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Abstract

We use a combination of both symbolic and numerical techniques to construct a degree bounded C^k-continuous, rational B-spline ϵ-approximations of real algebraic surface-surface intersection curves. The algebraic surfaces could be either in implicit or rational parametric form. At singular points we use the classical Newton power series factorizations to determine the distinct branches of the space intersection curve. Besides singular points we obtain an adaptive selection of regular points about which the curve approximation yields a small number of curve segments yet achieves C^k continuity between segments. Details of the implementation of these algorithms and approximation error bounds are also provided.

1 Introduction

It is well known that the set of parametric algebraic curves and surfaces are a subset of algebraic curves and surfaces of the same degree. In particular the intersection curve of two parametric surfaces may not be parametric. See [2] for a discussion of these facts and the desired use of parametric representations for certain geometric modeling and display operations. To compute parametric representations for non-parametric algebraic intersection curves requires approximation. In this paper we present algorithms to construct a piecewise rational B-spline (a degree bounded, piecewise parametric, C^k continuous) approximation and a piecewise standard NURBS (rational B-splines with positive denominator polynomial) approximation to a space curve, which comes from the intersection of two implicit defined surfaces (IIS); or the intersection of two parametric defined surfaces (IPS). Though we restrict discussions to implicit and parametric algebraic surface-surface intersections, the algorithms can be directly extended to the intersection curve of

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arbitrary analytic surfaces. The results of this paper are a natural extension of piecewise rational approximation of non-parametric algebraic plane curves[3]. Piecewise rational B-spline approximations find increasing use in interactive geometric design[18] and in the contouring of scattered data[20].

There are mainly two solution approaches to the handling of the piecewise rational B-spline approximation problem. One is based on the subdivision of the enclosing three dimensional space into small cells where the intersection curve is evaluated in each cell by first transforming it into Bernstein-Bezier (BB) form. For a topologically correct approximation this method can only cope with a restricted class of point singularities with distinct tangents. The other method of tackling the approximation problem is based on a tracing of the intersection curve[4, 19] without any special consideration to singularities. In this paper, we too adaptively march along the intersection curve, paying special heed to singular points, followed by a corresponding stitching together of the approximating rational B-spline curve segments. We construct a piecewise rational B-spline approximations to two kinds of space intersection curves (IIS, IPS). Except at singular points, the composite curve keeps a simpler variant of Frenet frame continuity based on the curve arc length as the parameter. Among the various local parameterizations of the space curve, taking arc length as parameter has several advantages as indicated in section 3.

In recent years, several authors have discussed the notion of geometric continuity at a common point for two incident space curves[11, 13, 22, 23]. Furthermore there are a plenty of references which use different continuity criteria and construct parametric B-splines to approximate an ordered list of points (see for e.g. [7, 9, 23]). The frame continuity used in this paper is a simpler form of geometric continuity with the connection matrix being diagonal, and differs from the well known Frenet-frame continuity that has a lower triangular connection matrix. We also exhibit how Padé approximation can be adapted to yield very natural Hermite approximations.

This paper is organized as follows: Section 2 formally defines the piecewise approximation problem and an outline of our piecewise rational B-spline approximation algorithm. Section 3 presents some mathematical preliminaries and also motivates our choice of frame continuity based on arc length parameterizations. Section 4 and 5 discuss the expansion of the two different surface-surface intersection curves into power series based on arc length. In Section 6, we illustrate how to use Newton iterations to cope with the solution of under-determined system of non-linear equations at regular curve points and over-determined system of non-linear equations at singularities. Section 7 presents details of several approaches to construct a rational parametric curve segment which is C^k frame continous at end points. Here we extend traditional Padé approximation techniques [17] to develop a two point Padé rational parametric curve interpolant. In section 8, we transform the piecewise rational approximation into piecewise rational B-splines and standard NURBS representations. Sections 9 and 10 treat the problem of isolating singular points on the intersection curve and the use of Newton factorizations to construct the approximation of the distinct branches of the curve at these singular points. Finally, in Section 11 we discuss the implementation of our algorithm and present several examples.

2 The Problem and the outline of the Algorithm

The Rational Approximation Problem

Given a real intersection space curve SC which is either

(a). the intersection of implicit surfaces (IIS) defined by $f_1(x,y,z)=0$, $f_2(x,y,z)=0$, and within a bounding box $B=\{(x,y,x): x_0 \le x \le x_1,\quad y_0 \le y \le y_1,\quad z_0 \le z \le z_1\}$

(b). the intersection of parametric surfaces(IPC) defined by

$$\begin{aligned} X_1(u_1,v_1) &= [G_{11}(u_1,v_1)\; G_{21}(u_1,v_1),\; G_{31}(u,v_1)]^T \\ X_2(u_2,v_2) &= [G_{12}(u_2,v_2)\; G_{22}(u_2,v_2),\; G_{32}(u_2,v_2)]^T \end{aligned}$$

and within a bounding box

$$\begin{aligned} B=\{(u_1,v_1,u_2,v_2) \;:\; & u_{10} \le u_1 \le u_{11},\quad v_{10} \le v_1 \le v_{11} \\ & u_{20} \le u_2 \le u_{21},\quad v_{20} \le v_2 \le v_{21}\} \end{aligned}$$

and an error bound $\epsilon > 0$, a continuity index k, construct a C^k (or G^k) continuous piecewise parametric rational ϵ-approximation of all portions of SC within the given bounding box B.

The Outline of the Algorithm

The approximation process is a tracing procedure along the curve. It consists of the following steps:

1. Form a starting point list (SPL) by computing the boundary points containing the intersection points of the curve SC and the bounding box B. Further SPL is made to contain at least one point for each inner loop component of SC i.e. a curve loop completely inside the given box B. Tracing direction are also provided at each of these points in SPL. (See section 11 for implementation details).

2. Test if SPL is empty. If yes, the tracing is finished. Otherwise, starting from a point p in SPL, trace the curve along the given direction until either of the following tests in step 3 or step 4 are true. The tracing step consists of the following sub-steps:

   (a). Compute an arc length based power series expansion (see sections 4 and 5) up to $k+1$ terms at the given point p.

   (b). Determine a step-length and a point $\hat{q}$ on the above expansion curve in the tracing direction within a step-length of p. and then starting from $\hat{q}$ refine to a new point q on the curve SC by Newton iterations (see section 6).

   (c). Compute an arc length power series expansion up to $k+1$ terms at the new point q.

   (d). Construct an approximating rational parametric curve segment by C^k Hermite interpolation (see section 7) of the two end points p and q and convert it into a rational B-spline or stand NURB with positive denominator polynomial (see section 8).

(e). Add the rational curve approximant into the piecewise approximation list and return to step 2, to continue the tracing from the the newly constructed point q.

3. Test if a singular point is met. If yes, stop the present tracing and put the end point of the tracing into SPL (we may delete a few approximation segments from the present approximation list, because the step length near a singular point is small). Then locate the singular point(see section 9), obtain a finite set of power series expansion at the singular point corresponding to the distinct curve branches (see section 10). Trace each branch one or two steps, and then put the end points of the tracing into SPL. Then return to Step 2.

4. Test if another point in SPL is met(see §12). If yes, we have stitched together one continuous segment of the curve. Delete the two end points of the traced segment from SPL and return to Step 2.

3 Mathematical Preliminaries

In this paper, we will express a space curve as a power series, locally at at point and with its arc length as a parameter. We refer to [8] for some intrinsic parameters of space curves. Let $r(s) = [x(s), y(s), z(s)]^T$ be a space curve, where s is arc length of the curve measured from some fixed point. The tangent vector $t(s) = r'(s)$ has unit length; $k(s) = \|r''(s)\|$ is the curvature, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. Further, $n(s) = r''(s)/k(s)$ is the principle normal; $b(s) = t(s) \times n(s)$ is the binormal, where $\times$ denotes the cross product of two vectors. Finally, the number $T(s)$ defined by $b'(s) = -T(s)n(s)$ is the torsion. The three orthogonal vectors $t(s), n(s)$ and $b(s)$ form the so called Frenet frame. These vectors are related by the following Frenet formulas

$$t' = kn, \qquad b' = -Tn, \qquad n' = -kt + Tb$$

The derivatives of $r(s)$ are therefore given by

$$r'(s) = t, \qquad r''(s) = kn, \qquad r'''(s) = k'n + kTb - k^2 t \tag{3.1}$$

Since $t = r'(s)$, $k = \|r''(s)\|$ and $T = r'(s) \times r''(s) \cdot r'''(s)/\|r''(s)\|^2$ then the curve is obviously tangent, or curvature or torsion continuous if $r'(s)$, or $r'(s)$ and $r''(s)$, or $r'(s)$, $r''(s)$ and $r'''(s)$ is continuous respectively. In this paper, we construct a piecewise approximation of the given curve such that the composite curve is tangent ($t(s)$), normal ($n(s)$) and binormal ($b(s)$) continuous.

Among the various local parameterizations of the space curve, taking arc length as parameter has several advantages.

A. If $r(s)$ is the parameterization of the given curve and s is arc length start from some point, the $r'(s)$, $r''(s)$, $r'''(s)$ is equivalent to $t(s)$, $n(s)$, $b(s)$ in the sense that the continuity of $r'(s)$, $r''(s)$, $r'''(s)$ are equivalent to the continuity of $t(s)$, $n(s)$, $b(s)$ where the triple $t(s)$, $n(s)$, $b(s)$ is the Frenet frame. Therefore, we need only to force the composite curve's first three derivatives to be continuous at the break points without considering the connection matrix as in the case of geometric continuity.

B. Since the arc length of the curve is independent of any coordinate system, then the expansion of power series may have larger convergence radius. This will, in turn, lead to less segments of approximation.

C. In geometry point of view, the frame of Frenet continuous is the most natural and useful requirement. It keeps the tangent, principle normal and binormal varying continuously, while other geometric continuity can not achieve this conclusion.

4 Local Expansion of the Intersection Curve of Implicit Surfaces

Let $f_1(p), f_2(p)$ be two algebraic polynomials with $p = [x, y, z]^T \in I\!R^3$. The intersection of implicit defined the surfaces(IIS) $f_1(p_1) = 0$, $f_2(p_1) = 0$ is defined by $f_1(p) = f_2(p) = 0$. In this paper, we assume the defining surfaces are smooth, i.e., the normals of the surfaces are not equal to zero at any point on the surface. Now let $F(p) = [f_1(p), f_2(p)]^T$, $p_0 \in I\!R^3$ be a point on the intersection curve $r(s)$, where s is the arc length measured from $p_0 = r(0)$ with prescribed direction. Then, as in [4], $r'(0), r''(0)$ and $r'''(0)$ are computed as follows:

$$F(r)(s) = F(r)(0) + s\frac{dF(r)(0)}{ds} + \frac{s^2}{2!}\frac{d^2F(r)(0)}{ds^2} + \dots \tag{4.1}$$

where

$$\frac{d^kF(r)(0)}{ds^k} = V_k(0) + \nabla F(p_0)r^{(k)}(0) \tag{4.2}$$

$$\begin{aligned} V_1(s) &= 0 \\ V_k(s) &= V'_{k-1}(s) + [\nabla F(r)]' r^{(k-1)}(s), \quad k = 1, 2, \dots \\ \nabla F(p) &= \left[\frac{\partial F(p)}{\partial x}, \frac{\partial F(p)}{\partial y}, \frac{\partial F(p)}{\partial z} \right] \in I\!R^{2\times 3} \end{aligned} \tag{4.3}$$

It follows from $F(r(s)) \equiv 0$ that

$$\nabla F(p_0)r^{(k)}(0) = -V_k(0) \tag{4.4}$$

The system of equation (4.4) has three unknowns and two equations. It has in general infinite many solutions. Now we assume $\nabla f_1(p_0)$ and $\nabla f_2(p_0)$ are linearly independent and illustrate how to get $r^{(k)}(0)$ such that the equations in the last section are satisfied.

Let t be a vector such that

$$t \in \nabla F(p_0)^{\perp}, \qquad \|t\| = 1 \tag{4.5}$$

and its sign is so chosen that t gives the correct direction along the same line we are going. Then for any vector $x \in I\!R^3$ there exist unique $\alpha \in I\!R$ and $y \in \text{range}(\nabla F(p_0)^T)$, such that $x = \alpha t + y$. Let

$$r^{(k)}(0) = \alpha_m t + \nabla F(p_0)^T \beta_m \tag{4.6}$$

Then by (4.4), we have β_m is uniquely defined by

$$\nabla F(p_0)\nabla F(p_0)^T\beta_m = -V_k(0) \tag{4.7}$$

and α_m is arbitrary. Now we determine α_m ($m = 1, \ldots, 4$), such that $r^{(m)}(0)$ ($m = 1, \ldots, 4$) satisfy (3.1).

A. m = 1. Since $V_1(0) = 0$, then $\beta_1 = 0$. Hence $r'(0) = \alpha_1 t$. According to the definition of t, we choose $\alpha_1 = 1$.

B. m = 2. Since we want the $r''(0)$ orthogonal to $r'(0) = t$, i.e., $r''(0) \in \text{range}(\nabla F(p_0)^T)$, the only choice is $\alpha_2 = 0$. We then have $k = |r''(0)|$.

C. m = 3. It follows from (3.1) that $k'n + kTb \in \text{range}(\nabla F(p_0)^T)$. Then $\alpha_3 = -k^2$, and further $k' = r''(0)^T r'''(0)/k$

D. m = 4. From (3.1)

$$r^{(4)}(s) = (k'' - kT^2 - k^3)n + [k'T + (kT)']b - 3kk't.$$

Then $\alpha_4 = -3kk'$.

Finally we obtain the approximate expansion $r(s) \approx \sum_{i=0}^{4}(r^{(i)}(0)/i!)s^i$

5 Local Expansion of the Intersection Curve of Parametric Surfaces

Let

$$\begin{aligned} X_1(u_1, v_1) &= [G_{11}(u_1, v_1)\ G_{21}(u_1, v_1),\ G_{31}(u, v_1)]^T \\ X_2(u_2, v_2) &= [G_{12}(u_2, v_2)\ G_{22}(u_2, v_2),\ G_{32}(u_2, v_2)]^T \end{aligned}$$

be two parametric surface, where G_{ij} are given smooth functions. The intersection curve of the parametric surface (IPS) is defined by

$$r(s) = X_1(u_1(s), v_1(s)) \qquad (\text{or } X_2(u_2(s), v_2(s)))$$

with $X_1(u_1(s), v_1(s)) = X_2(u_2(s), v_2(s))$ where the parameter s is the arc length measured from some point on the curve.

Let $Q_1 = (u_1, v_1)^T$, $Q_2 = (u_2, v_2)^T$, and Q_1^*, Q_2^* be the points in $\mathbb{R}^2$ such that $X_1(Q_1^*) = X_2(Q_2^*)$. At point $X_1(Q_1^*)$, we want to expand $r(s)$ into power series $r(s) = r(0) + r'(0)s + \frac{r''(0)}{2}s^2 + \ldots$ On the curve $r(s)$, Q_1 and Q_2 are functions of s, we can express them as

$$Q_j(s) = \sum_{i=0}^{\infty} \frac{Q_j^{(i)}(0)}{i!}\, s^i, \qquad j = 1, 2 \tag{5.1}$$

As the case of IIS, we expand $X_j(Q_j(s))$

$$X_j(Q_j(s)) = X_j(Q_j)(0) + \frac{dX_j(Q_j)(0)}{ds}\, s + \frac{d^2X_j(Q_j)(0)}{ds^2}\,\frac{s^2}{2} + \dots$$

for $j = 1, 2$,

$$\frac{d^kX_j(Q_j)(0)}{ds^k} = V_{kj}(0) + \nabla X_j(Q_j)Q_j^{(k)}(0), \qquad j = 1, 2 \tag{5.2}$$

where

$$\begin{aligned} V_{1j}(s) &= 0, \qquad j = 1, 2 \\ V_{kj}(s) &= \tfrac{d}{ds}V_{k-1,j}(s) + \tfrac{d}{ds}[\nabla X_j(Q_j)]Q_j^{(k-1)}(s). \end{aligned}$$

By $X_1(Q_1(s)) \equiv X_2(Q_2(s))$, we have

$$\nabla X_1 Q_1^{(m)} - \nabla X_2 Q_2^{(m)}(0) = V_{m1}(0) - V_{m2}(0).$$

Let

$$n_i = X_{iu_i} \times X_{iv_i}, \qquad X_{iu_i} = \left[\frac{\partial G_{1i}}{\partial u_i}, \frac{\partial G_{2i}}{\partial u_i}, \frac{\partial G_{3i}}{\partial u_i}\right]$$

Then n_1, n_2 are the normals of the two surfaces. Suppose n_1 and n_2 are linearly independent. Let $t \in I\!R^3$ such that $t \in [n_1 n_2]^\perp$, $\quad \|t\| = 1$, and its sign is properly chosen such that it points to the correct direction. Then we have the expression

$$\begin{aligned} \nabla X_1 Q_1^{(m)}(0) + V_{m1}(0) &= \nabla X_2 Q_2^{(m)}(0) + V_{m2}(0) \\ &= \alpha_m t + [n_1, n_2]\beta_m \end{aligned} \tag{5.3}$$

Since $n_1^T \nabla X_1 = 0, \quad n_2^T \nabla X_2 = 0$, we have from (5.3)

$$[n_1, n_2]^T\ [n_1, n_2]\beta_m = \begin{bmatrix} n_1^T V_{m1} \\ n_2^T V_{m2} \end{bmatrix}. \tag{5.4}$$

Therefore β_m is uniquely determined by the nonsingularity of the matrix $[n_1, n_2]^T[n_1, n_2]$, and α_m is arbitrary. From (5.3), we determine α_m, such that $r^{(m)}(0) = \alpha_m t + [n_1, n_2]\beta_m$. This can be done exactly the same as the case of IIS by regarding $[n_1, n_2]$ as $\nabla F(p_0)^T$.

After $r^{(m)}(0)$ are received, we can compute $Q_j^{(m)}(0)$, $j = 1, 2$. From (5.3),

$$\nabla X_j^T \nabla X_j Q_j^{(m)}(0) = \nabla X_j^T (r^{(m)}(0) - V_{mj}(0)), \qquad j = 1, 2. \tag{5.5}$$

Solving these equations, we get $Q_j^{(m)}(0)$.

The purpose of computing $Q_j^{(m)}(0)$ is to compute the approximate value of $Q_j(s)$ by (5.1). This approximate value serves as the initial value for the Newton method to get accurate value Q_j on the curve.

6 Newton Iterations

While tracing a surface-surface intersection curve SC, at simple (regular) points of SC we need to solve an undetermined nonlinear system that has more unknowns than equations. At singular points on the curve, we need to solve an overdetermined nonlinear system that has more equations and less unknowns. Consider in general an arbitrary system of nonlinear equations

$$F(x) = \begin{bmatrix} f_1(x_1, \ldots, x_m) \\ \cdots\cdots\cdots\cdots \\ f_n(x_1, \ldots, x_m) \end{bmatrix}$$

We need to determine solution of the system $F(x) = 0$ by Newton iterations from a given initial value $p_0 \in \mathbb{R}^m$. In our tracing procedure these initial values are points on the local expansion curves, within an adaptively computed step length. These initial values are then refined back to the original intersection curve SC to yield the actual interpolating points for the rational curve segment approximation. The Newton iteration used is

$$\nabla F(p_k)\Delta_k = -F(p_k), \quad p_{k+1} = p_k + \Delta_k \tag{6.1}$$

where $\nabla F = \left[\frac{\partial F}{\partial x_1} \frac{\partial F}{\partial x_2} \cdots \frac{\partial F}{\partial x_m} \right] = \begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{bmatrix}$ is a $n \times m$ matrix.

Case A: $\mathbf{m = n + 1}$. Here equation (6.1) is a under-determined linear system. Suppose the set of ∇f_i is linearly independent, then the general solution of (6.1) is

$$\Delta_k = \alpha_k t + \nabla F(p_k)^T \beta_k \tag{6.2}$$

where $t \in \nabla F(p_k)^{\perp}$, $\alpha_k \in \mathbb{R}$ is arbitrary and $\beta_k \in \mathbb{R}^n$ satisfies the following equation

$$\nabla F(p_k)\nabla F(p_k)^T X = -F(p_k) \tag{6.3}$$

This has a unique solution since $\nabla F(p_k)$ is of full rank. Finally, α_k is chosen as follows:

1. IIS case

   In this case, $m = 3$, $n = 2$ and t in (6.2) is the tangent direction of the curve. The change of p_k in the direction of t should be as small as possible. Therefore, we set $\alpha_k = 0$ ([4]).
2. IPS case

   Now $m = 4$, $n = 3$ and $p = (x_1, x_2, x_3, x_4)^T := (u_1, v_1, u_2, v_2)^T$

   $$f_i(x_1, x_2, x_3, x_4) = G_{i1}(x_1, x_2) - G_{i2}(x_3, x_4), \qquad i = 1, 2, 3$$

   The initial value is given by (5.1), i.e., $p_0 = (Q_1(s_0)^T, Q_2(s_0)^T)^T$, where s_0 is the step length of the approximation of $r(s)$. In order to determine α_k in (6.2), we project $p_{k+1} \in \mathbb{R}^4$ (domain space) into $\mathbb{R}^3$ (value space) by

   $$X_1(p_k^{(1)}) + \nabla X_1(p_k^{(1)})\Delta_k^{(1)}$$

where $p_k^{(1)}$ (or $p_k^{(2)}$) and $\Delta_k^{(1)}$ (or $\Delta_k^{(2)}$) are the first (or last) two components of p_k and Δ_k, respectively. Let

$$n_1 = X_{1u_1}(p_k^{(1)}) \times X_{1v_1}(p_k^{(1)}), \qquad n_2 = X_{2u_2}(p_k^{(2)}) \times X_{2v_2}(p_k^{(2)})$$

and $n_3 = n_1 \times n_2$. Then there exist $\bar{\alpha}_k \in I\!R$, $\bar{\beta}_k \in I\!R^2$ such that

$$X_1(p_k^{(1)}) + \nabla X_1(p_k^{(1)})\Delta_k^{(1)} = \bar{\alpha}_k n_3 + [n_1, n_2]\tilde{\beta}_k$$

and $\tilde{\beta}_k$ is determined uniquely by

$$\begin{bmatrix} n_1^T X_1(p_k^{(1)}) \\ n_2^T X_2(p_k^{(2)}) \end{bmatrix} = [n_1, n_2]^T [n_1, n_2]\tilde{\beta}_k$$

and

$$\begin{aligned} \bar{\alpha}_k &= n_3^T X_1(p_k^{(1)}) + n_3^T \nabla X_1(p_k^{(1)})\Delta_k^{(1)} \\ &= n_3^T X_1(p_k^{(1)}) + n_3^T \nabla X_1(p_k^{(1)})[\alpha_k t^{(1)} + \nabla F^{(1)}(p_k)^T \beta_k] \\ &= a(p_k)\alpha_k + b(p_k) \end{aligned}$$

where $a(p_k)$ and $b(p_k)$ are constants depending on p_k. For the same reason as IIS, we take $\bar{\alpha}_k = 0$. Hence $\alpha_k = -\frac{b(p_k)}{a(p_k)}$.

Case B: n > m. This case happens when we arrive at a singular point on the intersection curve SC (see Section 9). Now system (6.1) is over-determined. So we find the least squares approximate solution, i.e.,

$$\nabla F(p_k)^T \nabla F(p_k)\Delta_k = -\nabla F(p_k)^T F(p_k) \tag{6.4}$$

7 Rational Curve Hermite Interpolation between Simple Points

Let $r_1(u)$, $r_2(v)$ be two space curves, where u and v are arc lengths of the curves measured from some point on the respective curve. At point $u = u_0$, $v = v_0$, if

$$r_1^{(i)}(u_0) = r_2^{(i)}(v_0), \qquad i = 0, 1, \ldots, k$$

we say that r_1 and r_2 are k-frame connected, or the composite curve is k-frame continuous. In particular, if $k = 3$, we say the curve is frame continuous.

Given a point p_0 on the curve $r(s)$, which is either IIS, IPS or PC, the arc length s is measured from p_0 (i.e., $r(0) = p_0$) in the given direction.

Step Length

From the approximation $r(s) \approx \sum_{i=0}^{k+1} r^{(i)}(0)\frac{s^i}{i!}$, we compute a trial step length $\beta > 0$ such that

$$\frac{\|r^{(k+1)}(0)\|\beta^{k+1}}{(k+1)!} \;/\; \|\sum_{i=0}^{k} \frac{r^{(i)}(0)\beta^i}{i!}\| < \epsilon \tag{7.1}$$

For such a β, using $\sum_{i=0}^{k+1} r^{(i)}(0)\beta^i/i!$ (for IIS), or $[\sum_{i=0}^{k+1} Q_1^{(i)}(0)^T\beta^i/i!, \ \sum_{i=0}^{k+1} Q_2^{(i)}\beta^i/i!]^T$ (for IPS) as initial value, we compute a new point p_l on the curve by Newton iterations (section 6).

We then construct rational approximations as follows:

A. Rational Hermite interpolation

Let m, n be two nonnegative integers and $m + n = 2k + 1$. We construct a rational vector function $R(s) = [R_1(s), R_2(s), R_3(s)]^T$, where $R_i(s) = P_{mi}(s)/Q_{ni}(s)$, $i = 1,2,3$ are (m,n) type rational functions, such that

$$R^{(i)}(s) = r^{(i)}(s), \quad i = 0, 1, \ldots, k \tag{7.2}$$

for $s = 0$ and $s = \beta$.

If either $Q_{ni}(s)$ has zeros in $[0, \beta]$ or the error $\max_{s\in[0,\beta]} \|r(s) - R(s)\| > \epsilon$, we halve the β. The approximation error is bounded in the following way:

Since

$$e_i(s) = r_i(s)Q_{ni}(s) - P_{mi}(s) = O(s^{k+1}(s-\beta)^{k+1}),$$

by the remainder formula of Hermite interpolation [7], we have

$$e_i(s) = [s(s-\beta)]^{k+1}(r_iQ_{ni})[\underbrace{0,\ldots,0}_{k+1}\underbrace{\beta,\ldots,\beta}_{k+1},\ s],$$

where $f[t_0,\ldots,t_r]$ stands for divided difference of f on $t_0,\ldots,t_r$. Hence

$$|r_i(s) - R_i(s)| \le \left(\frac{\beta}{2}\right)^{2k+2} \frac{|D_{ki}(s)|}{\min_{s\in[0,\beta]}|Q_{ni}(s)|} \tag{7.3}$$

where $D_{ki}(s) = (r_iQ_{ni})[0,\ldots,0,\beta,\ldots,\beta,s]$ is a function in s. That can be bounded approximately by either

$$|D_{ki}(0)| + |D_{ki}(\beta)| \quad \text{or} \quad \max_{s\in[0,\beta]} |\bar{D}_{ki}(s)|$$

where $\bar{D}_{ki}(s)$ is the interpolation polynomial of degree 2 at $D_{ki}(0)$, $D_{ki}(\frac{\beta}{2})$ and $D_{ki}(\beta)$. Let $g = r_i/Q_{ni}$. Then the divided difference can be computed by the following well known recurrence

$$g[t_0,\ldots,t_k] = \begin{cases} g^{(k)}(t_0)/k! & \text{if } t_0 = \ldots = t_k \\ \dfrac{g[t_0,\ldots,t_{r-1}t_{r+1},\ldots,t_k] - g[t_0,\ldots,t_{s-1},t_{s+1},\ldots,t_k]}{t_s - t_r} & \text{if } t_r \ne t_s \end{cases}$$

B. Rational Vector Hermite Interpolation

We construct a rational function

$$R(s) = [P_{m1}(s), P_{m2}(s), P_{m3}(s)]^T/Q_n(s)$$

such that (7.2) holds and

$$m + n/3 = 2k + 1 \tag{7.4}$$

where n is divisible by 3. Now each component of the vector rational function has the same denominator. But the degree $m+n$ of each component is higher than the previous case. However, if we transform the vector rational function in case 1 into a rational function that has common denominator, then the degree is higher than in case 2. This transform is necessary when we represent the curve in rational Bernstein-Bézier form.

The error bound of the approximation can be estimated in the same way as before.

C. Two Point Padé Approximation

The two point Padé approximation method discussed here consists of the following two steps. First, compute the Padé approximation $P_{m_1i}(s)/Q_{n_1i}(s)$ at $s = 0$, such that

$$r_i(s) - P_{m_1i}(s)/Q_{n_1i}(s) = O(s^{k+1}), \qquad i = 1,2,3$$

and

$$m_1 + n_1 = k. \tag{7.5}$$

Second, compute the Padé approximation $P_{m_2i}(s)/Q_{n_2i}(s)$ at $s = \beta$ to the function $\bar{r}_i(s) = (r_i(s)$ $Q_{n_1i}(s) -$ $P_{m_1i}(s))/s^{k+1}$ such that

$$\bar{r}_i(s) - P_{m_2i}(s)/Q_{n_2i}(s) = O((s-\beta)^{k+1}), \qquad i = 1,2,3$$

and

$$m_2 + n_2 = k \tag{7.6}$$

The required two point approximation is

$$R_i(s) = \frac{P_{m_1i}(s)Q_{n_2i}(s) - s^{k+1}P_{m_2i}(s)}{Q_{n_1i}(s)Q_{n_2i}(s)}$$

which is $(\max\{m_1+n_2, k+m_2+1\},\ n_1+n_2)$ type rational function and satisfies condition (7.2). For example, if $k=3$, take $m_1 = m_2 = 2$, $n_1 = n_2 = 1$, then $R_i(s)$ is a (6.2) type rational function. Since the denominator of $R_i(s)$ is a product of two polynomials, it is easy to check the appearance of the poles of $R_i(s)$ in $[0,\beta]$ when n_i is small, say $n_i \le 2$.

Denote $Q_{ni}(s) = Q_{n_1i}(s)Q_{n_2i}(s)$ $(n = n_1 + n_2)$, the error can be estimated as in the rational Hermite interpolation case.

D. Two Point Vector Padé Approximation

Similar to the rational vector Hermite interpolation, we can also consider a two point vector Padé approximation. Now conditions (7.5) and (7.6) should be replaced by

$$m_1 + n_1/3 = k, \qquad m_2 + n_2/3 = k$$

respectively, and further we require that n_1 and n_2 are divisible by 3. The error can be computed as before.

8 Rational B-spline Representation

To interactively control the shape of the piecewise approximating curve or to interface to existing B-spline modelers, we represent each of the rational functions as rational B-splines. The first step is to transform the rational function into Bernstein-Bézier form. Let

$$r(s) = [x(s),\ y(s),\ z(s)]^T / w(s)$$

be a space curve on the interval $[a, b]$, where $x(s), y(s), z(s)$ and $w(s)$ are polynomials of degree n. Since

$$t^i = \sum_{j=i}^{n} \frac{C_i^j}{C_i^n} B_j^n(t)$$

with

$$t = \frac{s-a}{b-a} \in [0,1], \quad B_j^n(t) = C_i^n t^j (1-t)^{n-j}, \quad C_i^n = \frac{n!}{i!(n-i)!}$$

we have, for any polynomial $p(s)$ of degree n

$$\begin{aligned} p(s) &= \textstyle\sum_{i=0}^{n} c_i t^i \\ &= \textstyle\sum_{i=0}^{n} (\sum_{j=0}^{i} \frac{C_j^i}{C_j^n} c_j) B_i^n(t) \\ &= \textstyle\sum_{i=0}^{n} b_i' B_i^n(t) \end{aligned}$$

where $b_i' = \sum_{j=0}^{i} \frac{C_j^i}{C_j^n} c_j$. Therefore $r(s)$ can be expressed as

$$r(s) = \sum_{i=0}^{n} w_i b_i B_i^n(t) / \sum_{i=0}^{n} w_i B_i^n(t) = \sum_{i=0}^{n} w_i b_i N_i^n(s) / \sum_{i=0}^{n} w_i N_i^n(s)$$

where $w_i \in I\!R$, $b_i \in I\!R^3$ is Bézier point and $N_i^n(s) = B_i^n(t)$.

Let $T = \{t_0, ..., t_n, t_{n+1}, ..., t_{2n+1}\}$, where $t_i = a$ for $i = 0, ..., n$, $t_i = b$ for $i = n+1, ..., 2n+1$. Then it is easy to show that the normalized B-spline over T is $N_i^n(s)$ defined above. Therefore, the Bézier point is also the de Boor point in this special case. For the general B-spline

$$F(s) = \sum_{i=0}^{m} d_i N_i^n(s) \tag{8.1}$$

over

$$T = \{t_0 = ... = t_n \le t_{n+1} \le ... \le t_{m+1} = ... t_{m+n+1}\}$$

with $m \ge n$ and $t_i < t_{i+n+1}$. Most operations on splines, such as evaluation by the de Boor algorithm and knot insertion, do not need the explicit expression of $N_i^n(s)$ but the knot sequence T and the de Boor points. So these two sets are enough to represent the B-spline. For example, the evaluation of the B-spline $F(s)$ in (8.1) goes as follows: For $s \in [t_l, t_{l+1})$,

$$\left\{ \begin{array}{l} b_i^0 = d_i, \quad i = 0, 1, ..., m \\ d_i^r = \left(1 - \dfrac{s - t_i}{t_{i+n+1-r} - t_i}\right) d_{i-1}^{r-1} + \dfrac{s - t_i}{t_{i+n+1-r} - t_i} d_i^{r-1} \\ l - n + r \le i \le l, \quad r = 1, 2, ..., n \\ F(s) = d_l^n \end{array} \right.$$

and inserting a point t with $t_l \le t < t_{l+1}$ to T, we have the following algorithm for the new de Boor points d_i^*, for $i = 0, 1, ..., m+1$:

$$d_i^* = a_i d_i + (1 - a_i) d_{i-1}$$

where

$$a_i = \begin{cases} 1 & \text{if} \quad i \le l - n \\ \frac{t - t_i}{t_{i+n} - t_i} & \text{if} \quad l - n + 1 \le i \le l \\ 0 & \text{if} \quad l + 1 \le i \end{cases}$$

Standard NURB Representation

Quite often geometric designers and engineers using NURBS (Rational B-splines with non-uniform knot spacing) like to have NURBS in a standard form, where the denominator polynomial has only positive coefficients. This assumption is quite strong, but rids the curve of real poles (roots of the denominator polynomial) and gives the rational B-spline its convex hull property. In this subsection we show how to convert a curve in BB form (or normalized B-spline form) into a finite number of C^∞ standard NURB curve segments. We also show that for a degree d B-spline the number of NURB segments is bounded above by $\frac{n(n-1)}{2}$.

We only need to show the transformation for the denominator polynomial of the rational curve. Given a denominator polynomial $P(t) = \sum_{i=0}^{n} b_i B_i^n(t) \quad t \in [0,1]$ we divide the interval [0,1] into subintervals, say, $0 = t_0 < t_1 < \ldots < t_k = 1$, such that the BB-form of $P(t)$ on each of the subintervals $P(t)|_{[t_i t_{i+1}]} = P_i(t) \to P_i\left(\frac{s - t_i}{t_{i+1} - t_i}\right) = \bar{P}_i(s) = \sum b_j^i B_i^n(s)$ has positive coefficients. Without loss of generality we assume $P(t) > 0$ over [0,1], as this can be achieved for any polynomial by a simple translation. First we show how to compute the first breakpoint $t_1 = c$. By the subdivision formula $B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t)$ We have on [0,c], ($s = ct; \quad t \in [0,1]$)

$$\begin{aligned} P(s) = P(ct) &= \textstyle\sum_{i=0}^{n} b_i B_i^n(ct) \\ &= \textstyle\sum_{i=0}^{n} b_i \sum_{j=0}^{n} B_i^j(c) B_j^n(t) \\ &= \textstyle\sum_{j=0}^{n} \underbrace{\left(\sum_{i=0}^{n} b_i B_i^j(c) \right)} B_j^n(t) \qquad (B_i^j = 0 \quad if \quad i > j) \\ &= \textstyle\sum_{j=0}^{n} q_j(c) B_j^n(t) \end{aligned}$$

where $q_j(c) = \sum_{i=0}^{j} b_i B_i^j(c)$ is a degree j polynomial in BB form.

Note that the $\lim_{c \to 0} q_j(c) = b_0$. This is because $B_0^j(0) = 1$, $B_i^j(0) = 0$, $i > 0$. Therefore if we assume $P(t) > 0$ for $t \in [0,1]$ then $p(0) = b_0 > 0$. Hence find a root of $q_j(c)$ in [0,1] and take $c < \min\{\text{all roots of } q_j(c) \text{ in } [0,1]\}$. This c will guarantee all $q_j(c)$ are positive. The number of roots of all $q_j(c)$ is bounded by $\frac{n(n-1)}{2}$ which is then also a bound on the number of subintervals required.

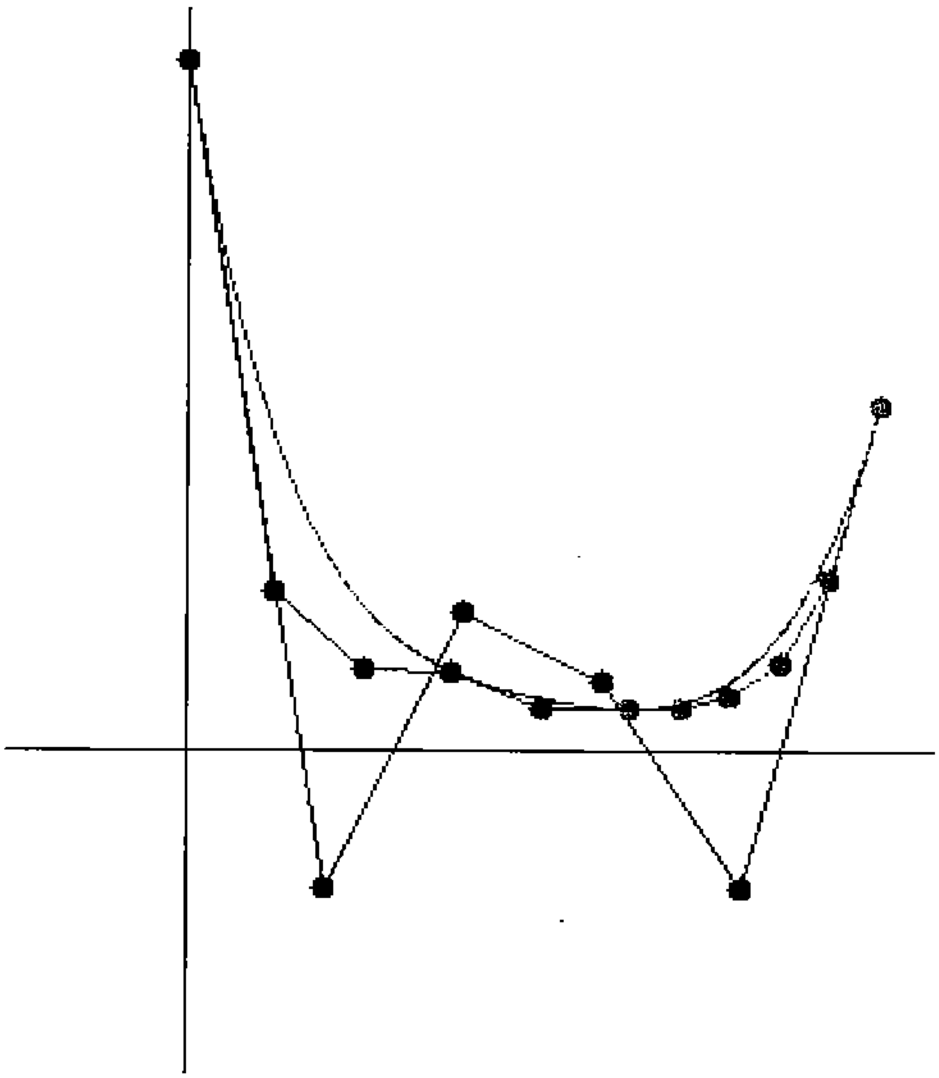

Figure 1: Denominator Polynomial with Positive Bezier Coefficients

Example 8.1 *Figure 1 shows an example of this conversion for the denominator polynomial* $(1-x)^5 - x*(1-x)^4 + 2*x^2*(1-x)^3 + x^3*(1-x)^2 - x^4*(1-x) + 0.5*x^5$

The initial Bezier or de Boor coefficients over [0,1] are

$$bb[0] = 1.000000 \quad bb[1] = -0.200000 \quad bb[2] = 0.200000$$
$$bb[3] = 0.100000 \quad bb[4] = -0.200000 \quad bb[5] = 0.500000$$

of which two coefficients are negative. The above conversion yields two pieces in standard NURB over [0,1] with 0.640072 as the breakpoint. The new coefficients of the two NURB pieces are

$$bb[0] = 1.000000 \quad bb[1] = 0.231913 \quad bb[2] = 0.119335$$
$$bb[3] = 0.111575 \quad bb[4] = 0.060781 \quad bb[5] = 0.060781$$

and

$$bb[0] = 0.060781 \quad bb[1] = 0.060781 \quad bb[2] = 0.076842$$
$$bb[3] = 0.125649 \quad bb[4] = 0.248051 \quad bb[5] = 0.500000$$

9 Isolating the Singular Points

During the tracing of an intersection space curve, one may encounter singular points. Near these points, the coefficient matrix of the systems (3.8) for IIS, (5.4) and (5.5) for IPS, are nearly singular. When a near singular condition of the coefficient matrix is detected, the tracing procedure is temporarily suspended and the singular point is accurated isolated as follows. **A. Singular Point of IIS**

Let $p_0 = (x_0, y_0, z_0)^T \in I\!R^3$ be a singular point of the intersection curve of $f_i(p) = 0$, $i = 1, 2$. That is $f_i(p_0) = 0$, $i = 1, 2$ and

$$\alpha_1 F_1^{(1)}(p - p_0) = \alpha_2 F_2^{(1)}(p - p_0) \tag{9.1}$$

where α_1, α_2 are constants, $|\alpha_1| + |\alpha_1| \neq 0$, $f_i(p) = \sum_{s=0} F_i^{(s)}(p - p_0)$ and $F_i^{(s)}(u, v, w)$ is a homogeneous polynomial of degree s. If the order of the singularity is greater than one, then equation (9.1) is replaced by

$$\alpha_1 F_1^{(s)}(p - p_0) = \alpha_2 F_2^{(s)}(p - p_0), \quad s = 1, 2, \ldots, L$$

or equivalently

$$\alpha_1 \frac{\partial^s f_1(p_0)}{\partial x^i \partial y^j \partial z^k} = \alpha_2 \frac{\partial^s f_2(p_0)}{\partial x^i \partial y^j \partial z^k}, \qquad \forall (i, j, k) \tag{9.2}$$

with $i + j + k = s$, $\quad s = 1, 2, \ldots, L$.

In order to eliminate α_1 and α_2, use one equation, of (9.2 $\alpha_2 \dfrac{\partial f_2(p_0)}{\partial x} = \alpha_1 \dfrac{\partial f_1(p_0)}{\partial x}$ to obtain

$$f_{i,j,k}(p_0) = \frac{\partial f_2(p_0)}{\partial x}\,\frac{\partial^s f_1(p_0)}{\partial x^i \partial y^j \partial z^k} - \frac{\partial f_1(p_0)}{\partial x}\,\frac{\partial^2 f_2(p_0)}{\partial x^i \partial y^j \partial z^k} = 0$$

for $\forall (i,j,k) \in \{(i,j,k) : i+j+k = s,\ s = 1,2,\ldots,L\} \setminus \{1,0,0\}$.

Now use Newton iterations (Section 6) to solve the system of equations

$$\begin{cases} f_i(p) &= 0 \\ f_{i,j,k}(p) &= 0, \qquad i+j+k \le s \end{cases} \tag{9.3}$$

Use $s = 1$ if the resulted matrix is nonsingular, otherwise increase s by 1 until the matrix is nonsingular.

B. Singular Points of IPS

Let $Q_1^*, Q_2^* \in \mathbb{R}^2$ be the points such that $X_1(Q_1^*) = X_2(Q_2^*)$, i.e., $p^* = X_1(Q_1^*)$ on the intersection curve. We use the definition of the singularity for IIS curve to define the singularity for an IPS curve. For this we need to determine the partial derivatives of parametric surfaces, as described below. We exhibit this for for surface X_1. Surface X_2 can be treated in the same way.

Suppose $\dfrac{\partial X_1(Q_1^*)}{\partial u_1}$ and $\dfrac{\partial X_1(Q_1^*)}{\partial v_1}$ are linearly independent. For smooth, parametric surfaces with faithful parametrizations, the Jacobian matrix

$$J(G_{11}, G_{21}) = \begin{bmatrix} \frac{\partial G_{11}(Q_1^*)}{\partial u_1} & \frac{\partial G_{11}(Q_1^*)}{\partial v_1} \\ \\ \frac{\partial G_{21}(Q_1^*)}{\partial u_1} & \frac{\partial G_{21}(Q_1^*)}{\partial v_1} \end{bmatrix}$$

is nonsingular and invertible. The inverse functions of

$$x = G_{11}(u_1, v_1), \qquad y = G_{21}(u_1, v_1) \tag{9.4}$$

also exist and are given by

$$u_1 = \tilde{G}_{11}(x, y) \qquad v_1 = \tilde{G}_{21}(x, y) \tag{9.5}$$

around Q_1^*. Substitute (9.5) into $z = G_{31}(u_1, v_1)$, to obtain an implicit representation of the parametric surface.

$$f_1(x, y, z) = G_{31}(\tilde{G}_{11}(x, y), \quad \tilde{G}_{21}(x, y)) - z = 0 \tag{9.6}$$

Now compute the partial derivatives of f_1. The derivative about z is trivial, so consider $\frac{\partial f_1}{\partial x}$ first. It follows from (9.6) and (9.4) that

$$\frac{\partial f_1}{\partial x} = \frac{\partial G_{31}}{\partial u_1}\,\frac{\partial u_1}{\partial x} + \frac{\partial G_{31}}{\partial v_1}\,\frac{\partial v_1}{\partial x} \tag{9.7}$$

and

$$J(G_{11}, G_{21}) \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial v_1}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{9.8}$$

Solving (9.8), we get $\frac{\partial u_1}{\partial x}$, $\frac{\partial v_1}{\partial x}$, from (9.7) we get $\frac{\partial f_1}{\partial x}$. Similarly, $\frac{\partial f_1}{\partial y}$ can be computed.

Knowing the partials one can compute the singular points as in in the IIS case. For higher order singularities the higher order partial derivatives can be computed similar to the computation of second order derivatives shown below.

From (9.7), we have

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x \partial y} &= \left(\frac{\partial^2 G_{31}}{\partial u_1^2} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{31}}{\partial u_1 \partial v_1} \frac{\partial v_1}{\partial y} \right) \frac{\partial u_1}{\partial x} + \frac{\partial G_{31}}{\partial u_1} \frac{\partial^2 u_1}{\partial x \partial y} \\ &+ \left(\frac{\partial^2 G_{31}}{\partial u_1 \partial v_1} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{31}}{\partial v_1^2} \frac{\partial v_1}{\partial y} \right) \frac{\partial v_1}{\partial x} + \frac{\partial G_{31}}{\partial v_1} \frac{\partial^2 v_1}{\partial x \partial y} \end{aligned} \tag{9.9}$$

and from (9.4), we have

$$J(G_{11}G_{21}) \begin{bmatrix} \frac{\partial^2 u_1}{\partial x \partial y} \\ \\ \frac{\partial^2 v_1}{\partial x \partial y} \end{bmatrix} = - \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} \tag{9.10}$$

where

$$v_i = \left(\frac{\partial^2 G_{i1}}{\partial u_1^2} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{i1}}{\partial u_1 \partial v_1} \frac{\partial v_1}{\partial y} \right) \frac{\partial u_1}{\partial x} + \left(\frac{\partial^2 G_{i1}}{\partial u_1 \partial v_1} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{i1}}{\partial v_1^2} \frac{\partial v_1}{\partial y} \right) \frac{\partial v_1}{\partial x}$$

From (9.10) we get $\frac{\partial^2 u_1}{\partial x \partial y}$, $\frac{\partial^2 v_1}{\partial x \partial y}$, from (9.9) we get $\frac{\partial^2 f_1}{\partial x \partial y}$.

10 The Local Approximation at Singular Points

At the singular points, simple Taylor series expansions fail and we must use special methods to tackle the approximation problem.

1. IIS.

   Let $p_0 = (x_0, y_0, z_0)^T \in I\!R^3$ be a singular point on the curve. since the matrix $\nabla f_i(p_0) \neq 0$, we may assume, WLG, that $\frac{\partial f_1}{\partial z} \neq 0$. Then we can express z by a power series $z = \phi(x, y)$ in x and y from $f_1(x, y, z) = 0$ around the point p_0. Substitute z into $f_2(x, y, z) = 0$, we get $h(x, y) = f(x, y, \phi(x, y)) = 0$ As in the plane curve case [3], expanding $h(x, y) = 0$ at point $(x_0, y_0)^T$ by Weierstrass and Newton factorization, we obtain

   $$\left\{ \begin{array}{lcl} x &=& x_0 + t^{k_i} \\ y &=& \psi_i(t) \end{array} \right. \qquad i = 0, 1, \ldots, m$$

where $\psi_i(t)$ is a power series in t and m is the number of the branches of the curve $h(x,y)=0$. We then have

$$\begin{aligned} z &= \phi(x_0 + t^{k_i}, \psi_i(t)) \qquad\qquad i = 0, 1, \ldots, m \\ &= \theta_i(t), \end{aligned}$$

Therefore we get the local parametric form of the space curve as

$$\left\{ \begin{array}{lcl} x &=& x_0 + t^{k_i} \\ y &=& \psi_i(t) \qquad i = 0, 1, \ldots, m \\ z &=& \theta_i(t) \end{array} \right.$$

For each branch, use the two point interpolating condition to get a rational approximation.

2. IPS

Let $Q_1^* = (u_1^*, v_1^*)^T$, $Q_2^* = (u_2^*, v_2^*)$ be the points in $\mathbb{R}^2$ such that $X_1(Q_1^*) = X_2(Q_2^*)$ and $X_1(Q_1^*)$ is a singular point of the curve IPS. Since the matrices $\nabla X_1(Q_1^*)$ and $\nabla X_2(Q_2^*)$ are full rank in column, we may assume $J(G_{11}, G_{21})$ is not singular at Q_1^*. By one of the first two equations, say the first, $G_{11}(u_1, v_1) = G_{12}(u_2, v_2)$, we can express u_1 as

$$u_1 = \phi^{(1)}(v_1, u_2, v_2) \tag{10.1}$$

Substituting it into another equation of the first two, we get

$$v_1 = \phi^{(2)}(u_2, v_2) \tag{10.2}$$

Substituting u_1 and then v_1 into the last equation $G_{31}(u_1, v_1) = G_{32}(u_2, v_2)$, we have $\phi^{(3)}(u_2, v_2) = 0$. Now, use plane curve factorization techniques for dealing with the singularities, we get

$$\begin{aligned} u_2 &= u_2^* + t^{k_i} \\ v_2 &= \phi_i(t), \qquad i = 0, 1, \ldots, m \end{aligned}$$

Substitute then back to (10.2) and (8.1), we have

$$\begin{aligned} v_1 &= \phi^{(2)}(u_2^* + t^{k_i}, \phi_i(t)) \\ &= \psi_i(t) \end{aligned}$$

$$\begin{aligned} u_1 &= \phi^{(1)}(\psi_i(t), u_2^* + t^{k_i}, \phi_i(t)) \\ &= \theta_i(t) \end{aligned}$$

Then the local parameterization is obtained by

$$\begin{aligned} r_i(t) &= X_1(\theta_i(t), \psi_i(t)) \\ &\quad \text{or } X_2(u_2^* + t^{k_i}, \phi_i(t)), \qquad i = 0, 1, \ldots, m \end{aligned}$$

The next step for getting approximation is the same as IIS.

11 Implementation Details and Examples

1. Starting Points

   In order to trace the intersection curve SC, we need to provide a starting point on each real component of the curve. Besides the boundary points which are straightforward roots of univariate or coupled bivariate polynomial equations[1] one computes a starting point on each real component completely inside the given box. For IIS this can be done by projecting the intersection curve (via resultant elimination) into a plane and then finding a coordinate axis extreme point on the projection curve of that component. See [1] for details of such resultant elimination schemes. For IPS, papers [6] [16] provide some numerical methods for computing these starting points.

2. Curve Interpolations Points

   When we march along the curve, we encounter precomputed points on the way. An encountered point may be a boundary point, a starting point on a closed loop or may be an end point of the prior segment tracing. Suppose $p_0 \in \mathbb{R}^3$ is a point on the curve, $r(s)$ ($s \in [0,\beta]$) is a segment of the curve, which approximates the original curve. Then a possible question is whether $r(s)$ passes through p_0 within the allowable error? We answer this question by computing the distance between p_0 and $r(s)$:

   $$\mathrm{dis}(p_0, r) = \min_{s\in[0,\beta]} \|r(s) - p_0\| \tag{11.1}$$

   Since $r(s)$ is a rational function in s, the minimum point of (11.1) can be computed by $\frac{d}{ds}\|r(s) - p_0\|^2 = 0$. If $s = s^* \in [0,\beta]$ is the minimum point, then if $\|r(s^*) - p_0\| < \epsilon$, $r(s)$ passes through p_0. Then we modify $r(s)$ such that $r(s)$ is frame continuous at s^* and replaces β by s^*. Otherwise, $r(s)$ does not pass through p_0.

3. Solving Linear System of Equations

   In all the cases in this paper (see (4.7), (5.4), (5.5), (6.3) and (6.4)), we always solve the linear system $Ax = b$ with a positive definite coefficient matrix A. The size of matrix A is as small as one, and as large as four. A stable method to solve this equation is to use singular value decomposition $A = U^T\Sigma U$, where U is an orthogonal matrix and Σ is a diagonal matrix. The solution is $x = U^T\Sigma^{-1}Ub$.

4. Tangent Direction

   In Sections 4 and 5, we have mentioned that the sign of the tangent vector t at an expansion point should be properly chosen. Now we will make this point clear.

   a. If the expansion point is a boundary point, then t points to the interior of the box.

   b. If the point is a starting point on a loop, then the sign can be any.

c. If the point is an end point of a previous approximation $\bar{r}(s) = \sum_{i=0}^{k+1} r^{(i)}(0)s^i / i!$ $(s \in [0, \beta])$, then we choose the sign of t such that $\bar{r}'(\beta)^T t \geq 0$.

Examples

We present several examples of piecewise rational approximations of implicit algebraic surface-surface intersection curves and parametric space curves. The implementation of parametric surface-surface intersection curves is in progress, and examples shall be included in the final version of the paper. Though we did not discuss the approximation of high degree parametric space curves, the derivation follows similar lines to the IIS and IPS cases.

Example 11.1 *The example is shown in Figure 2. The intersection curve is given by the two surfaces* $x^2 + y^2 - 1 = 0$ *and* $y^2 + z^2 - 1 = 0$ *within the bounding box* $[-2, 2 - 2, 2, -2, 2]$. *The approximation error epsilon is* 0.1 *for each of the piecewise rational B-spline approximations. The left window is a* C^1 *continuous piecewise parametric cubic B-splines while the right window is a a* C^3 *continuous piecewise parametric degree* 7 *B-splines.*

Example 11.2 *The example is shown in Figure 3. The intersection curve is given by the two surfaces* $x^2 + z^2 + 2 * z = 0$ *and* $y^2 + z^2 + 4 * z = 0$ *within the bounding box* $[-2, 2 - 2, 2, -2, 2]$. *The approximation error epsilon is* 0.1 *for the piecewise rational B-spline approximation in the top right and the bottom left window and* 0.01 *in the bottom right window. The top right window is a* C^1 *continuous piecewise parametric cubic B-splines while the bottom windows are* C^3 *continuous piecewise parametric degree* 7 *B-splines.*

Example 11.3 *The example is shown in Figure 4. The intersection curve is given by the two surfaces* $z^1 - 2 * x^4 - y^4 = 0$ *and* $z^1 - 3 * x^2 * y + y^2 - 2 * y^3 = 0$ *within the bounding box* $[-2, 2 - 2, 2, -2, 2]$. *The approximation error epsilon is* 0.01 *for each of the piecewise rational B-spline approximations. The top two windows are* C^2 *continuous piecewise parametric quintic B-splines while the bottom windows are* C^3 *continuous piecewise parametric degree* 7 *B-splines.*

Example 11.4 *This series of examples are shown in Figure 5. The original parametric space curve of the top left window is given by* $(x = t, y = t^2, z = t^3)$. *The original parametric space curve of the top right window is given by* $(x = t^2, y = t^3, z = t^4)$. *The original parametric space curve of the bottom left window is given by* $(x = t * (t - 1)/(1 + t^2), y = t * (1 - t)(1 + t), z = t * (1 - t) * t^2/(1 + 2 * t^2))$. *The original parametric space curve of the bottom right window is given by* $(x = t * (t - 2) * (t + 2), y = t * (t - 2) * (t + 2) * (1 + t), z = t * (t - 2) * (t + 2) * (1 - t))$. *For each of the windows the approximation error epsilon is* 0.01 *and the approximation is by* C^3 *continuous piecewise parametric degree* 7 *B-splines.*

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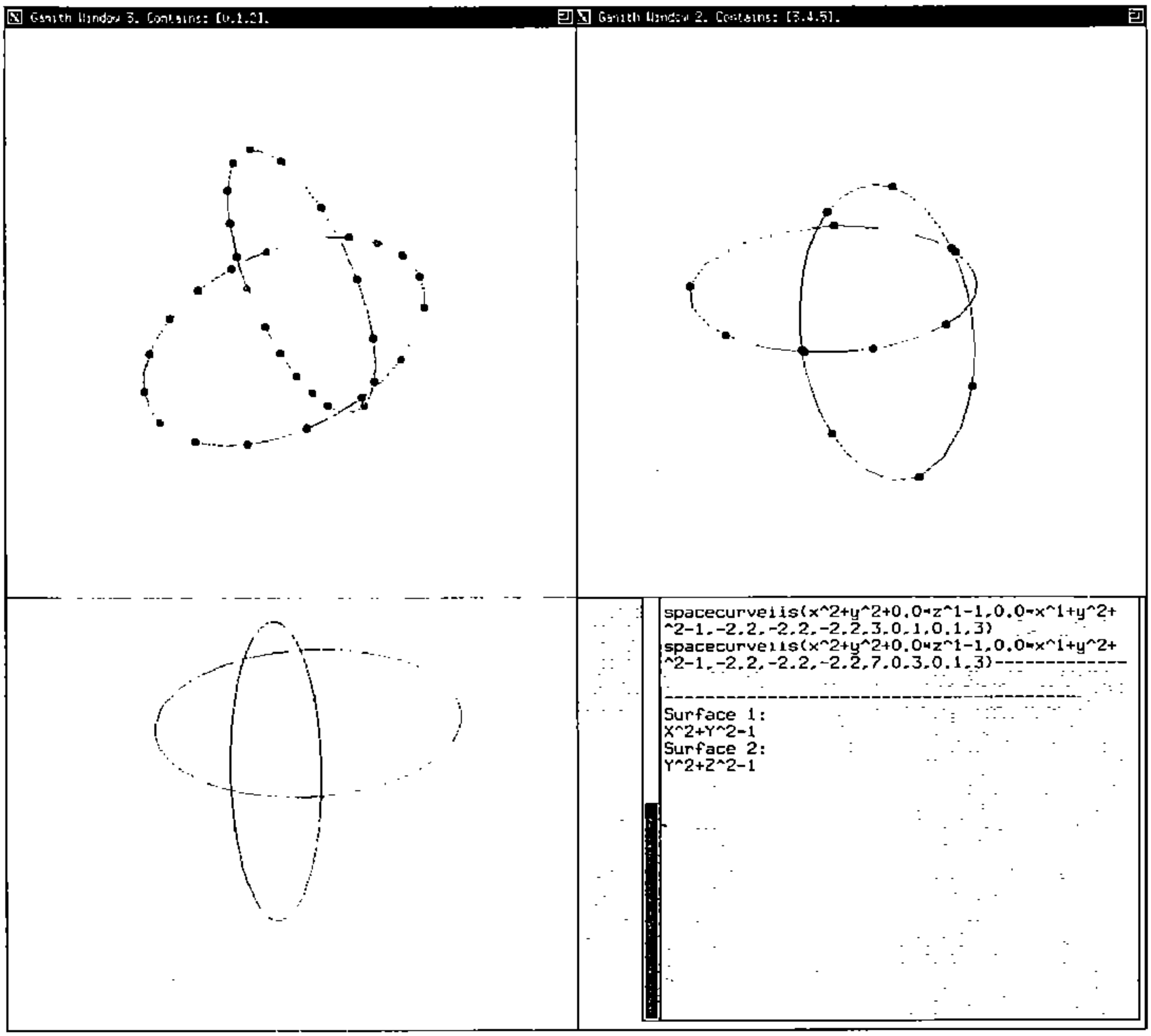

Figure 2: Piecewise Rational B-spline Approximation of the Intersection Curve of Two Surfaces

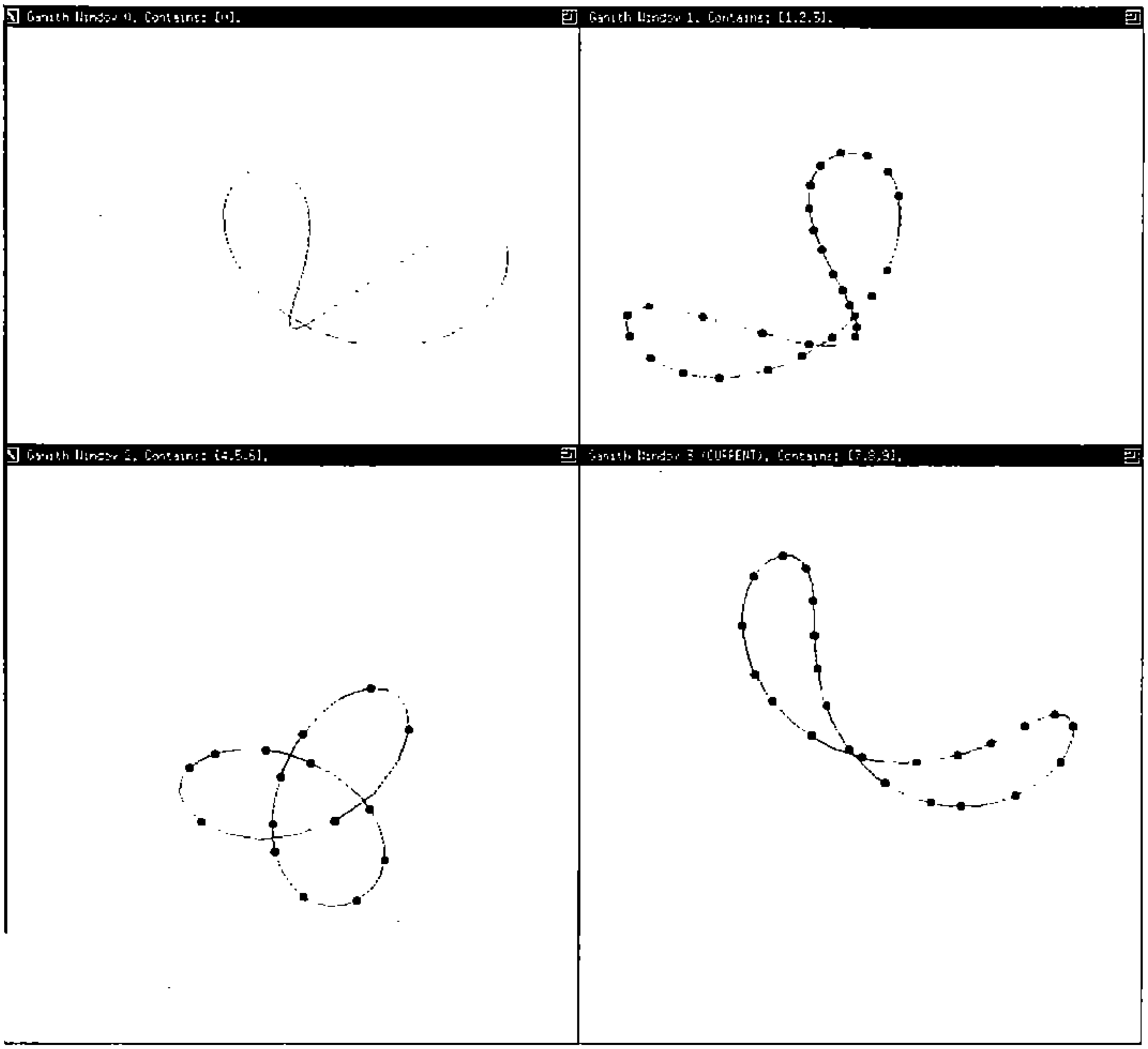

Figure 3: Piecewise Rational B-spline Approximation of the Intersection Curve of Two Surfaces

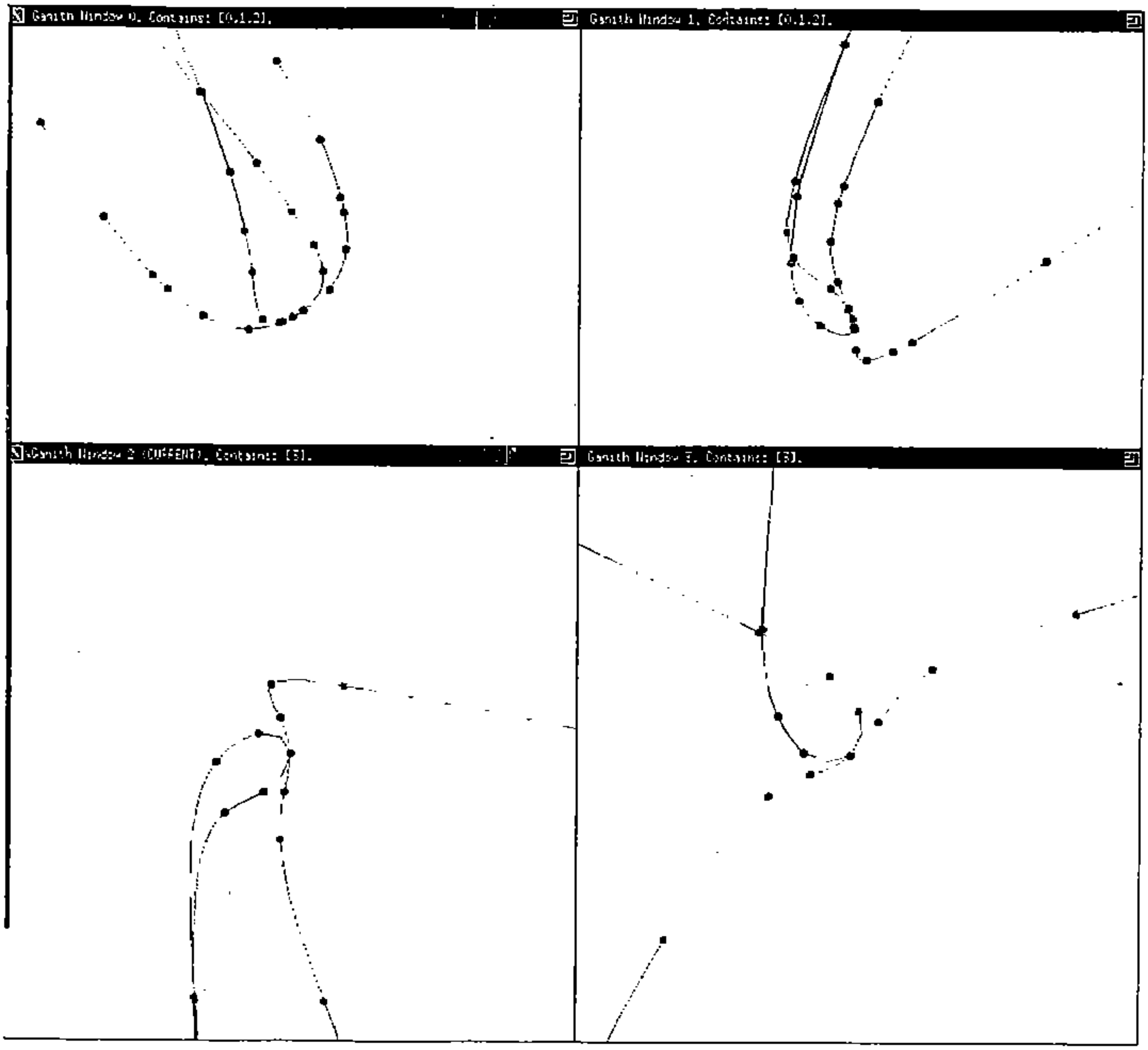

Figure 4: Piecewise Rational B-spline Approximation of the Intersection Curve of Two Surfaces

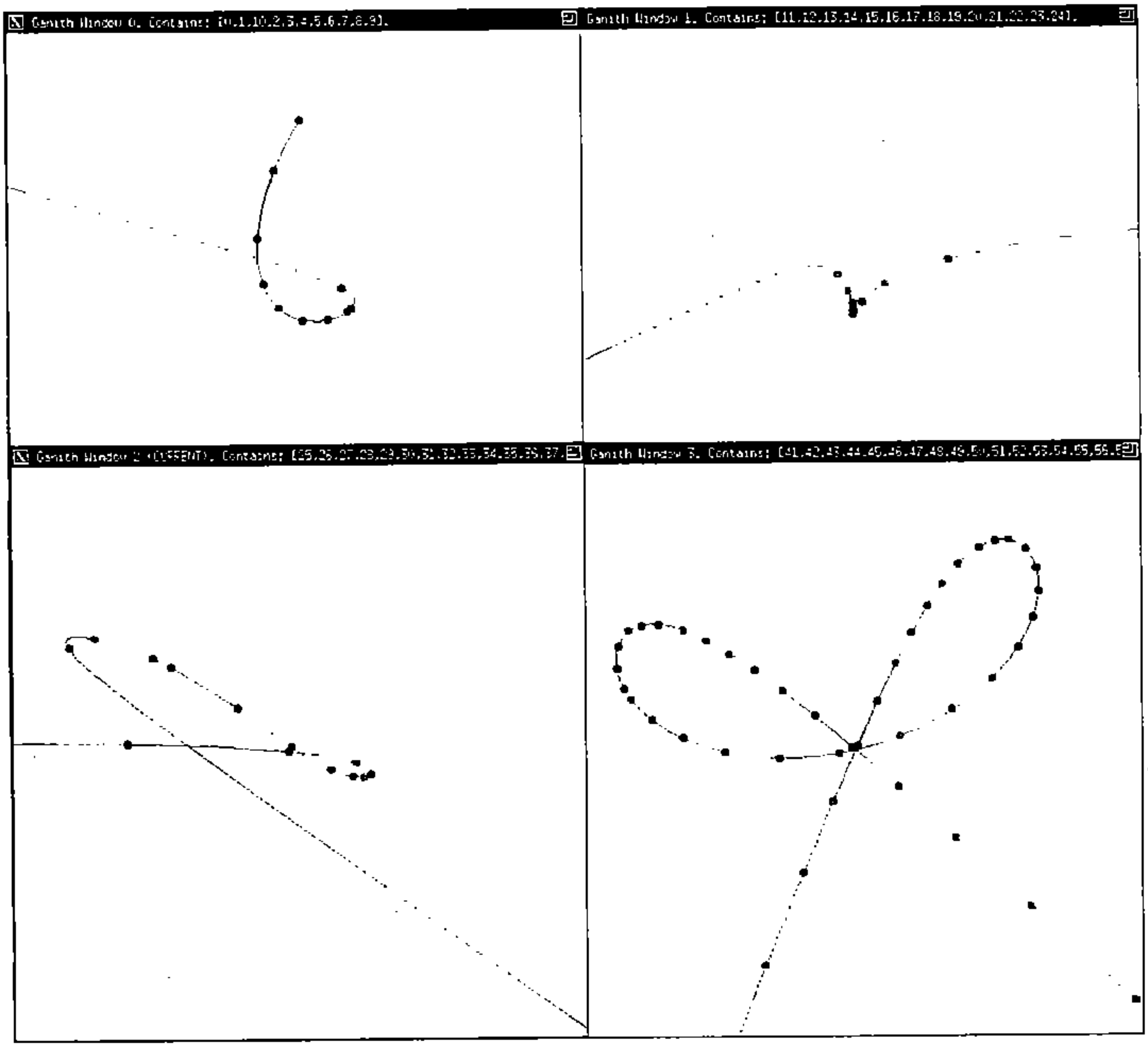

Figure 5: Piecewise Rational B-spline Approximation of Parametric Space Curves