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Systemic risk in financial networks

Peng-Chu Chen

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Is approved by the final examining committee:

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Approved by Major Professor(s): THOMAS MORIN

Approved by: ABHIJIT DESHMUKH 7/20/2016

Head of the Departmental Graduate Program

Date
SYSTEMIC RISK IN FINANCIAL NETWORKS

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Submitted to the Faculty
of
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by
Peng-Chu Chen

In Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

August 2016
Purdue University
West Lafayette, Indiana
To my family
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SYMBOLS

Scalars

\( b_j^i (i) \) \quad \text{equilibrium price paid by bank} \ j \ \text{to purchase the assets of bank} \ i \ \text{at time} \ t

\( n \) \quad \text{number of banks (nodes) in the financial system}

\( r \) \quad \text{interbank interest rate}

\( r_c \) \quad \text{interest rate of liquidity assistant loan}

\( SYS^t \) \quad \text{systemic risk allocated to time} \ t

\( SYS \) \quad \text{total systemic risk generated in time horizon} \ T

\( T \) \quad \text{time horizon}

\( UL^t \) \quad \text{ratio between aggregate unpaid liabilities and aggregate liabilities at time} \ t

\( w_{ij}^t \) \quad \text{unpaid debt of bank} \ i \ \text{to} \ j \ \text{at time} \ t

\( x_i^t \) \quad \text{reservation price of the auction on bank} \ i^{'s} \ \text{assets at time} \ t

\( \alpha_i^t \) \quad \text{auction price of bank} \ i^{'s} \ \text{assets at time} \ t

\( \beta_j^i (i) \) \quad \text{private valuation of bidder} \ j \ \text{on bank} \ i^{'s} \ \text{assets at time} \ t

\( \gamma \) \quad \text{bankruptcy cost factor}

\( \rho \) \quad \text{percentage of tolerated loss}

\( \varpi_i^t \) \quad \text{unpaid debt of bank} \ i \ \text{to all other banks at time} \ t

Vectors

\( c \) \quad \text{outside asset vector}

\( c^t \) \quad \text{operating cash inflow vector at time} \ t

\( d^t \) \quad \text{default indicator vector at time} \ t

\( e \) \quad \text{outside liability vector}

\( \ell \) \quad \text{total liability vector}
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<th>Symbol</th>
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<tr>
<td>ℓ</td>
<td>reduced-liability vector</td>
</tr>
<tr>
<td>ℓ^t</td>
<td>total liability vector at time t</td>
</tr>
<tr>
<td>o^t</td>
<td>liquidity assistance loan vector provided at time t</td>
</tr>
<tr>
<td>p^*</td>
<td>clearing payment vector</td>
</tr>
<tr>
<td>p^t</td>
<td>clearing payment vector at time t</td>
</tr>
<tr>
<td>p</td>
<td>payment vector</td>
</tr>
<tr>
<td>q^t</td>
<td>net worth vector at time t</td>
</tr>
<tr>
<td>s</td>
<td>systemic loss vector</td>
</tr>
<tr>
<td>s^t</td>
<td>state of financial network at time t</td>
</tr>
<tr>
<td>v^t</td>
<td>available cash vector before clearing at time t</td>
</tr>
<tr>
<td>κ^t</td>
<td>available cash vector after clearing at time t</td>
</tr>
<tr>
<td>ξ^t</td>
<td>non-cash asset vector at time t</td>
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**Matrices**

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<tr>
<td>I</td>
<td>identity matrix</td>
</tr>
<tr>
<td>L</td>
<td>interbank liability matrix</td>
</tr>
<tr>
<td>L^t</td>
<td>interbank liability matrix at time t</td>
</tr>
<tr>
<td>S</td>
<td>doubly stochastic matrix</td>
</tr>
<tr>
<td>U^t</td>
<td>ownership matrix</td>
</tr>
<tr>
<td>Π</td>
<td>relative liability matrix</td>
</tr>
<tr>
<td>Π^t</td>
<td>relative liability matrix at time t</td>
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**Sets/Singletons**

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<th>Symbol</th>
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<tr>
<td>m^t_i</td>
<td>winning bidder purchasing bank i’s assets at time t</td>
</tr>
<tr>
<td>Q^t_i</td>
<td>set of eligible bidders participating to bank i’s auction at time t</td>
</tr>
<tr>
<td>V</td>
<td>set of systemically important banks</td>
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ABBREVIATIONS

BIS       Bank for International Settlements
FDIC      Federal Deposit Insurance Corporation
LOLR      Lender of Last Resort
ML        Max-Liquidity
P&A       Purchase and Assumption
SID       Systemic Importance Driven
This thesis extends the literature of systemic risk in financial networks in two directions.

First, we develop a majorization-based tool to compare financial networks in terms of systemic losses with a focus on the implications of liability concentration. Specifically, we quantify liability concentration by applying the majorization order to the liability matrix that captures the interconnectedness of banks in a financial network. We develop notions of balancing and unbalancing networks to bring out the qualitatively different implications of liability concentration on the system’s loss profile. An empirical analysis of the network formed by the banking sectors of eight representative European countries suggests that the system is either unbalancing or close to it. This empirical finding, along with the majorization results, supports regulatory policies aiming at limiting the size of gross exposures to individual counterparties.

Second, we propose a multi-period clearing framework, where the level of systemic risk is mitigated through provision of liquidity assistance. The interbank liability network evolves stochastically over time, and assets of defaulted banks are sold to qualified banks within the network through a first-price sealed-bid auction. We find that policies targeting systemically important banks are more effective in core-periphery network structures, whereas those maximizing the total liquidity in the system are preferred in random network configurations. We assess sensitivity of systemic risk to variations in interbank liabilities as well as to their correlation structure.
CHAPTER 1. INTRODUCTION

1.1 Background

Financial institutions are connected to each other via a sophisticated network of multilateral exposures. Through these linkages, distress or failure of a financial institution triggering large unexpected losses on its trades can seriously affect the financial status of its counterparties. It is such negative externalities and the significant spillovers to the real economy that are the essence of systemic risk, see also [Caruana(2010)] and [Markose et al.(2012)].

The intricate structure of linkages can be naturally captured via a network representation of the financial system. Such a network models the interlinking exposures between financial institutions, and can thus assist in detecting important shock transmission mechanisms. This is also stressed by [Allen and Babus(2008)], who discuss how the use of network theories can enrich our understanding of financial systems, helping to answer questions related to how resilient financial networks are to contagion, and how financial institutions form connections when exposed to risk of contagion. Such a machinery becomes even more relevant if we consider the current post-crisis regime, where a series of decisions are being taken by governmental authorities to better monitor systemically important entities.

Starting with the seminal paper by [Allen and Gale(2001)], who employed an equilibrium approach to model the propagation of financial distress in a credit network, many other approaches have recently been proposed to explain systemic risk. [Gai and Kapadia(2010)] use statistical techniques from network theory to model how contagion spreads, and analyze how the knock-on effects of distress at some financial institutions can force other entities to write down the value of their assets. [Battiston et al.(2012a)] introduce the financial accelerator to characterize the feedback effect
arising from changes in the financial conditions of an agent. [Battiston et al.(2012b)] demonstrate that systemic risk does not necessarily decrease if the connectivity of the underlying financial network increases.

This thesis belongs to the stream of literature generated from the seminal work of [Eisenberg and Noe(2001)], who developed a clearing system framework consistent with bankruptcy law, to analyze systemic risk in interbank networks. To be more specific, they consider a financial network consisting of \( n \) nodes which represent financial institutions. For each node \( i \), let \( l_{i,j} \) be the amount of liabilities owed by \( i \) to \( j \), \( c_i \) the value of outside (non-interbank) assets held by node \( i \), and \( p_i^* \) the clearing payments made by node \( i \) to repay its liabilities. \( p_i^* \) must satisfy the conditions imposed by bankruptcy law which include proportional repayments of liabilities, limited liability, and absolute priority. This means that, for \( i = 1, \ldots, n \), \( p_i^* \) must satisfy the following equation:

\[
p_i^* = \min \left\{ \sum_{j=1}^{n} l_{i,j}, \sum_{j=1}^{n} \left( \frac{l_{j,i}}{\sum_{k=1}^{n} l_{j,k}} \right) p_j^* + c_i \right\}.
\]

[Eisenberg and Noe(2001)] propose the fictitious default algorithm to solve the above fixed point equations iteratively. It first determines the payments made by each node \( i \), assuming that all of node \( i \)'s debtors repay in full. If all nodes meet their liabilities, then the algorithm terminates and the clearing payment vector equals the total liability vector; otherwise, the set of defaulted nodes is identified and the payment vector is updated. The algorithm continues until the payment vector satisfies the above equation. Then, each node \( i \)'s exposure to the systemic risk is measured as the number of iterations needed for node \( i \) to default. The fewer iterations node \( i \) needs, the more systemic risk node \( i \) is exposed to.

Such a framework was utilized and extended along several directions. [Staum and Liu(2012)] analyze how systemic risk in financial networks should be quantified and allocated to individual institutions. [Rogers and Veraart(2013)] relax the assumption made in [Eisenberg and Noe(2001)] that a defaulting bank can liquidate its assets at
face value to repay due liabilities, and identify circumstances under which banks have incentives to rescue others. [Blanchet and Shi(2012)] consider a financial network with insurance and reinsurance companies, and provide a model to capture the total losses generated from default cascades originated from a reinsurance company. [Elliott et al.(2013)] study the impact of diversification and integration of a financial network on cascades of default events. On a more empirical side, [Cont et al.(2013)] and [Angelini et al.(1996)] analyze, respectively, Brazilian and Italian interbank systems, showing how defaults transmit through the payment system and originate systemic crisis.

For ease of terminology, throughout the thesis we will use “financial system” and “financial network” interchangeably and refer to its comprising entities as “financial institutions”, “banks” or “nodes”.

1.2 Contributions of the Thesis

The main contributions of this thesis are

- Development of a majorization-based tool to compare financial networks with different levels of liability concentration. We apply the majorization order to the liability matrix to quantify liability concentration. The notions of balancing and unbalancing networks are introduced to highlight the implications of liability concentration on the system’s loss profiles, depending on the capitalization level of the network.

We provide numerical examples of financial networks with empirically driven structures, such as perfect and imperfect tiering schemes. We conduct an empirical analysis of the network formed by the banking sectors of eight representative European countries. The analysis reveals that such a financial system is always either unbalancing or close to it. Together with the theoretical results, this empirical finding supports regulatory policies put forward by Basel Committee to limit the size of gross exposures to individual counterparties. These
results are presented in Chapter 2 and have been published in the INFORMS journal *Operations Research* (see [Capponi et al. (2015)]).

- Analysis of systemic risk mitigation policies. We propose a novel multi-period clearing framework, where the lender of last resort is introduced to mitigate the level of systemic risk through provision of liquidity assistance. Over time, the interbank liability network evolves stochastically. When a bank defaults, its assets are sold to qualified entities within the network through a first-price sealed-bid auction.

We measure the systemic risk reduction obtained by different mitigation policies. The simulation results show that policies maximizing the total liquidity reduce larger systemic risk in random network configurations, whereas those targeting systemically important banks are preferred in core-periphery network structures. Both types of policies become more effective when the variations in interbank liabilities increase or the correlation between interbank liabilities decreases. These results are presented in Chapter 3 and have been published in the finance journal *Journal of Economic Dynamics and Control* (see [Capponi and Chen (2015)]).

Chapter 4 summarizes the main results. The technical proofs of the results in Chapter 2 and 3 are delegated to the appendices.
CHAPTER 2. LIABILITY CONCENTRATION AND SYSTEMIC LOSSES IN FINANCIAL NETWORKS

2.1 Introduction

The financial industry is fraught with cases of bank failures due to large exposures to certain counterparties. One such example is Johnson Matthey Bankers in the United Kingdom in 1984. Bank assets more than doubled between 1980 and 1984, and loans became concentrated only to a few borrowers, including Mahmoud Sipra, Rajendra Sethia and ESAL Commodities, and Abdul Shamji. The quality of some of these loans turned out to be worse than expected, and Bank of England had to intervene to prevent a financial crisis. Another example is the Korean banking system during the crisis in the late 1990s, when the country’s bank assets were concentrated on five largest banks. These cases have prompted regulatory authorities in recent years to impose limits on banks’ exposures. For example, the Core Principles for Effective Banking Supervision (Core Principle 19) sets prudent limits on large exposures to a single borrower.

The goal of this chapter is to develop an analytical framework to assess how concentration of liabilities affects the exposures towards individual entities, and in turn induces losses in a financial system. Our starting point is the basic model of [Eisenberg and Noe(2001)], enhanced with bankruptcy costs as in [Glasserman and Young(2015)], which captures the interlinking exposures among financial institutions. The model yields the loss contributed by each node (bank) in the network, computed as the difference between its total liability and payment. We then use the majorization order ([Arnold et al.(2011)]) among vectors to express preferences between losses, i.e. one loss profile (vector) is preferred to another only if it is majorized by the latter. This allows us to capture the desired preference via a broad class of functions known
as Schur convex functions ( [Arnold et al.(2011)]), which preserve the majorization order. Such a class includes functions such as summation and max; thus, a loss profile is preferred to another if it results in a smaller total loss or a smaller worst-case loss.

More importantly, we use matrix majorization to compare relative liability matrices in terms of concentration of liabilities. When the relative liability matrix of network $a$ is majorized by the corresponding matrix of network $b$, it means that in network $a$ the interbank liabilities are more evenly distributed across the nodes in the network, as compared with $b$, where the liabilities are more concentrated. (Refer to Figure 2.1, where the relative liability matrix of the network on the left is majorized by the one on the right, in both upper and lower panels.)

In addition, we develop two new notions of financial networks, referred to as balancing and unbalancing. Note, the two notions are not orthogonal to each other, notwithstanding what their names may suggest. Both notions are defined in terms of the primitive data from the financial network; in particular, they both concern the pre-clearing equity position of the banks (nodes), with respect to their liabilities. The balancing notion is defined in terms of the base-liability configuration, i.e., assuming all nodes will pay their full liabilities; in which case, it stipulates that a node with a smaller liability is associated with a larger equity. This, of course, needs not be the case post-clearing, as some nodes might default. The unbalancing notion, instead, is defined in terms of a (minimally) reduced level of liability, which guarantees that all nodes will make full payment at clearing, and stipulates that a node with a smaller liability has a smaller equity. Thus, in an unbalancing system, a node with a smaller liability is more likely to default, and so does a node with a larger liability in a balancing system. (Both are the consequence of a smaller equity before clearing.)

Liability concentration has a knock-on effect when compounded with balancing and unbalancing systems. A high concentration of liability means that a node with a large (small) liability will receive more (less) payments relative to the case of low concentration. For example, when a balancing system is coupled with a lower concentration (of liability), a node with a larger liability tends to receive less payments
and thus incurs a larger loss. Moreover, a low concentration means that the payment flows among the nodes are more uniform; hence, the loss at one node is more likely to propagate to its neighboring nodes, and their neighboring nodes, and so forth. Thus, the low concentration implies a (potentially) more serious systemic consequence in the balancing case. In an unbalancing system, on the other hand, a higher liability concentration will yield the similar systemic consequence. In this case, a node with a smaller liability will tend to receive less payments, resulting in a larger loss. Since liabilities are more concentrated, the loss induced by a defaulted bank will have to be absorbed by a smaller number of its creditors, which may induce more defaults; hence, the contagion effect.

Our results are related to [Amini et al.(2010)], who show that a network with higher concentration of exposures is less resilient to shocks. In their framework, the loss incurred by the creditors of each node does not depend on the interbank liability structure. Rather it is given by the exposure of the creditor to the defaulted node multiplied by an exogenously specified loss given default rate. Moreover, their analysis is performed asymptotically as the number of institutions grows to infinity.

We use consolidated banking sector data of the eight largest European countries, consisting of balance sheet data and interbank exposures, to investigate the state of the network. Our analysis reveals that it is either unbalancing or close to it (see Table 2.3 for details) persistently over different time periods. These empirical findings, along with the above discussed theoretical results, support regulatory policies of the Basel Committee ([BCBS(2014)]) aiming at limiting the size of gross exposures to individual counterparties. Moreover, our results add to the understanding of preventive policies in bringing out their consequences and implications on the network. The regulator will monitor the interbank system and limit gross exposures toward banks that have small capital. In an unbalancing system, banks with smaller outstanding liabilities will incur larger losses. Hence, reducing gross exposures to them means to push banks with larger liabilities to lend less to those with smaller liabilities, making the matrix of interbank liabilities less concentrated and driving the network closer to
a balancing state. When this transition happens, banks with larger outstanding liabilities will suffer larger losses. The regulator would then need to incentivize them to lend more to banks with smaller liabilities. This will reduce the net exposure of banks with smaller liabilities to those with large liabilities, and thereby make liabilities more concentrated.

A brief overview of related literature is in order. Most studies on interbank networks have focused on understanding the impact of shocks, originating in a specific part of the network, on the overall financial system. [Allen and Gale(2001)] employ an equilibrium approach to model the propagation of financial distress in a credit network. [Gai and Kapadia(2010)] model how contagion spreads in a random network, and analyze the knock-on effects of distress. [Battiston et al.(2012a)] and [Battiston et al.(2012b)] characterize feedback effects arising from changing financial conditions of the network nodes. [Capponi and Chen(2015)] develop a multi-period extension of the Eisenberg-Noe model, and analyze the impact of different mitigation policies. [Rogers and Veraart(2013)] improve the realism of the [Eisenberg and Noe(2001)] model by including liquidation costs at default. [Elsinger et al.(2006)] distinguish between fundamental and contagious defaults in the [Eisenberg and Noe(2001)] framework, and analyze feedback and domino effects via an empirical analysis. [Glasserman and Young(2015)] show that, under a wide range of shock distributions, the contagion effects via network spillovers are usually small for realistic interbank networks. [Furfine(2003)] provides an empirical analysis quantifying contagion risk resulting from interbank federal funds exposures data. [Haldane and May(2011)] draw analogies with ecosystems and analyze how growth in interbank claims leads to instability. Other studies have explored the relation between the topological structure of the network and the magnitude of defaults it experiences. [Gai et al.(2011)] analyze the degree to which networks with a smaller number of key strongly interconnected players is affected by target shocks. [Elliott et al.(2013)] discuss the dependence of the probability of default cascades on integration and diversification. [Acemoglu et al.(2015)] develop a theoretical framework to explain the robust-yet-fragile tendency
of financial networks. On the empirical side, [Cont et al.(2013)] and [Angelini et al.(1996)] analyze, respectively, Brazilian and Italian interbank systems, and show how contagion through the payment system can originate systemic crisis.

The rest of this chapter is organized as follows. We start with preliminaries on both the Eisenberg-Noe model and the majorization order in Section 2.2, and formalize the notion of a loss profile and loss preference in a financial network. We then spell out in Section 2.3 the technical details in modeling a) the concentration of liabilities using matrix majorization, and b) the notion of balancing versus unbalancing networks and their implications on loss preference. Section 2.4 presents concrete examples to illustrate and enhance the notions of balancing and unbalancing systems. In particular, we make connections to the studies on German and Italian banks respectively presented in [Craig and Von Peter(2014)] and in [Fricke and Lux(2015)], where a tiering (or, core-periphery) structure has been identified as the primary configuration of interbank liabilities. We apply the balancing/unbalancing notions to the tiering structure and bring out the distinction between perfect and imperfect tiering schemes. Section 2.5 provides an empirical analysis of the network induced by the eight largest European banking sectors and develops policy implications. Proofs of technical results are delegated to Appendix A.

2.2 Loss Preferences

We describe the majorization method used to express loss preferences in Section 2.2.1. We recall the Eisenberg-Noe framework enhanced with bankruptcy costs in Section 2.2.2. We describe the objective of the study in Section 2.2.3.

2.2.1 Loss Comparison Using Majorization

We start by providing basic notations and definitions related to majorization and refer to [Arnold et al.(2011)] for a complete treatment of the subject.
Majorization is a preorder on vectors of real numbers, which measures the dispersion among the elements in a vector. For any vector \( x \in \mathbb{R}^n \), we use \( x[1], \ldots, x[n] \) to denote the ordered entries of \( x \) from largest to smallest (\( x[1] \) being the largest and \( x[n] \) the smallest). Moreover, we use \( x(1), \ldots, x(n) \) to denote the ordered entries of \( x \) from smallest to largest (\( x(1) \) being the smallest and \( x(n) \) the largest).

**Definition 2.2.1.** \( x \) is majorized by \( y \), denoted by \( x \prec y \), if
\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \text{ for } k = 1, \ldots, n-1, \quad \sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i],
\]

or equivalently,
\[
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i) \text{ for } k = 1, \ldots, n-1, \quad \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i).
\]

\( x \prec y \) indicates that the vector \( x \) is more evenly distributed than \( y \). Replacing the equality in Eq. (2.1) and (2.2) with \( \leq \) and \( \geq \) respectively leads to the notion of weak submajorization and weak supermajorization.

**Definition 2.2.2.** For \( x, y \in \mathbb{R}^n \), \( x \) is weakly submajorized by \( y \), denoted by \( x \prec_w y \), if
\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \text{ for } k = 1, \ldots, n.
\]

\( x \) is weakly supermajorized by \( y \), denoted by \( x \prec^w y \), if
\[
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i) \text{ for } k = 1, \ldots, n.
\]

Interchangeably, we denote \( x \succ^w y \) if \( y \prec^w x \) and \( x \succ_w y \) if \( y \prec_w x \).

We next explain how we express preferences between losses using majorization. Let \( x \in \mathbb{R}^n \) be a vector, whose \( i \)-th component \( x_i \) is interpreted as the loss generated by entity \( i \) in a financial network. We say that the loss vector \( x \) is preferred to the loss vector \( y \) if \( x \prec_w y \). Our choice is driven by the following consideration. Consider two networks \( a \) and \( b \) consisting of the same set of entities. We now think of \( x \) as the loss vector associated with network \( a \) and of \( y \) as the loss vector associated with
network \( b \). The difference between \( a \) and \( b \) lies in the interbank structure. If, for any \( k \in \{1, \ldots, n\} \), the sum of the \( k \) largest losses generated by entities in network \( a \) never exceeds the corresponding quantity in network \( b \), then we prefer the interbank network \( a \) to \( b \). In particular, this means that the maximum loss generated by a node in the network \( a \) does not exceed the maximum loss generated by a node in the network \( b \) \((k = 1)\). Further, it also implies that the total loss in the network \( a \) never exceeds the corresponding quantity in the network \( b \) \((k = n)\). Our preference criterion is also related to the monitoring mechanism proposed by [Duffie(2011)], where each systemically important entity is suggested to report the identities of the ten counterparties against which it has the largest gains or losses, under a set of stressful scenarios. While in [Duffie(2011)] the regulator is interested in monitoring losses on an individual basis, i.e. separately for each bank, in our case he would be concerned about the aggregate loss generated by those banks with the \( k \) highest shortfalls.

Notice that our objective is to measure the size of losses, and not in which specific nodes of the network they occurred. This property is preserved when weak submajorization is used to express preferences. Recall that weak submajorization is a preorder, i.e. \( x \prec_w y \) and \( y \prec_w x \) together imply that \( x = yP \) for some permutation matrix \( P \), but not that \( x = y \). Hence, a permutation of the loss vector is equally preferred to the original loss vector.

### 2.2.2 The Eisenberg-Noe Framework with Bankruptcy Costs

We define the loss vector associated with a financial system using an extended Eisenberg-Noe model, where losses due to bankruptcy are modeled as in [Glasserman and Young(2015)]. We consider a network of interbank liabilities consisting of \( n \) nodes, where each node represents a financial institution. Let \( L \in \mathbb{R}^{n \times n}_{\geq 0} \) be the interbank liability matrix with \( l_{i,j} \) denoting the amount of liabilities owed by \( i \) to \( j \), and \( c \in \mathbb{R}^{1 \times n}_{\geq 0} \) be the outside asset vector, in which each component \( c_i \) represents the
value of outside assets held by node $i$. Each node also has liabilities towards entities which are not part of the interbank network. More specifically, we let $e \in \mathbb{R}^{1 \times n}_{\geq 0}$ be the outside liability vector, where the entry $e_i$ denotes the amount of liabilities of node $i$ towards entities outside the network. These liabilities are assumed to have equal priority to the interbank liabilities.

The total liability vector is denoted by $\ell \in \mathbb{R}^{1 \times n}_{\geq 0}$, with $\ell_i := \sum_{j=1}^{n} l_{i,j} + e_i$ being the total amount of obligations from node $i$ to all other nodes and to the outside network. Further, we denote by $\pi_{i,j} := \begin{cases} \frac{l_{i,j}}{\ell_i} & \text{if } \ell_i > 0 \\ 0 & \text{if } \ell_i = 0, \end{cases}$ the relative size of liabilities owed by $i$ to $j$. Here, $\Pi$ is the interbank relative liability matrix. Because $i$ may also owe to entities outside the network, $\sum_{j=1}^{n} \pi_{i,j} \leq 1$ for each $i$, i.e. $\Pi$ is row substochastic.

The approach used by [Glasserman and Young(2015)] to model bankruptcy costs captures the fact that large shortfalls are more costly than small shortfalls. Concretely, when a node $i$ defaults its assets are reduced by the amount

$$\gamma \max \left\{ \ell_i - \left( \sum_{j=1}^{n} p_{j}^{\ast} \pi_{j,i} + c_i \right), 0 \right\}.$$  

Above, the term in curly brackets is the shortfall of node $i$ at default. Multiplying this quantity by the factor $\gamma$ gives the bankruptcy costs incurred by node $i$ at default. After accounting for these deadweight losses, the assets of node $i$ are distributed proportionally to its creditors. Hence, the clearing payment vector $p^\ast \in \mathbb{R}^{1 \times n}_{\geq 0}$ is a solution to the system

$$p^\ast = \left( [\ell \wedge (p^\ast \Pi + c)] - \gamma [\ell - (p^\ast \Pi + c)]^+ \right)^+,$$

where for any two vectors $x, y \in \mathbb{R}^{n}$,

$$x \wedge y := \left( \min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\} \right),$$
and \( x^+ := \max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\} \). Using the equality \((x - y)^+ = x - (x \wedge y)\), the above equation may be re-written as

\[
p^* = \left( (1 + \gamma) [\ell \wedge (p^*\Pi + c)] - \gamma \ell \right)^+ = \ell \wedge [p^*(1 + \gamma)\Pi + (1 + \gamma)c - \gamma \ell]^+.
\]

From the above expression, it is obvious that if \( \gamma = 0 \), then \( p^* \) coincides with the clearing payment vector in the basic Eisenberg-Noe model.

We use the 4-tuple \((\Pi, \ell, c, \gamma)\) to identify the financial system. We make the following assumption, supported by empirical evidence provided in section 2.5.

**Assumption 2.2.1.** For any financial system \((\Pi, \ell, c, \gamma)\), \((1 + \gamma)c - \gamma \ell \geq 0\).

[Glasserman and Young(2015)] mention that the case to be expected in practice is \( \gamma < 0.5 \). When \( \gamma = 0.5 \), the above condition reduces to \( c \geq \ell/3 \). We conduct an empirical study in Section 2.5 showing that, in this case, the previous inequality always holds (see Table 2.2 for details). Then the fixed point equation yielding the clearing payments can be simplified to

\[
p^* = \ell \wedge [p^*(1 + \gamma)\Pi + (1 + \gamma)c - \gamma \ell].
\]

[Glasserman and Young(2015)] show that the clearing payment vector is uniquely determined if the spectral radius of the matrix \((1 + \gamma)\Pi\) is smaller than 1. This turns out to be the case in most financial networks. (For instance, [Glasserman and Young(2015)] use European Banking Authority’s 2011 stress test data and find that the fraction of liabilities within the interbank network is 0.43.)

In the sequel, we use \( p^*(\Pi, \ell, c, \gamma) \) to emphasize the dependence of the clearing payment vector on the financial system \((\Pi, \ell, c, \gamma)\). We also distinguish between the asset value of a node before clearing and after clearing. The vector of asset values before clearing consists of the values of assets held by each node if all liabilities are repaid in full, and no bankruptcy costs are incurred. This is given by \( \ell\Pi + c \). The vector of asset values after clearing has components given by the values of assets held by each node after interbank clearing occurred. This is given by \((p^*\Pi + c) - \gamma[\ell - (p^*\Pi + c)]^+\).
2.2.3 Objective of this Study

We aim at understanding how losses are affected by concentration of liabilities. The loss vector associated with the financial system \((\Pi, \ell, c, \gamma)\) is defined as the difference between the total liability and clearing payment vector, i.e.

\[
s(\Pi, \ell, c, \gamma) := \ell - p^*(\Pi, \ell, c, \gamma).
\]

The \(i\)-th component of the above vector denotes the amount of losses generated by node \(i\). We illustrate in Figure 2.1 the behavior which we aim at capturing. The top graphs give interbank networks where the node with the largest outstanding liabilities (node 4) has the smallest equity value. The bottom graphs give interbank networks where the node with the smallest outstanding amount of liabilities (node 1) has the smallest equity value. Both top and bottom panels have in common that the node with the smallest equity value generates the largest loss in the system. Then the network with the smallest net exposure to this node is always the most preferred in terms of losses. However, there is a distinguishing feature between the interbank network structures. In the top panels, the undesired system is the network whose liabilities are less concentrated. In the bottom panels, instead, the network with higher concentration of liabilities is the undesired one. Our objective is to capture this behavior quantitatively.

The next section defines balancing systems to capture the network behavior reported in the top panels of Figure 2.1, and unbalancing systems to capture the network behavior in the bottom panels.

2.3 Concentration of Liabilities

The objective of this section is to quantitatively analyze how concentration of liabilities affects the loss profile of a financial system. Preliminaries on majorization, along with related results that will be extensively used later, are summarized in Section 2.3.1. A relaxed equivalent version of the relative liability matrix is introduced
(a) Networks in which the node with the largest outstanding liabilities (node 4) has the smallest equity value. Higher liability concentration generates smaller loss, hence it is preferred.

**Less concentrated liabilities**  
<table>
<thead>
<tr>
<th>$c$</th>
<th>(15, 25, 30, 125)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>(20, 120, 180, 400)</td>
</tr>
</tbody>
</table>

*Preferred*

```
<table>
<thead>
<tr>
<th>Equity†</th>
<th>(180, 65, −10, −190)</th>
<th>(45, 10, 10, −20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*$</td>
<td>(20, 110, 120, 190)</td>
<td>(20, 120, 180, 380)</td>
</tr>
<tr>
<td>$s$</td>
<td>(0, 10, 60, 210) ≥</td>
<td>(0, 0, 0, 20)</td>
</tr>
</tbody>
</table>
```

(b) Networks in which the node with the smallest outstanding liabilities (node 1) has the smallest equity value. Lower liability concentration generates smaller loss, hence is preferred.

**Less concentrated liabilities**  
<table>
<thead>
<tr>
<th>$c$</th>
<th>(20, 70, 100, 300)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>(120, 125, 150, 270)</td>
</tr>
</tbody>
</table>

```
<table>
<thead>
<tr>
<th>Equity†</th>
<th>(41, 85, 86, 146)</th>
<th>(0, 55, 135, 168)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*$</td>
<td>(110, 125, 150, 270)</td>
<td>(20, 80, 150, 270)</td>
</tr>
<tr>
<td>$s$</td>
<td>(10, 0, 0, 0) ≤</td>
<td>(100, 45, 0, 0)</td>
</tr>
</tbody>
</table>
```

Figure 2.1. Networks illustrating the objective of study. We set $\gamma = 0$. † The equity under the base-liability configuration is given by $\ell \Pi + c - \ell$. ‡ The equity under the reduced-liability configuration is given by $\ell \Pi + c - \ell$, where $\ell$ is later defined in Eq. (2.3).
in Section 2.3.2, and this proves to be the key to characterizing liability concentration via matrix majorization. The notions of balancing and unbalancing systems and results concerning loss preferences are presented in Section 2.3.3.

2.3.1 Preliminary Results

We start recalling the definition of similarly ordered vectors. Two vectors $x$ and $y$ are similarly ordered if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j$ (see also [Arnold et al.(2011)] Ch.6 Proposition A.1.a).

Next, we define what it means for a matrix to be order preserving.

**Definition 2.3.1.** Let $D \in \mathbb{R}^{n \times n}$ and $A \subset \mathbb{R}^n$ be a subset of the space of $n$ dimensional real-valued vectors. $D$ is order preserving w.r.t $A$ if for $x \in A$, $xD$ and $x$ are similarly ordered.

The next lemma characterizes the set of matrices which are order preserving w.r.t. a set of nonnegative similarly ordered vectors. Let $\mathcal{A}$ be a set of nonnegative similarly ordered vectors and $x$ be an arbitrary element of $\mathcal{A}$. We denote by $\nu(\mathcal{A})$ the mapping defined by $\nu_i(\mathcal{A}) = j$ if $x(i) = x_j$. Given a matrix $D := (d_{ij})_{i,j=1}^n$, we use the abbreviated notation $d_{\nu(A)}^{\nu(A)} := d_{\nu_i(A),\nu_j(A)}$.

**Lemma 2.3.1.** Fix a vector $z \in \mathbb{R}_{>0}^n$ and let $\mathcal{A} := \{x | x \text{ is similarly ordered to } z, 0 \leq x \leq z\}$. $D \in \mathbb{R}^{n \times n}$ is order preserving w.r.t. $\mathcal{A}$ if and only if $$\sum_{i=k}^n d_{\nu(A)}^{\nu(A)} \leq \sum_{i=k}^n d_{\nu(A)}^{\nu(A)} + 1, \quad k = 1, \ldots, n, j = 1, \ldots, n - 1.$$ 

Next, we define the class of matrices which preserves the weak majorization order.

**Definition 2.3.2.** Let $D \in \mathbb{R}^{n \times n}$ and $A \subset \mathbb{R}^n$ be a subset of the space of $n$ dimensional real-valued vectors.

- $D$ is weak submajorization preserving w.r.t $A$ if for $x, y \in A$, $x \prec_w y$ implies $xD \prec_w yD$. 

• D is weak supermajorization preserving w.r.t. A if for x, y ∈ A,

\[ x \prec^w y \text{ implies } xD \prec^w yD. \]

The set of order preserving matrices which are also weak submajorization or weak supermajorization preserving are characterized in the following lemma.

**Lemma 2.3.2.** Fix a vector \( z \in \mathbb{R}_{>0}^n \) and let \( A := \{x | x \text{ is similarly ordered to } z, 0 \leq x \leq z\} \). Assume that \( D \in \mathbb{R}^{n \times n} \) is an order preserving matrix w.r.t. A. The following statements hold:

• D is weak submajorization preserving w.r.t. A if and only if

\[
\sum_{j=k}^{n} d_{i,j}^{\nu(A)} \leq \sum_{j=k}^{n} d_{i+1,j}^{\nu(A)}, \quad k = 1, \ldots, n, i = 1, \ldots, n-1.
\]

• D is weak supermajorization preserving w.r.t. A if and only if

\[
\sum_{j=1}^{k} d_{i,j}^{\nu(A)} \geq \sum_{j=1}^{k} d_{i+1,j}^{\nu(A)}, \quad k = 1, \ldots, n, i = 1, \ldots, n-1.
\]

### 2.3.2 Liability Matrix: Relaxation and Concentration

The presence of zero elements on the diagonal of the matrix \( \Pi \) (i.e. \( \pi_{i,i} = 0 \)) restricts drastically the set of matrices \( \Pi \) that are order preserving. Indeed, multiplying a vector \( x \) by a relative liability matrix \( \Pi \), we can obtain a vector \( x\Pi \) that has the opposite ordering of \( x \). Such a situation can be avoided by replacing \( \Pi \) with a suitably chosen relaxed version defined as follows.

**Definition 2.3.3.** A 3-tuple \( (\Pi_{a,\gamma}, \ell, c_{a,\gamma}) \), \( \alpha \in [0, 1) \), is called the \( \alpha \)-relaxed equivalent version of a financial system \( (\Pi, \ell, c, \gamma) \), if \( \Pi_{a,\gamma} = (1 - \alpha) [(1 + \gamma)\Pi] + \alpha I \) and \( c_{a,\gamma} = (1 - \alpha) [(1 + \gamma)c - \gamma \ell] \). Here \( I \) is the identity matrix.

As an example, consider the case of no bankruptcy costs. Choosing \( \alpha = 0.5 \), we obtain that the vector \( x\Pi_{0.5,0} \) preserves the ordering of \( x \).
The clearing payment vector is invariant with respect to this relaxation, as stated in the following lemma.

**Lemma 2.3.3.** Let \((\Pi, \ell, c, \gamma)\) be a financial system. Then it holds that \(p^*(\Pi, \ell, c, \gamma) = p^*(\Pi_{\alpha, \gamma}, \ell, c_{\alpha, \gamma})\) for any \(\alpha \in [0, 1)\).

Last, in order to compare two network topologies with different liability configurations, the notion of majorization for vectors is generalized to *majorization for matrices*.

**Definition 2.3.4.** Let \(X\) and \(Y\) be \(m \times n\) matrices. \(X\) is said to be majorized by \(Y\), \(X \prec Y\), if there exists a doubly stochastic matrix \(S\) such that \(X = YS\).

If \(X \prec Y\), each row in \(X\) is more evenly distributed than the corresponding row in \(Y\). Indeed, when \(m = 1\), it is well known that this definition is equivalent to Definition 2.2.1 (see [Arnold et al.(2011)] Ch.2 Theorem B.2). This leads us to use the following criteria to compare networks in terms of liability concentration.

**Definition 2.3.5.** Given two financial systems \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\), we say that \(b\) has a higher liability concentration than \(a\) if there exists \(\alpha \in [0, 1)\) such that \(\Pi^a_{\alpha, \gamma} \prec \Pi^b_{\alpha, \gamma}\).

The above definition is consistent with intuition. All systems whose relative liability matrices are permutations of each other have the same liability concentration. In all other cases, if \(b\) has higher liability concentration than \(a\), then \(a\) cannot have higher concentration than \(b\). The precise statement and proof of this fact is given in Lemma A.1 in the appendix.
Using the above definition of liability concentration, we can verify that the networks in the left panels of Figure 2.1, here denoted by $a$, have lower liability concentrations than those in the right panels, here denoted by $b$. For the top panels, we have that $\Pi_{0.2,0}^{a} \prec \Pi_{0.2,0}^{b}$, given that

\[
\begin{pmatrix}
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2
\end{pmatrix}
\begin{pmatrix}
\mathbf{n}_{0.2,0}^{a}
\end{pmatrix}
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}
\begin{pmatrix}
0.2 & 0 & 0 & 0.6 \\
0.2 & 0 & 0 & 0.6 \\
0.2 & 0 & 0.2 & 0.6 \\
0.2 & 0.2 & 0.2 & 0.2
\end{pmatrix}
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}
\begin{pmatrix}
\mathbf{s}
\end{pmatrix}
\]

For the bottom panels, we have that $\Pi_{0.14,0}^{a} \prec \Pi_{0.14,0}^{b}$, i.e.,

\[
\begin{pmatrix}
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14
\end{pmatrix}
\begin{pmatrix}
\mathbf{n}_{0.14,0}^{a}
\end{pmatrix}
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}
\begin{pmatrix}
0.14 & 0.42 & 0 & 0 \\
0 & 0.14 & 0 & 0.42 \\
0 & 0 & 0.14 & 0.42 \\
0 & 0 & 0.42 & 0.14
\end{pmatrix}
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}
\begin{pmatrix}
\mathbf{s}
\end{pmatrix}
\]

**Remark 2.3.1.** [Amini et al. (2010)] use the proportion of edges which are contagious to assess the resilience of a financial network to shocks. The weight of the edge directed from $i$ to $j$ represents the exposure of $i$ to $j$. The edge from $i$ to $j$ is contagious if node $j$’s default will cause $i$ to default. Fix a level of total exposure within the network (given by the sum of bilateral exposures). In their model, losses are proportional to the size of exposures. Hence, a larger fraction of contagious edges implies a higher concentration of exposures within the network, and in turn a larger default cascade. Conversely, a smaller fraction of contagious edges implies a lower concentration of exposures and a smaller default cascade. Asymptotically, their analysis shows that a financial network with fewer contagious links is more resilient to shocks. Using the above mentioned implications, their results also indicate that a financial network with lower concentration of exposures is more resilient to shocks. Their results are
consistent with ours for the case of unbalancing systems, where higher systemic losses arise in the network with higher concentration of liabilities (see Theorem 2.3.2 for details).

2.3.3 Loss Preferences in Balancing and Unbalancing Systems

This section defines the class of balancing and unbalancing financial systems and provides quantitative statements relating liability concentration to loss profiles. We make the following empirically supported assumption (see Section 2.5, Table 2.2, for details).

Assumption 2.3.1. For any financial system \((\Pi, l, c, \gamma)\), \(l\) and \([(1 + \gamma)c - \gamma l]\) are similarly ordered.

This assumption directly implies that \(l\) and \(c\) are similarly ordered, i.e. that nodes with larger liabilities also have larger outside asset values.

The definition of balancing and unbalancing systems depends on the nodes’ equity under the base and reduced-liability configurations. In the balancing system, the equity of node \(i\) is

\[
\left[ l\Pi + c \right]_i \text{ assets under base-liability configuration} - l_i.
\]

The above can be viewed as the pre-clearing equity at node \(i\) (or equivalently, its post-clearing equity assuming all nodes pay their full liabilities). In the unbalancing system, the equity is computed by reducing each node’s liability to a level as follows:

\[
\ell := l - \delta 1,
\]

where

\[
\delta := \max \left\{ l_{(1)} - [(1 + \gamma)c - \gamma l]_{(1)}, 0 \right\}
\]

and 1 denotes a row vector with all entries equal to one. Thus, the equity of node \(i\) is given by

\[
\left[ l\Pi + c \right]_i \text{ assets under reduced-liability configuration} - \ell_i.
\]
We show in the following Lemma 2.3.4 that $\ell$ coincides with the maximum liability vector under which all nodes can repay their liabilities in full.

Given a financial system $(\Pi, \ell, c, \gamma)$, we set $\mu := \nu(\{\ell\})$, so that $\mu_i = j$ if $\ell_i = \ell_j$. We define balancing and unbalancing financial systems and illustrate their structural properties in Figure 2.2.

**Definition 2.3.6.** Let $(\Pi, \ell, c, \gamma)$ be a financial system.

(I) It is balancing if, for $j = 1, \ldots, n - 1$,

\[
\sum_{i=1}^{n} \ell(i) \pi_{ij} \mu + c_{j+1} - \ell(j+1) \leq \sum_{i=1}^{n} \ell(i) \pi_{ij} \mu + c_j - \ell(j),
\]

where we recall that $\ell$ is defined in Eq. (2.3).

(II) It is unbalancing if, for $j = 1, \ldots, n - 1$,

\[
\sum_{i=1}^{n} \ell(i) \pi_{ij} \mu + c_{j+1} - \ell(j+1) \geq \sum_{i=1}^{n} \ell(i) \pi_{ij} \mu + c_j - \ell(j),
\]

where we recall that $\ell$ is defined in Eq. (2.3).

Intuitively, in a balancing system, a node with a larger liability ($\ell_{j+1}$) has a smaller equity before clearing, as represented by the inequality in (2.4); in an unbalancing system, if a node has a larger liability then it also has a larger equity before clearing under the reduced-liability configuration, per the inequality in (2.5). This captures precisely the behavior of the networks illustrated in Figure 2.1, where the node with the largest liability has the smallest equity value in the top two graphs (balancing system), while the same node has the largest equity value in the bottom two graphs (unbalancing system).

The next lemma shows that under the reduced-liability configuration, all nodes repay their liabilities in full; and the reduction as specified in (2.3) is minimal — short of which the absence of default is no longer guaranteed.

**Lemma 2.3.4.** For any unbalancing system $(\Pi, \ell, c, \gamma)$, it must hold that $p^*(\Pi, \ell, c, \gamma) = \ell$. For any vector $\epsilon$ with strictly positive entries, there exists at least one unbalancing system $(\Pi, \ell, c, \gamma)$ such that $p^*(\Pi, \ell + \epsilon, c, \gamma) \geq \ell + \epsilon$. 

Figure 2.2. Balancing: if a node has larger liabilities than another, then its equity is smaller under the base-liability configuration. Unbalancing: if a node has larger liabilities than another, then its equity is larger under the reduced-liability configuration.
To proceed further, we need to first specify the set of payment vectors used to analyze the loss generated in balancing and unbalancing systems:

$$\mathcal{P} = \{ p | p \text{ is similarly ordered to } \ell, 0 \leq p \leq \ell \}.$$  

Note that in characterizing \( \mathcal{P} \), the condition of \( p \) being similarly ordered to \( \ell \) is the more substantive one. It requires that the nodes making larger payments are also those with larger outstanding liabilities; and Proposition 2.3.1 gives a sufficient condition for this to hold. The other condition \( 0 \leq p \leq \ell \) says that a node does not pay more than its nominal liabilities, but allows for a large class of payment vectors including those for which absolute priority or limited liability is violated.

The following proposition characterizes the relations between clearing payments, liabilities and loss vectors, both in balancing and unbalancing financial systems.

**Proposition 2.3.1.** Let \((\Pi, \ell, c, \gamma)\) be a financial system. Suppose there exists \( \alpha \in [0, 1) \) such that \( \Pi_{\alpha, \gamma} \) is order preserving w.r.t. \( \mathcal{P} \). Then,

(1) \( p^* \) is similarly ordered to \( \ell \).

(II) If \((\Pi, \ell, c, \gamma)\) is balancing, then \( \ell_1 - p^*_1 \geq \ell_2 - p^*_2 \geq \cdots \geq \ell_n - p^*_n \).

(III) If \((\Pi, \ell, c, \gamma)\) is unbalancing, then \( \ell_1(1) - p^*_1(1) \geq \ell_2(2) - p^*_2(2) \geq \cdots \geq \ell_n(n) - p^*_n(n) \).

The above proposition indicates that when there exists an \( \alpha \)-relaxed equivalent version that preserves the order of payments, the nodes making larger payments are also those with larger liabilities. Moreover, larger shortfalls occur at nodes with larger liabilities in balancing systems, and at those with smaller liabilities in unbalancing systems.

The next theorem compares clearing payment vectors and loss profiles in balancing systems based on liability concentration. It concludes that the system with a higher concentration is preferred as it results in a smaller loss.

**Theorem 2.3.1.** Let \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) be two balancing financial systems. Suppose there exists \( \alpha \in [0, 1) \) such that both \( \Pi^a_{\alpha, \gamma} \) and \( \Pi^b_{\alpha, \gamma} \) are order preserving w.r.t. \( \mathcal{P} \) and
(I) $\Pi^a_{\alpha,\gamma}$ or $\Pi^b_{\alpha,\gamma}$ is weak submajorization preserving w.r.t. $\mathcal{P}$,

(II) $\Pi^a_{\alpha,\gamma} \prec \Pi^b_{\alpha,\gamma}$.

Then, $p^*(\Pi^a, \ell, c, \gamma) \prec_w p^*(\Pi^b, \ell, c, \gamma)$ and $s(\Pi^a, \ell, c, \gamma) \succ_w s(\Pi^b, \ell, c, \gamma)$.

Here is the intuition behind the above results. First, from the earlier Proposition 2.3.1, we know that in the balancing case larger shortfalls occur at nodes with larger liabilities; and this holds for both systems $a$ and $b$. Moreover, in one of the systems, those nodes with larger liabilities will make larger payments due to the weak submajorization preserving condition in (I) (this can be seen from Lemma 2.3.2 setting $D = \Pi^a_{\alpha,\gamma}$), and they will receive smaller payments in system $a$ than in system $b$, as the payments in system $a$ are more evenly distributed due to the majorization order in (II). Consequently, nodes with large liabilities in system $a$ will make less payments and have larger shortfalls relative to those in system $b$.

Next, we give the corresponding result for unbalancing systems. In this case, the system with a lower liability concentration is preferred.

**Theorem 2.3.2.** Let $(\Pi^a, \ell, c, \gamma), (\Pi^b, \ell, c, \gamma)$ be two unbalancing financial systems. Suppose there exists $\alpha \in [0, 1)$ such that both $\Pi^a_{\alpha,\gamma}$ and $\Pi^b_{\alpha,\gamma}$ are order preserving w.r.t. $\mathcal{P}$ and

(I) $\Pi^a_{\alpha,\gamma}$ or $\Pi^b_{\alpha,\gamma}$ is weak supermajorization preserving w.r.t. $\mathcal{P}$,

(II) $\Pi^a_{\alpha,\gamma} \prec \Pi^b_{\alpha,\gamma}$.

Then, $p^*(\Pi^a, \ell, c, \gamma) \prec_w p^*(\Pi^b, \ell, c, \gamma)$ and $s(\Pi^a, \ell, c, \gamma) \succ_w s(\Pi^b, \ell, c, \gamma)$.

Here the intuition parallels the balancing case. In the unbalancing case larger shortfalls occur at nodes with smaller liabilities, per Proposition 2.3.1; and the losses are exacerbated by the fact that those nodes tend to make larger payments, due to the weak supermajorization preserving condition in (I) (this can be seen from Lemma 2.3.2 setting $D = \Pi^b_{\alpha,\gamma}$). Moreover, those same nodes in system $b$ will receive smaller payments than in system $a$, due to the majorization order in (II).
Figure 2.3. Two financial systems consisting of six nodes whose liabilities are $\ell_6 > \ell_5 > \cdots > \ell_1$. For brevity, $p^*(\Pi^a, \ell, c, \gamma)$ and $p^*(\Pi^b, \ell, c, \gamma)$ are denoted, respectively, by $p^a*$ and $p^b*$. The graphs illustrate the relation between asset values of defaulted nodes after clearing (when a node $i$ defaults, it must hold that $p^*_i = \left[ (p^*\Pi + c) - \gamma \left[ \ell - (p^*\Pi + c) \right]^+ \right]_i = \left[ p^*\Pi_{\alpha,\gamma} + c_{\alpha,\gamma} \right]_i$) and liability concentration. The left panel illustrates that in balancing systems, losses occur at the nodes with larger liabilities and larger losses are generated in system $a$ where liabilities are less concentrated. The right panel illustrates that in unbalancing systems, losses occur at the nodes with smaller liabilities and larger losses occur in system $b$ where liabilities are more concentrated.

Consequently, nodes with smaller liabilities in system $b$ can only make less payments and have larger shortfalls relative to those in system $a$.

Figure 2.3 illustrates how liability concentration affects the asset values of nodes after clearing, and consequently the loss profile in both types of systems.

We conclude this section with a result indicating that systemic losses would be reduced if the financial system is both balancing and unbalancing. As the next lemma shows, if at least one node does not default and order preserving relations of the relative liability matrix are maintained, there would be zero loss in such a financial system. From a regulatory perspective, this suggests that it is beneficial to drive the network towards such a state.
Proposition 2.3.2. Let \((\Pi, \ell, c, \gamma)\) be a financial system such that \(\sum_{i=1}^{n} \pi_{i,j+1}^\mu - \sum_{i=1}^{n} \pi_{i,j}^\mu \geq 0\) for \(j = 1, \ldots, n-1\), and at least one node repays its liabilities in full. If \((\Pi, \ell, c, \gamma)\) is both balancing and unbalancing, then \(s(\Pi, \ell, c, \gamma) = 0\).

We next provide an example showing the existence of a system satisfying the conditions in the above proposition.

Example 2.3.1. Consider a balancing (respectively unbalancing) system \((\Pi, \ell, c, \gamma)\) where the column sums of \(\Pi\) are identical and the inequalities in (2.4) (respectively (2.5)) become equalities. Then the system is both balancing and unbalancing. One instance of such a system is given by

\[
\Pi = \begin{pmatrix}
0 & 0.27 & 0.27 & 0.24 \\
0.26 & 0 & 0.25 & 0.28 \\
0.26 & 0.25 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0 \\
\end{pmatrix}, \quad \ell = \begin{pmatrix} 4 & 6 & 8 & 50 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 3.6 & 6.1 & 59.8 \end{pmatrix}, \quad \gamma = 0.
\]

Clearly, the difference between two consecutive rank ordered components in \(\ell \Pi + c = \begin{pmatrix} 18.7 & 20.7 & 22.7 & 64.7 \end{pmatrix}\) and \(\ell\) are the same. The same applies to \(\ell \Pi + c = \begin{pmatrix} 16.3 & 18.3 & 20.3 & 62.3 \end{pmatrix}\) and \(\ell = \begin{pmatrix} 1 & 3 & 5 & 47 \end{pmatrix}\).

2.4 Application to Core-Periphery Network

This section develops numerical examples of financial networks which mimic the structure of tiered systems. Such systems have been identified by empirical research as good descriptors of interbank activity, see the study of [Craig and Von Peter(2014)] on the German banking system from 1999 to 2007, and of [Fricke and Lux(2015)] on the overnight interbank transactions in the Italian market from 1999 to 2010. Our objective is to apply the results developed in the previous sections to analyze which tiered structure is preferred depending on whether the system is balancing or unbalancing.

In a tiered financial system the pattern of interbank liabilities follows a core-periphery structure. The network is centered around a set of core nodes which in-
intermediate between numerous smaller nodes in the periphery. In particular, such a
network has following properties:

(I) The size of the core node is significantly larger than that of peripheral nodes.
[Craig and Von Peter(2014)] find that the average size of core banks is 51 times
that of peripheral banks.

(II) The peripheral nodes do not borrow from or lend to other peripheral nodes.

A financial system satisfying (I) and (II) is called \textit{perfectly tiered}, while a financial
system which only satisfies (I) is called \textit{imperfectly tiered}. For illustration purposes,
we set the number of nodes $n = 4$, with nodes 1, 2, 3 being peripheral and node 4 core. This means that the relative liability matrix $\Pi^a$ of an imperfectly tiered
financial system is given by

$$
\Pi^a = \begin{pmatrix}
0 & \pi^a_{1,2} & \pi^a_{1,3} & \pi^a_{1,4} \\
\pi^a_{2,1} & 0 & \pi^a_{2,3} & \pi^a_{2,4} \\
\pi^a_{3,1} & \pi^a_{3,2} & 0 & \pi^a_{3,4} \\
\pi^a_{4,1} & \pi^a_{4,2} & \pi^a_{4,3} & 0
\end{pmatrix},
$$

where $\pi^a_{i,j} \geq 0$ for $i, j = 1, \ldots, 3$, and at least one entry $\pi^a_{i,j} > 0$. The relative liability
matrix $\Pi^b$ of a perfectly tiered financial system, instead, has the form given by

$$
\Pi^b = \begin{pmatrix}
0 & 0 & 0 & \pi^b_{1,4} \\
0 & 0 & 0 & \pi^b_{2,4} \\
0 & 0 & 0 & \pi^b_{3,4} \\
\pi^b_{4,1} & \pi^b_{4,2} & \pi^b_{4,3} & 0
\end{pmatrix},
$$

where $\pi^b_{i,4} \geq 0$ for $i = 1, \ldots, 3$, and $\pi^b_{4,j} \geq 0$ for $j = 1, \ldots, 3$.

We next develop numerical examples of networks that are balancing or unbalanc-
ing; and for both cases, we analyze whether the tiered or imperfectly tiered structure
is preferred.
2.4.1 Balancing Networks

Consider the three financial systems, \((\Pi^a, \ell, c, \gamma), (\tilde{\Pi}^a, \ell, c, \gamma), (\Pi^b, \ell, c, \gamma)\), given by

\[
\Pi^a = \begin{pmatrix}
0 & 0.25 & 0.25 & 0.24 \\
0.23 & 0 & 0.23 & 0.28 \\
0.23 & 0.23 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0 \\
\end{pmatrix}, \quad \tilde{\Pi}^a = \begin{pmatrix}
0 & 0.24 & 0.25 & 0.25 \\
0.18 & 0 & 0.28 & 0.28 \\
0.18 & 0.28 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0 \\
\end{pmatrix},
\]

\[
\Pi^b = \begin{pmatrix}
0 & 0 & 0 & 0.74 \\
0 & 0 & 0 & 0.74 \\
0 & 0 & 0 & 0.74 \\
0.28 & 0.28 & 0.28 & 0 \\
\end{pmatrix}, \quad \tilde{\Pi}^b = \begin{pmatrix}
0 & 0.25 & 0.23 & 0.23 \\
0.21 & 0.25 & 0.21 & 0.25 \\
0.21 & 0.21 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
\end{pmatrix},
\]

\[
\ell = \begin{pmatrix} 4g, \ 6g, \ 8g, \ 50g \end{pmatrix}, \quad c = \begin{pmatrix} 3g, \ 4g, \ 5g, \ 25g \end{pmatrix}, \quad \text{and } \gamma = 1/5. \quad \text{Here, } g \text{ is a positive constant.}
\]

Choosing \(\alpha = 1/4\), we obtain

\[
\Pi^a_{1/4,1/5} = \begin{pmatrix}
0.25 & 0.23 & 0.23 & 0.21 \\
0.21 & 0.25 & 0.21 & 0.25 \\
0.21 & 0.21 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
\end{pmatrix}, \quad \tilde{\Pi}^a_{1/4,1/5} = \begin{pmatrix}
0.25 & 0.21 & 0.23 & 0.23 \\
0.17 & 0.25 & 0.25 & 0.25 \\
0.17 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
\end{pmatrix},
\]

\[
\Pi^b_{1/4,1/5} = \begin{pmatrix}
0.25 & 0 & 0 & 0.67 \\
0 & 0.25 & 0 & 0.67 \\
0 & 0 & 0.25 & 0.67 \\
0.25 & 0.25 & 0.25 & 0.25 \\
\end{pmatrix},
\]

where the relaxed versions \(\Pi^a_{1/4,1/5}\) and \(\tilde{\Pi}^a_{1/4,1/5}\) are both order and weak submajorization preserving w.r.t. \(\mathcal{P}\). This is because both systems satisfy the assumptions in lemmas 2.3.1 and 2.3.2. Moreover, \(\Pi^b_{1/4,1/5}\) is order preserving w.r.t. \(\mathcal{P}\) given that
it satisfies the conditions of Lemma 2.3.1. In addition, we can find doubly stochastic matrices

\[
S = \begin{pmatrix}
0.35 & 0.27 & 0.27 & 0.11 \\
0.2 & 0.31 & 0.17 & 0.32 \\
0.2 & 0.17 & 0.31 & 0.32 \\
0.25 & 0.25 & 0.25 & 0.25 \\
\end{pmatrix}, \quad \tilde{S} = \begin{pmatrix}
0.47 & 0.11 & 0.21 & 0.21 \\
0.17 & 0.31 & 0.26 & 0.26 \\
0.17 & 0.31 & 0.26 & 0.26 \\
0.19 & 0.27 & 0.27 & 0.27 \\
\end{pmatrix}
\]

such that \( \Pi_{1/4,1/5}^a S = \Pi_{1/4,1/5}^a \) and \( \Pi_{1/4,1/5}^b \tilde{S} = \Pi_{1/4,1/5}^b \).

We next show that \((\Pi^a, \ell, c, \gamma), (\tilde{\Pi}^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) are balancing. For these systems, the equity vectors under the base-liability configuration are given by

\[
\ell \Pi^a + c - \ell = (16 g, 15 g, 13 g, -20 g) \\
\ell \tilde{\Pi}^a + c - \ell = (16 g, 15 g, 14 g, -20 g) \\
\ell \Pi^b + c - \ell = (13 g, 12 g, 11 g, -12 g).
\]

These vectors are all reversely ordered to the liability vector \( \ell \), which implies that the three systems are balancing from Definition 2.3.6.

By taking convex combinations of the financial systems \((\Pi^a, \ell, c, \gamma)\) and \((\tilde{\Pi}^a, \ell, c, \gamma)\), we can generate a large class of balancing systems. Concretely, let \((\hat{\Pi}^a, \ell, c, \gamma) = \lambda(\Pi^a, \ell, c, \gamma) + (1 - \lambda)(\tilde{\Pi}^a, \ell, c, \gamma)\) for some \( \lambda \in [0, 1] \). Then it is clear that \((\hat{\Pi}^a, \ell, c, \gamma)\) is balancing because the class of balancing systems is closed under convex combination. Moreover, \(\hat{\Pi}^a_{1/4,1/5}\) is order and weak submajorization preserving w.r.t. \( \mathcal{P} \) since both of these relations are preserved for convex combinations. Additionally \(\hat{\Pi}^a_{1/4,1/5}\) is majorized by \(\Pi^b_{1/4,1/5}\), given that \(\hat{\Pi}^a_{1/4,1/5} = \lambda \Pi^b_{1/4,1/5} S + (1 - \lambda) \Pi^b_{1/4,1/5} \tilde{S}\), and \(\lambda S + (1 - \lambda) \tilde{S}\) is doubly stochastic.

Since both \((\hat{\Pi}^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) satisfy the assumptions of Theorem 2.3.1, we deduce that \(s(\hat{\Pi}^a, \ell, c, \gamma) \succ_w s(\Pi^b, \ell, c, \gamma)\). Therefore, the perfectly tiered network is preferred to the imperfectly tiered network if both are balancing.

This can be understood as follows. When a network is balancing, nodes with larger liabilities will incur larger losses, i.e. the core nodes, see also left panel of Figure 2.3. Moreover, from Theorem 2.3.1 we obtain that the clearing payments in
the imperfectly tiered network are more evenly distributed than the corresponding payments in the perfectly tiered network. This is because in the imperfectly tiered structure, larger payments are made to periphery as opposed to core nodes, while in the perfectly tiered structure all payments from peripheral nodes are directed to core nodes. Hence, core nodes are more likely to default and generate larger losses in the imperfectly tiered network.

2.4.2 Unbalancing Networks

Let the three financial systems, \((\Pi^a, \ell, c, \gamma), (\tilde{\Pi}^a, \ell, c, \gamma), (\Pi^b, \ell, c, \gamma)\), be specified by

\[
\Pi^a = \begin{pmatrix} 0 & 0.32 & 0.33 & 0.25 \\ 0.27 & 0 & 0.33 & 0.3 \\ 0.27 & 0.25 & 0 & 0.38 \\ 0.25 & 0.27 & 0.27 & 0 \end{pmatrix}, \quad \tilde{\Pi}^a = \begin{pmatrix} 0 & 0.33 & 0.32 & 0.25 \\ 0.25 & 0 & 0.32 & 0.33 \\ 0.25 & 0.31 & 0 & 0.34 \\ 0.25 & 0.27 & 0.27 & 0 \end{pmatrix},
\]

\[
\Pi^b = \begin{pmatrix} 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0.9 \\ 0.19 & 0.3 & 0.3 & 0 \end{pmatrix},
\]

\[\ell = \left(24g, \ 26g, \ 28g, \ 60g\right), \ c = \left(3g, \ 8g, \ 13g, \ 56g\right), \ \text{and} \ \gamma = 1/10. \ \text{Choosing} \ \alpha = 1/4, \ \text{we obtain}
\]

\[
\Pi^a_{1/4,1/10} = \begin{pmatrix} 0.25 & 0.26 & 0.27 & 0.21 \\ 0.22 & 0.25 & 0.27 & 0.25 \\ 0.22 & 0.21 & 0.25 & 0.31 \\ 0.21 & 0.22 & 0.22 & 0.25 \end{pmatrix}, \quad \tilde{\Pi}^a_{1/4,1/10} = \begin{pmatrix} 0.25 & 0.27 & 0.26 & 0.21 \\ 0.21 & 0.25 & 0.26 & 0.27 \\ 0.21 & 0.25 & 0.25 & 0.28 \\ 0.21 & 0.22 & 0.22 & 0.25 \end{pmatrix},
\]

\[
\Pi^b_{1/4,1/10} = \begin{pmatrix} 0.25 & 0 & 0 & 0.74 \\ 0.25 & 0.25 & 0 & 0.74 \\ 0 & 0 & 0.25 & 0.74 \\ 0.15 & 0.25 & 0.25 & 0.25 \end{pmatrix}.
\]
The relaxed versions $\Pi^a_{1/4,1/10}$ and $\tilde{\Pi}^a_{1/4,1/10}$ are both order and weak supermajorization preserving w.r.t. $\mathcal{P}$ because they satisfy the assumptions of lemmas 2.3.1 and 2.3.2. Moreover, $\Pi^b_{1/4,1/10}$ is order preserving w.r.t. $\mathcal{P}$ because it satisfies the conditions in Lemma 2.3.1. The doubly stochastic matrices

$$S = \begin{pmatrix} 0.34 & 0.33 & 0.33 & 0 \\ 0.22 & 0.29 & 0.23 & 0.26 \\ 0.22 & 0.15 & 0.15 & 0.48 \\ 0.22 & 0.23 & 0.29 & 0.26 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0.37 & 0.33 & 0.3 & 0 \\ 0.21 & 0.2 & 0.23 & 0.36 \\ 0.21 & 0.2 & 0.2 & 0.39 \\ 0.21 & 0.27 & 0.27 & 0.25 \end{pmatrix}$$

are such that $\Pi^b_{1/4,1/10} S = \Pi^a_{1/4,1/10} S$ and $\Pi^b_{1/4,1/10} \tilde{S} = \Pi^a_{1/4,1/10} \tilde{S}$.

We next verify that $(\Pi^a, \ell, c, \gamma)$, $(\tilde{\Pi}^a, \ell, c, \gamma)$ and $(\Pi^b, \ell, c, \gamma)$ are unbalancing. The equity vectors under the reduced-liability configuration are given by

$$\ell \Pi^a + c - \ell = (14g, 16g, 19g, 22g)$$
$$\ell \tilde{\Pi}^a + c - \ell = (13g, 17g, 19g, 22g)$$
$$\ell \Pi^b + c - \ell = (9g, 16g, 19g, 27g),$$

and are all similarly ordered to the liability vector $\ell$. This implies that all three systems are unbalancing.

Consider a convex combination of the two imperfectly tiered systems, given by

$$(\hat{\Pi}^a, \ell, c, \gamma) = \lambda (\Pi^a, \ell, c, \gamma) + (1 - \lambda) (\tilde{\Pi}^a, \ell, c, \gamma),$$

where $\lambda \in [0, 1]$. Since the class of unbalancing systems is closed under convex combinations, we have that $(\hat{\Pi}^a, \ell, c, \gamma)$ is also unbalancing, while $\hat{\Pi}^a_{1/4,1/10}$ is order and weak supermajorization preserving w.r.t. $\mathcal{P}$ and majorized by $\Pi^b_{1/4,1/10}$. The majorization relation follows from the fact that $\hat{\Pi}^a_{1/4,1/10} = \lambda \Pi^b_{1/4,1/10} S + (1 - \lambda) \Pi^b_{1/4,1/10} \tilde{S}$.

Since both $(\hat{\Pi}^a, \ell, c, \gamma)$ and $(\Pi^b, \ell, c, \gamma)$ satisfy the assumptions of Theorem 2.3.2, we deduce that $s(\hat{\Pi}^a, \ell, c, \gamma) \prec_{w} s(\Pi^b, \ell, c, \gamma)$. Hence, if both networks are unbalancing, the imperfectly tiered is preferred to the perfectly tiered network. This can be explained in intuitive terms as follows. When a network is unbalancing, larger losses are incurred by nodes with smaller liabilities, i.e peripheral nodes, see also right panel.
of Figure 2.3. From Theorem 2.3.2, we obtain that the clearing payments in the perfectly tiered network are less evenly distributed than the corresponding payments in the imperfectly tiered network. This is because in the latter, larger payments are directed to periphery as opposed to core nodes, while in the perfectly tiered network peripheral nodes only receive payments from the core node. Hence, they are more likely to default and generate larger losses.

2.5 Empirical Analysis and Policy Implications

The objective of this section is to provide empirical evidence to (1) support the two assumptions made earlier in this chapter and (2) show that real-world financial networks often tend to be in an unbalancing state. We consider the system consisting of the banking sectors in eight European countries for seven years, starting from 2008 and ending in 2014. These countries are well representative of interbank activities in the European market as their liabilities account for 80% of the total liabilities of the European banking sector.

We use consolidated banking data released from the European Central Bank, reported in Table 2.2, and foreign claims data from the BIS (Bank for International Settlements) Quarterly Review, summarized in Table 2.1, to estimate the various parameters of the financial system.

Tables 2.1 and 2.2 show that both assumptions are satisfied in December 2009 and June 2010. Assumption 2.2.1 holds because the values in the fourth column of Table 2.2 are all positive. Moreover, such column is similarly ordered to the second column, hence showing that Assumption 2.3.1 also holds. Table 2.2 indicates that the financial system is in the unbalancing state at these two time points. This is because the column corresponding to the equity under the reduced-liability configuration is similarly ordered to the total liability vector in the second column.

We next analyze if the unbalancing state of the system is persistent over time. To this purpose, we recall that a financial system is unbalancing if all the inequalities
given in (2.5) are satisfied. Table 2.3 computes the degree of unbalance of the system, estimated as the ratio between the number of satisfied inequalities over the total number of inequalities. Large values of this ratio are indicative of a financial system which is close to an unbalancing state. The results in Table 2.3 confirm the findings from Tables 2.1 and 2.2 that the system is unbalancing in December 2009 and June 2010, and additionally indicate it is close to being unbalancing at the remaining points in time.

Since higher concentration of liabilities induce larger systemic losses in unbalancing systems, the theoretical findings of our study indicate that it is desirable for regulatory purposes to reduce gross exposures to individual counterparties and drive the network towards a state of smaller concentration. Policies of this type are already in place, see for instance the supervisory framework put forward by the Basel Committee ([BCBS(2014)]) which aims at limiting the size of gross exposures to individual counterparties.
Table 2.1. Banks’ consolidated foreign claims (in USD billion). Data source: BIS Quarterly Review Table 9B. The $ij$-th entry of each matrix denotes the interbank liabilities from the banking sector of country $i$ to the banking sector of country $j$.

<table>
<thead>
<tr>
<th></th>
<th>United Kindom</th>
<th>Germany</th>
<th>France</th>
<th>Spain</th>
<th>Netherland</th>
<th>Ireland</th>
<th>Belgium</th>
<th>Portugal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>December 2009</strong></td>
<td>(United Kindom)</td>
<td>0.00</td>
<td>500.62</td>
<td>341.62</td>
<td>409.36</td>
<td>189.95</td>
<td>231.97</td>
<td>36.22</td>
</tr>
<tr>
<td></td>
<td>(Germany)</td>
<td>172.97</td>
<td>0.00</td>
<td>292.94</td>
<td>51.02</td>
<td>176.58</td>
<td>36.35</td>
<td>20.52</td>
</tr>
<tr>
<td></td>
<td>(France)</td>
<td>239.17</td>
<td>195.64</td>
<td>0.00</td>
<td>50.42</td>
<td>92.73</td>
<td>20.60</td>
<td>32.57</td>
</tr>
<tr>
<td></td>
<td>(Spain)</td>
<td>114.14</td>
<td>237.98</td>
<td>219.64</td>
<td>0.00</td>
<td>119.73</td>
<td>30.23</td>
<td>26.56</td>
</tr>
<tr>
<td></td>
<td>(Netherland)</td>
<td>96.69</td>
<td>155.65</td>
<td>150.57</td>
<td>22.82</td>
<td>0.00</td>
<td>15.47</td>
<td>28.11</td>
</tr>
<tr>
<td></td>
<td>(Ireland)</td>
<td>187.51</td>
<td>183.76</td>
<td>60.33</td>
<td>15.66</td>
<td>30.82</td>
<td>0.00</td>
<td>64.50</td>
</tr>
<tr>
<td></td>
<td>(Belgium)</td>
<td>30.72</td>
<td>40.68</td>
<td>301.37</td>
<td>9.42</td>
<td>131.55</td>
<td>6.11</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(Portugal)</td>
<td>24.26</td>
<td>47.38</td>
<td>44.74</td>
<td>86.08</td>
<td>12.41</td>
<td>5.43</td>
<td>3.14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>United Kindom</th>
<th>Germany</th>
<th>France</th>
<th>Spain</th>
<th>Netherland</th>
<th>Ireland</th>
<th>Belgium</th>
<th>Portugal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>June 2010</strong></td>
<td>(United Kindom)</td>
<td>0.00</td>
<td>462.07</td>
<td>327.72</td>
<td>386.37</td>
<td>135.37</td>
<td>208.97</td>
<td>43.14</td>
</tr>
<tr>
<td></td>
<td>(Germany)</td>
<td>172.18</td>
<td>0.00</td>
<td>255.00</td>
<td>39.08</td>
<td>149.82</td>
<td>32.11</td>
<td>20.93</td>
</tr>
<tr>
<td></td>
<td>(France)</td>
<td>257.11</td>
<td>196.84</td>
<td>0.00</td>
<td>26.26</td>
<td>80.84</td>
<td>18.11</td>
<td>29.70</td>
</tr>
<tr>
<td></td>
<td>(Spain)</td>
<td>110.85</td>
<td>181.65</td>
<td>162.44</td>
<td>0.00</td>
<td>72.67</td>
<td>25.34</td>
<td>18.75</td>
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<td></td>
<td>(Netherland)</td>
<td>141.39</td>
<td>148.62</td>
<td>126.38</td>
<td>20.66</td>
<td>0.00</td>
<td>12.45</td>
<td>23.14</td>
</tr>
<tr>
<td></td>
<td>(Ireland)</td>
<td>148.51</td>
<td>138.57</td>
<td>50.08</td>
<td>13.98</td>
<td>21.20</td>
<td>0.00</td>
<td>53.99</td>
</tr>
<tr>
<td></td>
<td>(Belgium)</td>
<td>29.15</td>
<td>35.14</td>
<td>253.13</td>
<td>5.67</td>
<td>108.68</td>
<td>5.32</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(Portugal)</td>
<td>22.39</td>
<td>37.24</td>
<td>41.90</td>
<td>78.29</td>
<td>5.13</td>
<td>5.15</td>
<td>2.57</td>
</tr>
</tbody>
</table>
Table 2.2. Consolidated banking sector data of European countries. Data source: European Central Bank. The first two columns report the values of assets and liabilities of each banking sector. In our model, these correspond respectively to $\ell \Pi + c$ and $\ell$. We set $\gamma = 0.5$. The values reported in the third and fourth columns are computed as follows. We first estimate the relative liability matrix, $\Pi$, whose entries are obtained from the interbank liability matrix in Table 2.1 and the vector $\ell$ of total liabilities. We then compute $c$ by subtracting $\ell \Pi$ from the asset value vector given in the first column. The equity under the reduced-liability configuration is estimated using Eq. (2.5).

<table>
<thead>
<tr>
<th>Country</th>
<th>Assets (in USD billion)</th>
<th>Liabilities (in USD billion)</th>
<th>Equity† (1 + $\gamma$)c − $\gamma \ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UK</td>
<td>13,833</td>
<td>13,204</td>
<td>674</td>
</tr>
<tr>
<td>Germany</td>
<td>12,366</td>
<td>11,901</td>
<td>504</td>
</tr>
<tr>
<td>France</td>
<td>9,053</td>
<td>8,616</td>
<td>472</td>
</tr>
<tr>
<td>Spain</td>
<td>5,350</td>
<td>5,024</td>
<td>374</td>
</tr>
<tr>
<td>Netherlands</td>
<td>3,795</td>
<td>3,632</td>
<td>213</td>
</tr>
<tr>
<td>Ireland</td>
<td>1,919</td>
<td>1,828</td>
<td>149</td>
</tr>
<tr>
<td>Belgium</td>
<td>1,706</td>
<td>1,629</td>
<td>133</td>
</tr>
<tr>
<td>Portugal</td>
<td>732</td>
<td>686</td>
<td>104</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Country</th>
<th>Assets (in USD billion)</th>
<th>Liabilities (in USD billion)</th>
<th>Equity† (1 + $\gamma$)c − $\gamma \ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UK</td>
<td>13,956</td>
<td>13,258</td>
<td>736</td>
</tr>
<tr>
<td>Germany</td>
<td>11,533</td>
<td>11,126</td>
<td>443</td>
</tr>
<tr>
<td>France</td>
<td>8,485</td>
<td>8,077</td>
<td>439</td>
</tr>
<tr>
<td>Spain</td>
<td>4,765</td>
<td>4,482</td>
<td>325</td>
</tr>
<tr>
<td>Netherlands</td>
<td>3,506</td>
<td>3,366</td>
<td>184</td>
</tr>
<tr>
<td>Ireland</td>
<td>1,758</td>
<td>1,678</td>
<td>129</td>
</tr>
<tr>
<td>Belgium</td>
<td>1,530</td>
<td>1,464</td>
<td>116</td>
</tr>
<tr>
<td>Portugal</td>
<td>654</td>
<td>615</td>
<td>90</td>
</tr>
</tbody>
</table>

†The equity under the reduced-liability configuration.
Table 2.3.
The degree of unbalance of the financial system consisting of the eight representative banking sectors at ten different time points. In each time point, we sort the eight banking sectors by increasing liability size. We then check if the inequalities in Eq. (2.5) are satisfied for each pair of adjacent banking sectors. This yields a total of seven inequalities. We compute the degree of unbalance as the number of satisfied inequalities divided by the total number of inequalities.

<table>
<thead>
<tr>
<th></th>
<th>Degree of unbalance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec-2008</td>
<td>71%</td>
</tr>
<tr>
<td>Dec-2009</td>
<td>100%</td>
</tr>
<tr>
<td>Jun-2010</td>
<td>100%</td>
</tr>
<tr>
<td>Dec-2010</td>
<td>86%</td>
</tr>
<tr>
<td>Jun-2011</td>
<td>86%</td>
</tr>
<tr>
<td>Dec-2011</td>
<td>86%</td>
</tr>
<tr>
<td>Jun-2012</td>
<td>71%</td>
</tr>
<tr>
<td>Dec-2012</td>
<td>71%</td>
</tr>
<tr>
<td>Jun-2013</td>
<td>86%</td>
</tr>
<tr>
<td>Jun-2014</td>
<td>86%</td>
</tr>
</tbody>
</table>
CHAPTER 3. SYSTEMIC RISK MITIGATION IN FINANCIAL NETWORKS

3.1 Introduction

The prevailing literature on systemic risk have analyzed the consequences caused by defaults using a static model of counterparty exposures. Although static models provide insights about immediate consequences caused by defaults, they do not capture the propagation and aftershocks of default events. We advance the above literature by developing a multi-period clearing system, where the level of systemic risk can be controlled through provision of liquidity assistance loans. Such loans are provided by a regulatory entity, hereon referred to as the lender of last resort (LOLR), who has complete information on the liability structure of the network and mitigates systemic risk by providing liquidity loans to illiquid, but solvent banks. ¹ Such liquidity assistance policies are supported by the classical doctrine of the LOLR elaborated by [Bagehot(1873)], and theoretically justified by [Rochet and Vives(2004)].

We consider a stochastically evolving interbank liability structure where clearing, or settlement, of interbank claims is done using a dynamic extension of the algorithm originally proposed by [Eisenberg and Noe(2001)]. We show that if the subgraph of the financial network induced by the surviving banks is regular at all times, then the time sequence of clearing payments is unique when a liquidity loan allocation strategy is specified. The time evolving nature of the network allows modeling the dynamic component of systemic risk. [Merton et al.(2013)] also analyze macro risk changes occurring over time and find that interbank exposures are very sensitive to changes

¹If complete information of the financial network is not readily available, the regulator may also construct accurate proxies for interbank liabilities using standard methodologies, see for instance [Docherty and Wang(2010)]. Such procedures use transaction data from the U.S. Federal Fund Markets as proxies for decomposing the observed total liabilities into interbank liabilities.
in values of the underlying assets. The time related perspective of systemic risk is also emphasized in [ECB(2009)], see page 127 therein.

Our network model is designed to analyze systemic risk in networks of insured commercial banks. The latter are supervised by the Federal Deposit Insurance Corporation (FDIC), which usually uses the Purchase and Assumption (P&A) transaction to sell non-cash assets of defaulted banks in the network. Following [Giliberto and Varaiya(1989)], we assume that the sale is done through a first-price sealed-bid auction using the common value model. We assess systemic risk mitigation on the core-periphery network model. [Craig and Von Peter(2014)] performed an empirical analysis using bilateral interbank data from German banks from 1999 to 2007 and found that the matrix of interbank liabilities follows a tiered core-periphery structure. Such findings are also confirmed by [Fricke and Lux(2015)] who employed a detailed dataset containing all overnight interbank transactions in the Italian market from 1999 to 2010, and found that a core-periphery structure provides the best fit for these interbank data, with high degree of persistence over time.

We analyze the performance of two liquidity assistance strategies. The first, called \textit{Systemic Importance Driven} and abbreviated with SID, provides liquidity assistance loans to systemically important banks. The second policy, called \textit{Max-Liquidity} and abbreviated with ML, maximizes the instantaneous total liquidity in the network. We find that policies rescuing systemically important banks are preferred to those maximizing the total liquidity in the system when the network has a core-periphery structure. The reason is that, under the ML policy, there is a higher probability for systemically important banks to default. Failures of core banks use up the cash of solvent banks which need to purchase them through the auction, and hence become less liquid. This in turn increases the likelihood of defaults occurring in later periods and results in large increases of systemic risk in the network. When the variation in interbank liabilities is higher, a larger systemic risk reduction is obtained by SID relative to ML policy, given that the systemic consequences when systemically important banks fail are stronger. We find that systemic risk levels tend to maintain
similar levels over time when the SID policy is employed, while they manifest higher fluctuations in time under the ML strategy given the network is subject to a latency before stabilizing after early failures of systemically important banks. We also provide comparisons with a baseline random network and find that the ML policy is slightly preferred in this case. This is because all banks are equally systemically important and hence reducing systemic risk by maximizing the total liquidity in the system is preferable to targeting a specific set of banks to which provide liquidity assistance if the need arises.

The rest of this chapter is organized as follows. Section 3.2 introduces the building blocks of our framework and develops the controlled multi-period clearing payment system. Section 3.3 introduces the mitigation strategies. Section 3.4 discusses systemic risk under the core-periphery and random network topology.

3.2 The Framework

3.2.1 Financial Network

We fix a finite time horizon \( T \) divided into discrete intervals \([t, t+1), t \in \{0, 1, \ldots, T-1\}\), corresponding to times when interbank payments occur. We consider a network consisting of \( n \) nodes representing financial institutions which are insured commercial banks.

The state of the financial network is characterized by a time sequence of 2-tuples, \((L^t, c^t)\). Here, \( L^t \in \mathbb{R}_{\geq 0}^{n \times n} \) is the interbank liability matrix at \( t \), with \( l^t_{i,j} \) denoting the amount of liabilities owed by \( i \) to \( j \) at \( t \) for \( i \neq j \), and \( l^t_{i,i} = 0 \). We use \( c^t \in \mathbb{R}_{\geq 0}^n \) to denote the operating cash inflow vector, i.e. \( c^t_i \) quantifies the proceeds generated from operating activities of bank \( i \) at \( t \). We will later specify how \( L^t \) and \( c^t \) are formed, taking into account acquisitions arising in the network when banks default.
We use $\ell^t$ to denote the total liability vector, with $\ell^t_i = \sum_{j=1}^n l^t_{i,j}$ being the total amount of obligations from bank $i$ to all other banks at time $t$. Further, we denote by

$$\pi^t_{i,j} = \begin{cases} \frac{l^t_{i,j}}{\ell^t_i} & \text{if } \ell^t_i > 0 \\ 0 & \text{if } \ell^t_i = 0 \end{cases}$$

the relative size of liabilities owed by $i$ to $j$ at $t$. Here, $\Pi^t$ is the associated liability proportion matrix.

In our model each bank can neither sell illiquid assets nor borrow from the private sector. Such assumptions are mild, if we consider that our model is designed to understand systemic implications in crisis periods, as opposed to normal times.

The network is stochastic and the randomness comes from the uncertain nature of interbank liabilities and operating cash inflows before any acquisition takes place. We denote them by $\tilde{L}^t$ and $\tilde{c}^t$ respectively, to distinguish from the corresponding quantities $L^t$ and $c^t$ corresponding to the after acquisition stage. Both $\tilde{l}^t_{i,j}$ and $\tilde{c}^t_i$ are considered to be discrete time stochastic processes.

### 3.2.2 Lender of Last Resort (LOLR)

We introduce an outside entity, who has complete information on state of the network, and whose goal is to provide liquidity assistance loans to illiquid yet solvent banks according to a specified policy.

A bank is said to be rescued if it receives a loan from the LOLR. We denote by $o^t_i \geq 0$ the liquidity assistance loan granted by the LOLR to bank $i$ at $t$; $o^t_i$ is the associated vector. We distinguish the interbank interest rate $r$ from the rate $r_c$ at which liquidity loans need to be repaid. \(^2\) Liquidity loans are assumed to be longer-term than the time horizon, i.e. they are returned after $T$.

\(^2\)All loans granted from the Fed under the emergency program were repaid with an interest rate ranging between 0.5\% to 3.5\%, which was different from the prevailing market rate.
3.2.3 Illiquidity and Insolvency

We denote by \( p^t_i \) the total payment made by bank \( i \) at time \( t \). We use \( v^t_i \geq 0 \) to denote the available cash before clearing to \( i \) at \( t \). We then have that the available cash after clearing to bank \( i \) is given by

\[
\kappa^t_i = \sum_{j=1}^{n} \pi^t_{j,i} p^t_j + c^t_i + v^t_i - \ell^t_i. \tag{3.1}
\]

A bank \( i \) is defined to be illiquid at time \( t \) if \( \kappa^t_i < 0 \). Next, we introduce two other quantities of interest, \( \xi^t_i \) representing the total amount of non-cash assets belonging to bank \( i \), and \( q^t_i \) indicating the net assets of bank \( i \). The first quantity is given by the present value of operating cash inflows of \( i \) and book value of liabilities owed by other banks in the network to \( i \) in subsequent periods, while the second quantity is the residual asset value after deduction of due liabilities and debt to the LOLR, i.e.

\[
\xi^t_i = \mathbb{E}_t \left[ \sum_{\tau=t+1}^{T-1} (1 + r)^{-(\tau-t)} \left( c^\tau_i + \sum_{j=1}^{n} \ell^\tau_{j,i} \right) \right],
\]

\[
q^t_i = \kappa^t_i + \xi^t_i - \left[ \sum_{\tau=t+1}^{T-1} (1 + r)^{-(\tau-t)} \mathbb{E}_t [\ell^\tau_i] + \sum_{\tau=0}^{t-1} (1 + r_c)^{t-\tau} \sigma^\tau_i \right].
\]

Here, for a given random variable \( X \) we denote by \( \mathbb{E}_t [X] \) the expectation of \( X \) conditioned on the information set available at time \( t \) under the actual measure. In addition, the adjustment for the obligor’s credit risk is not included in the value of non-cash assets, i.e. we deal with accounting-based valuation. The information set is generated by the sample path of interbank liabilities and operating cash inflows up to time \( t \). A bank \( i \) is said to be insolvent at time \( t \) if \( q^t_i < 0 \).

3.2.4 Default

**Definition 3.2.1.** A bank is said to default at time \( t \) if it is (1) illiquid and insolvent at \( t \) or (2) illiquid and solvent at \( t \) but not rescued by the LOLR.

The default indicator vector at time \( t \), denoted by \( d^t \), is defined as \( d^t_i = 1 \) if bank \( i \) defaulted before \( t \), and 0 otherwise, with \( d^0 \) being the zero vector as all banks are
assumed to be alive at time zero. For future references, we use $\Delta d^t := d^{t+1} - d^t$ to denote the vector indicating the banks defaulting in the time period $[t, t + 1)$.

Next, we describe the mechanism triggered upon the default of a bank, with each bank supervised by the Federal Deposit Insurance Corporation (FDIC). The most common method used by the FDIC to resolve defaults is the Purchase and Assumption (P&A) transaction \(^3\), where the non-cash assets of the defaulted bank are sold to an eligible buyer satisfying the minimum capital and cash requirements \(^4\) under the following rules.

**Case 1. More than one eligible buyers in the financial sector.** The FDIC sells the defaulted bank to the winning bidder of a first-price sealed-bid auction.

**Case 2. Only one eligible buyer in the financial sector.** The FDIC sells the defaulted bank to it through a bargaining process.

**Case 3. No eligible buyer in the financial sector.** The FDIC sells the defaulted bank to a company in the real economy through another bargaining process.\(^5\)

After the defaulted bank is sold, its non-cash assets will be transferred to its buyer, and the auction payment will be distributed to its creditors based on the relative fraction of liabilities owed to them.

Next, we explain how to model the above described mechanism starting from the first case. We follow [Giliberto and Varaiya(1989)] and choose the common value

\(^3\)Indeed, from 1980 through 1994, the FDIC used P&A transactions to resolve 1,188 out of 1,617 total failures and assistance transactions, or 73.5 percent according to [FDIC(2013)].

\(^4\)The capital requirement is used by FDIC to determine the bidder eligibility, as described in [Giliberto and Varaiya(1989)]. In general, if a bank does not have sufficient cash to purchase the non-cash assets of the defaulted bank, it can raise cash from other banks or private sector. We do not model such a mechanism in this chapter, and hence require that eligible bidders must also have sufficient cash.

\(^5\)These cases are used to compute the auction price of a defaulted bank. Depending on the number of eligible buyers and which sector (financial sector or real economy) those buyers belong to, the auction price is determined through different mechanisms. The auction price is then redistributed proportionally to the creditors of the defaulted bank, hence it contributes to reduce the amount of unrepaid debt from the defaulted bank to its creditors in the network. The difference between the original liability exposure of bank $j$ to a defaulted bank $i$ and the auction proceeds received by $j$ when the auction on the assets of bank $i$ is completed may be interpreted as the loss of $j$ given the default of bank $i$. 
model to determine the auction price and winning bidder of a first-price sealed-bid auction. Such a model is justified by their claims that (1) the true value of the charter of the defaulted bank is nearly the same across bidders, (2) the eligible bidders may be equally capable of exploiting the defaulted bank’s franchise. Moreover, we assume that the FDIC selects the eligible bidders using the minimum capital and cash requirements, set equal to the present value of non-cash assets of bank $i$, i.e. to $\xi^t_i$. 6

We use $Q^t_i$ to denote the set of eligible bidders participating to bank $i$’s auction at time $t$, i.e.

$$Q^t_i = \{ j \in \{1,\ldots,n\} \mid d^t_j = 0, \min\{q^t_j, \kappa^t_j\} \geq \xi^t_i \}.$$  

We use $\alpha^t_i$ to denote the auction price of bank $i$, and $m^t_i$ to denote the winning bidder of bank $i$. For each bank $j \in Q^t_i$, let $b^t_j(i)$ be the equilibrium price paid by bank $j$ to purchase the assets of bank $i$ at $t$.

Denote by $x^t_i$ the reservation price announced at the beginning of the auction by the FDIC, computed as the fair market value of the non-cash assets of bank $i$ minus the costs of disposition and marketing. The private valuations of bank $i$ from each bidder $j \in Q^t_i$, denoted by $\beta^t_j(i)$, are assumed to be independently and uniformly distributed in the interval $[x^t_i, y^t_i]$, where $y^t_i \leq \xi^t_i$. Both $x^t_i$ and $y^t_i$ are common knowledge across all bidders. 7 Using the formula of the equilibrium bid in a first-price sealed-bid auction, see [Easley and Kleinberg(2010)] Chapter 9, we obtain

$$b^t_j(i) = x^t_i + \frac{|Q^t_i| - 1}{|Q^t_i|} (\beta^t_j(i) - x^t_i).$$

The bidder who places the highest bid wins the auction, i.e. $m^t_i = \arg\max_{j \in Q^t_i} b^t_j(i)$, and the auction price is

$$\alpha^t_i = b^t_{m^t_i}(i).$$ (3.2)

6Such a quantity is usually computed during an asset valuation review performed by the FDIC, where the liquidity of all potential bidders is also assessed, see [FDIC(2013)] Chapter 2 for details.

7Notice that each bidder knows the capital and cash requirements, set equal to $\xi^t_i$, as well as the reservation price $x^t_i$, given that these are revealed during the information meeting held by the FDIC before the auction takes place.
For the second case, while the reservation price of the FDIC for the asset of the defaulted bank $i$ at time $t$ is $x^t_i$ as in the first case, the reservation price of a buyer $j$ on bank $i$’s assets equals its private valuation of bank $i$, $\beta^t_j(i)$. Define the surplus each party gains by the end of a bargaining process as the difference between their reservation prices and the equilibrium price. Assuming that both parties have equal bargaining power, the equilibrium price is defined as the price resulting when each party takes equal surplus, i.e.

$$b^t_j(i) - x^t_i = \beta^t_j(i) - b^t_j(i).$$

The equilibrium price is then given by

$$b^t_j(i) = \frac{1}{2} (\beta^t_j(i) + x^t_i).$$

We remark that such a price coincides with the equilibrium price buyer $j$ would bid in an auction with two participants.

In the third case, we model companies in the real economy via a sink node in the network. Whenever a real economy company buys a defaulted bank, the sink node receives the liabilities transferred from the defaulted bank’s creditors, and the operating cash inflows generated by the defaulted bank. As in the second case, the sink node pays the same amount to buy a defaulted bank as when there is only one buyer in the financial sector.

It remains to specify how non-cash assets are transferred and the auction payment distributed. In cases when more than one bank defaults at a specific time, our model assumes that the bank with larger non-cash assets is auctioned first, and all auction payments are distributed after all non-cash assets are transferred. The ownership matrix at time $t$, $U^t$, is given by $u^t_{i,j} = 1$ if bank $i$ is owned by $j$ at time $t$, and $u^t_{i,j} = 0$ otherwise. By definition, $U^0 = I_n$, i.e. the $n \times n$ identity matrix. Denote
the \(i\)-th column of matrix \(U\) by \(U_i\). After the auction is completed, the ownership
matrix is updated according to

\[
U_{t+1}^i = \begin{cases} 
U_t^i + \sum_{k : m_k^t = i} U_k^t & \text{if } \Delta d_i^t = 0 \\
[0]^{n \times 1} & \text{if } \Delta d_i^t = 1,
\end{cases}
\]

where column \(k\) is added to column \(i\), and column \(k\) is set to zero if \(i\) acquires \(k\) at
time \(t\).

If bank \(i\) defaults at \(t\), its receivable liabilities and operating cash inflows are
transferred to the winning bidder after the auction is completed. Hence, when taking
into account the default events, the operating cash inflows and liabilities of each bank
\(i\) are determined by

\[
c_t^i = \left(U_t^i\right)^T \tilde{c}^t, \quad l_{i,j}^t = \left(\tilde{L}_t U_t^i\right)_{i,j}, \quad l_{i,i}^t = 0.
\]

Here, we recall that \(\tilde{c}^t\) and \(\tilde{L}_t\) are the realizations of the operating cash inflow vector
and interbank liability matrix at \(t\). In words, if bank \(i\) has not defaulted by time \(t\),
its operating cash inflows also include the ones generated by the banks it buys. If it
defaults, its operating cash inflows are transferred to its buyer. A similar argument
applies to the liabilities. The unpaid debt of bank \(i\) to \(j\) at \(t\) is defined as

\[
w_{i,j}^t = 1_{\Delta d_i^t = 1} \left(\sum_{\tau = t}^{T-1} (1 + r)^{-(\tau - t)} \mathbb{E}_t[l_{i,j}^\tau] - \pi_{i,j}^t \right), \quad \varpi_t^i = \sum_{j=1}^n w_{i,j}^t,
\]

i.e. the present value of the debt owed by \(i\) to \(j\) netted of the payment done at
the default time. Here, \(1_A\) is the indicator function of the event \(A\). The following
inductive relationship allows computing the available cash of bank \(i\) before clearing
iteratively

\[
v_{i}^{t+1} = 1_{d_{i}^{t+1} = 0}(1 + r) \left(\kappa_i^t - \sum_{j=1}^n 1_{m_j^t = i} \alpha_j^t + \sum_{j=1}^n \frac{w_{i,j}^t}{\varpi_j^t} \min \{\alpha_j^t, \varpi_j^t\} \right)
\]

for \(t \in \{0, 1, \ldots, T-2\}\) with \(v_i^0 \geq 0\). If bank \(i\) has not defaulted by \(t\), the cash available
to bank \(i\) before clearing at \(t + 1\) includes the cash available to it after clearing at
\(t\) netted of the amount spent to acquire banks which have defaulted at \(t\), plus the
auction payments proportionally distributed from the FDIC to repay partially or in full the debt of the defaulted banks at $t$. On the other hand, if bank $i$ defaults by $t$, it does not have any cash available since it has been acquired.

### 3.2.5 The Controlled Clearing Payment System

We introduce a multi-period controlled clearing payment system, which generalizes the single period clearing system in [Eisenberg and Noe (2001)]. In each time period, clearing payments satisfy the standard conditions imposed by bankruptcy laws: limited liability of equity, priority of liability over equity, and proportional repayments of liabilities in default. This leads to the following

**Definition 3.2.2.** Given a dynamic network, $\{(\tilde{L}_t, \tilde{c}_t)\}_{t=0}^{T-1}$, a time sequence $\{p_t^i\}_{t=0}^{T-1}$ is a clearing sequence of payments controlled by $\sigma^i$ if it satisfies the following conditions:

- **a. Proportional repayment of liabilities.** A bank $i \in \{1, \ldots, n\}$ pays $\pi_{i,j}^t p_t^i$ to bank $j$ at time $t$, were bank $i$ not to default before $t$.

- **b. Absolute priority.** For each $t$, if a bank $i$ does not default at time $t$, it pays its liabilities in full. If it defaults, its repaid liabilities are reduced by the illiquidity amount remaining after provisions of liquidity loans. If it defaults before time $t$, it does not make any payment. Formally, for $i \in \{1, \ldots, n\}$, $t \in \{0, 1, \ldots, T-1\}$, we have

$$p_t^i = \begin{cases} 
\ell_t^i + \min\{0, \kappa_t^i + o_t^i\}, & \text{if } d_t^i = 0 \\
0, & \text{if } d_t^i = 1.
\end{cases}$$

- **c. Admissible liquidity loans.** The LOLR provides loans only to illiquid yet solvent banks, i.e. for $i \in \{1, \ldots, n\}$ at time $t$, $o_t^i > 0 \Rightarrow \kappa_t^i < 0$ and $q_t^i \geq 0$ and $d_t^i = 0$. 
When a liquidity loan allocation policy \( \{o^t\} \) is specified, the sequence \( \{p^t\} \) is uniquely determined provided that the subgraph of the financial network induced by the set of non-defaulted banks satisfies, at all times, the regularity condition introduced in [Eisenberg and Noe(2001)]. Such a condition states that a financial system is regular if the risk orbit of each bank \( i \), consisting of all banks \( j \) reachable from \( i \) via a directed path, is a surplus set. This means that every bank in the set is not liable to any bank outside it and total operating cash flows of all banks in the set is positive. A simple sufficient condition guaranteeing this is that operating cash inflows of all banks are strictly positive at all times. This will be satisfied in our numerical simulations. We then have the following result, whose proof is reported in Appendix B.

**Lemma 3.2.1.** Let the financial network be such that the subgraphs induced by the non-defaulted banks are regular for all \( t \). Then, if a liquidity loan allocation policy \( \{o^t\} \) is specified, the sequence \( \{p^t\} \) is uniquely determined.

### 3.3 Liquidity Assistance Policies

We analyze systemic risk mitigation resulting from two liquidity assistance strategies. The first, referred to as *Systemic Importance Driven* (SID), provides liquidity assistance to a selected set of banks considered systemically important given the potential losses to the system induced by their defaults. We compare it with a benchmark policy, called *Max-Liquidity* (ML), focusing on maximizing the instantaneous total liquidity in the system.

In the following, we use \( s^t = (v^t, d^t, U^t) \) to denote the state of the network. Here, we recall that the first two components are \( n \)-dimensional vectors denoting, respectively, cash and default indicators associated with each bank, while the last component is the ownership matrix. Further, \( X^t = (\bar{L}^t, \bar{c}^t) \) is the stochastic process capturing the randomness of the network.
3.3.1 Systemic Importance Driven (SID)

This policy identifies a set of systemically important financial institutions (SIFIs). Our definition of SIFIs is consistent with [FSB(2011)], which defines financial institutions to be systemically important if their distress or disorderly failure, because of their size, complexity and systemic interconnectedness, would cause significant disruption to the wider financial system and economic activity. However, differently from [FSB(2011)] which suggests to take preventive measures against these institutions, our policy only provides liquidity assistance to them if they are close to default.

We next describe how the set of SIFIs is constructed. We first simulate sample paths of interbank liabilities and operating cash inflows. For each sample path $a$, denote by $\ell_{i}^{t,a}$, $p_{i}^{t,a}$, $\kappa_{i}^{t,a}$, and $d_{i}^{t,a}$ the liabilities, clearing payment, available cash after clearing, and default indicator of bank $i$, respectively generated or computed at time $t$ on the sample path $a$. We then solve for the clearing payment

$$p_{i}^{t,a} = \begin{cases} 
\ell_{i}^{t,a} + \min\{0, \kappa_{i}^{t,a}\} & \text{if } d_{i}^{t,a} = 0 \\
0, & \text{if } d_{i}^{t,a} = 1,
\end{cases}$$

for $t = 0, \ldots, T - 1$. Then compute $\bar{\varpi}_{i} = E\left[\sum_{t=0}^{T-1}(1+r)^{-t}\mathbb{1}_{\Delta d_{i}^{t,a}=1}\bar{\varpi}_{i}^{t,a}\right]$, i.e. the expected amount of unpaid debt of bank $i$ in the absence of mitigation. Here, $\bar{\varpi}_{i}^{t,a}$ is the amount of unpaid debt of bank $i$ on the sample path $a$ computed using Eq. (3.3) (which in turn uses $p_{i}^{t,a}$ computed above), while $\Delta d_{i}^{t,a} = 1$ if bank $i$ defaulted in the time period $[t, t+1)$ on the sample path $a$. We set $z$ as the threshold value so that

$$\frac{1}{n} \left| \{i : \bar{\varpi}_{i} > z\} \right| = 1 - \rho$$

where $\rho$ is a policy parameter governing the percentage of tolerated loss. The set $\mathcal{V}$ of systemically important banks consists of all banks for which the amount of unpaid debt generated by their default exceeds $z$, i.e. $\mathcal{V} = \{i \in \{1, \ldots, n\} | \bar{\varpi}_{i} \geq z\}$. Liquidity assistance will only be provided to banks belonging to this set. Next, we describe the
main step of the procedure. Denote the liquidity loan vector by $o^t_{SI}$. The clearing payment is determined by solving the equations below. For $i = 1, \ldots, n$,

$$
p^t_i = \begin{cases} 
\ell^t_i & \text{if } i \in \mathcal{V}, d^t_i = 0 \text{ and } q^t_i \geq 0 \\
\ell^t_i + \min\{0, \kappa^t_i\} & \text{if } (i \in \mathcal{V}, d^t_i = 0 \text{ and } q^t_i < 0) \text{ or } (i \notin \mathcal{V} \text{ and } d^t_i = 0) \\
0 & \text{if } d^t_i = 1,
\end{cases}
$$

i.e. liquidity loans are only provided to solvent systemically important banks who are unable to currently fulfill liability obligations. Concretely, each component of the liquidity loan vector is given by

$$
o^t_{SI,i} = 1_{i \in \mathcal{V}, d^t_i = 0, q^t_i \geq 0} \max\{-\kappa^t_i, 0\}, i = 1, \ldots, n,
$$

where $1_{i \in \mathcal{V}, d^t_i = 0, q^t_i \geq 0} = 1$ if $i \in \mathcal{V}, d^t_i = 0$ and $q^t_i \geq 0$; 0 otherwise.

### 3.3.2 Maximum Liquidity (ML)

This policy mimics the behavior adopted by regulators when they immediately want to restore financial stability following a period of financial distress, as also discussed by [Hoggarth et al.(2004)]. It is myopic, and in each time step selects a vector $o^t_{ML}$ of liquidity assistance loans so to maximize the total flow of payments, i.e.

$$
o^t_{ML} = \arg \max_{o^t \in \mathcal{O}^t} \sum_{i=1}^{n} \left[ p^t_i(s^t, o^t, X^t) - c^t_i \right].
$$

Here, $\mathcal{O}^t$ represents the feasible region satisfying the above definition of controlled clearing payment sequence. In order to provide a fair comparison between this policy and SID, we set the initial budget at disposal of the LOLR to be the same as the total amount of liquidity loans used by the SID policy, i.e. its time zero value is given by $\sum_{\tau=0}^{T-1} \sum_{i=1}^{n} o^t_{SI,i,(1 + r_c)^{-\tau}}$. Here, the sequence of vectors $\{o^t_{SI}\}$ is obtained by applying the SID policy on the same network simulation.
The LOLR will allocate this budget to financially distressed banks ensuring that the amount of liquidity loans granted in each time period does not exceed the currently available budget, i.e.

\[
\sum_{i=1}^{n} o_{ML,i}^t \leq \sum_{\tau=0}^{T-1} \sum_{i=1}^{n} o_{SL,i}^\tau (1 + r_c)^{t-\tau} - \sum_{\tau=0}^{t-1} \sum_{i=1}^{n} o_{ML,i}^\tau (1 + r_c)^{t-\tau}, \quad t \in \{0, \ldots, T-1\}.
\]

(3.4)

Notice that \(\kappa_i^t\) is the available cash to bank \(i\) after clearing occurs at \(t\), and summing these amounts over the banks in the network, we have

\[
\sum_{i=1}^{n} \kappa_i^t = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \pi_{j,i} p_j^t + c_i^t + v_i^t - \ell_i^t \right) = \sum_{i=1}^{n} \left( p_i^t + c_i^t + v_i^t - \ell_i^t \right).
\]

From here, we can see that this policy maximizes the total instantaneous liquidity in the network.

### 3.4 Systemic Risk Analysis

We develop a simulation based study to assess the amount of systemic risk generated by the network after controlling for the liquidity assistance policy. First, we define the fraction of unpaid liabilities caused by defaults occurring in decision epoch \(t\) relative to the total size of liabilities as

\[
UL^t = \frac{\sum_{i=1}^{n} \Delta d_i^t (1 + r)^{-t} \left[ (\ell_i^t - p_i^t(s^t, o^t, X^t)) + \max\{\varpi_i^t - \alpha_i^t, 0\} \right]}{\sum_{i=1}^{n} \sum_{t=0}^{T-1} (1 + r)^{-t} \ell_i^t}.
\]

We recall from Section 3.2 that \(\varpi_i^t\) denotes the amount of unpaid debt at time \(t\) of bank \(i\) to its creditors and \(\alpha_i^t\) the price that the FDIC receives from the winner of the auction on bank \(i\), if \(i\) defaults at \(t\). The amount \(\alpha_i^t\) is redistributed proportionally to bank \(i\)'s creditors, hence it contributes to reduce the amount of unrepaid debt from \(i\) to its creditors in the network. Hence, \(UL^t\) is a random variable representing the fraction of liabilities unpaid by the banks in the network at time \(t\) relative to the amount of total interbank liabilities over the entire time horizon, and after accounting
for the assistance policy $o^t$ by the lender of last resort. Clearly, the above quantity is zero if no default occurs at $t$. In this case, $\ell^t_i = p^t_i(s^t, o^t, X^t)$ and $\varpi^t_i = \alpha^t_i = 0$ for each $i$, given that no auction takes place. The systemic risk allocated to time $t$ is defined as the expected value of such percentage, i.e.

$$SYS^t = E[UL^t].$$

The total amount of systemic risk is given by $SYS = \sum_{t=0}^{T-1} SYS^t$.

### 3.4.1 Core-Periphery and Random Networks

We fix the number of network nodes to $n = 21$, including twenty bank nodes and a sink non-bank node as defined in Section 3.2. We consider two network topologies, core-periphery and random networks. The findings of [Craig and Von Peter(2014)] and [Fricke and Lux(2015)] guide the design of the core-periphery network structure. Following [Craig and Von Peter(2014)], we introduce the blockmodel given by

$$M = \begin{pmatrix} 1 & RR \\ CR & 0 \end{pmatrix}.$$

It consists of a complete block (denoted by $1$) and a zero block ($0$) on the diagonal, specifying relations within the core and peripheral tiers. More specifically, each core bank transacts with any other core bank, but peripheral banks do not directly interact with each other. The two off-diagonal blocks specify relations between the tiers: the core-to-periphery block ($CP$) must be row-regular ($RR$), i.e. it must have at least a one in every row, while the periphery-to-core block ($PC$) must be column-regular ($CR$), i.e. have at least a one in every column. This means that each core bank is liable to at least one peripheral bank, and each core bank must lend to at least one peripheral bank. Notice that such a model only specifies the market structure since the size of $M$ and its blocks can vary over time. We initialize the network with 4 core banks and 16 peripheral banks, see Figure 3.1 for an illustration of the network configuration. Then, we use the following transition probability matrix to determine
migrations from core to periphery and vice versa. Let $\psi, \psi' \in \{\text{Core, Periphery}\}$. The transition matrix is given by 

$$
\mathbf{P}(\psi|\psi') = \begin{pmatrix}
\text{Core} & \text{Periphery} \\
0.9 & 0.1 \\
0.1 & 0.9
\end{pmatrix}.
$$

[Fricke and Lux(2015)] find that the core banks are significantly larger and more active than peripheral banks. Further, [Craig and Von Peter(2014)] estimate the ratio of core to peripheral average banks’ asset sizes to be 51 (see section 4.1 therein). We recall that the initial book value of bank $i$’s assets is given by its currently available cash plus the present values of its operating cash inflows and total lending amount. We thus choose the total lending amount, operating cash inflows, and initial cash of core banks to be 51 times as large as the corresponding quantities of peripheral banks. Moreover, we set the size of interbank loans between core banks to be 50 times as large as the lending amount from peripheral to core banks. The lending amount from core to peripheral banks is chosen to be of the same size as the lending amount from peripheral to core banks.

It is well understood that the interbank liability matrix cannot be reconstructed with 100% accuracy since the only publicly available information are total interbank liabilities and claims. For this reason, we estimate the relative liability matrix using the entropy maximization method proposed by [Upper and Worms(2004)]. This approach effectively assumes that banks diversify their claims and liabilities by spreading their lending and borrowing across all other banks in the system. In the absence of any prior information about the interbank liability matrix, this method provides a consistent way to estimate it by minimizing the amount of prior information built into the distribution of interbank liabilities. We report the details of such estimation method in Appendix C.

The numerical values closely resemble the estimates provided by [Craig and Von Peter(2014)] (see equation (7) therein). While they also assign a positive probability to exit the network to each bank, in our model the exit of bank from the network is determined by the occurrence of a default event. We adjust for it by redistributing the probability of exiting network to migrating to core or periphery status.
As a benchmark comparison, we consider a random network of interbank loans, which decouples the relation between the distribution of banks’ sizes and the network structure. To facilitate the comparison, we take the number of banks to be the same as in the core-periphery network, and choose the total liabilities and claims of each bank to be the average of the corresponding quantities in the core block of the core-periphery network. As for the core-periphery network, we use the method of entropy maximization\(^9\) to generate the interbank liability matrix.

Under both network topologies, during any auction each bidder’s private valuation of the non-cash assets of a defaulted bank is assumed to be uniformly distributed in an interval whose right extreme is the mean of its non-cash assets, and whose left extreme is the mean minus one standard deviation.

We use the coefficient of variation, given by the ratio of the standard deviation of interbank liabilities to its mean, to measure the normalized dispersion of the probability distribution of liabilities. We assume it to be the same for all banks.

\(^9\)See Appendix C for details.
3.4.2 Simulation Results

We fix the number of Monte-Carlo runs to 2,000. We set the time horizon to $T = 20$. The time step used in the simulation is $\Delta t = 1$, resulting in twenty payment periods. The time sequences of outstanding liabilities and claims of each bank in the network are chosen to be two sequences of independent Gaussian distributed random variables. For each bank, its operating cash inflows are a time sequence of independently distributed Gaussian random variables. Operating cash inflows of any pair of banks in the network are independent. More specifically, in core-periphery networks we set the means of total liabilities of a periphery bank to $5,000$ and of a core bank to $255,000 = 51 \times 5,000$. Further, the means of operating cash inflows are set to $500$ for periphery banks and to $25,500 = 51 \times 500$. We set the means of total liabilities and of operating cash inflows of each bank to $55,000$ and $5,500$, respectively, in the random network. These values are the average of the means of total liabilities and operating cash inflows over all banks in the core-periphery network.

In all plots, the systemic risk reduction is computed as the difference between the average systemic risk obtained after policy control and the corresponding quantity in the absence of mitigation. Default reduction is the difference between the average number of defaulted banks after policy control and without mitigation.

We illustrate the sensitivity of systemic risk to the different structural network parameters. In both network configurations, as variation in interbank liabilities increases, the difference between received assets and due liabilities of each bank experiences higher fluctuations, making each bank more vulnerable and increasing systemic risk. In the core-periphery network, if the budget is sufficient, ML can rescue all banks and hence reduce systemic risk. On the other hand, if the budget is not sufficient, ML may choose to rescue banks which are not systemically important in order to maximize the total liquidity in the system. When liabilities have larger variation, a higher number of banks becomes systemically important and consequently the sys-
Systemic risk resulting from their failures gets larger. This is reflected in Figure 3.1(a) showing that the SID policy achieves a better reduction in systemic risk relatively to the ML policy when the coefficient of variation becomes sufficiently high.

Conversely, in the random network, the ML policy achieves better systemic risk reduction as shown in Figure 3.1(b). Since all banks are equally systemically important, it is preferable to utilize in full the available budget to rescue banks and maximize the total liquidity in the network rather than spending the budget only to assist a specific set of target banks identified by the SID policy. On the other hand, the ML policy is not as effective in reducing the number of defaults. Under SID, a larger number of banks default in early periods (see Figure 3.5(b)) because they may not be systemically important and hence are not rescued. The cash proceeds coming from the liquidation of their noncash assets through the auction make surviving banks able to later fulfill liability obligations even without liquidity assistance.

As correlation increases and interbank liabilities manifest stronger comovement, systemic risk tends to decrease. This is because, in aggregate, the payment amount that each bank receives from others in the network approaches the total value of its outstanding liabilities hence resulting in smaller systemic risk. As a consequence, the impact of mitigation becomes less important. This is reflected in Figure 3.3 showing that the systemic risk reduction monotonically decreases with correlation. Nevertheless, SID still achieves a superior performance both in terms of systemic risk reduction and diminished number of defaults in the core-periphery network, while ML has better reduction in systemic risk and worse reduction in number of defaults in random networks. Note that as correlation increases, SID achieves a similar performance to ML in the core-periphery network. This is because the stronger comovement in liabilities decreases the difference in systemic importance of the banks. As a result, both SID and ML target banks of similar systemic importance.

As expected, systemic risk increases if the percentage $\rho$ of tolerated loss increases (when the tolerated loss rate approaches 100% no mitigation is applied). Figure 3.4 shows that the gap between the systemic risk reduction obtained by SID and ML is
very small when no loss or full loss is tolerated. In the first case, the budget used by SID is large enough to allow both policies to completely eliminate systemic risk. In the second case, the opposite happens and both policies use very little budget resulting in high systemic risk. However, when the tolerated amount of loss is at an intermediate level hence requiring a careful budget allocation, rescuing systemically important banks achieves higher systemic risk reduction in the core-periphery network.

In order to illustrate the dependence of systemic risk on the topology of core-periphery networks, we consider what happens when the persistence over time of the network structure decreases. If the probability that a bank migrates from core to periphery increases, the network would have a higher number of non-systemically important banks. As a consequence, we expect smaller systemic risk levels and a decreasing need of using mitigation policies. This is reflected in Figure 3.5 which further confirms that systemic risk is driven by heterogeneity in sizes and connectedness of banks.

Figure 3.5(a) shows the contagious effect exerted by the core banks in the core-periphery network. The SID policy focuses on rescuing core banks. On the other hand, ML can choose to rescue periphery banks in the intent of maximizing the total liquidity in the system sacrificing systemically important core banks. The right graph of Figure 3.6(a) indicates that this can generate failures of other core banks in the network and present higher systemic risk. Notice that when the auction proceeds coming from the failure of these banks are redistributed to others in the network, it increases their cash reserves and make them able to later fulfill liability obligations. This explains why, under the ML policy and in the absence of mitigation, the level of systemic risk decreases and the cumulative number of defaults stays constant after a certain time epoch. Unlike the core-periphery network, in the random network the cash proceeds resulting from the auction of noncash assets of defaulted banks do not sufficiently increase the cash reserves of surviving banks. Hence, default keep occurring at later times as Figure 3.6(b) shows.
Figure 3.2. The left panels report the dependence of systemic risk reduction on the coefficient of variation of liabilities. The right panels show the same dependence for the default reduction. Interbank liabilities are assumed to be uncorrelated, the percentage $\rho$ of tolerated loss is 20%, and the probability of migrating from core to periphery is 0.1. The 95% confidence band is superimposed in all plots.
Figure 3.3. The left panels report the dependence of systemic risk reduction on the correlation of interbank liabilities, while the right panels show the same dependence for the default reduction. We fix the coefficient of variation to 0.5, the percentage $\rho$ of tolerated loss to 20%, and the probability of migrating from core to periphery to 0.1. The 95% confidence band is superimposed in all plots.
Figure 3.4. The left panels report the systemic risk reduction as the percentage $\rho$ of tolerated loss increases. The right panels show the same dependence for the default reduction. We set the coefficient of variation to 0.5, interbank liabilities correlation to 0, and the probability of migrating from core to periphery to 0.1. The 95% confidence band is superimposed in all plots.
Figure 3.5. (a): The left panel shows the systemic risk with respect to the probability of migrating from core to periphery. The right panel shows the same dependence for the number of defaults. (b): The left panel shows the systemic risk reduction with respect to this migration probability. The right panel shows the same dependence for the default reduction. We fix the coefficient of variation to 0.5, interbank liabilities correlation to 0, and percentage \( \rho \) of tolerated loss to 20%. The 95% confidence band is superimposed in all plots.
Figure 3.6. (a): The left panel shows the evolution of systemic risk over time for a core-periphery network. The right panel shows the systemic risk breakdown into contributions from core and peripheral banks. Core and peripheral banks are denoted by $C$ and $P$ respectively. (b): Evolution of systemic risk over time for a random network. The middle plot zooms on the curves between times 2 and 20 so to illustrate the difference between the policies in these time periods. We set the coefficient of variation to 0.5, zero correlation between inter-bank liabilities, $\rho = 20\%$, and the probability of migrating from core to periphery to 0.1.
Figure 3.7. (a): The left panel shows the cumulative number of defaults over time for a core-periphery network. The right panel gives a breakdown into defaults of core and peripheral banks. Core and peripheral banks are denoted by $C$ and $P$ respectively. (b): Cumulative number of defaults over time for a random network. We set the coefficient of variation to 0.5, no correlation between interbank liabilities, $\rho = 20\%$, and the probability of migrating from core to periphery to 0.1.
CHAPTER 4. SUMMARY

This thesis extends the systemic risk literature in two directions. Chapter 2 provides an analytical tool to compare financial networks. We have focused on the relation between the topology of the network and the loss profile of the financial system. To be more specific, we have used vector majorization to express preferences between losses and matrix majorization to compare networks in terms of interbank liabilities concentration. The notions of balancing and unbalancing network have been developed to illustrate the different impact that concentration of interbank liabilities can have on systemic risk.

Our main result is that higher liability concentration leads to larger systemic losses in unbalancing (highly capitalized) financial networks, while the opposite is true in balancing (lowly capitalized) systems. For regulatory purposes, it is desirable for the network to be in a state which is both balancing and unbalancing as this would reduced losses to the minimum extent. We have shown how our framework can be specialized to reproduce tiered systems, identified by recent empirical studies as good descriptors of real-world financial networks. Our results indicate that an imperfectly tiered is preferred to a perfectly tiered structure if the state of the financial network is unbalancing, whereas the opposite preference relation holds if the state is balancing.

We have conducted an empirical analysis of the network formed by the eight largest European banking sectors and found that the state of the financial network is either unbalancing or close to it, consistently over a period of seven years. Such an analysis, along with the theoretical predictions of our study, indicates that it is advisable to avoid concentration of gross exposures, as they would have serious systemic effects in unbalancing systems.

Chapter 3 develops a multi-period controlled clearing payment system building on the framework of [Eisenberg and Noe(2001)]. We have introduced an outside entity
whose goal is to provide liquidity assistance loans to financially distressed banks so to reduce the level of systemic risk within the network. Our network consists of insured commercial banks supervised by the FDIC, which sells non-cash assets of defaulted banks via a first-price sealed-bid auction.

We have focused our analysis on two network configurations, the empirically driven core-periphery topology, and a baseline random network. We have illustrated the systemic risk mitigation effect of two liquidity assistance policies, Systemic Importance Driven and Maximum Liquidity. Our simulation based study shows that in the core-periphery network, strategies maximizing the total liquidity of the system may not reduce systemic risk and number of defaults. This is because the failure of systemically important banks can have contagious effects which propagate wider in the financial system. Such systemic effects would be mitigated by the SID policy which targets a class of banks with the highest default consequences in the system. If each bank is equally systemically important, as it is the case in a random network, the Max-Liquidity policy achieves better systemic risk reduction than SID given that it greedily allocates its available budget to prevent any failure rather than failures of banks belonging to a specific subset.
REFERENCES
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APPENDICES
APPENDIX A. CHAPTER 2 PROOFS AND AUXILIARY RESULTS

Proof of Lemma 2.3.1. We first prove the sufficient condition. Let \(x \in A\). For notational convenience, we omit the dependence of \(\nu(A)\) on \(A\). For \(j = 1, \ldots, n - 1\),

\[
[xD]_{\nu_{j+1}} - [xD]_{\nu_j} = \sum_{i=1}^{n} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
= x(n) (d_{n,j+1}^\nu - d_{n,j}^\nu) + \sum_{i=1}^{n-1} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq x(n-1) \sum_{i=n-1}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) + \sum_{i=1}^{n-2} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq x(n-2) \sum_{i=n-2}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) + \sum_{i=1}^{n-3} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq \cdots \geq x(1) \sum_{i=1}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) \geq 0
\]

where each inequality above follows from the inequality \(\sum_{i=k}^{n} d_{i,j}^\nu \leq \sum_{i=k}^{n} d_{i,j+1}^\nu\) for \(k = 1, \ldots, n\). This shows that \(xD\) is similarly ordered to \(x\) and \(D\) is order preserving with respect to \(A\). Next, we prove the necessary condition. Let \(m = \min_{i \in \{1, \ldots, n\}} \{z_i\}\).

For \(k = 1, \ldots, n\), choose \(x \in A\) such that \(x(n) = x(n-1) = \cdots = x(k) = m\) and \(x(k-1) = x(k-2) = \cdots = x(1) = 0\) if \(k > 1\), and \(x(n) = x(n-1) = \cdots = x(1) = m\) if \(k = 1\).

Because \(xD\) is similarly ordered to \(x\), it holds that for \(j = 1, \ldots, n - 1\),

\[
0 \leq [xD]_{\nu_{j+1}} - [xD]_{\nu_j} = \sum_{i=1}^{n} x(i) d_{i,j+1}^\nu - \sum_{i=1}^{n} x(i) d_{i,j}^\nu = \left(\sum_{i=k}^{n} d_{i,j+1}^\nu - \sum_{i=k}^{n} d_{i,j}^\nu\right) m.
\]

Since \(m > 0\), the above inequality implies that \(\sum_{i=k}^{n} d_{i,j+1}^\nu - \sum_{i=k}^{n} d_{i,j}^\nu \geq 0\) holds for \(j = 1, \ldots, n - 1, k = 1, \ldots, n\). \(\square\)
Proof of Lemma 2.3.2. First, we prove the sufficient condition in the first statement. Let $x, y \in \mathcal{A}, x \prec_w y$. For notational convenience, we omit the dependence of $\nu(A)$ on $A$. For $k = 1, \ldots, n$,
\[
\sum_{j=k}^{n} [xD]_{\nu_j} - \sum_{j=k}^{n} [yD]_{\nu_j} = \sum_{j=k}^{n} \left( \sum_{i=1}^{n} x(i) d_{i,j}^{\nu} - \sum_{i=1}^{n} y(i) d_{i,j}^{\nu} \right) \\
= \sum_{i=1}^{n} \left( x(i) - y(i) \right) \sum_{j=k}^{n} d_{1,j}^{\nu} + \sum_{i=2}^{n} \left( x(i) - y(i) \right) \sum_{j=k}^{n} \left( d_{2,j}^{\nu} - d_{1,j}^{\nu} \right) \\
+ \cdots + \left( x(n) - y(n) \right) \sum_{j=k}^{n} \left( d_{n,j}^{\nu} - d_{n-1,j}^{\nu} \right) \leq 0,
\]
where the first equality follows from the fact that $D$ is order preserving w.r.t. $A$. The last inequality follows from $x \prec_w y$ along with the assumption that $\sum_{j=k}^{n} d_{i-1,j}^{\nu} \leq \sum_{j=k}^{n} d_{i,j}^{\nu}$ for $i = 2, \ldots, n$. Hence, using the definition of weak submajorization, $xD \prec_w yD$.

We then prove the necessary condition in the first statement. Choose $x, y \in \mathcal{A}$ such that $0 \leq x(t) < y(t), y(t) - x(t) = x(t-1) - y(t-1)$ for some $t \in \{2, \ldots, n\}$ and $x(u) = y(u)$ for $u = 1, \ldots, n, u \notin \{t-1, t\}$. Clearly,
\[
\sum_{i=t}^{n} x(i) < \sum_{i=t}^{n} y(i), \quad \sum_{i=u}^{n} x(i) = \sum_{i=u}^{n} y(i) \quad \text{for} \quad u = 1, \ldots, n, u \neq t, \tag{A.2}
\]
and $x \prec_w y$. Because $D$ is order and weak submajorization preserving w.r.t. $A$, the inequality in (A.1) must hold. Using (A.2) the inequality in (A.1) may be simplified to
\[
\sum_{i=t}^{n} \left( x(i) - y(i) \right) \sum_{j=k}^{n} \left( d_{t,j}^{\nu} - d_{t-1,j}^{\nu} \right) \leq 0,
\]
for $k = 1, \ldots, n$. Because $\sum_{i=t}^{n} \left( x(i) - y(i) \right) < 0$, we obtain that $\sum_{j=k}^{n} \left( d_{t,j}^{\nu} - d_{t-1,j}^{\nu} \right) \geq 0$ for $k = 1, \ldots, n$. This concludes the proof of the first statement.

The proof for the second statement follows using similar arguments as above. To prove the sufficiency, we need to show that $\sum_{j=1}^{k} [xD]_{\nu_j} - \sum_{j=1}^{k} [yD]_{\nu_j} \geq 0$ for $k = 1, \ldots, n, x, y \in \mathcal{A}$. This can be done by expanding vector-matrix products and
combining terms in a similar way as done for the inequality (A.1). Next, we show
the necessity of the condition. Choose \(x, y \in A\) such that \(x(t) > y(t) \geq 0\), \(x(t) - y(t) = y(t+1) - x(t+1)\) for some \(t \in \{1, \ldots, n-1\}\) and \(x(u) = y(u)\) for \(u = 1, \ldots, n, u \notin \{t, t+1\}\). Then we obtain
\[
\sum_{i=1}^{t} (x(i) - y(i)) \sum_{j=1}^{k} (d_{ij} - d_{t+1,j}) \geq 0,
\]
for \(k = 1, \ldots, n\). We conclude the proof following similar arguments as in the proof of the first statement.

\[\square\]

**Proof of Lemma 2.3.3.** For notational convenience, we set \(\bar{\Pi} := (1 + \gamma)\Pi\) and \(\bar{c} := (1+\gamma)c - \gamma\ell\). As pointed out in [Eisenberg and Noe(2001)], the clearing payment vector \(p^*\) is obtained as the solution to the following optimization problem:
\[
\max_x f(x), \quad \text{s.t.} \quad x(I - \bar{\Pi}) \leq \bar{c}, \quad 0 \leq x \leq \ell,
\]
where the objective function \(f\) is any real valued increasing function of the vector \(x\). Multiplying both sides of the first constraint by \((1 - \alpha)\), with \(\alpha \in [0, 1]\), will lead to an equivalent optimization problem. But this leads to
\[
x[I - (1 - \alpha)\bar{\Pi} - \alpha I] \leq (1 - \alpha)\bar{c}.
\]
That is, if we replace \(\bar{\Pi}\) by the matrix \(\Pi_{\alpha,\gamma} := (1 - \alpha)\bar{\Pi} + \alpha I\) and \(\bar{c}\) by \(c_{\alpha,\gamma} := (1 - \alpha)\bar{c}\), the clearing payment vector stays the same.

\[\square\]

**Lemma A.1.** Let \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) be two financial systems. If there exists \(\alpha\) so that \(\Pi^a_{\alpha,\gamma} \prec \Pi^b_{\alpha,\gamma}\) and \(\Pi^a_{\alpha,\gamma} \not\succ \Pi^b_{\alpha,\gamma}\), then it must hold that \(\Pi^a_{\beta,\gamma} \not\succ \Pi^b_{\beta,\gamma}\) for all \(\beta \in [0, 1], \beta \neq \alpha\).

**Proof.** Assume the existence of \(A := \Pi^a_{\alpha,\gamma}, B := \Pi^b_{\alpha,\gamma}, C := \Pi^a_{\beta,\gamma}, D := \Pi^b_{\beta,\gamma}, \beta \neq \alpha\), so that \(A \prec B\) and \(A \not\succ B\), but \(C \succ D\). Denote by \(X^k\) the \(k\)-th row of the matrix \(X\). Because \(A \prec B\) and \(A \not\succ B\), there must exist \(g\) and \(i\) such that
\[
\sum_{j=1}^{g} A^i_{(j)} > \sum_{j=1}^{g} B^i_{(j)}.
\]
(A.3)
Moreover, since \( C \succ D \), from the definition of majorization the following inequality must hold:

\[
\sum_{j=1}^{k} C_{(j)}^i \leq \sum_{j=1}^{k} D_{(j)}^i \quad \text{for any } k = 1, \ldots, n. \quad (A.4)
\]

Next, we show that Eq. (A.3) and Eq. (A.4) cannot hold simultaneously. Let \( h, m, w, z \) be such that \( A_{(h)}^i = B_{(m)}^i = \alpha \) and \( C_{(w)}^i = D_{(z)}^i = \beta \). We first discuss the implications of Eq. (A.3) using a case-by-case analysis based on \( g \).

- **\( g > \max\{h, m\} \).** Using the definition of relaxed equivalent version, we obtain

  \[
  \sum_{j=1, j \neq h}^{g} A_{(j)}^i + \alpha = \sum_{j=1}^{g} A_{(j)}^i > \sum_{j=1}^{g} B_{(j)}^i = \sum_{j=1, j \neq m}^{g} B_{(j)}^i + \alpha \\
  \Rightarrow \sum_{j=1}^{g} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{a,i} > \sum_{j=1}^{g} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{b,i}. \quad (A.5)
  \]

- **\( h \geq g \geq m \).** We obtain

  \[
  \sum_{j=1}^{g-1} A_{(j)}^i + \alpha \geq \sum_{j=1}^{g} A_{(j)}^i > \sum_{j=1}^{g} B_{(j)}^i = \sum_{j=1, j \neq m}^{g} B_{(j)}^i + \alpha
  \]

  leading to the inequality (A.5).

- **\( h \leq g \leq m \).** Eq. (A.3) implies that

  \[
  \sum_{j=1}^{m} A_{(j)}^i = \sum_{j=1, j \neq h}^{g} A_{(j)}^i + \alpha + \sum_{j=g+1}^{m} A_{(j)}^i > \sum_{j=1}^{g} B_{(j)}^i + \sum_{j=g+1}^{m-1} B_{(j)}^i + \alpha = \sum_{j=1}^{m} B_{(j)}^i
  \]

  \[
  \Rightarrow \sum_{j=1}^{m} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{a,i} > \sum_{j=1}^{m} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{b,i}. \quad (A.6)
  \]

- **\( g < \min\{h, m\} \).** Eq. (A.3) directly leads to

  \[
  \sum_{j=1}^{g+1} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{a,i} > \sum_{j=1}^{g+1} (1 - \alpha)(1 + \gamma) \Pi_{(j)}^{b,i}, \quad g < \min\{h, m\}. \quad (A.7)
  \]

Next, we discuss the implications of Eq. (A.4) and show that it leads to

\[
\sum_{j=1}^{k} (1 - \beta)(1 + \gamma) \Pi_{(j)}^{a,i} \leq \sum_{j=1}^{k} (1 - \beta)(1 + \gamma) \Pi_{(j)}^{b,i}, \quad k = 1, \ldots, n. \quad (A.8)
\]

This is done via a case-by-case analysis based on \( k \).
\( k > \max\{w, z\} \). We obtain
\[
\sum_{j=1, j \neq w}^{k} C_{(j)}^i + \beta = \sum_{j=1}^{k} C_{(j)}^i \leq \sum_{j=1}^{k} D_{(j)}^i = \sum_{j=1, j \neq z}^{k} D_{(j)}^i + \beta
\]
hence implying the inequality (A.8).

\( k < \min\{w, z\} \). Eq. (A.4) directly leads to inequality (A.8).

\( w \geq k \geq z \). Eq. (A.4) implies the following inequality
\[
\sum_{j=1}^{k} C_{(j)}^i + \beta \leq \sum_{j=1}^{k} D_{(j)}^i + \beta \leq \sum_{j=z+1}^{k} D_{(j)}^i,
\]
which further leads to the inequality (A.8).

\( w \leq k \leq z \). We obtain
\[
\sum_{j=1, j \neq w}^{k} C_{(j)}^i + \beta = \sum_{j=1}^{k} C_{(j)}^i \leq \sum_{j=1}^{k} D_{(j)}^i \leq \sum_{j=D_{(k)}}^{k} D_{(j)}^i + \beta,
\]
which again leads to the inequality (A.8).

Setting \( k = g \) in Eq. (A.8) shows that Eq. (A.5) and Eq. (A.8) cannot hold simultaneously. Setting \( k = m \), we obtain that Eq. (A.6) and Eq. (A.8) cannot hold simultaneously. Setting \( k = g + 1 \), we obtain that Eq. (A.7) and Eq. (A.8) cannot hold simultaneously. This ends the proof.

\[
\square
\]

**Proof of Lemma 2.3.4.** We prove the first statement by showing that \( \ell \Pi + c \geq \ell \).

The asset value of the node with the smallest liability is given by
\[
\sum_{i=1}^{n} \ell_{(i)} \pi_{i,1}^\mu + c_{(1)} \geq c_{(1)} \geq \mathbb{1}_{c_{(1)} < \ell_{(1)}} \left[ c_{(1)} + \gamma \left( c_{(1)} - \ell_{(1)} \right) \right] + \mathbb{1}_{c_{(1)} \geq \ell_{(1)}} \ell_{(1)}
\]
\[
\geq \mathbb{1}_{c_{(1)} < \ell_{(1)}} \left[ \ell_{(1)} - \max \left\{ \ell_{(1)} - \left[ (1 + \gamma)c - \gamma \ell_{(1)}, 0 \right], 0 \right\} \right] + \mathbb{1}_{c_{(1)} \geq \ell_{(1)}} \ell_{(1)}
\]
\[
\geq \mathbb{1}_{c_{(1)} < \ell_{(1)}} \ell_{(1)} + \mathbb{1}_{c_{(1)} \geq \ell_{(1)}} \ell_{(1)} \geq \ell_{(1)} ,
\]
where \( 1_A \) denotes the indicator function of the event \( A \). Because \((\Pi, \ell, c, \gamma)\) is unbalancing, by definition it satisfies the inequalities (2.5). Combining those with the above inequality leads to
\[
\sum_{i=1}^{n} \ell(i) \pi_{i,n} + c(n) - \ell(n) \geq \cdots \geq \sum_{i=1}^{n} \ell(i) \pi_{i,2} + c(2) - \ell(2) \geq \sum_{i=1}^{n} \ell(i) \pi_{i,1} + c(1) - \ell(1) \geq 0,
\]
which proves the first statement.

Given \( \epsilon \), we can choose the entries of \( \Pi \) to be small enough so that
\[
\sum_{i=1}^{n} (\ell(i) + \epsilon \mu_i) \pi_{i,1} + c(1) < \ell(1) + \epsilon \mu_1 \quad \text{and} \quad c_{j+1} - c(j) \geq \ell_{j+1} - \ell_j + \max_{k=1,\ldots,n} \{\sum_{i=1}^{n} \ell(i) \pi_{ik}\}
\]
for \( j = 1, \ldots, n-1 \) (recall here that \( \mu \) has been defined above Definition 2.3.6). Such a system \((\Pi, \ell, c, \gamma)\) satisfies the inequalities (2.5), but \((\ell + \epsilon)\Pi + c \not\geq \ell + \epsilon\). Hence, this system is unbalancing but \( p^*(\Pi, \ell + \epsilon, c, \gamma) \not\geq \ell + \epsilon\).

\[\square\]

For any vector \( x \in \mathbb{R}^n \), define
\[
\Delta x := \left( 0, x(2) - x(1), x(3) - x(2), \ldots, x(n) - x(n-1) \right),
\]
i.e., the vector whose components are the increments from one component to the next rank ordered component in the original vector \( x \).

**Lemma A.2.** Let \( x, y \in \mathbb{R}^n_{\geq 0} \) such that \( x \) and \( y \) are similarly ordered. If \( \Delta x \leq \Delta y \), then \( \Delta (x \wedge y) \leq \Delta y \). Vice versa, if \( \Delta x \geq \Delta y \), then \( \Delta (x \wedge y) \geq \Delta y \).

**Proof.** We first prove that \( \Delta x \leq \Delta y \Rightarrow \Delta (x \wedge y) \leq \Delta y \). For \( i = 1, \ldots, n-1 \),
\[
\Delta (x \wedge y)_{i+1} = (x \wedge y)_{(i+1)} - (x \wedge y)_{(i)}
\]
\[
= \min \{x_{(i+1)}, y_{(i+1)}\} - \min \{x_{(i)}, y_{(i)}\}
\]
\[
= \begin{cases} 
  x_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\
  x_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)} \\
  y_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)}
\end{cases}
\]
\[
\leq y_{(i+1)} - y_{(i)} = \Delta y_{i+1},
\]
where we have used the assumption that $x$ is similarly ordered to $y$. Notice that the case $x_{(i+1)} \geq y_{(i+1)}$ and $x_{(i)} \leq y_{(i)}$ is not listed because it violates the assumption that $\Delta x \leq \Delta y$. Hence, $\Delta (x \land y) \leq \Delta y$.

We next show that $\Delta x \geq \Delta y \Rightarrow \Delta (x \land y) \geq \Delta y$. This holds because

$$
\Delta (x \land y)_{i+1} = (x \land y)_{(i+1)} - (x \land y)_{(i)}
= \min \{x_{(i+1)}, y_{(i+1)}\} - \min \{x_{(i)}, y_{(i)}\}
= \begin{cases} 
    x_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\
    y_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\
    y_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)}
\end{cases}
\geq y_{(i+1)} - y_{(i)} = \Delta y_{i+1},
$$

where the second equality follows from the assumption that $x$ is similarly ordered to $y$. The third equality does not include the case $x_{(i+1)} \leq y_{(i+1)}$ and $x_{(i)} \geq y_{(i)}$ because such a case violates the assumption that $\Delta x \geq \Delta y$.

□

The next lemma shows that if a payment vector has smaller (larger) variation than the liability vector, such a relation is preserved if both vectors are multiplied by an order preserving relaxed equivalent version of the relative liability matrix.

**Lemma A.3.** If $\Pi_{\alpha,\gamma} := (\pi_{\alpha,\gamma,i,j})_{i,j=1}^{n}$ is order preserving w.r.t. $P$, for any $p \in P$ it must hold that

(I) $\Delta p \leq \Delta \ell$ implies that $\Delta (p \Pi_{\alpha,\gamma}) \leq \Delta (\ell \Pi_{\alpha,\gamma})$.

(II) $\Delta p \geq \Delta \ell$ implies that $\Delta (p \Pi_{\alpha,\gamma}) \geq \Delta ([\ell - (\ell_{(1)} - p_{(1)})] \Pi_{\alpha,\gamma})$. 
Proof. For \( j = 1, \ldots, n - 1, p \in \mathcal{P} \),

\[
[\Delta(\ell \Pi_{\alpha,\gamma})]_{j+1} - [\Delta(p \Pi_{\alpha,\gamma})]_{j+1} = \sum_{i=1}^{n} (\ell(i) - p(i)) \left( \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \right)
\]

\[
= (\ell(1) - p(1)) \left( \pi^\mu_{\alpha,\gamma,1,j+1} - \pi^\mu_{\alpha,\gamma,1,j} \right) + \left( [\ell(1) + \Delta \ell_2 - (p(1) + \Delta p_2)] \left( \pi^\mu_{\alpha,\gamma,2,j+1} - \pi^\mu_{\alpha,\gamma,2,j} \right) + \ldots \right.
\]

\[
+ \left[ (\ell(1) + \sum_{k=2}^{n} \Delta \ell_k) - (p(1) + \sum_{k=2}^{n} \Delta p_k) \right] \left( \pi^\mu_{\alpha,\gamma,n,j+1} - \pi^\mu_{\alpha,\gamma,n,j} \right)
\]

\[
= (\ell(1) - p(1)) \left( \sum_{i=1}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \right) + (\Delta \ell_2 - \Delta p_2) \left( \sum_{i=2}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \right) + \ldots
\]

\[
+ (\Delta \ell_n - \Delta p_n) \left( \pi^\mu_{\alpha,\gamma,n,j+1} - \pi^\mu_{\alpha,\gamma,n,j} \right).
\]

Applying Lemma 2.3.1 with \( \mathcal{A} = \mathcal{P} \) we deduce that \( \sum_{i=1}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). If \( \Delta p \leq \Delta \ell \), then \([\Delta(\Pi_{\alpha,\gamma})]_{j+1} - [\Delta(\Pi_{\alpha,\gamma})]_{j+1} \geq 0 \) because \( \ell \geq p \) and \( \sum_{i=1}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). This proves (I). Vice versa, if \( \Delta p \geq \Delta \ell \) then

\[
[\Delta(\ell \Pi_{\alpha,\gamma})]_{j+1} - [\Delta(p \Pi_{\alpha,\gamma})]_{j+1} - (\ell(1) - p(1)) \left( \sum_{i=1}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \right) \leq 0
\]

because \( \sum_{i=1}^{n} \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). This proves (II).

The next lemma shows that if the financial system is balancing (unbalancing), the vector of asset values before clearing under the base (reduced) liability configuration has smaller (larger) variation than the liability vector. Moreover, if the payment vector has smaller (larger) variation than the liability vector, then the assets after payments are settled also have smaller (larger) variation than the liabilities.

Lemma A.4. Let \((\Pi, \ell, c, \gamma)\) be a financial system and \(\Pi_{\alpha,\gamma}\) be an \(\alpha\)-relaxed equivalent version which is order preserving w.r.t. \(\mathcal{P}\).

(I) If \((\Pi, \ell, c, \gamma)\) is balancing, then

- \(\Delta(\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell\).
- \(\Delta p \leq \Delta \ell \) implies that \(\Delta(p \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) for \( p \in \mathcal{P} \).
(II) If \((\Pi, \ell, c, \gamma)\) is unbalancing, then

- \(\Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta \ell\).
- \(\Delta p \geq \Delta \ell\) implies that \(\Delta (p \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta \ell\) for \(p \in \mathcal{P}\) such that \(p \geq [(1 + \gamma)c - \gamma \ell] \wedge \ell\).

**Proof.** (I)

\[
[\Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma})]_j
= [\Delta (\alpha \ell + (1 - \alpha) [\ell(1 + \gamma) \Pi + (1 + \gamma)c - \gamma \ell])]_j
= \alpha (\ell(j) - \ell(j-1))
+ (1 - \alpha) \left[ \sum_{i=1}^{n} \ell(i)(1 + \gamma) (\pi^\mu_{i,j} - \pi^\mu_{i,j-1}) + (1 + \gamma) (c_{(j)} - c_{(j-1)}) - \gamma (\ell(j) - \ell(j-1)) \right]
\leq \alpha (\ell(j) - \ell(j-1)) + (1 - \alpha) \left[ (1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma (\ell(j) - \ell(j-1)) \right] = [\Delta \ell]_j,
\]

for \(j = 2, \ldots, n\). The second equality holds because \(\Pi_{\alpha,\gamma}\) is order preserving w.r.t. \(\mathcal{P}\) and \((1 + \gamma)c - \gamma \ell\) is similarly ordered to \(\ell\). The inequality follows from the fact that the system is balancing.

Next, by Lemma A.3 (I), \(\Delta p \leq \Delta \ell\) implies that \(\Delta (p \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell\), where the first inequality holds because \(\Pi_{\alpha,\gamma}\) is order preserving w.r.t. \(\mathcal{P}\), and \(c_{\alpha,\gamma}\) and \(p\) are similarly ordered to \(\ell\).

(II)

\[
[\Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma})]_j
= [\Delta (\alpha \ell + (1 - \alpha) [\ell(1 + \gamma) \Pi + (1 + \gamma)c - \gamma \ell])]_j
= \alpha (\ell(j) - \ell(j-1))
+ (1 - \alpha) \left[ \sum_{i=1}^{n} \ell(i)(1 + \gamma) (\pi^\mu_{i,j} - \pi^\mu_{i,j-1}) + (1 + \gamma) (c_{(j)} - c_{(j-1)}) - \gamma (\ell(j) - \ell(j-1)) \right]
\geq \alpha (\ell(j) - \ell(j-1)) + (1 - \alpha) \left[ (1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma (\ell(j) - \ell(j-1)) \right]
= \alpha (\ell(j) - \ell(j-1)) + (1 - \alpha) \left[ (1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma (\ell(j) - \ell(j-1)) \right] = [\Delta \ell]_j,
\]
for \( j = 2, \ldots, n \). The second equality follows from the fact that \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. \( P \), the fact that both \((1 + \gamma)c - \gamma \ell\) and \( \ell \) are similarly ordered to \( \ell \), and the equality \( \Delta \ell = \Delta \ell \). The inequality comes from the fact that the system is unbalancing.

Next, by Lemma A.3 (II), \( \Delta p \geq \Delta \ell \) implies

\[
\Delta (p \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta ((\ell - (\ell_1 - p_1)) \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta \ell,
\]

where the first inequality follows because \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. \( P \), and \( c_{\alpha,\gamma} \), \( p \), and \( \ell - (\ell_1 - p_1) \) are similarly ordered to \( \ell \). The second inequality follows from the inequality

\[
\ell_1 - p_1 \leq \ell_1 - \left[ ((1 + \gamma)c - \gamma \ell) \wedge \ell_1 \right] = \max \left\{ \ell_1 - ((1 + \gamma)c - \gamma \ell)_1, 0 \right\},
\]

the inequality \( \Delta ((\ell - (\ell_1 - p_1)) \Pi_{\alpha,\gamma}) \geq \Delta (\ell \Pi_{\alpha,\gamma}) \) which is implied by Lemma 2.3.1 (choosing \( D = \Pi_{\alpha,\gamma} \) and \( A = P \) therein), and using the assumption that \( p \geq \left[ (1 + \gamma)c - \gamma \ell \right] \wedge \ell \).

\[
\square
\]

The following lemma gives some useful properties of the vector sequence converging to the clearing payment vector.

**Lemma A.5.** Let \((\Pi, \ell, c, \gamma)\) be a financial system. Suppose there exists \( \alpha \in [0, 1) \) such that \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. to \( P \). Define the vector valued function

\[
F(p; \Pi_{\alpha,\gamma}, \ell, c_{\alpha,\gamma}) := \ell \wedge (p \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}),
\]

and a sequence of vectors \( \{f_u\}_{u=0}^{\infty} \) given by \( f_u := F(f_{u-1}) \) and \( f_0 := \ell \). The following statements hold:

(I) \( f_u \) is similarly ordered to \( \ell \) and \( \{f_u\}_{u=0}^{\infty} \) decreasingly converges to \( p^*(\Pi, \ell, c, \gamma) \).

(II) If \((\Pi, \ell, c, \gamma)\) is balancing, then \( \Delta f_u \leq \Delta \ell \) and \( \Delta (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \).

(III) If \((\Pi, \ell, c, \gamma)\) is unbalancing, then \( \Delta f_u \geq \Delta \ell \) and \( \Delta (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta \ell \).

**Proof.** (I) It has been proven in [Eisenberg and Noe(2001)], Lemma 5, that \( f_u \) decreasingly converges to \( p^*(\Pi, \ell, c, \gamma) \). The statement that \( f_u \) is similarly ordered to \( \ell \)
follows from the fact that \( \Pi_{\alpha,\gamma} \) is order preserving and from the assumption that \( c_{\alpha,\gamma} \) is similarly ordered to \( \ell \).

(II) We prove that \( \Delta f_u \leq \Delta \ell \) and \( \Delta (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) by induction. For \( u = 0 \), we know that \( f_0 = \ell \). Clearly, \( \Delta f_0 \leq \Delta \ell \). From the assumptions that \((\Pi, \ell, c, \gamma)\) is balancing and \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), it follows that \( \Delta (\ell \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) from Lemma A.4 (I). Hence, \( \Delta (f_0 \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \). Next, we prove the statement for \( u + 1 \) assuming that it holds for \( u \). Since \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), \( f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma} \) is similarly ordered to \( \ell \). Hence \( f_{u+1} = \ell \land (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \) is similarly ordered to \( \ell \). Using the induction hypothesis that \( \Delta (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) and Lemma A.2, we obtain

\[
\Delta f_{u+1} = \Delta [\ell \land (f_u \Pi_{\alpha,\gamma} + c_{\alpha,\gamma})] \leq \Delta \ell.
\]

Since \((\Pi, \ell, c, \gamma)\) is balancing, \( \Pi_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), and \( f_{u+1} \in \mathcal{P} \), we can apply Lemma A.4 (I) and deduce that \( \Delta (f_{u+1} \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \). This concludes the induction.

(III) Using the fact that \( \{f_u\} \) is a decreasing sequence converging to \( p^* \) and \( p^* \geq [(1 + \gamma)c - \gamma \ell] \land \ell \), we deduce that \( f_u \geq [(1 + \gamma)c - \gamma \ell] \land \ell \). Then, applying Lemma A.4 (II), we can use similar arguments as in (II) to conclude the proof.

\(\square\)

**Proof of Proposition 2.3.1.** Letting \( u \to \infty \) in Lemma A.5 (I) leads to \( p^* \) being similarly ordered to \( \ell \), hence yielding (I). Moreover, we obtain \( \Delta p^* \leq \Delta \ell \) for balancing financial systems and \( \Delta p^* \geq \Delta \ell \) for unbalancing financial systems from (II) and (III) in Lemma A.5. It then follows that

\[
el[i] - p^*_i \geq \ell[i] - p^*_i - ([\Delta \ell]_{n-i+1} - [\Delta p^*]_{n-i+1}) = \ell[i+1] - p^*_{i+1}, \quad i = 1, \ldots, n - 1
\]

for balancing financial systems, and

\[
el(i) - p^*_i \geq \ell(i) - p^*_i + ([\Delta \ell]_{i+1} - [\Delta p^*]_{i+1}) = \ell(i+1) - p^*_{i+1}, \quad i = 1, \ldots, n - 1
\]

for unbalancing financial systems.

\(\square\)
The following lemma provides sufficient conditions under which the minimum operation preserves the weak majorization relation. This is needed in the following proofs, given that the clearing payment is given by the minimum between two vectors.

**Lemma A.6.** Let \( x, y, z \in \mathbb{R}_{\geq 0}^n \) such that \( x \) and \( y \) are similarly ordered to \( z \).

(I) If \( z[i] \leq a[i] \) implies \( z[k] \leq a[k] \) for \( k > i, a \in \{x, y\} \), then \( x \prec_w y \) implies \( (x \land z) \prec_w (y \land z) \).

(II) If \( z(i) \leq a(i) \) implies \( z(k) \leq a(k) \) for \( k > i, a \in \{x, y\} \), then \( x \prec_w y \) implies \( (x \land z) \prec_w (y \land z) \).

**Proof.** (I) Because \( x \) and \( y \) are similarly ordered to \( z \), clearly, \((x \land z)\) and \((y \land z)\) are similarly ordered to \( z \). Hence, proving \((x \land z) \prec_w (y \land z)\) is equivalent to show that

\[
\sum_{i=1}^{k} \min\{x[i], z[i]\} \leq \sum_{i=1}^{k} \min\{y[i], z[i]\} \quad \text{for} \quad k = 1, \ldots, n. \tag{A.9}
\]

Let \( m_x = \min\{i = 1, \ldots, n | z[i] \leq x[i]\} \) and \( m_y = \min\{i = 1, \ldots, n | z[i] \leq y[i]\} \). It must hold that for \( k = 1, \ldots, m_y - 1, \)

\[
\sum_{i=1}^{k} \min\{x[i], z[i]\} \leq \sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] = \sum_{i=1}^{k} \min\{y[i], z[i]\}, \tag{A.10}
\]

where the second inequality follows from \( x \prec_w y \). Moreover, for \( k = m_y, \ldots, n, \)

\[
\sum_{i=1}^{k} \min\{x[i], z[i]\} = \mathbb{1}_{m_x < m_y} \left( \sum_{i=1}^{m_x - 1} \min\{x[i], z[i]\} + \sum_{i=m_y}^{k} z[i] \right) + \mathbb{1}_{m_x \geq m_y} \left( \sum_{i=1}^{m_y - 1} x[i] + \sum_{i=m_y}^{k} \min\{x[i], z[i]\} \right) \leq \sum_{i=1}^{m_y - 1} y[i] + \sum_{i=m_y}^{k} z[i] = \sum_{i=1}^{k} \min\{y[i], z[i]\}, \tag{A.11}
\]

where \( \mathbb{1}_A \) denotes the indicator function of the event \( A \). The above inequality follows from \( x \prec_w y \). Using inequalities (A.10) and (A.11), we obtain the inequality in (A.9).
(II) Because \( x \) and \( y \) are similarly ordered to \( z \), clearly, \( (x \land z) \) and \( (y \land z) \) are similarly ordered to \( z \). Hence, proving \( (x \land z) \prec^w (y \land z) \) is equivalent to show
\[
\sum_{i=1}^{k} \min \{x_{(i)}, z_{(i)}\} \geq \sum_{i=1}^{k} \min \{y_{(i)}, z_{(i)}\} \quad \text{for } k = 1, \ldots, n. \tag{A.12}
\]

Let \( m_x = \min \{i = 1, \ldots, n \mid z_{(i)} \leq x_{(i)}\} \) and \( m_y = \min \{i = 1, \ldots, n \mid z_{(i)} \leq y_{(i)}\} \). For \( k = 1, \ldots, m_x - 1 \), we must have
\[
\sum_{i=1}^{k} \min \{x_{(i)}, z_{(i)}\} = \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \geq \sum_{i=1}^{k} \min \{y_{(i)}, z_{(i)}\}, \tag{A.13}
\]
where the first inequality follows from \( x \prec^w y \). Moreover, for \( k = m_x, \ldots, n \),
\[
\sum_{i=1}^{k} \min \{x_{(i)}, z_{(i)}\} = \sum_{i=1}^{m_x - 1} x_{(i)} + \sum_{i=m_x}^{k} z_{(i)} \geq \mathbb{1}_{m_x < m_y} \left( \sum_{i=1}^{m_x - 1} y_{(i)} + \sum_{i=m_x}^{k} \min \{y_{(i)}, z_{(i)}\} \right) \tag{A.14}
\]
\[
+ \mathbb{1}_{m_x \geq m_y} \left( \sum_{i=1}^{m_x - 1} \min \{y_{(i)}, z_{(i)}\} + \sum_{i=m_x}^{k} z_{(i)} \right)
\]
where the inequality follows from \( x \prec^w y \). Using inequalities (A.13) and (A.14), we obtain the inequality (A.12).

\[\square\]

**Proof of Theorem 2.3.1.** Recall from Lemma A.5 that a sequence of vectors
\[
\{f_u(\Pi_{\alpha, \gamma})\}_{u=0}^\infty,
\]
where
\[
F(p, \Pi_{\alpha, \gamma}; \ell, c_{\alpha, \gamma}) := \ell \land (p\Pi_{\alpha, \gamma} + c_{\alpha, \gamma})
\]
\[
f_u(\Pi_{\alpha, \gamma}) := F(f_{u-1}(\Pi_{\alpha, \gamma}), \Pi_{\alpha, \gamma})
\]
\[
f_0(\Pi_{\alpha, \gamma}) := \ell,
\]
converges to the clearing payment vector $p^*$. Hence, proving that $p^*(\Pi_{a,\gamma}^a, \ell, c_{a,\gamma}) \prec_w p^*(\Pi_{a,\gamma}^b, \ell, c_{a,\gamma})$ is equivalent to showing that $f_u(\Pi_{a,\gamma}^a) \prec_w f_u(\Pi_{a,\gamma}^b)$ for $u = 1, 2, \ldots$. For brevity, we denote hereafter $f_u(\Pi_{a,\gamma}^x)$ by $f_u^x$ and $p^*(\Pi_{a,\gamma}^x, \ell, c_{a,\gamma})$ by $p^{x*}$.

Next, we prove that $f_u^a \prec_w f_u^b$ by induction. Without loss of generality, we take $\Pi_{a,\gamma}^b$ to be weak submajorization preserving w.r.t. $P$. (If it were the case that $\Pi_{a,\gamma}^b$ is weak submajorization preserving w.r.t. $P$, we would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the roles of $a$ and $b$.) For $u = 0$, we have $f_0^a = \ell \prec_w \ell = f_0^b$. Assume $f_u^a \prec_w f_u^b$. Then we want to prove the statement for $u + 1$. First, notice that we have

$$(f_u^a \Pi_{a,\gamma}^a + c_{a,\gamma}) \prec_w (f_u^b \Pi_{a,\gamma}^a + c_{a,\gamma}) \prec_w (f_u^b \Pi_{a,\gamma}^b + c_{a,\gamma}).$$

The first inequality follows from the assumption that $\Pi_{a,\gamma}^a$ is order and weak submajorization preserving, the fact that $f_u^a, f_u^b$ are similarly ordered to $\ell$ by Lemma A.5, and Assumption 2.3.1 that $c_{a,\gamma}$ is similarly ordered to $\ell$. The second inequality follows from the majorization inequality $\Pi_{a,\gamma}^a \prec \Pi_{a,\gamma}^b$, along with the fact that $\Pi_{a,\gamma}^a$ and $\Pi_{a,\gamma}^b$ are both order preserving w.r.t. $P$. For $z \in \{a, b\}$, since $\Delta(f_u^z \Pi_{a,\gamma}^z + c_{a,\gamma}) \leq \Delta \ell$ (by Lemma A.5 (II)), $\ell_{[i]} \leq (f_u^z \Pi_{a,\gamma}^z + c_{a,\gamma})_{[i]}$ we must have that $\ell_{[k]} \leq (f_u^z \Pi_{a,\gamma}^z + c_{a,\gamma})_{[k]}$ for $k > i$. Moreover, $f_u^z \Pi_{a,\gamma}^z + c_{a,\gamma}$ is similarly ordered to $\ell$. Using Lemma A.6 (I), we deduce

$$f_{u+1}^a = [\ell \wedge (f_u^a \Pi_{a,\gamma}^a + c_{a,\gamma})] \prec_w [\ell \wedge (f_u^b \Pi_{a,\gamma}^b + c_{a,\gamma})] = f_{u+1}^b$$

by taking $x = f_u^a \Pi_{a,\gamma}^a + c_{a,\gamma}$, $y = f_u^b \Pi_{a,\gamma}^b + c_{a,\gamma}$, and $z = \ell$. This concludes the proof that $f_u^a \prec_w f_u^b$ for $u = 1, 2, \ldots$.

By definition of weak submajorization, this means that

$$\sum_{i=1}^k f_{u,[i]}^b - \sum_{i=1}^k f_{u,[i]}^a \geq 0 \text{ for } k = 1, \ldots, n.$$

Letting $u \to \infty$, the above inequality leads to

$$\sum_{i=1}^k p_{[i]}^{b*} - \sum_{i=1}^k p_{[i]}^{a*} \geq 0 \text{ for } k = 1, \ldots, n, \text{ hence, } p^{a*} \prec_w p^{b*}.$$
Together with the above inequalities, Proposition 2.3.1 (II) implies that
\[
\sum_{i=1}^{k} [\ell - p^a_{\ast}]_{i} = \sum_{i=1}^{k} \ell_{i} - p^a_{\ast} \geq \sum_{i=1}^{k} \ell_{i} - p^b_{\ast} = \sum_{i=1}^{k} [\ell - p^b_{\ast}]_{i}
\]
for \( k = 1, \ldots, n \), or equivalently \( s(\Pi^a_\alpha, \ell, c, \gamma) \succ_w s(\Pi^b_\alpha, \ell, c, \gamma) \).

\[\Box\]

**Proof of Theorem 2.3.2.** Similarly to the proof for Theorem 2.3.1, proving that \( p^*(\Pi^a_{\alpha,\gamma}, \ell, c, \alpha, \gamma) \prec_w p^*(\Pi^b_{\alpha,\gamma}, \ell, c, \alpha, \gamma) \) is equivalent to showing that \( f_u(\Pi^a_{\alpha,\gamma}) \prec_w f_u(\Pi^b_{\alpha,\gamma}) \) for \( u = 1, 2, \ldots \). For brevity, we denote hereafter \( f_u(\Pi^a_{\alpha,\gamma}) \) by \( f_u^a \) and \( p^*(\Pi^a_{\alpha,\gamma}, \ell, c, \alpha, \gamma) \) by \( p^a_{\ast} \).

Next, we prove \( f_u^a \prec_w f_u^b \) by induction. Without loss of generality, we take \( \Pi^a_{\alpha,\gamma} \) to be weak supermajorization preserving w.r.t. \( \mathcal{P} \). (If \( \Pi^b_{\alpha,\gamma} \) were to be weak supermajorization preserving w.r.t. \( \mathcal{P} \), we would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the roles of \( a \) and \( b \).) For \( u = 0 \), by definition, \( f_0^a = \ell \prec_w \ell = f_0^b \). Assume \( f_u^a \prec_w f_u^b \). Then we want to prove the statement for \( u + 1 \). First, we notice that the following majorization inequalities hold:
\[
(f_u^a \Pi^a_{\alpha,\gamma} + c_{\alpha,\gamma}) \prec_w (f_u^a \Pi^a_{\alpha,\gamma} + c_{\alpha,\gamma}) \prec_w (f_u^b \Pi^b_{\alpha,\gamma} + c_{\alpha,\gamma}),
\]
where the first inequality follows from the assumption that \( \Pi^a_{\alpha,\gamma} \) is order and weak supermajorization preserving w.r.t. \( \mathcal{P} \) and the fact that \( f_u^a \) and \( f_u^b \) are similarly ordered to \( \ell \) by Lemma A.5 and \( c_{\alpha,\gamma} \) is similarly ordered to \( \ell \) in light of Assumption 2.3.1; the second inequality is due to that \( \Pi^a_{\alpha,\gamma} \prec \Pi^b_{\alpha,\gamma} \), and \( \Pi^a_{\alpha,\gamma} \) and \( \Pi^b_{\alpha,\gamma} \) are order preserving w.r.t. \( \mathcal{P} \). For \( z \in \{a, b\} \), because \( \Delta(f_u^z \Pi^z_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta \ell \) (by Lemma A.5 (III)), \( \ell_{(i)} \leq (f_u^z \Pi^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{(i)} \) must imply that \( \ell_{(k)} \leq (f_u^z \Pi^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{(k)} \) for \( k > i \). Moreover, \( f_u^z \Pi^z_{\alpha,\gamma} + c_{\alpha,\gamma} \) is similarly ordered to \( \ell \). Applying Lemma A.6 (II) with \( x = f_u^a \Pi^a_{\alpha,\gamma} + c_{\alpha,\gamma} \), \( y = f_u^b \Pi^b_{\alpha,\gamma} + c_{\alpha,\gamma} \), and \( z = \ell \), we deduce
\[
f_{u+1}^a = [\ell \wedge (f_u^a \Pi^a_{\alpha,\gamma} + c_{\alpha,\gamma})] \prec_w [\ell \wedge (f_u^b \Pi^b_{\alpha,\gamma} + c_{\alpha,\gamma})] = f_{u+1}^b.
\]
This concludes the proof that \( f_u^a \prec_w f_u^b \) for \( u = 1, 2, \ldots \).
By definition of weak supermajorization,
\[
\sum_{i=1}^{k} f_{u_i(i)}^b - \sum_{i=1}^{k} f_{u_i(i)}^a \leq 0 \quad \text{for } k = 1, \ldots, n.
\]
Letting \( u \to \infty \), the above inequality leads to
\[
\sum_{i=1}^{k} p_{(i)}^{bs} - \sum_{i=1}^{k} p_{(i)}^{as} \leq 0 \quad \text{for } k = 1, \ldots, n, \text{ hence, } p_{(i)}^{as} \prec_w p_{(i)}^{bs}.
\]
Together with the above inequalities, Proposition 2.3.1 (III) leads to
\[
\sum_{i=1}^{k} \left[ \ell - p_{(i)}^{as} \right] \pi_{i,j}^\mu + c_{(j+1)} \geq \ell_{(j+1)} - \ell_{(j)} = \ell_{(j+1)} - \ell_{(j)} - \max \left\{ \ell_{(1)} - (1 + \gamma)c - \gamma \ell_{(1)}, 0 \right\} \geq \ell_{(j+1)} - \ell_{(j)}
\]
for \( k = 1, \ldots, n \), or equivalently \( s(\Pi^a, \ell, c, \gamma) \prec_w s(\Pi^b, \ell, c, \gamma) \).

\[\square\]

Proof of Proposition 2.3.2. Because \((\Pi, \ell, c, \gamma)\) is unbalancing, it must hold that for \( j = 1, \ldots, n - 1, \)
\[
\sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^\mu + c_{(j+1)} - \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^\mu + c_{(j)} \geq \ell_{(j+1)} - \ell_{(j)} = \ell_{(j+1)} - \ell_{(j)}
\]
\[
\Rightarrow \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^\mu + c_{(j+1)} - \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^\mu + c_{(j)} \geq \ell_{(j+1)} - \ell_{(j)},
\]
where the last inequality follows from the assumption that \( \sum_{i=1}^{n} \pi_{i,j+1}^\mu - \sum_{i=1}^{n} \pi_{i,j}^\mu \) is nonnegative. Moreover, since \((\Pi, \ell, c, \gamma)\) is also balancing, we deduce
\[
\sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^\mu + c_{(j+1)} - \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^\mu + c_{(j)} = \ell_{(j+1)} - \ell_{(j)},
\]
for \( j = 1, \ldots, n - 1 \). By the assumption of the lemma, at least one node repays its liabilities in full, hence it must hold that
\[
\sum_{i=1}^{n} \ell_{(i)} \pi_{i,k}^\mu + c_{(k)} \geq \sum_{i=1}^{n} p_{(i)}^{as} \pi_{i,k}^\mu + c_{(k)} \geq \ell_{(k)} \text{ for some } k \in \{1, \ldots, n\},
\]
where we recall that $p^*_{\mu_i}$ is the clearing payment made by node $j$ if $\ell_{(i)} = \ell_j$. Together with Eq. (A.15), the above inequality implies
\[
\left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^\mu + c_{(j)} \right] - \ell_{(j)} \geq 0 \text{ for } j = 1, \ldots, n. ,
\]

But this means that $p^* = \ell = \ell \land (\ell \Pi + c)$. Hence, it must hold that $s(\Pi, \ell, c, \gamma) = 0$.\qed
APPENDIX B. CHAPTER 3 PROOF OF UNIQUENESS OF CLEARING PAYMENT VECTOR

Proof of Lemma 3.2.1. Consider a generic epoch $t$. We need to show that there exists a unique solution $p^t$ to the system of equalities in the absolute priority constraint, which may be rewritten as

$$p^t_i = \begin{cases} 
\min\{\ell^t_i, \sum_{j=1}^n \pi^t_{ji} p^t_j + c^t_i + v^t_i + o^t_i\}, & \text{if } d^t_i = 0, \\
0, & \text{if } d^t_i = 1.
\end{cases}$$

Let $i \in \{k \in \{1, \ldots, n\} | d^t_k = 0\}$. Since $o^t$ is specified, $c^t_i$, and $v^t_i$ are known, then we can define $c^t_i + v^t_i + o^t_i$ as $c^t_i$ and simplify the first equation as

$$p^t_i = \min \left\{ \ell^t_i, \sum_{j \in \{k \in \{1, \ldots, n\} | d^t_k = 0\}} \pi^t_{ji} p^t_j + c^t_i \right\}.$$ 

Since the subgraph induced by set $\{k \in \{1, \ldots, n\} | d^t_k = 0\}$ is regular, it was shown in [Eisenberg and Noe(2001)] that the above system of equations has a unique solution, $p^t_i$. \qed
APPENDIX C. CHAPTER 3 ENTROPY MAXIMIZATION METHOD

Denote the set of core and peripheral banks by $\mathcal{C}$ and $\mathcal{P}$ respectively. We next describe how we apply the entropy maximization method of [Upper and Worms(2004)] to generate the matrix of interbank liabilities. This consists of the following steps:

**Step 1.** Distribute the total liabilities to the three blocks in the blockmodel. Denote the total amount that a bank $i \in \mathcal{C}$ borrows from the banks in $\mathcal{C}$ and $\mathcal{P}$ by $\tilde{\ell}^c_i$ and $\tilde{\ell}^p_i$ respectively. Furthermore, the total amount which a bank $i \in \mathcal{P}$ borrows from $\mathcal{C}$ is denoted by $\tilde{\ell}^{cp}_i$. Denote the total amount lent by a bank $i \in \mathcal{C}$ to $\mathcal{C}$ and $\mathcal{P}$ by $\tilde{a}^c_i$ and $\tilde{a}^{cp}_i$ respectively, and the total lending amount of a bank $i \in \mathcal{P}$ to $\mathcal{C}$ by $\tilde{a}^{pc}_i$. Following the guideline that the size of interbank loans between core banks is 50 times as large as the lending amount from peripheral to core banks and core to peripheral banks, we set

$$
\tilde{\ell}^c_i = \frac{50}{51} \tilde{\ell}^c_i \quad \tilde{a}^c_i = \frac{50}{51} \tilde{a}^c_i \quad \text{for } i \in \mathcal{C},
$$

$$
\tilde{\ell}^{cp}_i = \frac{1}{51} \tilde{\ell}^{cp}_i \quad \tilde{a}^{cp}_i = \frac{1}{51} \tilde{a}^{cp}_i \quad \text{for } i \in \mathcal{C},
$$

$$
\tilde{\ell}^{pc}_i = \tilde{\ell}^{pc}_i \quad \tilde{a}^{pc}_i = \tilde{a}^{pc}_i \quad \text{for } i \in \mathcal{P}.
$$

**Step 2.** Normalize total lending and borrowing amount within each block. This yields

$$
\hat{\ell}^c_i = \frac{\tilde{\ell}^c_i}{\sum_{i \in \mathcal{C}} \tilde{\ell}^c_i} \quad \hat{a}^c_i = \frac{\tilde{a}^c_i}{\sum_{i \in \mathcal{C}} \tilde{a}^c_i},
$$

$$
\hat{\ell}^{cp}_i = \frac{\tilde{\ell}^{cp}_i}{\sum_{i \in \mathcal{C}} \tilde{\ell}^{cp}_i} \quad \hat{a}^{cp}_i = \frac{\tilde{a}^{cp}_i}{\sum_{i \in \mathcal{C}} \tilde{a}^{cp}_i},
$$

$$
\hat{\ell}^{pc}_i = \frac{\tilde{\ell}^{pc}_i}{\sum_{i \in \mathcal{P}} \tilde{\ell}^{pc}_i} \quad \hat{a}^{pc}_i = \frac{\tilde{a}^{pc}_i}{\sum_{i \in \mathcal{P}} \tilde{a}^{pc}_i}.
$$

**Step 3.** Generate the matrices of relative liabilities within each block, denoted by $\hat{\Pi}^{cc}$, $\hat{\Pi}^{cp}$ and $\hat{\Pi}^{pc}$. Assume the normalized lending and borrowing amounts of each
bank are independent. We set $\hat{\Pi}^{CC}_{ij} = \hat{\ell}^{CC}_i \hat{a}^{CC}_j$ for $i, j \in C$, which amounts to maximizing the entropy of $\hat{\Pi}^{CC}$. Similarly, we set $\hat{\Pi}^{CP}_{ij} = \hat{\ell}^{CP}_i \hat{a}^{CP}_j$ for $i \in C, j \in P$ and $\hat{\Pi}^{PC}_{ij} = \hat{\ell}^{PC}_i \hat{a}^{PC}_j$ for $i \in P, j \in C$.

**Step 4.** Adjust the matrix of relative liabilities in the complete block, i.e. the core-to-core block. Notice that the elements on the main diagonal of $\hat{\Pi}^{CC}$ can be non-zero. We use the RAS algorithm, given in [Bacharach(1965)], and derive $\hat{\Pi}^{CC*}$ so that the sum of rows and columns of $\hat{\Pi}^{CC*}$ are equal to the corresponding quantities in $\hat{\Pi}^{CC}$, the elements on the main diagonal of $\hat{\Pi}^{CC*}$ are zeros, and the cross entropy between $\hat{\Pi}^{CC}$ and $\hat{\Pi}^{CC*}$ is minimized. The RAS algorithm consists of a sequence of iterations yielding successive refinements of the matrix $\hat{\Pi}^{CC*}$. Let $Diag(\psi)$ be the diagonal matrix with vector $\psi$ on the diagonal. In the initialization step, we set $\Pi^{(0)} = \hat{\Pi}^{CC} - Diag \left( \left( \hat{\pi}^{CC}_{11}, \hat{\pi}^{CC}_{22}, \ldots, \hat{\pi}^{CC}_{nn} \right) \right)$, i.e to the matrix $\hat{\Pi}^{CC}$ whose diagonal elements are replaced by zero. For $m \in \mathbb{Z}_{\geq 0}$, set

$$\Pi^{(2m+1)} = Diag(\gamma^{(m+1)}) \Pi^{(2m)}$$

$$\Pi^{(2m+2)} = \Pi^{(2m+1)} Diag(\theta^{(m+1)})$$

where

$$\gamma^{(m+1)}_i = \frac{\hat{\ell}^{CC}_i}{\sum_{j=1}^{n} \hat{\pi}^{(2m)}_{ij}}$$

and

$$\theta^{(m+1)}_j = \frac{\hat{a}^{CC}_j}{\sum_{i=1}^{n} \hat{\pi}^{(2m+1)}_{ij}}.$$

Iterating $\Pi^{(m)}$ according to the above relation, we obtain $\hat{\Pi}^{CC*} = \lim_{m \to \infty} \Pi^{(m)}$.

**Step 5.** Combine $\hat{\Pi}^{CC*}$, $\hat{\Pi}^{CP}$, and $\hat{\Pi}^{PC}$ into $\hat{\Pi}'$ such that the sum of each row in $\hat{\Pi}'$ equals 1, and the loans between core banks to be 50 times as large as the lending amount from peripheral to core banks or from core to periphery. Concretely, let

$$F = \begin{pmatrix} \frac{50}{51} \hat{\Pi}^{CC*} & \frac{1}{51} \hat{\Pi}^{CP} \end{pmatrix}$$

and

$$G = \begin{pmatrix} \frac{1}{\sum_{j=1}^{n} f_{1,j}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sum_{j=1}^{n} f_{2,j}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sum_{j=1}^{n} f_{n,j}} \end{pmatrix},$$

where $0$ denotes a zero matrix, we derive $\hat{\Pi}' = GF$. Then, $\hat{L}' = Diag(\hat{\ell}') \hat{\Pi}'$. 
VITA

Peng-Chu received his Master’s degree in Transportation Engineering from University of California, Berkeley in 2003 and his Bachelor’s degree in Civil Engineering from National Taiwan University in 2001. Before studying in Purdue University, he was a senior flight network engineer at FedEx Express and senior industrial engineer at AU Optronics Corp.

His research interests are in performance analysis of complex network infrastructures. His primary objective is to develop a new class of performance metrics and novel tools for simultaneous comparison of networks along multiple measures of performance. The main applications of his research are in systemic risk and contagion in financial networks. His research contributes to the understanding of the financial contagion mechanisms and the impact of network topology on systemic losses, and to assess the impact of prescriptive policies for systemic risk.