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Abstract

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m, n) rational ϵ-approximation of a real algebraic plane curve of degree d. At singular points we use an effective version of the Weierstrass Preparation Theorem and Newton power series factorizations, based on the technique of Hensel lifting. These, together with rational Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. Details of the implementation of these algorithms are also provided.

1 Introduction

An algebraic plane curve C of degree d in $\mathbb{R}^2$ is implicitly defined by a single polynomial equation $f(x, y) = 0$ of degree d with coefficients in $\mathbb{R}$. A rational algebraic curve of degree d in $\mathbb{R}^2$ can additionally be defined by rational parametric equations which are given as $(x = G_1(u), y = G_2(u))$, where G_1 and G_2 are rational functions in u of degree d, i.e., each is a quotient of polynomials in u of maximum degree d with coefficients in $\mathbb{R}$. Rational curves are only a subset of implicit algebraic curves of the same degree. While all degree two curves (conics) are rational, only a subset of degree three (cubics) and higher degree curves are rational. In general, a necessary and sufficient condition for the global rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: a curve is rational if and only if $g = 0$, where g, the genus of the curve is a measure of the deficiency of the curve's singularities from its maximum allowable limit [18].

The Rational Approximation Problem

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Given a real algebraic plane curve C: $f(x,y) = 0$ of degree d and of arbitrary genus, a box B defined by $\{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, an error bound $\epsilon > 0$, and integers m, n with $m + n \leq d$ construct a C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of all portions of C within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \leq m$ and degree $Q_i \leq n$. Here C^{-1} means no continuity condition is imposed between the different pieces, C^0 implies there are no gaps and C^1 implies that the derivatives are continuous at the common end points of adjacent pieces.

Results:

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m,n) rational ϵ-approximation of a real algebraic plane curve. At singular points we use an effective version of the Weierstrass Preparation Theorem and Newton power series factorizations, based on the technique of Hensel lifting. These, together with rational Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. Details of the implementation of all these algorithms in GANITH [6] are also provided.

Applications:

In geometric design and computer graphics one often uses rational algebraic curves and surfaces because of the advantages obtained from having both the implicit and rational parametric representations [3], [15]. While the rational parametric form of representing a curve allows efficient tracings, ease for transformations and shape control, the implicit form is preferred for testing whether a point is on the given curve and is further conducive to the direct application of algebraic techniques. Simpler algorithms are also possible when both representations are available. For example, a straightforward method exists for computing curve - curve and surface - surface intersection approximations when one of the curves, respectively surfaces, is in its implicit form and the other in its parametric form. Global parameterization algorithms exist for implicit algebraic curves of genus zero [1, 2] which allows one to compute this dual representation. A solution to our rational approximation problem yields a rational representation, although approximate, and with all the above advantages for arbitrary genus algebraic plane curves.

Prior Work

In [5, 14], power series are constructed to locally approximate plane algebraic curves and surface intersections at simple points. The method of [14] technically relies on the Implicit Function Theorem, seeking to represent a curve branch explicitly in one coordinate as function of the other coordinate(s), while [5] uses a Taylor series expansion. Both these methods however do not seem to have a natural extension that handles singular points. Papers [5] and [12] survey a number of techniques for generating a piecewise linear approximation of an algebraic curve. Further, [16, 17] present techniques for parametric curve approximations which work only for special cases. In this paper we extend these to higher order rational function approximations of the curve as well as deal with all real singularities of the given algebraic curve.

Methods for computing local branch parameterizations at singular points have been presented in [9, 10, 11], based on the Newton polygon, see [18]. We instead use the iterative lifting technique of Hensel together with the fast univariate Padé algorithm of [7]. Additional efficient power series manipulations we use are based on algorithms in [8, 9, 13].

2 Sketch of Algorithm

Input Given a curve C of degree d, a bounding box B, a a finite precision real number ϵ and integers m, n with $m+n \leq d$.

Output A C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of all portions of C within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \leq m$ and degree $Q_i \leq n$ and $m+n \leq d$.

Algorithm We state the algorithm for a C^1 continuous piecewise rational ϵ-approximation. The C^{-1} and C^0 are similar and simpler.

1. Compute all intersections of the given real plane curve C within the given bounding box B. Let the curve within the box B be denoted by C_B. Next, compute all singularities on the bounded plane curve C_B . Denote the set of all these points by V. The curve C_B yields a natural hypergraph[1] G having V as its vertex set and the set of curve segments of C_B joining any pair of points in V, as its edge set E. Now starting from a singular point (x_i, y_i) in V we trace out an Eulerian tour of the hypergraph G, approximating each of the edges by C^1 continuous piecewise rational curves.

2. Compute a Newton factorization for each singular point (x_i, y_i) in V and obtain a power series representation for each analytic branch of C at (x_i, y_i) and given by

$$\begin{cases} X(s) = x_i + s^{k_i} \\ Y(s) = \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = y_i \end{cases} \tag{1}$$

or

$$\begin{cases} Y(s) = y_i + s^{k_i} \\ X(s) = \sum_{j=0}^{\infty} \tilde{c}_j^{(i)} s^j, \quad c_0^{(i)} = x_i \end{cases} \tag{2}$$

3. Without loss of generality, we only consider the case where the analytic branch at the singularity is of type (1). Compute $\frac{P_m(s)}{Q_n(s)}$ the (m, n) Padé approximation of $Y(s)$. That is $\frac{P_m(s)}{Q_n(s)} - Y(s) = O(s^{m+n+1})$

4. Compute $\beta > 0$ a real number, corresponding to points $(\tilde{x}_i = X(\beta), \tilde{y}_i = Y(\beta))$ and $(\hat{x}_i = X(-\beta), \hat{y}_i = Y(-\beta))$ on the analytic branch of the original curve C, such that $\frac{P_m(s)}{Q_n(s)}$ is convergent for $s \in [-\beta, \beta]$

5. Determine a and b in

$$\frac{\tilde{P}_m(s)}{Q_n(s)} = \frac{P_m(s) + s^k(a + bs)}{Q_n(s)} \tag{3}$$

[1] A graph with perhaps multiple edges between a pair of vertices

for some k, $2 \leq k < m$ such that

$$a = \frac{(k+1)(YQ_n - P_m)(\beta) - \beta(YQ_n - P_m)'(\beta)}{\beta^k}, \tag{4}$$

$$b = \frac{\beta(YQ_n - P_m)'(\beta) - k(YQ_n - P_m)(\beta)}{\beta^{k+1}} \tag{5}$$

to achieve C^1 continuity at the point $(\tilde{x}_i, \tilde{y}_i)$ between the original curve and the ϵ-approximation

$$X(s) = x_i + s^{k_i}$$
$$Y(s) = \frac{\tilde{P}_m(s)}{Q_n(s)}$$

within the range $s \in [0, \beta]$. Similarly repeat the a, b computation to achieve C^1 continuity at the point $(\hat{x}_i, \hat{y}_i)$ between the original curve and another ϵ-approximation

$$X(s) = x_i + s^{k_i}$$
$$Y(s) = \frac{\hat{P}_m(s)}{Q_n(s)}$$

within the range $s \in [-\beta, 0]$.

6. Update the hypergraph G by adding two additional vertices $(\tilde{x}_i, \tilde{y}_i)$ and $(\hat{x}_i, \hat{y}_i)$ thereby splitting the original edges incident to the singularity (x_i, y_i), and tagging the new edges $[(x_i, y_i), (\tilde{x}_i, \tilde{y}_i)]$ and $[(x_i, y_i), (\hat{x}_i, \hat{y}_i)]$ as visited. The Eulerian tour is continued with vertex point $(\tilde{x}_i, \tilde{y}_i)$ in the direction away from the singularity.

7. For simple points $(\breve{x}_i, \breve{y}_i)$ on the curve a Taylor expansion yields (with loss of generality) the single analytic branch given, by

$$X(s) = \breve{x}_i + s$$
$$Y(s) = \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = \breve{y}_i$$

Exactly the same steps as above are used for determining the β a, and b for a C^1 ϵ-approximation of these analytic branches. The C^1 continuity here is achieved at the point $(\tilde{x}_i = X(\beta), \tilde{y}_i = Y(\beta))$ between the original curve and the ϵ-approximation and the hypergraph is updated with only this single vertex.

8. New starting vertex points are chosen in the hypergraph till all edges are visited exactly once in the Euler tour. For each visited edge the C^1 piecewise approximation rational curves are stored in a separate list and finally output.

3 Details and Correctness of Algorithm

3.1 Expansion at Simple Points

Let $f(x,y) = \sum a_{ij}, x^i y^j = 0$ be an algebraic curve and (x_0, y_0) be a simple point on it. By a simple translation $x = \tilde{x} - x_0, \ \ y = \tilde{y} - y_0$ we may assume that $(x_0, y_0) = (0,0)$, i.e., $f(0,0) = 0$. Since $(0,0)$ is a simple point of the curve, we assume without loss of generality, that $f_y(0,0) \neq 0$. Consider $f(x,y)$ in its recursive canonical form (RCF) form as a polynomial in y coefficients polynomials in x: $f(x,y) = a_0(x) + a_1(x)y + \cdots + a_d(x)y^d$, with $a_i(x) = \sum_{i=0}^{m_i} a_{ij} x^j, \ \ i = 0, 1, \ldots, d$ and by the earlier assumption $a_1(0) = a_{10} \neq 0$. Let $y(x) = \sum_{i=1}^{\infty} A_i x^i = \sum_{i=1}^{\infty} A(1)_i x^i$ Then

$$y^j(x) = \sum_{i=j}^{\infty} A(j)_i x^i \tag{6}$$

where $A(j)_i = \sum_{i_1 + i_j = i} A_{i1} A_{i2}, \ldots, A_{ij}$. Hence $A(j+1)_i = \sum k = j^{i-1} A(j)_k A_{ik}$ substituting (6) into $f(x,y) = 0$, we have

$$\begin{aligned} 0 &= a_0(x) + \textstyle\sum_{j=1}^{n} a_j(x) \sum_{i=j}^{\infty} A(j)_i x^i \\ &= a_0(x) + \textstyle\sum_{j=1}^{n} \ \sum_{i=j}^{\infty} B(j)_i x^i \\ &= \textstyle\sum_{i=1}^{m_1} a_{0i} x^i + \sum_{i=1}^{\infty} B(1)_i x^i + \sum_{i=2}^{\infty} \sum_{j=2}^{\min(i,n)} B(j)_i) x^i \end{aligned} \tag{7}$$

where $B(j)_i = \sum_{s=0}^{\min\{i-j, m_j\}} a_{j_s} A(j)_{i-s}, \ \ j = 1, 2, \ldots, i \geq j$ It follows from (7) that

$$B(i)_1 + a_{01} = 0 \tag{8}$$

$$B(1)_i + a_{0i} + \sum_{j=2}^{\min(i,n)} B(j)_i = 0, \ i = 2, 3, \ldots, \tag{9}$$

Since $B(1)_i = \sum_{s=0}^{\min\{i-1,m\}} a_{1s} A_{i-s} = a_{10} A_i + \sum_{s=1}^{m+n\{i-1,m_1]\}} a_{1s} A_{i-s}$, it follows from (8) and (9) that

$$\begin{aligned} A_1 &= -a_{01}/a_{10} \\ A_i &= -\left[\textstyle\sum_{s=1}^{\min\{i-1,m_1\}} a_{1s} A_{i-s} + a_{0i} + \sum_{j=2}^{\min(i,n)} B(j)_i \right] / a_{10}, \quad i = 2, 3, \ldots, \end{aligned}$$

3.2 Expansion at Singular Points

3.2.1 Hensel Lifting

Consider $f(x,y)$ of degree d. Assume it is monic in y .

$$f(x,y) \ = \ f_0(y) \ + \ f_1(y)x \ + \ \cdots \ + \ f_k(y)x^k \ + \ \cdots$$

We wish to compute real power series factors $g(x,y)$ and $h(x,y)$ where $f(x,y) = g(x,y)h(x,y)$. The technique of Hensel lifting allows one to reconstruct the power series factors

$$\begin{aligned} g(x,y) &= g_0(y) + g_1(y)x + \cdots + g_i(y)x^i + \cdots \\ h(x,y) &= h_0(y) + h_1(y)x + \cdots + h_j(y)x^j + \cdots \end{aligned} \tag{10}$$

from initial factors $f(0,y) = f_0(y) = g_0(y)h_0(y)$.

Consider the factorization of $f(0,y) = f_0(y)$ as the base case of $k = 0$. Assume $f_0(y)$ is of degree d. Choose real coprime factors $g_0(y)$ of degree p and $h_0(y)$ of degree q satisfying: $p + q = d$. Real coprimeness is achieved by ensuring that g_0 and h_0 contain distinct real roots of f_0 and that complex conjugate pairs are not split up. For the case $d = 2$ however, it may arise that the only coprime factors of f_0 are complex, i.e., the distinct roots are complex conjugates. In that case there only exist complex power series solutions. Since $GCD(g_0(y), h_0(y)) = 1$ using the fast GCD algorithm we can also compute $\alpha(y)$ and $\beta(y)$ such that $\alpha(y)g_0(y) + \beta(y)h_0(y) = 1$

In the iterative Case of $k \geq 1$, we compute $g_k(y)$ and $h_k(y)$ of the desired factorization (10), with degree of $g_k(y) < p$ and degree of $h_k(y) < q$, as follows. We note from (10) that

$$f_k(y) = \sum_{i+j=k} g_i(y)h_j(y)$$

and additionally

$$f_k(y) - \sum_{i<k \wedge j<k} g_i(y)h_j(y) = g_0(y)h_k^*(y) + h_0(y)g_k^*(y) \tag{11}$$

Hence,

$$\begin{aligned} h_k^*(y) &= \alpha(y)[f_k(y) - \sum_{i<k \wedge j<k} g_i(y)h_j(y)] \\ g_k^*(y) &= \beta(y)[f_k(y) - \sum_{i<k \wedge j<k} g_i(y)h_j(y)] \end{aligned}$$

If degree $h_k^*(y) \geq q$ then compute $h_k(y) = h_k^*(y) \bmod h_0(y)$ and set $g_k(y) = \gamma(y)g_0(y) + g_k^*(y)$ where $h_k^*(y) = \gamma(y)h_0(y) + h_k(y)$.

$$f_k(y) - \sum_{i<k \wedge j<k} g_i(y)h_j(y) = g_0(y)h_k(y) + h_0(y)g_k(y) \tag{12}$$

Clearly degree $h_k(y)$ is $< q$. Additionally in (12) the degree of $g_k(y)$ must also be $< p$. This is so because in (12) the degree of the LHS is $< d$ and since degree $g_0(y)h_k(y)$ is $< d$ and degree $h_0(y)$ is $= q$, it must be that degree $g_k(y)$ is $< p$.

3.2.2 Weierstrass Factorization

Consider $f(x,y)$ with degree d and $ord_y\ f(0,y) = e < \infty$. An $ord_y\ f(0,y) = \infty$ corresponds to $f(0,y) = 0$. This can easily be rectified by a simple linear transformation of $f(x,y)$, which yields a nonzero $f(0,y)$ and hence a finite $ord_y\ f(0,y)$. We wish to compute a power series factorization of the form $f(x,y) = g(x,y)\underbrace{(y^e + a_{e-1}(x)y^{e-1} + \cdots + a_0(x))}_{h(x,y)}$ where $g(x,y)$ is a unit power series, i.e., $g(0,0) \neq 0$ while $h(x,y)$ is a polynomial in y with coefficients $a_i(x)$, $i = 0 \ldots e-1$ being non-unit power series, i.e., $a_i(0) = 0$. Such a factorization is known as a Weierstrass preparation and is always possible as we now show.

The Weierstrass preparation can efficiently be achieved via Hensel Lifting. Given

$$f(x,y) = f_0(y) + f_1(y)x + \cdots + f_k(y)x^k + \cdots$$

with

$$f(0,y) = f_0(y) = \underbrace{(a_0 + a_1 y + \cdots)}_{g_0(y)} \underbrace{y^d}_{h_0(y)}$$

in general for $k \geq 1$, we wish to compute $h_k(y)$ and $g_k(y)$ using Hensel, yielding factors similar to (10) such that

$$f_k(y) - \sum_{i+j=k} g_i(y)h_j(y) = g_0(y)h_k(y) + y^e g_k(y) \tag{13}$$

with degree $h_k(y) < e$.

To achieve this we compute $A(y) = \frac{f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y)}{g_0(y)}$ and then set $h_k(y) =$ terms of $A(y)$ with degree $< e$ and $g_k(y) =$ terms of $A(y)$ with degree $\geq e$.

3.2.3 Newton Factorization

Consider $f(x,y)$, a monic polynomial in y of degree d, with coefficients polynomial or power series or meromorphic series in x

$$f(x,y) = y^d + a_{n-1}(x)y^{d-1} + \cdots + a_0(x)$$

Then it is possible to factor $f(x,y)$ into linear factors

$$f(x,y) = \Pi_{i=1}^{d} (y - \eta_i(t))$$

with $x = t^m$ and m a positive integer and $\eta_i(t)$ power series or meromorphic series. This factorization can also be achieved via Hensel lifting. We precondition the curve so that it admits a non-trivial base factorization, i.e. having at least two coprime factors which can be lifted.

Step 1: Make $a_{d-1}(x) = 0$ via substitution $\tilde{y} = y + \frac{a_{d-1}(x)}{d}$

Step 2: Ensure some $a_{d-i}(0) \neq 0$ for $i \geq 2$ via substitution $\breve{y} = \frac{\tilde{y}}{x^\lambda}$ with $\lambda = min_{(2 \geq i \geq d)} \frac{\alpha_i}{i}$ and $\alpha_i = ord_x \alpha_{d-i}(x)$. Then $f(0,\breve{y}) = f_0(\breve{y})$ has at least two distinct roots.

Step 3: Now use Hensel lifting to lift the factorization $f_0(\breve{y}) = g_0(\breve{y})h_0(\breve{y})$ to $f(x,\breve{y}) = g(x,\breve{y})h(x,\breve{y})$. Repeat Steps 1-3 until all factors are linear or all real factors are obtained.

3.2.4 Local Parameterization

Consider an implicit plane algebraic curve $f(x,y) = 0$, with a singularity at the origin. For a non-rational singularity we compute a rational approximation to the singularity as well as determine the multiplicity order of the singularity. This order k is the minimum order of the partials which are greater than machine precision. On translating the curve to make the singularity to be at the origin, we also discard all monomial terms of degree less tahn k.

To compute a local parametric approximation of each of the curve's branches incident at the origin, we execute the following steps:

1. Compute a Weierstrass power series factorization of $f(x,y)$ into $f = gh$, where $g((x,y))$ is a unit power series and $h((x))(y)$ is a polynomial in y with coefficients non-unit power series in x. The equation $h = 0$ corresponds to the curve's branches at the origin while the power eries equation $g = 0$ corresponds to the portion of the plane curve away from the origin.

2. Recursively apply the Newton factorization to $h((x))(y)$ till all factors are linear in y or all real factors are obtained. Each of these power series factors represent a local branch parameterization of the type $x = t^k$ and $y = b_i((t))$ where b_i is a power series. The minimum of k and $ord_t(b_i)$, say e, is known as the order of the branch, with $e > 1$ implying a singular branch of the curve.

3.3 C^1 Continuous Padé Approximation

3.3.1 C^1-continuity

Let $P_m(s)/Q_n(s)$ be the (m,n) Padé approximation of $Y(s)$, That is $\frac{P_m(s)}{Q_n(s)} - Y(s) = O(s^{m+n+1})$. Let $\beta > 0$ be a real number, corresponding to a point on the analytic branch of the original curve C, such that $\frac{P_m(s)}{Q_n(s)}$ is convergent for $s \in [0,\beta]$. This β is determined in subsection 3.3.2. Consider

$$\frac{\tilde{P}_m(s)}{Q_n(s)} = \frac{P_m(s) + s^k(a+bs)}{Q_n(s)}, \qquad 2 \le k < m \tag{14}$$

Note that the above choice of $\tilde{P}_m(s)$ does not change either the degree of the approximation nor the order of the approximation error (shown in subsection 3.3.2). Any choice of k within the allowed range suffices and we have currently left that as a parameter in our implementation. For a fixed choice of k, determine a and b such that

$$\frac{\tilde{P}_m(\beta)}{Q_n(\beta)} = Y(\beta), \qquad C^0 \text{ continuity} \tag{15}$$

and further

$$\left(\frac{\tilde{P}_m}{Q_n}\right)'(\beta) = Y'(\beta), \qquad C^1 \text{ continuity} \tag{16}$$

Hence, for C^0 continuity, we have

$$a = \frac{Y(\beta)Q_n(\beta) - P_m(\beta)}{\beta^k}, \tag{17}$$

and $b = 0$. For C^1 continuity, it follows from (15) and (16) that

$$a + b\beta = \frac{Y(\beta)Q_n(\beta) - P_m(\beta)}{\beta^k}, \tag{18}$$

$$ka + (k+1)b\beta = \frac{YQ_n - P_m{}'(\beta)}{\beta^{k-1}}. \tag{19}$$

Since the matrix

$$\begin{bmatrix} 1 & \beta \\ k & (k+1)\beta \end{bmatrix} \tag{20}$$

is nonsingular for $\beta \neq 0$, equations (18) and (19) have a unique solution and

$$a = \frac{(k+1)(YQ_n - P_m)(\beta) - \beta(YQ_n - P_m)'(\beta)}{\beta^k},$$

$$b = \frac{\beta(YQ_n - P_m)'(\beta) - k(YQ_n - P_m)(\beta)}{\beta^{k+1}}$$

For C^{-1} continuity , i.e no continuity constraints, $a = b = k = 0$. For C^0 continuity , i.e no gaps, $b = 0$ and a is computed in a similar fashion as above for some fixed k such that $1 \leq k \leq m$.

3.3.2 Approximation Error Bound

We now compute $\beta > 0$ a real number, corresponding to a point on the analytic branch of the original curve C, such that $\frac{P_m(s)}{Q_n(s)}$ is convergent for $s \in [0, \beta]$.

Note that $\tilde{P}(s) = P_m(s) + s^k(a + bs)$, where a and b are choosen to enforce C^1 continuity . Since

$$Y(s) - \frac{\tilde{P}_m(s)}{Q_n(s)} = \frac{Y(s)Q_n(s) - P_m(s) - s^k(a+bs)}{Q_n(s)},$$

$s^k(a + bs)$ can be regarded as an C^1 interpolating polynomial of $Y(s)Q_n(s) - P_m(s)$ at points 0 and β. Hence we have

$$Y(s) - \frac{\tilde{P}_n(s)}{Q_n(s)} = \frac{(YQ_n - P_m)^{(k+2)}(\xi))}{Q_n(s)(k+2)!} s^k (s-\beta)^2, \quad \xi \in [0, \beta].$$

where $(YQ_n - P_m)^{(k+2)}$ is the $(k+2)^{th}$ derivative of the power series. Since

$$|s^k(s-\beta)^2| \leq \frac{4k^k\beta^{k+2}}{(k+2)^{(k+2)}}, \qquad s \in [0, \beta],$$

we have

$$\left| Y(s) - \frac{\tilde{P}(s)}{Q_n(s)} \right| \leq \left| \frac{(YQ_n - P_m)^{(k+2)}(\xi))}{Q_n(s)(k+2)!} \right| \frac{4k^k\beta^{k+2}}{(k+2)^{(k+2)}}$$

From $(YQ_n - P_m)(s) = \sum_{i=m+n+1}^{\infty} e_i s^i$, we have

$$\left| \frac{(YQ_n - P_m)^{(k+2)}(\xi)}{(k+2)!} \right| \leq \sum_{i=m+n+1}^{\infty} |e_i| C_{k+2}^i \beta^{i-k-2}.$$

Let $Q_n^{-1}(s) = \sum_{i=0}^{\infty} q_i s^i$ and $|Q_n^{-1}(s)| \leq \sum_{i=0}^{\infty} |q_i| \beta^i$, then

$$\left| \frac{(YQ_n - P_m)^{(k+2)}(\xi)}{Q_n(s)(k+2)!} \right| \leq \left| \frac{(YQ_n - P_m)^{(k+2)}(\xi)}{(k+2)!} \right| |Q_n^{-1}(s)| = (\sum_{i=0}^{\infty} r_i \beta^i) \beta^{m+n-k-1}$$

$$\left| Y(s) - \frac{\tilde{P}(s)}{Q_n(s)} \right| \leq (\sum_{i=0}^{\infty} r_i \beta^i) \frac{4k^k \beta^{m+n+1}}{(k+2)^{(k+2)}} \tag{21}$$

An interesting property of (21) is that the order of the approximation does not depends on k. For C^0 continuity, a similar bound can be obtained.

In our implementation, we take $\sum_{i=0}^{\infty} r_i\beta^i \approx r_0 + r_1\beta_1$ and then determine β_1 such that

$$(r_0 + r_1\beta_1)\frac{4k^k\beta_1^{m+n+1}}{(k+2)^{(k+2)}} \le \epsilon$$

Next compute the smallest pole of the rational function, i.e.

$$\beta_2 = c * \min\{z_i : Q_m(z_i) = 0, \quad z_i \in I\!R\}$$

for some positive constant $c < 1$ and take $\beta_3 = \min\{\beta_1, \beta_2\}$. From the point on the Padé approximation $\left\{ \begin{array}{lcl} x_i & = & X(\beta_3) \\ y_i & = & P_m(\beta_3)/Q_n(\beta_3) \end{array} \right\}$ we compute via Newton's method the nearest point $(\tilde{x}_i, \tilde{y}_i)$ on the analytic branch

$$X(s) = x_i + s^{k_i}$$
$$Y(s) = \sum_{j=0}^{\infty} c_j^{(i)} s^j$$

of the original curve $f(x, y) = 0$. Finally we determine β from the equation $\beta^{k_i} = \tilde{x}_i - x_i$.

4 Implementation Issues

The rational approximation algorithms has been implemented in its entirety as part of GANITH, an X-11 based interactive algebraic geometry toolkit, using Common Lisp for the symbolic computation and C for all numeric and graphical computation. The Hensel power series computations of section 3, as well as its use in sections Weierstrass and Newton factorizations are based on a robust implementation of the fast euclidean HGCD algorithm [7]. Rational Padé approximants are also computed based on the same HGCD algorithm, [7]. Power Series are stored as truncated sparse polynomials, as are the original algebraic curves, in recursive canonical form. Floating point coefficients are allowed in the input curve representations, which are then converted to rational numbers for the GCD and power series computations. In Newton factorizations, user options are provided to compute only real branch factorizations. This is achieved by not allowing complex conjugate roots of the appropriate univariate polynomial, to split in the base case of the Henselian computation. Singularity computations and intersection with the bounding box are done in GANITH using multivariate resultants and based on the method of birational maps [4].

Examples from the software implementation, are shown in Figures 1, 2, and 3, at the end of the paper. Figure 1, shows C^{-1}, C^0 and C^1 continuous piecewise rational cubic approximations of an implicitly defined sextic plane curve with a tacnodal singularity at the origin. Figure 2, shows the C^1 approximation of Figure 1, for different values of ϵ. Finally, Figure 3, show C^1 continuous piecewise rational quadratic and cubic approximations of several quartic and sextic curves.

5 Conclusions and Future Research

The results of this paper are being extended to deal with power series and Padé computations in two or more variables. In particular,our goals are to compute

1. power series expansions about singular points and curves on real algebraic surfaces, to yield bivariate local parameterizations. The case of singular points on surfaces is extremely difficult and Newton series factorizations are not always possible.

2. suitable simple points and curves about which Taylor expansions coupled with Padé approximations would yield a a piecewise continuous rational surface approximation of a real algebraic surface.

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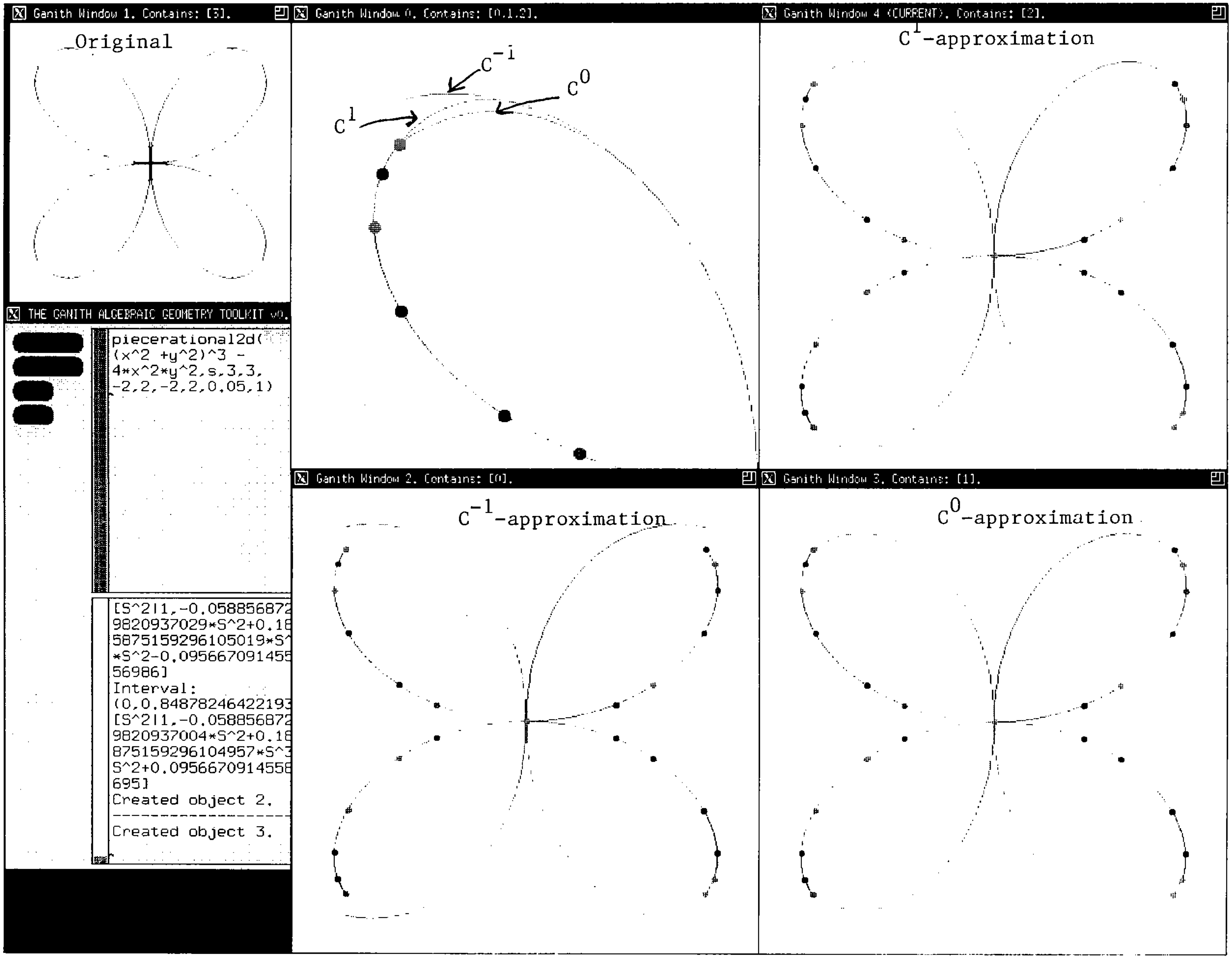

Figure 1: Piecewise rational cubic approximations of $(x^2 + y^2)^3 - 4x^2y^2$.

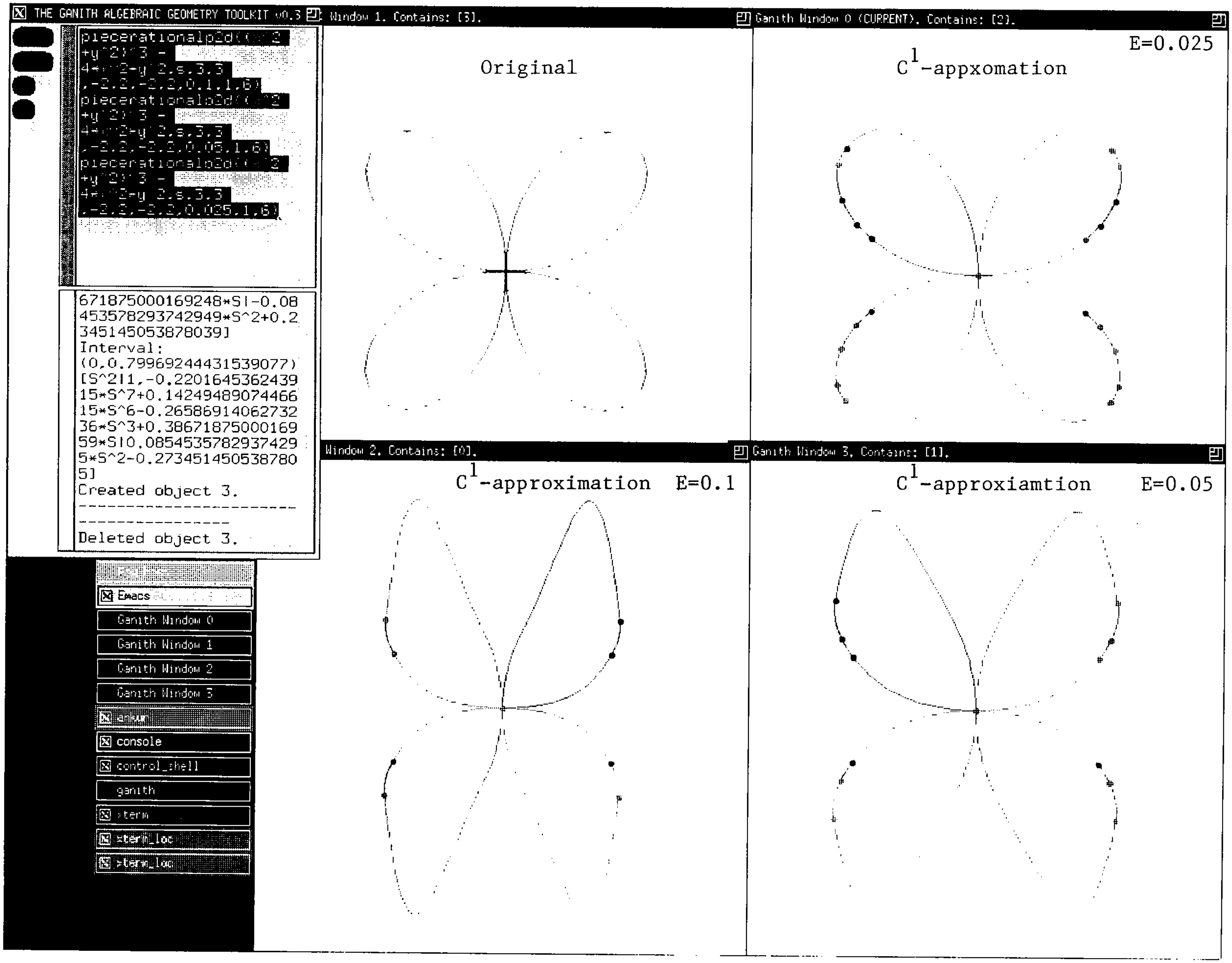

Figure 2: Piecewise C^1 rational cubic approxiamtions with different E.

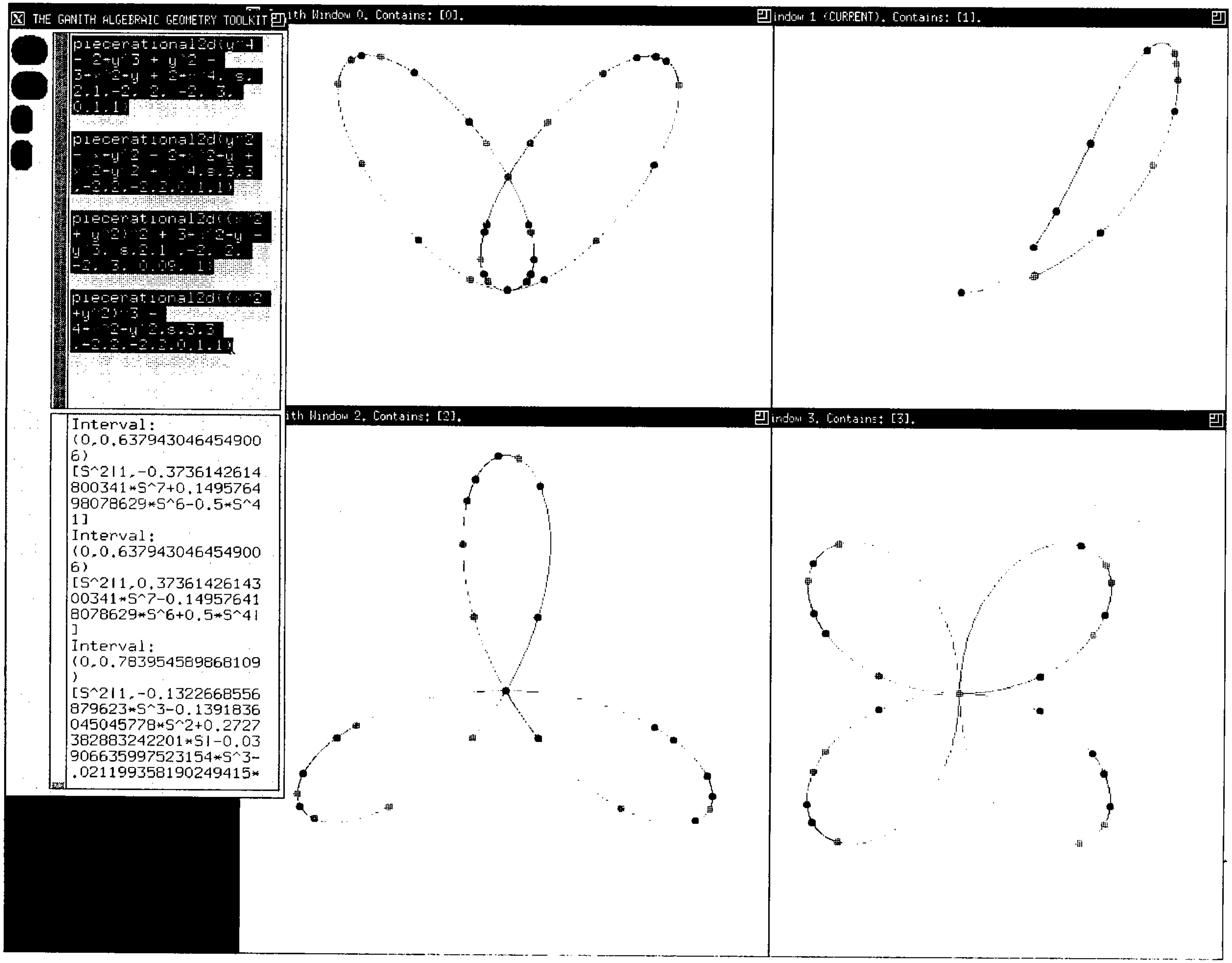

Figure 3: Examples of piecewise C^1 rational quadratic and cubic approximations.