, one for a_{k-1} , and another for a_{k-2} .

The former involves the unknowns a_{k-1} , $a_{|k-1-(2n+1)|}$, $a_{|k-1-2(2n+1)|}$, $\dots$, $a_{|k-1-s(2n+1)|}$, where

$$s = \begin{cases} 2 \left\lfloor \frac{k}{2n+1} \right\rfloor - 1 & \text{if } k = 0 \pmod{2n+1}, \\ \left\lfloor \frac{k}{2n+1} \right\rfloor & \text{if } k = 1 \pmod{2n+1}, \\ 2 \left\lfloor \frac{k}{2n+1} \right\rfloor & \text{if } 1 < k < n+2 \pmod{2n+1}, \\ 2 \left\lfloor \frac{k}{2n+1} \right\rfloor + 1 & \text{otherwise,} \end{cases} \quad (3.37)$$

and is obtained by setting $r = |k-1-(j-1)(2n+1)|$, $j = 1, 2, \dots, s+1$, in (3.36). All these values of r are strictly less than k , as required in (3.36). This is easily checked by examining each of the four cases in (3.37).

We shall first assume $k > 2n+2$. Note that this implies $s \geq 2$.

To determine the desired equations, it is necessary, similarly as in the proof of Theorem 3.3, to examine the three indices, r , $|r-2n-1|$ and $r+2n+1$ in the second factor of the integrand in (3.36), in particular, whether they are zero, positive but $\leq k$, or $> k$.

For $j = 1$, we have $r = k-1 > 0$, $|r-2n-1| = k-1-(2n+1) > 0$ by assumption, and $r+2n+1 = k+2n > k$. Accordingly, Eq. (3.36) becomes

$$2a_{k-1} + a_{|k-1-(2n+1)|} = 0. \quad (3.38_1)$$

When $2 \leq j \leq s$, we again have $r > 0$ (and $r < k$, as already observed), since $r = 0$ would imply $k = 1 \pmod{2n+1}$, hence by the second relation in (3.37), $k = 1 + s(2n+1)$. This in turn implies $j = s+1$, which is presently excluded. The other two indices compute to

$$|r - 2n - 1| = \begin{cases} |k - 1 - (j - 2)(2n + 1)| & \text{if } k < 1 + (j - 1)(2n + 1), \\ |k - 1 - j(2n + 1)| & \text{otherwise} \end{cases} \quad (3.39)$$

and

$$r + 2n + 1 = \begin{cases} |k - 1 - (j - 2)(2n + 1)| & \text{if } k \geq 1 + (j - 1)(2n + 1), \\ |k - 1 - j(2n + 1)| & \text{otherwise.} \end{cases} \quad (3.40)$$

Both of them (as observed previously) are $< k$, and the second is clearly positive. The first is nonnegative, and equal to zero if and only if $k = 1 \pmod{2n + 1}$ and $j = s$. Indeed, $r = 2n + 1$ implies $|k - 1 - (j - 1)(2n + 1)| = 2n + 1$, hence $k = 1 \pmod{2n + 1}$, which by (3.37) again implies $k = 1 + s(2n + 1)$. Therefore, $|s(2n + 1) - (j - 1)(2n + 1)| = 2n + 1$, that is, $(s - j + 1)(2n + 1) = 2n + 1$. Since $j \leq s$, this implies $j = s$. The converse is obvious. Note also that for $j = s$ and $k = 1 \pmod{2n + 1}$, we are in the second case for (3.39), and in the first for (3.40). Altogether, it follows that Eq. (3.36) takes the form

$$a_{|k-1-(j-2)(2n+1)|} + 2a_{|k-1-(j-1)(2n+1)|} + a_{|k-1-j(2n+1)|} = 0, \quad (3.38_j)$$

$$j = 2, 3, \dots, s - 1 \quad (\text{if } s \geq 3),$$

and for $j = s$ the form

$$a_{|k-1-(s-2)(2n+1)|} + 2a_{|k-1-(s-1)(2n+1)|} + \gamma a_{|k-1-s(2n+1)|} = 0, \quad (3.38_s)$$

where

$$\gamma = \begin{cases} 1 & \text{if } k \not\equiv 1 \pmod{2n + 1}, \\ 2 & \text{otherwise.} \end{cases} \quad (3.41)$$

Finally, for $j = s + 1$, we have for the first index

$$r = |k - 1 - s(2n + 1)| \begin{cases} = 0 & \text{if } k = 1 \pmod{2n + 1}, \\ > 0 & \text{otherwise.} \end{cases}$$

For the others, we note, first of all, that

$$k - 1 - s(2n + 1) \leq 0.$$

This is easily verified by examining separately the four cases in (3.37). It then follows that

$$\begin{aligned} |r - 2n - 1| &= ||k - 1 - s(2n + 1)| - (2n + 1)| \\ &= |s(2n + 1) - k + 1 - (2n + 1)| = |(s - 1)(2n + 1) - k + 1| \\ &= |k - 1 - (s - 1)(2n + 1)|. \end{aligned}$$

Furthermore, for the same reason,

$$\begin{aligned} r + 2n + 1 &= |k - 1 - s(2n + 1)| + 2n + 1 = s(2n + 1) - k + 1 + 2n + 1 \\ &= (s + 1)(2n + 1) - k + 1. \end{aligned}$$

By again examining the four cases in (3.37), one finds that $r + 2n + 1 > k$, unless either $k = 1 \pmod{2n + 1}$, in which case $r + 2n + 1 = |k - 1 - (s - 1)(2n + 1)|$, or $k = n + 1 \pmod{2n + 1}$, in which case $r + 2n + 1 = k$. As a result, the equation (3.36) gives

$$a_{|r-1-(s-1)(2n+1)|} + 2a_{|k-1-s(2n+1)|} = \delta, \quad (3.38_{s+1})$$

where

$$\delta = \begin{cases} 0 & \text{if } k \neq n + 1 \pmod{2n + 1}, \\ -1 & \text{otherwise.} \end{cases} \quad (3.42)$$

In summary, letting $\alpha_j = a_{|k-1-(j-1)(2n+1)|}$, the system for determining a_{k-1} is given by

$$\begin{aligned} 2\alpha_1 + \alpha_2 &= 0, \\ \alpha_{j-1} + 2\alpha_j + \alpha_{j+1} &= 0, \quad 2 \leq j \leq s-1, \\ \alpha_{s-1} + 2\alpha_s + \gamma\alpha_{s+1} &= 0, \\ \alpha_s + 2\alpha_{s+1} &= \delta, \end{aligned} \tag{3.43}$$

where γ and δ are defined in (3.41) and (3.42), respectively. This is exactly the same system as (3.22) in the proof of Theorem 3.3, except for the definition of γ and δ , and can be solved similarly as before. Specifically, if $k \not\equiv n+1 \pmod{2n+1}$, then $\delta = 0$ and all $\alpha_j = 0$. In particular, $\alpha_1 = 0$, that is, $a_{k-1} = 0$. If, on the other hand, $k \equiv n+1 \pmod{2n+1}$, then $\delta = -1$, and since $k \not\equiv 1 \pmod{2n+1}$, we have $\gamma = 1$ by (3.41). Solving (3.43) as a second-order linear difference equation with two boundary conditions yields

$$\alpha_1 = a_{k-1} = \frac{(-1)^{s+1}}{s+2}.$$

Since, by the third relation in (3.37), s is even, say $s = 2m$, where $k = m(2n+1) + n+1$, we get $a_{k-1} = -1/(2(m+1))$. Thus,

$$a_{k-1} = \begin{cases} 0 & \text{if } k \not\equiv n+1 \pmod{2n+1}, \\ -\frac{1}{2(1+(k-n-1)/(2n+1))} & \text{otherwise.} \end{cases} \tag{3.44}$$

We have shown (3.44) under the assumption $k > 2n+2$. One can verify directly that (3.44) holds also for $1 \leq k \leq 2n+2$ by considering separately the cases $1 \leq k \leq n+1$, $n+1 < k < 2n+1$, $k = 2n+1$ and $k = 2n+2$, putting in (3.36) $r = k-1$ in the first case, and $r = k-1$ and $r = |k-2-2n|$ in the others.

We now turn our attention to a_{k-2} . Here, we set up a system of linear equations for a_{k-2} , $a_{|k-2-2(2n+1)|}, \dots, a_{|k-2-s(2n+1)|}$, where

$$s = \begin{cases} 2 \left\lfloor \frac{k}{2n+1} \right\rfloor - 1 & \text{if } 0 \leq k \leq 1 \pmod{2n+1}, \\ \left\lfloor \frac{k}{2n+1} \right\rfloor & \text{if } k = 2 \pmod{2n+1}, \\ 2 \left\lfloor \frac{k}{2n+1} \right\rfloor & \text{if } 2 < k < n+2 \pmod{2n+1}, n \geq 2, \\ 2 \left\lfloor \frac{k}{2n+1} \right\rfloor + 1 & \text{otherwise, } n \geq 2, \end{cases} \quad (3.45)$$

by letting $r = |k - 2 - (j-1)(2n+1)|$, $j = 1, 2, \dots, s+1$, in (3.36). By a discussion very similar to the one carried out for the coefficient a_{k-1} , one obtains for $\alpha'_j = a_{|k-2-(j-1)(2n+1)|}$, assuming first $k > 2n+3$ (hence $s \geq 2$), the system

$$\begin{aligned} 2\alpha'_1 + \alpha'_2 &= 0, \\ \alpha'_{j-1} + 2\alpha'_j + \alpha'_{j+1} &= 0, \quad 2 \leq j \leq s-1, \\ \alpha'_{s-1} + 2\alpha'_s + \gamma'\alpha'_{s+1} &= 0, \\ \alpha'_s + 2\alpha'_{s+1} &= \delta', \end{aligned} \quad (3.46)$$

where γ' and δ' are defined by

$$\gamma' = \begin{cases} 1 & \text{if } k \not\equiv 2 \pmod{2n+1}, \\ 2 & \text{otherwise,} \end{cases} \quad \delta' = \begin{cases} 0 & \text{if } k \not\equiv 1 \pmod{2n+1}, \\ -1 & \text{otherwise.} \end{cases} \quad (3.47)$$

Straightforward solution yields, for $\alpha'_1 = a_{k-2}$,

$$a_{k-2} = \begin{cases} 0 & \text{if } k \not\equiv 1 \pmod{2n+1}, \\ \frac{1}{1 + 2(k-1)/(2n+1)} & \text{if } k \equiv 1 \pmod{2n+1}. \end{cases} \quad (3.48)$$

The same holds true if $2 \leq k \leq 2n+3$, as can be verified by considering separately the cases $2 \leq k < n+1$, $n+1 \leq k < 2n+2$, $k = 2n+2$ and $k = 2n+3$.

Returning now to (3.35), we can write

$$\hat{\pi}_{k,n}(t) = \hat{T}_k(t) + \frac{1}{2} a_{k-1} \hat{T}_{k-1}(t) + \cdots + \frac{1}{2^{k-1}} a_0 \hat{T}_0(t),$$

or, in view of (3.44) and (3.48),

$$\begin{aligned} \hat{\pi}_{k,n} &= \hat{T}_k + \frac{1}{4} \frac{1}{1 + 2(k-1)/(2n+1)} \hat{T}_{k-2} + \{\hat{T}_{k-3}, \hat{T}_{k-4}, \dots\} \\ &\quad \text{if } k = 1 \pmod{2n+1}, \end{aligned} \quad (3.49)$$

$$\begin{aligned} \hat{\pi}_{k,n} &= \hat{T}_k - \frac{1}{4} \frac{1}{1 + (k-n-1)/(2n+1)} \hat{T}_{k-1} + \{\hat{T}_{k-3}, \hat{T}_{k-4}, \dots\} \\ &\quad \text{if } k = n+1 \pmod{2n+1}, \end{aligned} \quad (3.50)$$

$$\hat{\pi}_{k,n} = \hat{T}_k + \{\hat{T}_{k-3}, \hat{T}_{k-4}, \dots\} \quad \text{otherwise.} \quad (3.51)$$

If $k = 0 \pmod{2n+1}$ ($k \neq 0$), hence $k+1 = 1 \pmod{2n+1}$, we obtain from (3.49) and (3.51)

$$\begin{aligned} \hat{\pi}_{k+1,n}(t) - t\hat{\pi}_{k,n}(t) &= \hat{T}_{k+1}(t) + \frac{1}{4} \frac{1}{1 + 2k/(2n+1)} \hat{T}_{k-1}(t) - t\hat{T}_k(t) \\ &\quad + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &= -\frac{1}{4} \left(1 - \frac{1}{1 + 2k/(2n+1)} \right) \hat{T}_{k-1}(t) + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &= -\frac{1}{4} \left(1 - \frac{1}{1 + 2k/(2n+1)} \right) \hat{\pi}_{k-1,n}(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\alpha}_{k,n} &= 0, \quad \hat{\beta}_{k,n} = \frac{1}{4} \left(1 - \frac{1}{1 + 2k/(2n+1)} \right) \quad \text{if } k = 0 \pmod{2n+1} \\ &\quad (k \neq 0). \end{aligned}$$

This agrees with the third relation in (3.33) and proves the second in (3.34).

Next, consider $k = 1 \bmod 2n + 1$ (and $n \geq 2$). Then, similarly as above, (3.51) and (3.49) yield

$$\begin{aligned}\hat{\pi}_{k+1,n}(t) - t\hat{\pi}_{k,n}(t) &= \hat{T}_{k+1}(t) - \frac{1}{4} \frac{1}{1 + 2(k-1)/(2n+1)} \hat{T}_{k-1}(t) - t\hat{T}_k(t) \\ &\quad + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &= -\frac{1}{4} \left(1 + \frac{1}{1 + 2(k-1)/(2n+1)}\right) \hat{\pi}_{k-1,n}(t),\end{aligned}\tag{3.52}$$

hence

$$\hat{\alpha}_{k,n} = 0, \quad \hat{\beta}_{k,n} = \frac{1}{4} \left(1 + \frac{1}{1 + 2(k-1)/(2n+1)}\right) \quad \text{if } k = 1 \bmod 2n + 1.$$

This agrees with the third relation in (3.33), if $n \geq 2$, and proves the third in (3.34). If $n = 1$, then (3.52) must be slightly modified, since (3.50) and (3.49) are now relevant, and one finds $\hat{\alpha}_{k,1} = \frac{1}{4} (1 + (k-1)/3)^{-1}$ and $\hat{\beta}_{k,1} = \frac{1}{4} (1 + (1 + 2(k-1)/3)^{-1})$, which is the desired result in the case $n = 1$.

If $k = n \bmod 2n + 1$ (and $n \geq 2$), then a similar argument, based on (3.50) and (3.51), gives

$$\hat{\pi}_{k+1,n}(t) - t\hat{\pi}_{k,n}(t) = -\frac{1}{4} \frac{1}{1 + (k-n)/(2n+1)} \hat{\pi}_{k,n}(t) - \frac{1}{4} \hat{\pi}_{k-1,n}(t),$$

and thus

$$\hat{\alpha}_{k,n} = \frac{1}{4} \frac{1}{1 + (k-n)/(2n+1)}, \quad \hat{\beta}_{k,n} = \frac{1}{4} \quad \text{if } k = n \bmod 2n + 1.$$

This proves the first formula in (3.33) and the last in (3.34), if $n \geq 2$. The case $n = 1$ again requires the use of (3.49) and (3.50), which gives $\hat{\alpha}_{k,1} = \frac{1}{4} (1 + (k-1)/3)^{-1}$, $\hat{\beta}_{k,1} = \frac{1}{4} (1 + (1 + 2(k-1)/3)^{-1})$, proving the first formula in (3.33) and the third in (3.34), both for $n = 1$.

Finally, if $k = n + 1 \bmod 2n + 1$, $n \geq 1$, then (3.51) and (3.50) produce

$$\begin{aligned}\hat{\pi}_{k+1,n}(t) - t\hat{\pi}_{k,n}(t) &= \hat{T}_{k+1}(t) + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &- t\hat{T}_k(t) + \frac{1}{4} \frac{1}{1 + (k - n - 1)/(2n + 1)} t\hat{T}_{k-1}(t) + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &= -\frac{1}{4} \hat{T}_{k-1}(t) + \frac{1}{4} \frac{1}{1 + (k - n - 1)/(2n + 1)} \hat{T}_k(t) + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\}.\end{aligned}$$

Applying (3.50) to the middle term then yields

$$\begin{aligned}\hat{\pi}_{k+1,n}(t) - t\hat{\pi}_{k,n}(t) &= -\frac{1}{4} \hat{T}_{k-1}(t) + \frac{1}{4} \frac{1}{1 + (k - n - 1)/(2n + 1)} \hat{\pi}_{k,n}(t) \\ &+ \frac{1}{16} \frac{1}{(1 + (k - n - 1)/(2n + 1))^2} \hat{T}_{k-1}(t) + \{\hat{T}_{k-2}, \hat{T}_{k-3}, \dots\} \\ &= -\frac{1}{4} \left(1 - \frac{1}{4} \frac{1}{(1 + (k - n - 1)/(2n + 1))^2}\right) \hat{\pi}_{k-1,n}(t) \\ &+ \frac{1}{4} \frac{1}{1 + (k - n - 1)/(2n + 1)} \hat{\pi}_{k,n}(t),\end{aligned}$$

that is,

$$\begin{aligned}\hat{\alpha}_{k,n} &= -\frac{1}{4} \frac{1}{1 + (k - n - 1)/(2n + 1)}, \\ \hat{\beta}_{k,n} &= \frac{1}{4} \left(1 - \frac{1}{4(1 + (k - n - 1)/(2n + 1))^2}\right) \quad \text{if } k = n + 1 \bmod 2n + 1.\end{aligned}$$

This proves the second relation in (3.33) and the fourth in (3.34). The proof is completed by noting that for all remaining values of k , Eq. (3.51) applies, both as written, and with k replaced by $k + 1$. From this, the last relations in (3.33) and (3.34) follow immediately. $\square$

In analogy to (3.12') and (3.28'), we have, for $k \neq 0$,

$$\hat{\alpha}_{k,n} = \begin{cases} \hat{\alpha}_{1+3(k-n)/(2n+1),1} & \text{if } k = n \bmod 2n + 1, \\ \hat{\alpha}_{2+3(k-n-1)/(2n+1),1} & \text{if } k = n + 1 \bmod 2n + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.33')$$

and

$$\hat{\beta}_{k,n} = \begin{cases} \hat{\beta}_{3k/(2n+1),1} & \text{if } k = 0 \bmod 2n+1, \\ \hat{\beta}_{1+3(k-1)/(2n+1),1} & \text{if } k = 1 \bmod 2n+1, \\ \hat{\beta}_{2+3(k-n-1)/(2n+1),1} & \text{if } k = n+1 \bmod 2n+1, \\ \frac{1}{4} & \text{otherwise .} \end{cases} \quad (3.34')$$

4. INTERLACING PROPERTIES FOR ZEROS

In some applications, for example to extended interpolation [2], [5], one is interested in the zeros τ_ν of $\pi_n(\cdot; d\sigma)$ and the zeros $\hat{\tau}_\mu$ of $\hat{\pi}_{n+1,n}(\cdot) = \pi_{n+1}(\cdot; d\hat{\sigma}_n)$. In particular, one would like to know whether they interlace, i.e., whether, if ordered decreasingly and contained in $(-1,1)$, they satisfy

$$-1 < \hat{\tau}_{n+1} < \tau_n < \hat{\tau}_n < \cdots < \hat{\tau}_2 < \tau_1 < \hat{\tau}_1 < 1. \quad (4.1)$$

This is the question to be studied in this section, first analytically for Chebyshev measures, and then numerically for Gegenbauer measures.

4.1. Chebyshev measures.

THEOREM 4.1. *For the Chebyshev measures $d\sigma^{[i]}$, $i = 1, 2, 3, 4$ (cf. (3.1)), and for any fixed $n \geq 1$, the zeros τ_ν of $\pi_n(\cdot; d\sigma^{[i]})$ interlace with the zeros $\hat{\tau}_\mu$ of $\hat{\pi}_{n+1,n}(\cdot) = \pi_{n+1}(\cdot; d\hat{\sigma}_n^{[i]})$.*

Proof. We carry out the proof only for $d\sigma^{[1]}$, that is $i = 1$, since the proof in the case $i = 3$ is similar, and in the case $i = 2$ trivial (since $\pi_n = \hat{T}_n$, $\hat{\pi}_{n+1,n} = \hat{T}_{n+1}$). The case $i = 4$ follows directly from the case $i = 3$ by virtue of (3.30).

For $\sigma^{[1]}$, one has $\pi_n = \hat{T}_n$ (the monic Chebyshev polynomial of the first kind) and, if $n \geq 2$,

$$\hat{\pi}_{n+1,n}(t) = t\hat{T}_n(t) - \frac{3}{8}\hat{T}_{n-1}, \quad (4.2)$$

as follows from Theorems 3.1 and 3.3. One easily calculates

$$\hat{\pi}_{n+1,n}(1) = \frac{1}{2^{n+1}}, \quad \hat{\pi}_{n+1,n}(-1) = \frac{(-1)^{n+1}}{2^{n+1}}, \quad (4.3)$$

and, since $\tau_\nu = \cos \theta_\nu$, $\theta_\nu = (2\nu - 1)\pi/2n$,

$$\hat{\pi}_{n+1,n}(\tau_\nu) = \frac{3}{2^{n+1}} (-1)^\nu \sin \theta_\nu, \quad \nu = 1, 2, \dots, n. \quad (4.4)$$

For $n = 1$, one has $\pi_1(t) = t$, $\hat{\pi}_{2,1}(t) = t^2 - 3/4$, from which the validity of (4.3), (4.4) follows also for $n = 1$. It is seen that $\hat{\pi}_{n+1,n}(t)$ takes on values with alternating signs as t assumes the values $1, \tau_\nu, \nu = 1, 2, \dots, n$, and -1 in this order. This proves (4.1). $\square$

4.2. Gegenbauer measures. These are the measures $d\sigma^\lambda(t) = (1 - t^2)^{\lambda-1/2}dt$ on $(-1,1)$, where $\lambda > -1/2$. They include the Chebyshev measures of the first and second kind as special cases, $\lambda = 0$ and $\lambda = 1$, respectively. The zeros τ_ν and $\hat{\tau}_\mu$ of $\pi_n(\cdot; d\sigma^\lambda)$ and $\hat{\pi}_{n+1,n}(\cdot) = \pi_{n+1}(\cdot; d\hat{\sigma}_n^\lambda)$ are now functions of λ , and we may study their motion induced by letting λ vary from $\lambda = 0$ to either side of 0. When $\lambda = 0$, we know from Theorem 4.1 that interlacing holds. As we increase (or decrease) λ from the starting value $\lambda = 0$, the interlacing property remains intact so long as no collision occurs between a zero τ_ν and a zero $\hat{\tau}_\mu$. Once such a collision occurs, the polynomials π_n and $\hat{\pi}_{n+1,n}$ have a common zero, and this is tantamount to the resultant $R(\pi_n, \hat{\pi}_{n+1,n})$ of π_n and $\hat{\pi}_{n+1,n}$ vanishing. This resultant, of course, is also a function (indeed a rational function) of λ . Therefore, to determine an interval $\lambda_n < \lambda < \Lambda_n$ in the parameter λ on which the interlacing property holds, we need to determine the smallest positive zero, Λ_n , and the largest negative zero, λ_n , if any, of $R(\pi_n, \hat{\pi}_{n+1,n})$. The same approach has previously been taken in [6] to study the interlacing property for Gauss-Kronrod quadrature formulae, and we refer to this work for computational details. The results obtained are summarized in Table 4.1. Evidently, the λ_n increase, and the Λ_n decrease, monotonically with increasing n . It is an interesting open problem to establish that the limits of λ_n and Λ_n as $n \rightarrow \infty$ indeed exist, and to determine their values.

n	λ_n	Λ_n	n	λ_n	Λ_n
1	$-\frac{1}{2}$	∞			
2	-.367035	∞	14	-.098522	3.010405
3	-.298848	∞	15	-.093053	2.938215
4	-.251041	9.094948	16	-.088177	2.879043
5	-.216429	5.753360	17	-.083804	2.829653
6	-.190358	4.670917	18	-.079856	2.787798
7	-.170044	4.134548	19	-.076275	2.751893
8	-.153770	3.813842	20	-.073010	2.720697
9	-.140435	3.600251	24	-.062397	2.628545
10	-.129301	3.447629	28	-.054536	2.561196
11	-.119861	3.333022	32	-.048469	2.501838
12	-.111750	3.215973	36	-.043641	2.458914
13	-.104704	3.100459	40	-.039703	2.426421

TABLE 4.1. *Intervals $\lambda_n < \lambda < \Lambda_n$ in which the interlacing property (4.1) holds for Gegenbauer measures $d\sigma^\lambda$*

The cases $1 \leq n \leq 3$ of Table 4.1 can be verified algebraically. To do so, one uses the well-known formula,

$$\int_{-1}^1 t^{2m} d\sigma^\lambda(t) = \frac{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(m + \lambda + 1)}, \quad m = 0, 1, 2, \dots,$$

to compute the coefficients of π_n and $\hat{\pi}_{n+1,n}$ from the respective orthogonality relations. The coefficients turn out to be rational functions of λ . One then computes the required resultant explicitly as a rational function of λ and determines its zeros.

For $n = 1$, for example, one finds

$$\pi_1(t) = t, \quad \hat{\pi}_{2,1}(t) = t^2 - \frac{3}{2(\lambda + 2)}, \quad (4.51)$$

and these polynomials obviously have no common zeros. Hence, $\lambda_1 = -\frac{1}{2}$ and $\Lambda_1 = \infty$.

For $n = 2$, one computes

$$\pi_2(t) = t^2 - \frac{1}{2(\lambda+1)}, \quad \hat{\pi}_{3,2}(t) = t^3 - \frac{3(13\lambda+7)}{2(\lambda+4)(5\lambda+3)}t. \quad (4.5_2)$$

Since $t = 0$ is not a common zero, we may consider the simpler polynomials

$$p_2(t) = \pi_2(\sqrt{t}), \quad \hat{p}_3(t) = \frac{1}{\sqrt{t}} \hat{\pi}_{3,2}(\sqrt{t}).$$

They have a common zero if and only if π_2 and $\hat{\pi}_{3,2}$ do. Their resultant computes to

$$R(p_2, \hat{p}_3) = \frac{34\lambda^2 + 37\lambda + 9}{2(\lambda+1)(\lambda+4)(5\lambda+3)},$$

where the denominator is positive for all $\lambda > -\frac{1}{2}$. The numerator has two negative zeros, the larger of which is $(-37 + \sqrt{145})/68 = -.3670353738\dots$ and defines λ_2 , since $R > 0$ for all $\lambda > \lambda_2$.

For $n = 3$, a more lengthy computation gives

$$\begin{aligned} \pi_3(t) &= t^3 - \frac{3}{2(\lambda+2)}t, \\ \hat{\pi}_{4,3}(t) &= t^4 - \frac{15(14\lambda^3 + 71\lambda^2 + 158\lambda + 72)}{(\lambda+6)(26\lambda^3 + 153\lambda^2 + 336\lambda + 160)}t^2 \\ &\quad + \frac{45(22\lambda^5 + 183\lambda^4 + 644\lambda^3 + 1460\lambda^2 + 1776\lambda + 640)}{4(\lambda+6)(\lambda+5)(\lambda+4)(\lambda+2)(26\lambda^3 + 153\lambda^2 + 336\lambda + 160)}. \end{aligned} \quad (4.5_3)$$

Again, these polynomials cannot have $t = 0$ as a common zero, since otherwise $\hat{\pi}_{4,3}$ would have a double zero at the origin, which is impossible ($\hat{\pi}_{4,3}$ is an orthogonal polynomial!). We therefore compute the resultant of $p_3(t) = \frac{1}{\sqrt{t}} \pi_3(\sqrt{t})$ and $\hat{p}_4(t) = \hat{\pi}_{4,3}(\sqrt{t})$ and obtain

$$R(p_3, \hat{p}_4) = -\frac{9(4\lambda^6 + 572\lambda^5 + 5105\lambda^4 + 17298\lambda^3 + 27256\lambda^2 + 17440\lambda + 3200)}{4(\lambda+6)(\lambda+5)(\lambda+4)(\lambda+2)^2(26\lambda^3 + 153\lambda^2 + 336\lambda + 160)}.$$

The cubic polynomial in the denominator has a pair of complex zeros and a negative zero at $\lambda = -.6447375076\dots$, so that the denominator is positive for all $\lambda > -\frac{1}{2}$. The numerator polynomial, on the other hand, has four

negative zeros and a pair of complex conjugate zeros, and is therefore positive for all $\lambda > \lambda_3$, where $\lambda_3 = -.2988479654 \dots$ is the largest negative zero.

5. THE POLYNOMIAL SEQUENCE $\{\hat{\pi}_{n,n-1}\}$

The special sequence of induced polynomials $\{\hat{\pi}_{n,n-1}\}$ (where $\pi_{0,-1} = 1$) whose degrees exceed the degrees of the generating polynomials by one, are of some interest in extended interpolation, as already mentioned. Here we ask the question whether they also form an orthogonal sequence, and if so, relative to what measure.

We restrict ourselves to the original measure being a Jacobi measure, and begin with the case of the Gegenbauer measure

$$d\sigma^\lambda(t) = (1 - t^2)^{\lambda-1/2} dt \text{ on } (-1, 1), \quad \lambda > -1/2. \quad (5.1)$$

Thus,

$$\hat{\pi}_{n,n-1}(t) = \pi_n(\cdot; d\hat{\sigma}_{n-1}^\lambda), \quad d\hat{\sigma}_{n-1}^\lambda = [\pi_{n-1}(\cdot; d\sigma^\lambda)]^2 d\sigma^\lambda. \quad (5.2)$$

In the special cases $\lambda = 0$ and $\lambda = 1$, the polynomials (5.2) are indeed orthogonal. For $\lambda = 1$, this is trivial, since $\hat{\pi}_{n,n-1} = \hat{T}_n$ (cf. Theorem 3.4). For $\lambda = 0$, it is known [5, Eq (2.3)] that $\hat{\pi}_{n,n-1} = 2^{-(n-1)} (T_n - \frac{1}{2} T_{n-2})$, from which it easily follows that

$$\hat{\pi}_{n+1,n}(t) = t\hat{\pi}_{n,n-1}(t) - \beta_n \hat{\pi}_{n-1,n-2}(t), \quad n \geq 1, \quad (5.3)$$

where

$$\beta_n = \begin{cases} 3/4 & \text{if } n = 1, \\ 1/8 & \text{if } n = 2, \\ 1/4 & \text{otherwise.} \end{cases} \quad (\lambda = 0) \quad (5.4)$$

Since $\beta_n > 0$ for all $n \geq 1$, Favard's theorem tells us that there exists a positive symmetric measure with respect to which the polynomial sequence $\{\hat{\pi}_{n,n-1}\}$ is orthogonal. From [7, Eq. (3.8) and Prop. 2.1] it follows indeed that

$$\hat{\pi}_{n,n-1}(\cdot) = \pi_n(\cdot; ds), \quad n \geq 0; \quad ds(t) = \frac{(1-t^2)^{-1/2}}{1 - \frac{8}{9}t^2} \text{ on } (-1, 1). \quad (5.5)$$

We believe that these two values, $\lambda = 0$ and $\lambda = 1$, are the only values of the parameter λ for which the sequence $\{\hat{\pi}_{n,n-1}\}$ in (5.2) defines an orthogonal sequence. The reason for this is as follows. Suppose there are positive constants β_n such that (5.3) holds. Then, putting $n = 3$ in (5.3), using (4.5₁₋₃), and comparing coefficients of equal powers on the left and right, gives an equation in λ , namely

$$a_{30}(\lambda) + [a_{21}(\lambda) - a_{32}(\lambda)]a_{10}(\lambda) = 0,$$

where $a_{ij}(\lambda)$ is the coefficient of t^j in $\hat{\pi}_{i+1,i}$ of Eq. (4.5_i). After some algebraic manipulations, this simplifies to

$$\lambda(\lambda - 1)(188\lambda^4 + 712\lambda^3 - 743\lambda^2 - 3202\lambda - 1680) = 0. \quad (5.6)$$

We again recognize the two known solutions, $\lambda = 0$ and $\lambda = 1$. Since the quartic in (5.6) is less than $188 \cdot 2^{-4} + 3202 \cdot 2^{-1} - 1680 = -67.25 < 0$ for $\lambda \in (-\frac{1}{2}, 0)$, it has no zero in that interval. Indeed, all its zeros are real, less than $-1/2$, except for one positive zero, $\lambda_0 = 2.2276142863\dots$. One could now in principle repeat the reasoning above with values $n \geq 4$ and arrive at a sequence of additional algebraic equations that λ would have to satisfy. It is very unlikely that the single root λ_0 found above is a root of all these equations, and therefore equally unlikely that a recurrence relation (5.3) exists for any λ other than $\lambda = 0$ or $\lambda = 1$. Because of the extreme complexity of the algebra involved, however, we have not carried through the calculations.

For Jacobi measures

$$d\sigma^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta \text{ on } (-1, 1), \quad \alpha > -1, \quad \beta > -1, \quad (5.7)$$

we recall from [5, Eq. (2.17)] that, when $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, then $\hat{\pi}_{n,n-1} = 2^{-(n-1)}(T_n - \frac{1}{2}T_{n-1})$, $n \geq 1$, from which there follows that

$$\hat{\pi}_{n+1,n}(t) = (t - \alpha_n)\hat{\pi}_{n,n-1}(t) - \beta_n\hat{\pi}_{n-1,n-2}(t), \quad n \geq 1, \quad (5.8)$$

holds, with

$$\alpha_n = \begin{cases} -1/4 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \beta_n = \begin{cases} 3/8 & \text{if } n = 1, \\ 1/4 & \text{otherwise.} \end{cases} \quad (5.9)$$

Moreover, from the first line of Table 1 in [7, p. 222], with $\mu = -\frac{1}{2}$, we find

$$\hat{\pi}_{n,n-1}(\cdot) = \pi_n(\cdot; ds), \quad n \geq 0; \quad ds(t) = \frac{(1-t^2)^{-1/2}}{\frac{5}{4} - t} \text{ on } (-1, 1). \quad (5.10)$$

Similar results of course hold for $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, with the signs of α_1 and t reversed in (5.9) and (5.10), respectively. For reasons analogous to those stated in the Gegenbauer case, we conjecture, however, that $\{\hat{\pi}_{n,n-1}\}$ for the general Jacobi measure $d\sigma = d\sigma^{\alpha,\beta}$ cannot be an orthogonal sequence, unless $|\alpha| = |\beta| = \frac{1}{2}$.

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