1992

Parameterization in Finite Precision

Chanderjit L. Bajaj

Andrew V. Royappa

Report Number:
92-005

Bajaj, Chanderjit L. and Royappa, Andrew V., "Parameterization in Finite Precision" (1992). *Department of Computer Science Technical Reports*. Paper 930.
<https://docs.lib.purdue.edu/cstech/930>

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries.
Please contact epubs@purdue.edu for additional information.

PARAMETERIZATION IN FINITIE PRECISION

**Chanderjit Bajaj
Andrew V. Royappa**

**CSD-TR 92-005
January 1992**

Parameterization In Finite Precision

Chanderjit L. Bajaj Andrew V. Royappa

Department of Computer Science,
Purdue University,
West Lafayette, IN 47907

October 31, 1991

Abstract

In computer graphics and geometric modeling, it is often useful to compute a rational parameterization of an implicit curve or surface. Current algorithms for this problem assume exact arithmetic; their behavior is analyzed when this assumption is removed. By backward error analysis, a simple and precise characterization of the error is derived. In the case of conics and quadrics, this is immediately utilized to compute geometrically meaningful error bounds. After analyzing the error in rational cubic parameterization, we show a simple algorithm for the exact parameterization of rational cubics.

1 Introduction

Techniques from algebraic geometry have recently been introduced in the geometric modeling of curved objects. These techniques usually involve computations with algebraic numbers and so assume exact computations. They sometimes remain viable when numerical approximations are made. We analyze some recently given algorithms in the context of such approximations.

Algebraic curves and surfaces are the most common representations for curved objects in geometric modeling. Algebraics satisfy *polynomial* equations: a curve satisfies $f(x, y) = 0$, and a surface satisfies $g(x, y, z) = 0$. An algebraic represented by its defining polynomial is said to be in *implicit* form. The curve described explicitly by rational functions of a parameter t :

$$x = \frac{f_1(t)}{f_3(t)}, y = \frac{f_2(t)}{f_3(t)},$$

where f_1, f_2, f_3 are polynomials in t , is also algebraic. In this case it is said to be in *rational parametric* form. Likewise, an algebraic surface is described explicitly by rational functions in two parameters s and t .

Each form has certain benefits and drawbacks. For instance, the parametric form is useful for rapidly generating points on the curve or surface for display; the implicit form is useful for point classification (deciding if a point is inside, outside, or on a surface), a fundamental operation in geometric modeling. There are many other tradeoffs. It is therefore very useful if both forms are known, for a given curve or surface. Certain operations such as intersection also become easier when both forms are known.

Mathematical techniques from algebraic geometry have recently been applied to the problem of converting between the two forms. While *implicitizing* a parametric curve or surface is always possible, the converse (*rational parameterization*) is not always possible. That is, all algebraics are not rational. However, important classes of curves and surfaces are rational and algorithms for their rational parameterization based on algebraic geometry have been given in [1],[2],[3], [4],[16], and some will be analyzed here. There is

also a numerical method due to Jacobi which works by iteratively converting a conic or quadric to standard form (see [11]).

The algorithms for rational parameterization involve computations with algebraic numbers. Such numbers are roots of polynomials (e.g. $2/3, \sqrt{2}, i$ etc.) and may have infinite, non-repeating expansions. While methods do exist for their exact manipulation ([15]), these techniques are costly and complex.

We will show that parameterization is viable in the presence of numerical (i.e., truncated, finite-precision) approximations to algebraic numbers.

This paper is organized as follows. In section two, an algorithm for the rational parameterization of conic curves ([1]) is briefly reviewed. The algorithm is then analyzed in section 3 for errors which arise due to numerical approximations. Using backward error analysis, we show that the output corresponds to an *exact* computation on a *perturbed* input. This information is in turn used to derive a geometrically meaningful error bound. Next in section 4, it is shown how this technique and the associated error analysis generalizes to quadric and higher-dimensional surfaces of degree two. In sections 5 and 6, an algorithm for parameterizing rational cubic curves ([2]) is similarly analyzed for errors due to numerical approximations. Finally, in section 7, a new, simple and exact algorithm for the parameterization of rational cubics is given, which will always produce the exact answer using only rational number arithmetic.

2 Conic Parameterization

We first show an algorithm ([1]) for conic parameterization. The algorithm is given for conics in homogeneous form; this allows the use of both projective and affine transformations. The algorithm is then analyzed for the error in its output when approximations are used for algebraic numbers.

Given the equation of a conic plane curve, parameter functions for the curve are derived. The parameter functions are given as closed form formulas in the parameter t , the coefficients of the curve, and the coordinates of a point on the curve.

Input. An irreducible conic curve given by the quadratic equation

$$f(x, y) = a_{20}y^2 + a_{11}xy + a_{02}x^2 + a_{10}y + a_{01}x + a_{00} = 0$$

Output. Rational functions $(x(t), y(t))$ of degree at most two, such that $f(x(t), y(t)) = 0$.

Algorithm.

1. Homogenize the conic. This yields the homogeneous equation

$$F(X, Y, W) = a_{20}Y^2 + a_{11}XY + a_{02}X^2 + a_{10}YW + a_{01}XW + a_{00}W^2 = 0$$

If the X^2 , Y^2 or W^2 term is missing from the conic's equation, then it will be linear in the corresponding variable, and can be immediately parameterized. Compute quadratic polynomials $X(t)$, $Y(t)$ and $W(t)$ such that $F(X(t), Y(t), W(t)) = 0$, and go to step 4.

2. If all squared terms are present, apply a linear transformation that cancels one of these terms.
3. Parameterize the transformed conic, and apply the inverse transformation to the parameterization; this yields three quadratic polynomials $X(t)$, $Y(t)$ and $W(t)$ such that $F(X(t), Y(t), W(t)) = 0$.
4. The parameterization for the affine conic is then given by

$$\begin{aligned} x(t) &= \frac{X(t)}{W(t)} \\ y(t) &= \frac{Y(t)}{W(t)} \end{aligned} \tag{1}$$

Transformations. If all three squared terms are present, then any one of the following three transformations may be used in step 2 of the conic parameterization algorithm.

- To cancel the X^2 term, use the transformation

$$\begin{aligned} X &= bX_1 \\ Y &= cX_1 + Y_1 \\ W &= dX_1 + W_1 \end{aligned} \quad (2)$$

where (b, c, d) are the homogeneous coordinates of some point on the curve. For the transformation to be well-defined, b must be non-zero. Then, if $d \neq 0$, the transformation is affine; otherwise it is projective. Since proportional projective coordinates represent the same point, we make the restriction $d = 0$ or $d = 1$. If $d = 0$, then we should also make a restriction $b = 1$ or $c = 1$; since $b \neq 0$ is required for the transformation to be well-defined, we will restrict $b = 1$ in this case.

Transforming F yields a new conic curve with equation $F_1(X_1, Y_1, W_1) = 0$, where

$$\begin{aligned} F_1(X_1, Y_1, W_1) &= F(bX_1, cX_1 + Y_1, dX_1 + W_1) \\ &= F(b, c, d)X_1^2 + F_2(X_1, Y_1, W_1) \end{aligned}$$

and F_2 is a conic in X_1, Y_1 and W_1 with no X_1^2 term:

$$F_2(X_1, Y_1, W_1) = (a_{10}d + 2a_{20}c + a_{11}b)X_1Y_1 + (2a_{00}d + a_{10}c + a_{01}b)X_1W_1 + a_{20}Y_1^2 + a_{10}Y_1W_1 + a_{00}W_1^2 \quad (3)$$

Therefore, F_2 is immediately parameterized by the following formulas:

$$\begin{aligned} X_1(t) &= a_{20}t^2 + a_{10}t + a_{00} \\ Y_1(t) &= -(a_{10}d + a_{11}b + 2a_{20}c)t^2 - (2a_{00}d + a_{01}b + a_{10}c)t \\ W_1(t) &= -(a_{10}d + a_{11}b + 2a_{20}c)t - (2a_{00}d + a_{01}b + a_{10}c) \end{aligned} \quad (4)$$

This symbolic parameterization for F_2 is independent of the specific values for b, c and d .

Now, recall that (b, c, d) is a point on the curve $F(X, Y, W) = 0$. Then $F(b, c, d) = 0$, and $F_1(X_1, Y_1, W_1) = F_2(X_1, Y_1, W_1)$. Hence the parameterization (4) is also one for $F_1(X_1, Y_1, W_1) = 0$. Applying the inverse linear transformation to this parameterization immediately yields one for the original conic:

$$\begin{aligned} X(t) &= b(a_{20}t^2 + a_{10}t + a_{00}) \\ Y(t) &= -(a_{10}d + a_{11}b + a_{20}c)t^2 - (2a_{00}d + a_{01}b)t + a_{00}c \\ W(t) &= a_{20}dt^2 - (a_{11}b + 2a_{20}c)t - (a_{00}d + a_{01}b + a_{10}c) \end{aligned} \quad (5)$$

- To cancel the Y^2 term, use the transformation

$$\begin{aligned} X &= X_1 + bY_1 \\ Y &= cY_1 \\ W &= dY_1 + W_1 \end{aligned}$$

Again, let (b, c, d) be the homogeneous coordinates of a point on the curve. For the transformation to be well-defined, c must be non-zero. As before, we make the restriction that $d = 0$ or $d = 1$. If $d = 0$, then then we further restrict $c = 1$.

The computations are symmetric to the first case, and yield a similar formula for the parameterization.

- To cancel the W^2 term, use the transformation below.

$$\begin{aligned} X &= X_1 + bW_1 \\ Y &= Y_1 + cW_1 \\ W &= dW_1 \end{aligned}$$

Let (b, c, d) be the homogeneous coordinates of a point on the curve. In this case $d \neq 0$ is required for the transformation to be well-defined; hence, we restrict $d = 1$. Therefore this transformation is always affine. It translates the (affine) point (b, c) to the (affine) origin, which is equivalent to cancelling the constant term in the affine conic equation. It corresponds to intersecting the conic with a pencil of lines through the affine point (b, c) . Then, as before, a formula is derived for the parameterization.

3 Error in Conic Parameterization

The only computation in the algorithm given above is to derive the coordinates of a point on the input conic curve. Once these coordinates are found, the parameterization is given as a closed-form formula in terms of those numbers and the coefficients of the input curve.

The coordinates of the point satisfy a polynomial equation (the implicit equation of the curve), and hence are algebraic numbers.

The algorithm takes as input a curve given by the affine equation $f(x, y) = 0$, and produces as output two rational functions $(x(t), y(t))$ that satisfy f . The functions $x(t)$ and $y(t)$ are specified in terms of algebraic numbers b and c (the value d is always either 0 or 1 exactly). When approximations $\tilde{b}$ and $\tilde{c}$ are used instead of b and c , the output of the algorithm will be approximate rational parametric functions $\tilde{x}(t)$ and $\tilde{y}(t)$.

3.1 Backward Error Analysis

The value $f(\tilde{x}(t), \tilde{y}(t))$ depends on which of the three transformations was used. Let $f(x, y) = 0$ be the equation of a conic curve, as before, and let $F(X, Y, W)$ be the homogeneous form of f . In the analysis the following fact will be used.

$$f\left(\frac{X}{W}, \frac{Y}{W}\right) = \frac{F(X, Y, W)}{W^2}$$

We now enumerate the three cases corresponding to the three transformations. Note that scaling the input polynomial $f(x, y)$ by a constant doesn't affect the output parametric functions $\tilde{x}(t)$ and $\tilde{y}(t)$, in any of the three cases. To start the analysis, the value of $f(\tilde{x}(t), \tilde{y}(t))$ is computed by back-substitution. Note that this value must vanish when exact arithmetic is used, since every point on the output (parametric) curve must be on the input (implicit) curve. However, in the presence of numerical approximations, it will be non-zero, and can be computed symbolically.

- In the first transformation, the X^2 term was cancelled. Suppose the algorithm computed $(\tilde{b}, \tilde{c}, d)$ as an approximation to a point on the curve, and output approximate rational functions $(\tilde{x}(t), \tilde{y}(t))$. We first

compute $f(\tilde{x}(t), \tilde{y}(t))$:

$$\begin{aligned}
f(\tilde{x}(t), \tilde{y}(t)) &= f\left(\frac{\tilde{X}(t)}{\tilde{W}(t)}, \frac{\tilde{Y}(t)}{\tilde{W}(t)}\right) && \text{by (1)} \\
&= \frac{F(\tilde{X}(t), \tilde{Y}(t), \tilde{W}(t))}{\tilde{W}^2(t)} && \text{by (3.1)} \\
&= \frac{F(b\tilde{X}_1(t), c\tilde{X}_1(t) + \tilde{Y}_1(t), d\tilde{X}_1(t) + \tilde{W}_1(t))}{\tilde{W}^2(t)} \\
&= \frac{F_1(\tilde{X}_1(t), \tilde{Y}_1(t), \tilde{W}_1(t))}{\tilde{W}^2(t)} \\
&= \frac{F(\tilde{b}, \tilde{c}, d)\tilde{X}_1^2(t) + F_2(\tilde{X}_1(t), \tilde{Y}_1(t), \tilde{W}_1(t))}{\tilde{W}^2(t)} \\
&= \frac{F(\tilde{b}, \tilde{c}, d)\tilde{X}_1^2(t)}{\tilde{W}^2(t)} && \text{by (3) and (4)} \\
&= \frac{F(\tilde{b}, \tilde{c}, d)\tilde{X}^2(t)}{\tilde{b}^2\tilde{W}^2(t)} \\
&= \frac{F(\tilde{b}, \tilde{c}, d)\tilde{x}^2(t)}{\tilde{b}^2} && \text{by (1)}
\end{aligned}$$

When $d = 1$, $F(\tilde{b}, \tilde{c}, d) = f(\tilde{b}, \tilde{c})$. When $d = 0$, $\tilde{b} = 1$, and $F(\tilde{b}, \tilde{c}, d) = a_{20}\tilde{c}^2 + a_{11}\tilde{c} + a_{02}$. Thus, when the algorithm uses the first transformation, $f(\tilde{x}(t), \tilde{y}(t))$ equals

$$\begin{aligned}
&\frac{f(\tilde{b}, \tilde{c})}{\tilde{b}^2}\tilde{x}^2(t) && \text{if } d = 1 \\
&(a_{20}\tilde{c}^2 + a_{11}\tilde{c} + a_{02})\tilde{x}^2(t) && \text{if } d = 0
\end{aligned} \tag{6}$$

When the transformation that cancels the Y^2 term is used, the computation is symmetric with the above one, and $f(\tilde{x}(t), \tilde{y}(t))$ is given by $\frac{f(\tilde{b}, \tilde{c})}{\tilde{c}^2}\tilde{y}^2(t)$ or $(a_{20} + a_{11}\tilde{b} + a_{02}\tilde{b}^2)\tilde{y}^2(t)$, if d equals 0 or 1, respectively.

Finally, when the transformation that cancels the W^2 term is used, the error computation is again similar; since $d = 1$ is always the case, there is only one result, $f(\tilde{x}(t), \tilde{y}(t)) = f(\tilde{b}, \tilde{c})$.

With the above above in hand, we can now describe exactly what happens to the original input. The algorithm starts with the implicit equation $f(x, y) = 0$ of a conic curve C , and produces as output the parametric equations $(\tilde{x}(t), \tilde{y}(t))$ of another conic curve $\tilde{C}$. Then $\tilde{C}$ will correspond also to some algebraic curve. Its implicit equation is found below.

Consider each transformation:

- Suppose transformation 1 was used. Let $\delta = \frac{f(\tilde{b}, \tilde{c})}{\tilde{b}^2}$ or $\delta = a_{20}\tilde{c}^2 + a_{11}\tilde{c} + a_{02}$, depending on whether the transformation was affine or projective, for some numbers $\tilde{b}, \tilde{c}$. Then by (6), $f(\tilde{x}(t), \tilde{y}(t)) = \delta\tilde{x}(t)^2$, and for infinitely many values t_0 , $f(\tilde{x}(t_0), \tilde{y}(t_0)) - \delta\tilde{x}(t_0)^2 = 0$. Thus $\tilde{C}$ is identical to the conic curve defined by the equation $f(x, y) - \delta x^2 = 0$, that is,

$$a_{20}y^2 + a_{11}xy + (a_{02} - \delta)x^2 + a_{10}y + a_{01}x + a_{00} = 0$$

since both are conics sharing infinitely many points.

- For transformation 2, $\delta = \frac{f(\tilde{b}, \tilde{c})}{\tilde{c}^2}$ or $\delta = a_{20} + a_{11}\tilde{b} + a_{02}\tilde{b}^2$, depending on whether the transformation was affine or projective, and we find that $\tilde{C}$ is exactly the curve

$$(a_{20} - \delta)y^2 + a_{11}xy + a_{02}x^2 + a_{10}y + a_{01}x + a_{00} = 0$$

- When transformation 3 is used, let $\delta = f(\bar{b}, \bar{c})$, and $\bar{C}$ is identical to the conic curve

$$a_{20}y^2 + a_{11}xy + a_{02}x^2 + a_{10}y + a_{01}x + (a_{00} - \delta) = 0$$

Thus the algorithm computes *exactly* the parameterization of a *perturbed* input curve. The input curve is perturbed in precisely one of the coefficients a_{20} , a_{02} , or a_{00} , depending on the transformation used. In each case a value for the perturbation was given; this value vanishes if approximations are not used.

Thus the error of the parameterization algorithm has been characterized as being equivalent to perturbing the input; all other computations being carried out in rational arithmetic, the error due to approximating algebraic numbers was localized in some sense, as the magnitude of the perturbation.

3.2 Geometric Interpretation of Error

Armed with the algebraic error analysis, we seek error bounds of a more geometric nature, by investigating the geometric effects of perturbing a single coefficient in the equation of a conic curve. The perturbations yield an entire family of conics, and geometric error bounds have immediate applications towards the accurate evaluation and display of curves and surfaces.

In particular, the effect of perturbing the constant coefficient is investigated. We then show how this leads to geometrically meaningful error bounds on conic parameterization in the presence of approximations. In [10], general bounds are given for geometric perturbations at a point on a curve due to perturbations in the coefficients of its equation; however, in the conic case, specific bounds can be derived that serve our purpose better.

It will be shown that perturbing the constant coefficient gives rise to a conic similar to the original conic. We then bound the maximum orthogonal distance between the original and perturbed conic.

3.2.1 Properties of Conics

Much is known about conics; we cite some relevant facts from [17] and [14]. Consider the affine quadratic equation of a conic curve C , in the form

$$F(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

The *discriminant* of C is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

The following facts about conics are known:

1. C degenerates to a pair of lines when $\Delta = 0$
2. C is a parabola when $ab - h^2 = 0$, an ellipse when $ab - h^2 < 0$, and a hyperbola when $ab - h^2 > 0$.
3. When C is not a parabola, its center is given by $(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2})$.
4. The axes of the conic are given by the equation $h(x^2 - y^2) - (a - b)xy = 0$.
5. The conic can be translated to have its center at the origin, and in this coordinate system its equation is

$$F(x, y) = ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0$$

6. A conic that has been translated to the center can further have its axes be rotated to become the principal axes, and in this coordinate system its equation is

$$F(x, y) = (a + b + R)x^2 + (a + b - R)y^2 + \frac{2\Delta}{ab - h^2} = 0$$

where $R^2 = (a - b)^2 + 4h^2$.

3.2.2 Bounding the Orthogonal Distance

The facts above make it clear that perturbing the constant term c in the equation of a conic will produce a new conic of the same type that is concentric and coaxial with the original (see also [7]). Figure 1 shows a family of conics perturbed in the constant coefficient. Perturbing the coefficients of x^2 or y^2 , on the other hand, can change all these quantities: figure 2 shows a family of conics perturbed only in the coefficient of x^2 ; they vary in type, center, and axis. We will therefore only consider the third transformation of the conic algorithm, which is always affine. However, some bounds can also be derived for the other cases if needed.

Even when only the constant coefficient is perturbed, the conic could still degenerate into a pair of lines. A large enough perturbation could turn a hyperbola into one that is concentric and coaxial to the original, but with transverse and conjugate axes reversed, as in Figure 3. Hence, an upper bound must be imposed on the perturbation. Since the constant coefficient c appears linearly in the discriminant Δ , so will the perturbed coefficient $c + \delta$, and hence one can immediately bound $|\delta|$ to avoid this case. If this bound is very small the conic will already be close to degenerate.

For perturbations smaller than this bound, then, we wish to geometrically describe the error. Define the (orthogonal) distance from a point p on one conic to the other conic as the shortest distance along the normal vector at p to the other conic. Then the maximum orthogonal distance from a point on one conic to the other will occur at one of the extreme points of the conic along its major axis, if ellipse, or transverse axis, if hyperbola (see figures 1, 3).

Now suppose one is given two conics $C, \tilde{C}$, where the second conic is derived by perturbing the constant coefficient in the equation of the first (if C is a parabola, some slight modifications will apply to the arguments below). Then they will be concentric and coaxial, and we can consider their equations in a coordinate system where their center is at the origin and their axes are aligned with the primary axes. In this coordinate system their equations will take the form

$$F(x, y) = Ax^2 + By^2 + C_1 = 0, \tilde{F}(x, y) = Ax^2 + By^2 + C_2 = 0$$

Referring to figure 4, let $d_x, \tilde{d}_x$ be the distances along the x -axis from the origin to $C, \tilde{C}$ respectively. Likewise, let $d_y, \tilde{d}_y$ be the distances along the y -axis. (In the case of a hyperbola, only one of these distances is finite). Then $\tilde{d}_x = d_x + p_x$ and $\tilde{d}_y = d_y + p_y$. One of p_x and p_y will be the maximum orthogonal distance between the two curves. We can solve for p_x and p_y directly.

To solve for p_x , put $y = 0$ in the curve equations. Then $d_x^2 = -\frac{C_1}{A}$ and $\tilde{d}_x^2 = -\frac{C_2}{A}$. Hence $\tilde{d}_x^2 - d_x^2 = (d_x + p_x)^2 - d_x^2 = \frac{C_1 - C_2}{A}$. So $p_x^2 + 2d_x p_x = \frac{C_1 - C_2}{A}$, and since $p_x = 0$ when $C_1 - C_2 = 0$, we find that

$$p_x = -d_x + \sqrt{d_x^2 + \frac{(C_1 - C_2)}{A}} \quad \text{and also} \quad p_y = -d_y + \sqrt{d_y^2 + \frac{(C_1 - C_2)}{B}}$$

Revert to the original coordinate system, where the conics have equations $ax^2 + by^2 + 2hxy + 2gx + 2fy + c_1 = 0, ax^2 + by^2 + 2hxy + 2gx + 2fy + c_2 = 0$; then, by the coordinate transformations of the previous section,

$$C_1 - C_2 = \frac{2\Delta_1}{ab - h^2} - \frac{2\Delta_2}{ab - h^2} = \frac{2}{ab - h^2}(\Delta_1 - \Delta_2)$$

By the definition of the discriminant, $\Delta_1 - \Delta_2 = (ab - h^2)(c_1 - c_2)$ and hence $C_1 - C_2 = 2(c_1 - c_2)$. Finally, using the definitions for A and B , and R as given in the previous section, and putting $\delta = c_1 - c_2$, we find that

$$p_x = -d_x + \sqrt{d_x^2 + \left(\frac{2}{a + b + R}\right)\delta} \quad \text{and also} \quad p_y = -d_y + \sqrt{d_y^2 + \left(\frac{2}{a + b - R}\right)\delta}$$

Now suppose it is desired that $|p_x| < \epsilon$ for some $\epsilon > 0$, and we wish to bound $|\delta|$. To have $|p_x| < \epsilon$, it is necessary that $p_x < \epsilon$ and $p_x > -\epsilon$. For reasons that will soon be clear, we will require $\epsilon < d_x$. Considering each case separately,

1. $p_x < \epsilon$ implies that $\left(-d_x + \sqrt{d_x^2 + \left(\frac{2}{a+b+R}\right)\delta}\right) < \epsilon$.

2. Similarly, $p_x > -\epsilon$ implies that $\left(-d_x + \sqrt{d_x^2 + \left(\frac{2}{a+b+R}\right)\delta}\right) > -\epsilon$, which can be

After considering both possibilities for the sign of $a + b + R$, the requirements above may be satisfied by taking

$$|\delta| < \epsilon(2d_x + \epsilon) \left| \frac{a+b+R}{2} \right| \quad \text{and} \quad |\delta| < \epsilon(2d_x - \epsilon) \left| \frac{a+b+R}{2} \right|$$

for cases (1) and (2) respectively.

Finally, recalling that $\epsilon < d_x$, choices (1) and (2) can be simultaneously satisfied by choosing

$$|\delta| < \epsilon \cdot d_x \cdot \left| \frac{a+b+R}{2} \right|$$

This is the only simplification made in the calculation, and at most a factor of two of accuracy (one bit) is lost.

The error along the y axis is bounded in an identical way. The quantities d_x and d_y are independent of any scaling of the coefficients of the original conic by a constant, but the scale factor will be linearly present in the quantities $a + b + R$ and $a + b - R$. However, from way δ was defined in the backward error analysis of the previous section, it will also contain this factor so no changes are necessary. Keeping this in mind, it suffices to compute δ such that

$$|\delta| < \epsilon \cdot \frac{\min(d_x \cdot |a+b+R|, d_y \cdot |a+b-R|)}{2}$$

Lower bounds are similarly calculated.

3.2.3 Summary

At this point we have shown the following:

1. Numerical approximations during parameterization introduce an error.
2. The error is algebraically a perturbation δ in one coefficient of the input implicit equation of the conic.
3. When this perturbation is in the constant coefficient of a conic, the perturbed conic is coaxial and concentric with the original, and the maximum orthogonal distance p between these two conics can be explicitly computed in terms of δ and the coefficients of the conic.
4. Given a geometric bound ϵ on the maximum orthogonal distance $|p|$ between the original and perturbed conics, an upper bound on the parameterization error $|\delta|$ can be derived in terms of ϵ and the coefficients of the conic. So the geometric constraint ϵ can be used to decide how small δ must be chosen, that is, how closely to approximate the required algebraic numbers.

4 Quadric Surface Parameterization

The parameterization algorithm [1] applies to all higher dimensional hypersurfaces of degree two. The quadric surfaces are an important class in geometric modeling, and the error analysis done for conics generalizes directly to higher dimensions. In the case of quadrics, one can cancel any of the X^2, Y^2, Z^2, W^2 terms, at the cost of a perturbation in the corresponding coefficient.

The geometry of quadric surfaces is also well-studied and similar geometric error bounds can be derived. In the quadric case, the discriminant of a quadric is a 4×4 matrix in its coefficients, and its sign distinguishes

the various types of conics (its sign is invariant under scaling of the quadric equation by a constant, since each term in the discriminant is a product of four coefficients).

If a perturbation δ in the constant coefficient is chosen small enough to preserve the sign of the discriminant and hence avoid degeneracies (this bound is easily computed as in the conic case, since δ once again appears linearly in the quadric discriminant), one can again consider families of concentric, coaxial quadrics of the same type, and derive geometric error bounds similar to the conic case.

5 Cubic Parameterization

Given the equation of a cubic plane curve, parameter functions for the curve are derived ([2]). The parameter functions could be given as closed form formulas in the parameter t , the coefficients of the curve, and the coordinates of a point on the curve. However, because of the size of these expressions, it is instead shown how they can be derived. The derivation assumes exact arithmetic, but a subsequent analysis reveals the error in the output when this assumption is not made.

Input. A cubic plane curve given by the cubic equation

$$f(x, y) = a_{30}y^3 + a_{21}xy^2 + a_{20}y^2 + a_{12}x^2y + a_{11}xy + a_{10}y + a_{03}x^3 + a_{02}x^2 + a_{01}x + a_{00}$$

Output. Rational functions $(x(t), y(t))$ of degree at most four, such that $f(x(t), y(t)) = 0$.

Algorithm. As in the conic case, the curve is transformed by birational transformations into a form that it is readily parameterizable. Each transformation preserves the rationality of the curve. The steps are as follows. If the cubic has a zero x^3 or y^3 term, the first step can be omitted. If both are present, then the first step below cancels y^3 , and the steps that follow assume that. The x^3 term could just as well be cancelled: the computation is symmetric.

1. Apply a transformation that removes the y^3 term of f . This can be done via the linear transformation

$$\begin{aligned} x &= x_1 + qy_1 \\ y &= y_1 \end{aligned} \tag{7}$$

When applied to the cubic equation $f(x, y) = 0$, this yields a new cubic curve with equation $f_1(x_1, y_1) = 0$, in which

$$\begin{aligned} f_1(x_1, y_1) &= f(x_1 + qy_1, y_1) \\ &= L(q)y_1^3 + f_2(x_1, y_1) \end{aligned} \tag{8}$$

where $L(q) = a_{03}q^3 + a_{12}q^2 + a_{21}q + a_{30}$. Choose q to be a root of L , i.e. $L(q) = 0$. Then $f_1(x_1, y_1) = f_2(x_1, y_1)$.

2. Parameterize the cubic curve with equation $f_2(x_1, y_1) = 0$, which has no y_1^3 term. In order to do this, f_2 is transformed into a quadratic. f_2 is of the form

$$f_2(x_1, y_1) = g_1(x_1)y_1^2 + g_2(x_1)y_1 + g_3(x_1) \tag{9}$$

where g_1, g_2, g_3 have degrees equal to their subscripts. The discriminant of f_2 is simply $g_4(x_1) = g_2(x_1)^2 - 4g_1(x_1)g_3(x_1)$. The roots of the discriminant are projections of the extreme and singular points of the curve; hence, if the curve has a singular point, the discriminant must have a multiple root. By performing the following substitution

$$y_2 = 2g_1y_1 + g_2 \tag{10}$$

we have

$$\begin{aligned}
4g_1f_2 &= 4g_1^2y_1^2 + 4g_1g_2y_1 + 4g_1g_3 \\
&= (2g_1y_1 + g_2)^2 - (g_2^2 - 4g_1g_3) \\
&= y_2^2 - g_4
\end{aligned} \tag{11}$$

Note that $g_4(x_1)$ is a polynomial in x_1 of degree at most four. The curve is singular (and hence rational) if and only if $g_4(x_1)$ has a multiple root. This repeated root can be real or complex; only the real case is considered here. Now for any number r , expand the polynomial $g_4(x_1)$ in a Taylor series at r . Then

$$g_4(x_1) = g_4(r) + g_4'(r)(x_1 - r) + \frac{g_4''(r)}{2}(x_1 - r)^2 + \frac{g_4^{(3)}(r)}{3!}(x_1 - r)^3 + \frac{g_4^{(4)}(r)}{4!}(x_1 - r)^4$$

The terms of order higher than 4 are zero since g_4 is a polynomial of degree 4. Collecting coefficients of $(x_1 - r)^2$, we see that

$$g_4(x_1) = q_2(x_1)(x_1 - r)^2 + g_4'(r)(x_1 - r) + g_4(r) \tag{12}$$

where $q_2(x_1)$ is of degree two. Now apply the substitution

$$y_3 = y_2/(x_1 - r) \tag{13}$$

together with (12) into the right-hand side of (9); this leads to

$$\begin{aligned}
4g_1f_1 &= y_2^2 - g_4(x_1) \\
&= (y_3^2 - q_2(x_1))(x_1 - r)^2 + g_4'(r)(x_1 - r) + g_4(r) \\
&= f_3(x_1, y_3)
\end{aligned} \tag{14}$$

Choose r to be a multiple root of $g_4(x_1)$; then $g_4(r) = g_4'(r) = 0$, therefore $f_3(x_1, y_3) = (y_3^2 - q_2(x_1))(x_1 - r)^2$. This suggests that to parameterize $f_3(x_1, y_3)$, we can simply parameterize the curve with equation $C(x_1, y_3) = y_3^2 - q_2(x_1) = 0$. This curve is a conic since $q_2(x_1)$ is of degree two.

3. Parameterize the conic with equation $C(x_1, y_3) = 0$ using the methods of the previous section. This yields a pair of rational functions $(x_1(t), y_3(t))$ such that $C(x_1(t), y_3(t)) = 0$. Then apply transformations (13), (10), (7) in reverse to find a pair of rational functions $(x(t), y(t))$ that parameterize the input curve. These rational functions will have a common denominator, with degree of numerators and denominator not exceeding four.

6 Error in Cubic Parameterization

The cubic parameterization calls for computing a root q of the cubic polynomial $L(q)$, a multiple root r of the quartic polynomial $g_4(x_1)$, and a parameterization $(x_1(t), y_3(t))$ of the conic with equation $C(x_1, y_3) = 0$. Assuming the last conic transformation was used, a pair of algebraic numbers (b, c) need to be computed.

If all algebraic computations were exact, i.e. $L(q) = g_4(r) = C(x_1(t), y_3(t)) = 0$, then the output will also be exact. However, one is usually forced to use approximations $\bar{q}, \bar{r}$ and $(\bar{x}_1(t), \bar{y}_3(t))$, which will lead to an approximate output parameterization $(\bar{x}(t), \bar{y}(t))$. In this case one must have measure the error incurred. Once again, a backward error analysis will be performed, by back-substitution.

6.1 Backward Error Analysis

Given the approximate output parameterization $(\tilde{x}(t), \tilde{y}(t))$, we compute the algebraic distance error $f(\tilde{x}(t), \tilde{y}(t))$. The subscript (t) is dropped for convenience.

$$f(\tilde{x}, \tilde{y}) = f(x_1 + \tilde{q}y_1, y_1) \quad \text{by (7)}$$

$$\begin{aligned} &= f_1(x_1, y_1) \\ &= L(\tilde{q})y_1^3 + f_2(x_1, y_1) \quad \text{by (8)} \end{aligned}$$

Continuing,

$$f_2(\tilde{x}_1, \tilde{y}_1) = f_2(\tilde{x}_1, \frac{\tilde{y}_2 - g_2(\tilde{x}_1)}{2g_1(\tilde{x}_1)}) \quad \text{by (10)}$$

$$= g_1(\tilde{x}_1) \left(\frac{\tilde{y}_2 - g_2(\tilde{x}_1)}{2g_1(\tilde{x}_1)} \right)^2 + g_2(\tilde{x}_1) \left(\frac{\tilde{y}_2 - g_2(\tilde{x}_1)}{2g_1(\tilde{x}_1)} \right) + g_3(\tilde{x}_1) \quad \text{by (9)}$$

$$= \frac{\tilde{y}_2^2 - g_4(\tilde{x}_1)}{4g_1(\tilde{x}_1)} \quad \text{by (11)}$$

$$= \frac{(\tilde{y}_3^2 - q_2(\tilde{x}_1))(\tilde{x}_1 - \tilde{r})^2 + g'_4(\tilde{r})(\tilde{x}_1 - \tilde{r}) + g_4(\tilde{r})}{4g_1(\tilde{x}_1)} \quad \text{by (12)}$$

Now $C(\tilde{x}_1(t), \tilde{y}_3(t)) = \tilde{y}_3^2 - q_2(\tilde{x}_1)$, and since we assumed that the third conic transformation was used to parameterize C , it follows that there is a point $(\tilde{b}, \tilde{c})$ such that $C(\tilde{x}_1(t), \tilde{y}_3(t)) = C(\tilde{b}, \tilde{c})$. Therefore,

$$\begin{aligned} f(\tilde{x}, \tilde{y}) &= L(\tilde{q})\tilde{y}_1^3 + \frac{(\tilde{y}_3^2 - q_2(\tilde{x}_1))(\tilde{x}_1 - \tilde{r})^2 + g'_4(\tilde{r})(\tilde{x}_1 - \tilde{r}) + g_4(\tilde{r})}{4g_1(\tilde{x}_1)} \\ &= L(\tilde{q})\tilde{y}_1^3 + \frac{C(\tilde{b}, \tilde{c})(\tilde{x}_1 - \tilde{r})^2 + g'_4(\tilde{r})(\tilde{x}_1 - \tilde{r}) + g_4(\tilde{r})}{4g_1(\tilde{x}_1)} \\ &= L(\tilde{q})\tilde{y}^3 + \frac{C(\tilde{b}, \tilde{c})(\tilde{x} - \tilde{q}\tilde{y} - \tilde{r})^2 + g'_4(\tilde{r})(\tilde{x} - \tilde{q}\tilde{y} - \tilde{r}) + g_4(\tilde{r})}{4g_1(\tilde{x} - \tilde{q}\tilde{y})} \end{aligned}$$

Thus we see that the points of the output parameterization lie on the curve

$$f(x, y) = L(\tilde{q})y^3 + \frac{C(\tilde{b}, \tilde{c})(x - \tilde{q}y - \tilde{r})^2 + g'_4(\tilde{r})(x - \tilde{q}y - \tilde{r}) + g_4(\tilde{r})}{4g_1(x - \tilde{q}y)}$$

If the values $\tilde{q}, \tilde{r}, \tilde{b}, \tilde{c}$ are chosen to be *exact*, then $L(\tilde{q}) = g_4(\tilde{r}) = g'_4(\tilde{r}) = C(\tilde{b}, \tilde{c}) = 0$, and it is clear that the parametric output curve coincides with the implicit input curve.

However, if the values are *not* exact, the output curve differs from the input curve; the factor $g_1(x - \tilde{q}y)$ in the denominator may be linear, and hence the output curve may even be quartic. The coefficient perturbations are now present in many terms, not just one.

7 Error Free Cubic Parameterization

Every rational cubic has a singular point. It is well-known (see, e.g. [19] for details) that such a cubic can be parameterized by a pencil of lines through the singularity, which then intersect the cubic at exactly one other point. The coordinates of the latter point parameterized by the slope of the line give parameter functions for the cubic curve. The parameter functions are given as closed form formulas in the parameter t , the coefficients of the curve, and the coordinates (b, c) of the singularity, as shown below:

$$\begin{aligned} x(t) &= \frac{a_{30}bt^3 - (3a_{30}c + a_{20})t^2 - (2a_{21}c + a_{12}b + a_{11})t - (2a_{03}b + a_{12}c + a_{02})}{a_{30}t^3 + a_{21}t^2 + a_{12}t + a_{03}} \\ y(t) &= -\frac{(2a_{30}c + a_{21}b + a_{20})t^3 + (a_{21}c + 2a_{12}b + a_{11})t^2 + (3a_{03}b + a_{02})t - a_{03}c}{a_{30}t^3 + a_{21}t^2 + a_{12}t + a_{03}} \end{aligned}$$

7.1 Error in Pencil Cubic Parameterization

If an approximation to the singular point is calculated and used to compute a parameterization, as above, then it is easy to show that the output curve will correspond to a perturbation in the input curve equation of the constant and linear terms.

However, it is also known that if a cubic curve is irreducible and its coefficients are rational, the coordinates of the singular point must be rational numbers¹. Let $f(x, y) = 0$ be the equation of such a curve. Then this fact can be informally shown as follows:

1. Let A be an automorphism of the field of complex numbers, that is, $A(p + q) = A(p) + A(q)$, $A(pq) = A(p)A(q)$, and $A(1) = 1$, where p, q are any complex numbers. If $h(x)$ is a polynomial with rational coefficients and $h(b) = 0$ for some number b , then also $h(A(b)) = 0$. This applies to any polynomial with rational coefficients. Since the coordinates (b, c) of a singular point of a curve with rational coefficients satisfy a system of such equations, $(A(b), A(c))$ will also be a singular point.
2. An irreducible cubic curve has at most singular point; if it had more, it would have more than three intersections with some line and hence would not be cubic.
3. Thus if (b, c) is the singular point of a cubic curve, then for any automorphism A of the complex numbers, $(A(b), A(c)) = (b, c)$. That is, A fixes b and c .
4. Any complex number fixed by an automorphism must be rational (see, e.g., [12]).

Therefore, one can parameterize an irreducible rational cubic curve with rational coefficients without error, by finding the singular point.

An affine singular point is found as a solution to the system of equations $f(x, y) = f_x(x, y) = f_y(x, y) = 0$. Note that f_x and f_y are quadratic. Using a subresultant remainder sequence [9], the x -coordinate of the singular point can be found as a rational root [13] of $h_4(x) = \text{Resultant}(f_x(x, y), f_y(x, y), y)$, which is a quartic polynomial in x . The subresultant may then be used to compute the y -coordinate ([4]). Each (x, y) pair found this way can be tested to satisfy $f(x, y) = 0$. The tests will be error-free and there can be at most four such tests. All operations can be carried out using only rational numbers. The rational root finder takes time quadratic in the length of the coefficients h_4 , and the coefficients of h_4 can be four times as long as those of f (by elementary properties of resultants).

8 Conclusions

Some recently developed mathematical methods in geometric modeling were studied without the assumption of exact arithmetic for algebraic number manipulations. Using backward error analysis, an algebraic characterization of the error is derived. For conic and quadric surfaces, this analysis is effectively used to impose bounds on the geometric distance error, which finds application in the accurate evaluation and display of curves and surfaces, among others. The backward error analysis is also performed for rational cubic curves, and for this case an *exact* algorithm is given that requires no algebraic number calculations.

References

- [1] Abhyankar, S. S., and Bajaj, C., (1987a), Automatic Parameterization of Rational Curves and Surfaces I: Conics and Conicoids, *Computer Aided Design*, 19, 1, 11 - 14.
- [2] Abhyankar, S. S., and Bajaj, C., (1987b), Automatic Parameterization of Rational Curves and Surfaces II: Cubics and Cubicoids, *Computer Aided Design*, 19, 9, 499 - 502.
- [3] Abhyankar, S. S., and Bajaj, C., (1988), Automatic Parameterization of Rational Curves and Surfaces III: Algebraic Plane Curves, *Computer Aided Geometric Design*, 5, 309 - 321.

¹We thank Allan Adler for this information.

- [4] Abhyankar, S., and Bajaj, C., (1989), "Automatic Rational Parameterization of Curves and Surfaces IV: Algebraic Space Curves", *ACM Transactions on Graphics*, (1989), 8, 4, 325-334.
- [5] Bajaj, C., (1990), "Rational Hypersurface Display," *Computer Graphics*, 24, 2, 117-128.
- [6] Bajaj, C., and Royappa, A., (1990) "The Ganith Algebraic Geometry Toolkit", in *DISCO '90*, Capri, Italy.
- [7] Bookstein, F. L., (1979), "Fitting Conic Sections to Scattered Data," *Computer Graphics and Image Processing*, 9, 56-71.
- [8] Canny, J. (1988) "Elimination Theory", *The Complexity of Robot Motion Planning*, ACM Doctoral Dissertation Series, MIT Press, chapter 3.
- [9] Collins, G. (1967), "Subresultants and Reduced Polynomial Remainder Sequences," *JACM*, Vol. 14, No. 1, Jan. 1967, pp. 128-142.
- [10] Farouki, R.T., (1987) On the Numerical Condition of Algebraic Curves and Surfaces 1. Implicit Equations, IBM Research Report RC 13263 (#59267), 1987.
- [11] Golub, G., and van Loan, C. (1983), *Matrix Computations*, Johns Hopkins Press, pp 444-448.
- [12] Hodge, W.V.D., and Pedoe, D. (1952), *Methods of Algebraic Geometry*, v.2, p. 69, Cambridge University Press.
- [13] Loos, R., (1983) "Computing Rational Zeroes of Integral Polynomials by p-Adic Expansion", *SIAM J. Computing*, Vol. 12, No. 2, May 1983, pp 286-293.
- [14] Salmon, G., (1879), *Conic Sections*, Longmans, Green and Co., London.
- [15] Schwartz, J. and Sharir, M., (1983), On the Piano Movers Problem: II, General Techniques for Computing Topological Properties of Real Algebraic Manifolds, *Adv. Appl. Math.* Vol. 4, pp. 298-351.
- [16] Sederberg, T., and Snively, J., (1987), "Parameterization of Cubic Algebraic Surfaces", *The Mathematics of Surfaces II*, ed. R. Martin, Oxford University Press, pp 299-321.
- [17] Sommerville, D.M.Y., (1924), *Analytical Conics*, G. Bell and Sons, London.
- [18] Sommerville, D.M.Y., (1951), *Analytical Geometry of Three Dimensions*, G. Bell and Sons, Cambridge University Press.
- [19] Walker, R., (1950), *Algebraic Curves*, Springer Verlag 1978, New York

Figure 1: Perturbing constant coefficient

Figure 2: Perturbing x^2 coefficient

Figure 3: Perturbing constant coefficient

Figure 4