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Abstract

We characterize the dimension of fixed degree functional and implicit algebraic splines in three dimensional (x, y, z) space. For a a given planar triangulation $\mathcal{T}$ both functional and implicit algebraic splines interpolate specified z_i values at the vertices $v_i = (x_i, y_i)$ of $\mathcal{T}$. For a three dimensional triangulation $\mathcal{ST}$ the implicit algebraic splines interpolate the boundary vertices $v_j = (x_j, y_j, z_j)$ of $\mathcal{ST}$. The main results are :-

1. lower bounds on the dimension of degree m rational function and implicit algebraic, C^0 and C^1 interpolatory splines over $\mathcal{T}$ and implicit algebraic C^1 interpolatory splines over $\mathcal{ST}$.
2. explicit C^0 interpolatory basis for degree 3 implicit algebraic splines over $\mathcal{T}$ and C^1 interpolatory bases for degree 5 and degree 7, implicit algebraic splines over convex $\mathcal{ST}$ and arbitrary $\mathcal{ST}$, respectively.

1 Introduction

Piecewise polynomial functions or surfaces of some fixed degree m and continuously differentiable upto some order r (i.e. C^r-continuous or C^r-smooth) are known as splines or finite elements. Splines are used in applications ranging from image processing, computer aided design, to the solution of partial differential equations via finite element analysis. C^r-continuous splines are traditionally defined over a given planar triangulation $\mathcal{T}$, with a bivariate polynomial function $z = f_1(x, y)$ for each triangular face. See the next section for a summary of prior results over $\mathcal{T}$ [1, 2, 3, 4, 6, 7, 8, 9, 10, 11].

In this paper we extend these past results in two directions. First we consider algebraic splines defined over $\mathcal{T}$, with either a rational bivariate function $z = \frac{f_1(x,y)}{f_2(x,y)}$ or an implicit algebraic surface, a trivariate polynomial $f_1(x, y, z) = 0$ for each triangular face. Next we also consider implict algebraic splines defined over triangulations in three dimensional space $\mathcal{ST}$, with a implicit algebraic surface $f_1(x, y, z) = 0$ for each triangular face of the boundary

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of $\mathcal{ST}$. Note that since functions are single valued, functional splines cannot be defined for general spatial triangulations $\mathcal{ST}$. The motivation and advantages of dealing with algebraic surfaces are detailed in [5].

In this paper we distinquish between polynomial functions $z = f_1(x,y)$, rational functions $z = \frac{f_1(x,y)}{f_2(x,y)}$ and algebraic surfaces $f_1(x,y,z) = 0$ where the f_i, $i = 1,2$ are polynomials of degree m, and term the respective set of all C^r-*smooth* algebraic splines of degree m over $\mathcal{T}$ to be $SP_m^r(\mathcal{T})$, $SR_m^r(\mathcal{T})$ and $SI_m^r(\mathcal{T})$. Similarly $SI_m^r(\mathcal{ST})$ is the set of C^r-*smooth* implicit algebraic splines of degree m over $\mathcal{ST}$. All these sets form a vector space over the field of coefficients of the polynomials.

There are two basic problems dealing with splines:

1. Estimate the dimension of the spline sets $SP_m^r(\mathcal{T})$, $SR_m^r(\mathcal{T})$, $SI_m^r(\mathcal{T})$, and $SI_m^r(\mathcal{ST})$ over triangulations $\mathcal{T}$ and $\mathcal{ST}$ respectively, for various $m \geq 1$ and $r \geq 0$.

2. Construct an explicit interpolatory basis for each of the $SP_m^r(\mathcal{T})$, $SR_m^r(\mathcal{T})$, $SI_m^r(\mathcal{T})$, and $SI_m^r(\mathcal{ST})$ spline sets for various $m \geq 1$ and $r \geq 0$.

The main results of this paper are as follows. In section 4 we show that (a) the dimension of $SR_m^0(\mathcal{T})$ is $\geq 2\binom{m+2}{2}f - (2m+1)e + (2m+1)\alpha - v + 2$ and (b) the dimension of $SR_m^1(\mathcal{T})$ is $\geq 2\binom{m+2}{2}f - 2(2m+1)e + 4m\alpha + v + 2$. In section 5.1 we show that (a) the dimension of $SI_m^0(\mathcal{T})$ is $\geq \binom{m+3}{3}f - (2m+1)e + (2m+1)\alpha - v + 2$ and (b) the dimension of $SI_m^1(\mathcal{T})$ is $\geq \binom{m+3}{3}f - (6m+1)e + 3(2m+1)\alpha + v + 2$. In section 5.2 we show that (a) the dimension of $SI_m^0(\mathcal{ST})$ is $\geq \binom{m+3}{3}f - (2m+1)e - 2v - 4g + 2$ and (b) the dimension of $SI_m^1(\mathcal{ST})$ is $\geq \binom{m+3}{3}f - (6m+1)e - 2v - 4g + 2$. Finally in section 6 we construct local triangular bases for (a) $SI_5^1(\mathcal{ST})$ over closed convex triangulations $\mathcal{ST}$ and (b) $SI_3^0(\mathcal{ST})$, $SI_7^1(\mathcal{ST})$ over arbitrary closed spatial triangulations $\mathcal{ST}$.

2 Facts & Prior Results

Let v, e and f be the number of faces, vertices and edges of a finite generic triangulation $\mathcal{T}$ where the outside face is not counted. Let α be the number of vertices of the outer face. Then from Euler's formula $v - e + f = 1$. Since each face other than the outside is triangular $3f + \alpha = 2e$. Thus $e = 3v - \alpha - 3$, and $f = 2v - \alpha - 2$.

Fact 2.1 *The dimension of* $SP_1^0(\mathcal{T})$ = *dimension of* $SR_1^0(\mathcal{T})$ = *dimension of* $SI_1^0(\mathcal{T}) = v$.

Lemma 2.1 **(Billera [7])** *For a generic*[1] *triangulation* $\mathcal{T}$, *the dimension of* $SP_m^0(\mathcal{T}) = v + (m-1)e + \binom{m-1}{2} f$.

Lemma 2.2 **(Billera [6])** *For a generic triangulation* $\mathcal{T}$, *the dimension of* $SP_2^1(\mathcal{T}) = 3v - 5e + 6f + 2\alpha = \alpha + 3$.

Lemma 2.3 **(Billera [6])** *For a generic triangulation* $\mathcal{T}$, *the dimension of* $SP_3^1(\mathcal{T}) = 3v - 7e + 10f + 4\alpha = f + 2\alpha + 3 = 2v + \alpha + 1$.

Lemma 2.4 **(Morgan & Scott [11])** *For* $m \geq 5$ *and any triangulation* $\mathcal{T}$, *the dimension of* $SP_m^1(\mathcal{T}) = 3v - (2m+1)e + \binom{m+2}{2} f + 2(m-1)\alpha + \beta$ *where* β = *the number of rectangles triangulated with cross diagonals.*

Morgan and Scott [11] also construct an explicit basis for $SP_m^1(\mathcal{T})$.

Lemma 2.5 **(Alfeld & Schumaker [3])** *For* $m \geq 4r + 1$ *and any triangulation* $\mathcal{T}$, *the dimension of* $SP_m^r(\mathcal{T}) = 3v - (2m+1)e + \binom{m+2}{2} f + 2(m-1)\alpha + \beta$ *where* β = *the number of rectangles triangulated with cross diagonals.*

Alfeld, Piper & Schumaker [2] also construct explicit bases for $SP_m^r(\mathcal{T})$ for $m \geq 4r + 1$ and for SP_4^1 [1].

3 Interpolatory Curves in Space

We state here some straightforward lemmas which we shall use in subsequent sections.

Lemma 3.1 *Circles can* C^0 *interpolate the two endpoints of any edge.*

Proof: The set of all circles form a vector space of dimension 3. The C^0 interpolation of the endpoints of an edge impose 2 linear conditions on the coefficients of any circle. $\diamondsuit$

A point $\mathbf{p} = (p^x, p^y, p^z)$ in space with a prespecified 'normal' vector $\mathbf{n} = (n^x, n^y, n^z)$ defines a unique oriented plane at $\mathbf{p}$ given by $T_{\mathbf{p}}(x,y,z) = n^x(x - p^x) + n^y(y - p^y) + n^z(z - p^z) = 0$.

Lemma 3.2 *Irreducible conics can* C^1 *interpolate the two endpoints* $\mathbf{p}_0$ *and* $\mathbf{p}_1$ *of any edge with prespecified 'normal' vectors* $\mathbf{n}_0$ *and* $\mathbf{n}_1$ *respectively, iff* $T_{\mathbf{p}_0}(\mathbf{p}_1) \cdot T_{\mathbf{p}_1}(\mathbf{p}_0) > 0$.

[1] A triangulation is considered generic if its vertex coordinates are not algebraic, i.e. the roots of any polynomial equation.

Proof: ($\Rightarrow$) From the definition of C^1 interpolation.

($\Leftarrow$) If $T_{\mathbf{p}_0}(\mathbf{p}_1) \cdot T_{\mathbf{p}_1}(\mathbf{p}_0) > 0$, then consider any plane containing both $\mathbf{p}_0$ and $\mathbf{p}_1$. By a simple affine rotation we can assume this to be the (x, y) plane. Then the irreducible conic $q(x,y) = L(x,y)^2 - \kappa \cdot T_{\mathbf{p}_0}(x,y) \cdot T_{\mathbf{p}_1}(x,y) = 0$ or $-q(x,y) = 0$ will C^1 smoothly interpolate the endpoints where $L(x,y) = 0$ is the line connecting $\mathbf{p}_0$ and $\mathbf{p}_1$, and κ is a positive constant. $\diamondsuit$

Lemma 3.3 *Irreducible space cubic curves can C^1 interpolate the two endpoints of any edge with any prespecified 'normal' vectors.*

Proof: The set of all (irreducible) parametric space cubic curves $x(t), y(t), z(t)$ form a vector space of dimension 12. The C^1 interpolation of the endpoints of an edge with prespecified 'normal' vectors impose 10 linear conditions on the coefficients of such cubics. $\diamondsuit$

Theorem 3.1 (Bezout) *An algebraic space curve of degree n intersects an algebraic surface of degree m in either at most $m * n$ points, or infinitely often.*

4 Rational Function Splines

Let v, e and f be the number of faces, vertices and edges of the triangulation $\mathcal{T}$ where the outside face is not counted. Let α be the number of vertices of the outer face. Then from Euler's formula $v - e + f = 1$. Since each face other than the outside is triangular $3f + \alpha = 2e$. Thus $e = 3v - \alpha - 3$, and $f = 2v - \alpha - 2$.

Consider first the question of constructing a C^0-continuous piecewise rational function over a triangulation $\mathcal{T}$ in the $z = 0$ plane. If over each triangular face of $\mathcal{T}$, we have a single rational function $z = \frac{f_1(x,y)}{f_2(x,y)} = F(x,y)$ with f_i , $i = 1, 2$, polynomials of degree m, then the set $SR_m(\mathcal{T})$ of all F over $\mathcal{T}$ is a vector space of dimension $k = 2\binom{m+2}{2} f - f$. If we require the rational functions F defined per face of $\mathcal{T}$ to be continuous across face boundaries, then the dimension of $SR_m(\mathcal{T})$ is further restricted and a lower bound is given by Lemma 4.1. This subspace is $SR_m^0(\mathcal{T})$.

Lemma 4.1 *The dimension of $SR_m^0(\mathcal{T})$, the set of C^0-continuous piecewise degree m rational functions is* $\geq 2\binom{m+2}{2} f - (2m+1)e + (2m+1)\alpha - v + 2$

Sketch of Proof: At each vertex the patches must have the same z value. For each interior vertex the number of incident patches is equal the to the number of incident edges. This number is one less for each exterior vertex. Denote the set of interior vertices by int and the set of exterior vertices by ext. Thus C^0 continuity at vertices requires satisfication of no more than $k_v = \sum_{int}(d_i - 1) + \sum_{ext}(d_i - 2)$ linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v = 2e - v - \alpha$. Since the intersection curve between any two adjacent functional patches varies as a rational function of degree at most m, C^0 continuity across each interior edge requires at most $(2m-1)$ additional linear constraints for a total of $k_e = (2m-1)(e-\alpha)$. The total number of independent linear constraints are thus at most $(2m+1)e - 2m\alpha - v$. ◇

If we further require that at each vertex v_i the functions F of incident faces have a specified value z_i, then the degrees of freedom of $SR_m^0(\mathcal{T})$ are further restricted by v.

Lemma 4.2 *The dimension of $SR_m^0(\mathcal{T})$, the set of C^0-continuous piecewise degree m rational functions having a specified z_i value at each vertex v_i is* $\geq 2\binom{m+2}{2} f - (2m+1)e + (2m+1)\alpha - 2v + 2$.

Consider next the question of constructing a C^1-continuous piecewise rational function over the triangulation $\mathcal{T}$ in the $z=0$ plane. Since we require the rational functions $z = \frac{f_1(x,y)}{f_2(x,y)} = F(x,y)$ defined per face of $\mathcal{T}$, to be C^1 continuous across face boundaries, the dimension of $SR_m^0(\mathcal{T})$ of lemma 4.1 is further restricted and a lower bound is given by Lemma 4.3. This subspace is $SR_m^1(\mathcal{T})$.

Lemma 4.3 *The dimension of $SR_m^1(\mathcal{T})$, the set of C^1-continuous piecewise degree m rational functions is* $\geq 2\binom{m+2}{2} f - 2(2m+1)e + 4m\alpha + v + 2$.

Sketch of Proof: At each vertex the patches of $SR_m^0(\mathcal{T})$ must now also have the same derivative value. Thus C^1 continuity at vertices requires satisfication of at most $k_v = 2\sum_{int}(d_i - 1) + 2\sum_{ext}(d_i - 2)$ additional linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v = 4e - 2v - 2\alpha$. Since the derivative of any functional patch varies as a rational function of degree at most $m-1$, C^1 continuity across each interior edge requires at most $(2m-3)$ additional linear constraints for a total of $k_e = (2m-3)(e-\alpha)$. The total number of additional linear constraints are thus no larger than $(2m+1)e - (2m-1)\alpha - 2v$. ◇

If we further require that at each vertex v_i the functions F of incident faces have a specified value z_i and a specified derivative vector $\mathbf{n}_i$, then the degrees of freedom of $SR_m^1(\mathcal{T})$ are further restricted by $3v$.

Lemma 4.4 *The dimension of $SR_m^1(\mathcal{T})$, the set of C^1-continuous piecewise degree m rational functions having a specified value z_i and a specified derivative vector $\mathbf{n}_i$ at each vertex v_i, is* $\geq 2\binom{m+2}{2} f - 2(2m+1)e + 4m\alpha - 2v + 2$.

5 Implicit Algebraic Splines

5.1 Planar Triangulations $\mathcal{T}$

Let v, e and f be the number of faces, vertices and edges of a finite generic triangulation $\mathcal{T}$ where the outside face is not counted. Let α be the number of vertices of the outer face.

Then from Euler's formula $v-e+f=1$. Since each face other than the outside is triangular $3f+\alpha=2e$. Thus $e=3v-\alpha-3$, and $f=2v-\alpha-2$.

Consider first the question of constructing a C^0-continuous piecewise algebraic surface over a triangulation $\mathcal{T}$ in the $z=0$ plane. If over each triangular face of $\mathcal{T}$, we have a single algebraic surface $F(x,y,z)=0$ with F a polynomial of degree m, then the set $SI_m(\mathcal{T})$ of all F over $\mathcal{T}$ is a vector space of dimension $k=\binom{m+3}{3}f-f$. If we require the surfaces F defined per face of $\mathcal{T}$ to be continuous across face boundaries, then the dimension of $SI_m(\mathcal{T})$ is further restricted and a lower bound is given by Lemma 5.1. This subspace is $SI_m^0(\mathcal{T})$.

Lemma 5.1 *The dimension of $SI_m^0(\mathcal{T})$, the set of C^0-continuous piecewise algebraic surfaces of degree $m>1$ is* $\geq \binom{m+3}{3}f-(2m+1)e+(2m+1)\alpha-v+2$.

Proof: At each vertex the patches must have the same z value. Thus C^0 continuity at vertices requires satisfication of at most $k_v=\sum_{int}(d_i-1)+\sum_{ext}(d_i-2)$ linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v=2e-v-\alpha$. Conics are the lowest degree curves which can form a triangular patch on a surface of degree $m>1$. By Lemma 3.1 circles suffice for C^0 interpolation of the endpoints of an edge. Since a circle can intersect a degree m algebraic surface in at most $2m$ points (via Bezout), C^0 continuity across each interior edge requires $(2m-1)$ additional linear constraints for a total of $k_e=(2m-1)(e-\alpha)$. The total number of independent linear constraints are thus no larger than $(2m+1)e-2m\alpha-v$. $\Diamond$

If we further require that at each vertex v_i the surfaces F of incident faces have a specified value z_i, then the degrees of freedom of $SI_m^0(\mathcal{T})$ are further restricted by v.

Lemma 5.2 *The dimension of $SI_m^0(\mathcal{T})$, the set of C^0-continuous piecewise algebraic surfaces of degree $m>1$ having a specified z_i value at each vertex v_i is* $\geq \binom{m+3}{3}f-(2m+1)e+(2m+1)\alpha-2v+2$.

Consider next the question of constructing a C^1-continuous piecewise algebraic surface over the triangulation $\mathcal{T}$ in the $z=0$ plane. Since we require each of the algebraic surfaces $F(x,y,z)=0$ defined per face of $\mathcal{T}$, to be C^1 continuous across face boundaries, the dimension of $SI_m^0(\mathcal{T})$ of lemma 5.1 is further restricted and a lower bound is now given by Lemma 5.3. This subspace is $SI_m^1(\mathcal{T})$.

Lemma 5.3 *The dimension of $SI_m^1(\mathcal{T})$, the set of C^1-continuous piecewise algebraic surfaces of degree $m>1$ is* $\geq \binom{m+3}{3}f-(6m+1)e+3(2m+1)\alpha+v+2$.

Sketch of Proof: At each vertex the patches of $SI_m^0(T)$ must now also have the same derivative value. Thus C^1 continuity (including C^0) at vertices requires in total the satisfication of at most $k_v = 3\sum_{int}(d_i - 1) + 3\sum_{ext}(d_i - 2)$ linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v = 6e - 3v - 3\alpha$. Now the gradient of an algebraic surface of degree m is a $(3 times 1)$ vector of algebraic surfaces of degree at most $m - 1$. Further by Lemma 3.3 degree three curves suffice for C^1 interpolation of the endpoints and normals of an edge. Since a cubic curve can intersect a degree m algebraic surface in at most $3m$ points (via Bezout), C^0 continuity across each interior edge requires at most $(3m - 1)$ linear constraints. Surface patches which have C^0 continuity of these cubics already satisfy one of the components of the gradient along the edge curves. Hence C^1 continuity across each interior edge requires $(3m - 4)$ additional linear constraints for a total of $k_e = (6m - 5)(e - \alpha)$. The total number of additional linear constraints are thus no larger than $(6m + 1)e - 2(3m - 1)\alpha - 3v$. $\diamondsuit$

If we further require that at each vertex v_i the functions F of incident faces have a specified value z_i and a specified derivative vector $\mathbf{n}_i$, then the degrees of freedom of $SI_m^1(T)$ are further restricted by $3v$.

Lemma 5.4 *The dimension of $SI_m^1(T)$, the set of C^1-continuous piecewise algebraic surfaces of degree $m > 1$ having a specified value z_i and a specified derivative vector $\mathbf{n}_i$ at each vertex v_i, is of dimension* $\geq \binom{m+3}{3} f - (6m + 1)e + 3(2m + 1)\alpha - 2v + 2$.

5.2 Spatial Triangulations $\mathcal{ST}$

We consider here the case of a spatial traingulation $\mathcal{ST}$ which is closed and of genus g, i.e. a simplicial polyhedron of genus g. The case of open spatial triangulations with holes can be dealt with in a similar fashion. Let v, e and f be the number of boundary faces, vertices and edges of $\mathcal{ST}$. Then from Euler's formula $v - e + f = 2 - 2g$. Since each face is triangular and each edge has exactly two incident faces $3f = 2e$. Thus $e = 3(v + 2g - 2)$, and $f = 2(v + 2g - 2)$.

Consider first the question of constructing a C^0-continuous piecewise algebraic surface over the boundary faces of a spatial triangulation $\mathcal{ST}$ in (x, y, z) space. If over each triangular face of $\mathcal{ST}$, we have a single algebraic surface $F(x, y, z) = 0$ with F a polynomial of degree m, then the set $SI_m(\mathcal{ST})$ of all F over $\mathcal{ST}$ is a vector space of dimension $k = \binom{m+3}{3} f - f$. If we require the surfaces F defined per face of $\mathcal{ST}$ to be continuous across face boundaries, then the dimension of $SI_m(\mathcal{ST})$ is further restricted amd a lower bound is given by Lemma 5.5. This subspace is $SI_m^0(\mathcal{ST})$.

Lemma 5.5 *The dimension of $SI_m^0(\mathcal{ST})$, the set of C^0-continuous piecewise algebraic surfaces of degree $m > 1$ is* $\geq \binom{m+3}{3} f - (2m + 1)e - 2v - 4g + 2$.

Sketch of Proof: At each boundary vertex the surface patches must be zero. Thus C^0 continuity at boundary vertices requires satisfication of at most $k_v = \sum_{boundary} d_i = 2e$ linear constraints where d_i is the degree of the ith vertex. Conics are the lowest degree curves which can form a triangular patch on a surface of degree $m > 1$. By Lemma 3.1 circles suffice for C^0 interpolation of the endpoints of an edge. Since a circle can intersect a degree m algebraic surface in at most $2m$ points (via Bezout), C^0 continuity across each boundary edge requires at most $(2m-1)$ additional linear constraints for a total of $k_e = (2m-1)e$. The total number of independent linear constraints are thus no larger than $(2m+1)e$. $\diamondsuit$

Consider next the question of constructing a C^1-continuous piecewise algebraic surface over the boundary faces of a spatial triangulation $\mathcal{ST}$. Since we require each of the algebraic surfaces $F(x, y, z) = 0$ defined per face of $\mathcal{ST}$, to be C^1 continuous across face boundaries, the the dimension of $SI_m^0(\mathcal{ST})$ of lemma 5.5 is further restricted and a lower bound is now given by Lemma 5.6. This subspace is $SI_m^1(\mathcal{ST})$.

Lemma 5.6 *The dimension of $SI_m^1(\mathcal{ST})$, the set of C^1-continuous piecewise algebraic surfaces of degree $m > 1$ is $\geq \begin{pmatrix} m+3 \\ 3 \end{pmatrix} f - (6m+1)e - 2v - 4g + 2$.*

Sketch of Proof: At each vertex the patches of $SI_m^0(\mathcal{ST})$ must now also have the same derivative value. Thus C^1 continuity (including C^0) at boundary vertices requires in total the satisfication of at most $k_v = 3\sum_{boundary} d_i = 6e$ linear constraints where d_i is the degree of the ith vertex. Now the gradient of an algebraic surface of degree m is a $(3times1)$ vector of algebraic surfaces of degree at most $m-1$. Further by Lemma 3.3 degree three curves suffice for C^1 interpolation of the endpoints and normals of an edge. Since a cubic curve can intersect a degree m algebraic surface in at most $3m$ points (via Bezout), C^0 continuity across each boundary edge requires $(3m-1)$ linear constraints. Surface patches which have C^0 continuity of these cubics already satisfy one of the components of the gradient along the edge curves. Hence C^1 continuity across each interior edge requires $(3m-4)$ additional linear constraints for a total of $k_e = (6m-5)e$. The total number of additional linear constraints are thus no larger than $(6m+1)e$. $\diamondsuit$

6 Explicit Bases for Low Degree Algebraic Splines

Prior work on explicit spline bases have dealt with polynomial functions over planar triangulations [1, 2, 10, 11]. Here we present some of the first results on explicit bases for low degree implicit algebraic splines over closed spatial triangulations.

Lemma 6.1 *There is a local triangular basis for $SI_3^0(\mathcal{ST})$ algebraic splines over arbitrary spatial triangulations $\mathcal{ST}$.*

Sketch of Proof: Replace each edge of $\mathcal{ST}$ by any circle interpolating the two endpoint vertices of that edge via Lemma 3.1. Now for each trio of circular arcs forming a curvilinear triangle, a local triangular cubic interpolant $f(x, y, z) = 0$ can be constructed as follows.

Select 5 additional points on each of the three circles forming the curvilinear triangle. Next compute a cubic algebraic surface which C^0 interpolates this collection of 18 points (including the vertices of the curvilinear triangle). Since point interpolation yields a single linear constraint on the coefficients of $f(x, y, z)$, and a cubic surface is a vector space of dimension 19, the above C^0 interpolation is always possible. Furthermore, since the cubic surface meets each circular arc at exactly 7 points, by Bezout's theorem it is guaranteed to C^0 interpolate each of the three boundary circles. ◇

Lemma 6.2 *There is a local triangular basis for $SI_5^1(ST)$ algebraic splines over convex spatial triangulations ST.*

Sketch of Proof: Select a single 'normal' vector at each of the vertices of ST. These 'normals' may be chosen by enclosing the convex triangulation by a sphere and then choosing the radial direction of each vertex from the centre of the sphere. Next, replace each edge of ST by a conic (for example an ellipse), C^1 interpolating the two endpoint vertices and prespecified normals. Note that is is always possible by Lemma 3.2 due to the compatible selection of vertex 'normals', possible because of the convexity of ST. Now for each trio of conic arcs forming a curvilinear triangle, a local triangular quintic interpolant $f(x, y, z) = 0$ can be constructed as follows. Select 9 additional points on each of the three conics forming the curvilinear triangle. For 7 of these points, on each of the three conics, select 'normal' vectors. These normals are chosen for a conic on a quadric (degree 2) algebraic surface, i.e. in a quadratic variation. Finally compute a quintic (degree 5) algebraic surface which interpolates this collection of 30 points (including the vertices of the curvilinear triangle) and the 21 specified 'normal' vectors. Since both point and 'normal' interpolation yields a single linear constraint each on the coefficients of $f(x, y, z)$, and a quintic surface is a vector space of dimension 55, the above C^1 interpolation is always possible. Furthermore, since the quintic surface meets each conic at exactly 11 points and its gradient matches the 'normals' along the conic at 9 points, by Bezout's theorem the quintic is guaranteed to C^1 interpolate each of the three boundary conics. ◇

Lemma 6.3 *There is a local triangular basis for $SI_7^1(ST)$ algebraic splines over arbitrary spatial triangulations ST.*

Sketch of Proof: Select a single 'normal' vector at each of the vertices of ST. These 'normals' may be chosen by a weighted average of the face normals of the incident faces at that vertex, so that the normals are all oriented positivley, relative to ST. Next, replace each edge of ST by a cubic C^1 interpolating the two endpoint vertices and prespecified normals. This is always possible by Lemma 3.3. Now for each trio of cubic arcs forming a curvilinear triangle, a local triangular degree seven interpolant $f(x, y, z) = 0$ can be constructed as follows. Select 20 additional points on each of the three cubics forming the curvilinear triangle. For 17 of these points, on each of the three cubics, select 'normal' vectors. These 'normals' are chosen for a cubic curve on a cubic (degree 3) algebraic surface, i.e. in a cubic variation. Finally compute a degree 7 algebraic surface which interpolates this collection of

63 points (including the vertices of the curvilinear triangle) and the 51 specified 'normal' vectors. Since both point and 'normal' interpolation yields a single linear constraint each on the coefficients of $f(x,y,z)$, and a degree 7 surface is a vector space of dimension 120, the above C^1 interpolation is always possible. Furthermore, since the degree 7 surface meets each cubic at exactly 22 points and its gradient matches the 'normals' along the cubic at 19 points, by Bezout's theorem the degree 7 surface is guaranteed to C^1 interpolate each of the three boundary cubics. $\diamond$

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