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Abstract

In this paper we present the convergence analysis of iterative schemes for solving linear systems resulting from discretizing multidimensional linear second order elliptic partial differential equations (PDEs) defined in a hyper-parallelepiped Ω and subject to Dirichlet boundary conditions on some faces of Ω and Neumann on the others, using a new class of line cubic spline collocation (LCSC) methods. These LCSC methods approximate the differential operator along lines in each dimension independently and then combine the resulting equations into one large non-symmetric linear system of equations which lacks many of the properties found in Ritz-Galerkin type finite element methods. Nevertheless, we derive analytic expressions for the spectral radius of the corresponding Jacobi iteration matrix and from this we determine the convergence ranges and compute the optimal parameters for the Extrapolated Jacobi and SOR methods. Experimental results presented confirm the theoretical convergence results and indicate that the latter hold for problems more general than Helmholtz problems.

1 Introduction

We consider the following second order linear elliptic partial differential equation (PDE)

$$Lu \equiv \sum_{i=1}^{k} \alpha_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{k} \beta_i \frac{\partial u}{\partial x_i} + \gamma u = f \text{ in } \Omega, \tag{1a}$$

subject to Dirichlet and/or Neumann boundary conditions

$$Bu = g \text{ on } \partial\Omega \equiv \text{boundary of } \Omega \tag{2a}$$

where Bu is u (or $\frac{\partial u}{\partial x_i}$), $\Omega \equiv \prod_{i=1}^{k} \otimes [a_i, b_i]$ is a rectangular domain in R^k (the space of k real variables) and $\alpha_i (< 0), \beta_i, \gamma (\geq 0), f$ and g are functions of k variables.

If Line Cubic Spline Collocation (LCSC) methods are used to solve (1a),(2a), then the differential operator is discretized along lines in each direction independently and then the line discretization stencils are combined into one large linear system. In Section 2 we briefly describe such discretization procedures. We also present and discuss the resulting coefficient linear system of collocation equations.

In [5], we were able to formulate and analyze iterative schemes for solving the LCSC linear systems in the case of Helmholtz problems, with Dirichlet boundary conditions and constant coefficients, that is

$$\begin{aligned} Lu &\equiv \textstyle\sum_{i=1}^{k} \alpha_i D_{x_i}^2 u + \gamma u = f \quad \text{in } \Omega & (1b) \\ u &= g \quad \text{on } \partial\Omega \; . & (2b) \end{aligned}$$

Unfortunately the convergence analysis presented in [5], as it stands, can not be applied in the case of the presence of Neumann boundary conditions on some of the faces of Ω. In Section 3 we present a convergence analysis of block Jacobi, Extrapolated Jacobi (EJ) and Successive Overrelaxation (SOR) schemes for the iterative solution of the collocation equations that arise from discretizing elliptic problems (1b) subject to Neumann (N) boundary conditions on one or more (**but not on all**) faces of Ω and Dirichlet (D) ones on the others. More specifically analytic expressions or sharp bounds for the spectral radius of the corresponding block Jacobi iteration matrix are derived and from these we determine the convergence ranges and compute the optimal parameters for the Extrapolated Jacobi and SOR methods.

Finally, in Section 4 we present the results of various numerical experiments designed for the verification of the theoretical behavior of the iterative LCSC solvers. The experiments show very good agreement with the theoretical predictions as regards the convergence of the iterative method used. Although the theoretical results presented here hold for the model problem (1b), (2b), our experimental results indicate that the behavior of the iterative LCSC solvers on the general problem (1a), (2a) is similar.

2 The Line Cubic Spline Collocation Method

In this section we briefly describe the LCSC discretization method and introduce some notation to be used later. We start by introducing one extra point beyond each end of the intervals $[a_i, b_i]$. Each of these enlarged intervals is then discretized uniformly with step size h_i by

$$\Delta_i \equiv \left\{ \tau_\ell^i = a_i + \ell h_i \ ; \ \ell = -1, \ldots, N_i + 1 \text{ with } h_i = \frac{b_i - a_i}{N_i} \right\}.$$

A complete discretization of Ω is obtained by taking the tensor product of these discretized lines, so the mesh $\Delta \equiv \prod_{i=1}^{k} \otimes \Delta_i$ provides an extended uniform partition of Ω. A collocation approximation u_Δ of u in the space $S_{3,\Delta}$ of cubic splines in k dimensions is defined by requiring that it satisfies the equation (1) at all the mesh points of Δ and the equation (2) on the boundary mesh points.

In the line cubic spline collocation methods we consider in this paper the collocation approximation is made on each set $\mathcal{L}^i$ of lines parallel to the x_i axis. More specifically, this collocation approximation $u_\Delta^i \in S_{3,\Delta}$ on each line in $\mathcal{L}^i$ is represented as follows

$$u_\Delta^i(\mathbf{x}) = \sum_{\ell=-1}^{N_i+1} U_\ell^i(\mathbf{x}^i) B_\ell^i(x_i), \tag{3}$$

where $B_\ell^i(x_i)$ are the B-spline basis defined on Δ_i, $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{x}^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$. Observe that (3) is the sum of one dimensional splines in the x_i variable whose coefficients $U^i(\mathbf{x}^i)$ are functions of the other $k-1$ variables. Furthermore, this approximation is *redundant* in that there are k choices for representing $u_\Delta^i(\mathbf{x})$, one for each coordinate direction.

2.1 The Second Order Line Cubic Spline Collocation (LCSC) Method

From the assumed representation (3) of u^i_Δ and the nature of the B-spline basis functions we conclude easily (see also ([9] and [5]) that

$$D^2_{x_i} u^i_\Delta|_{T^i_\ell} = [U^i_{\ell-1}(\mathbf{n}_i) - 2U^i_\ell(\mathbf{n}_i) + U^i_{\ell+1}(\mathbf{n}_i)]/h_i^2, \tag{4}$$

$$D_{x_i} u^i_\Delta|_{T^i_\ell} = [U^i_{\ell-1}(\mathbf{n}_i) - U^i_{\ell+1}(\mathbf{n}_i)]/2h_i, \tag{5}$$

and

$$u^i_\Delta|_{T^i_\ell} = [U^i_{\ell-1}(\mathbf{n}_i) + 4U^i_\ell(\mathbf{n}_i) + U^i_{\ell+1}(\mathbf{n}_i)]/6 \tag{6}$$

at the mesh (collocation) points T^i_ℓ for $\ell = 1, 2, \ldots, N_i - 1$ on each line of $\mathcal{L}^i$.

First, we observe that the collocation equations obtained from the boundary conditions and the differential equation at the end points of each line can be explicitly determined as follows

Dirichlet (D) boundary conditions on T^i_0

$$U^i_0 = \frac{h_i^2 f(\mathrm{T}^i_0)}{6\alpha_i}, \qquad U^i_{-1} = -U^i_1 + \frac{2h_i^2 f(\mathrm{T}^i_0)}{3\alpha_i}, \tag{7}$$

Neumann (N) boundary conditions on T^i_0

$$U^i_0 = U^i_1 + \frac{h_i^2 f(\mathrm{T}^i_0)}{2\alpha_i}, \qquad U^i_{-1} = U^i_1 \tag{8}$$

Dirichlet (D) boundary conditions on $\mathrm{T}^i_{N_i}$

$$U^i_{N_i} = \frac{h_i^2 f(\mathrm{T}^i_{N_i})}{6\alpha_i}, \quad U^i_{N_i+1} = -U^i_{N_i-1} + \frac{2h_i^2 f(\mathrm{T}^i_{N_i})}{3\alpha_i}, \tag{9}$$

Neumann (N) boundary conditions on $\mathrm{T}^i_{N_i}$

$$U^i_{N_i} = U^i_{N_i-1} + \frac{h_i^2 f(\mathrm{T}^i_{N_i})}{2\alpha_i}, \quad U^i_{N_i+1} = U^i_{N_i-1} \tag{10}$$

where for simplicity we have assumed that all boundary conditions are homogeneous. The rest of the unknowns are determined by solving the so called **interior collocation** equations, that is, on those lines of $\mathcal{L}^i$ not in $\partial\Omega$,

$$\left(\sum_{i=1}^{k} \alpha_i \, D^2_{x_i} \, u^i_\Delta + \gamma u^1_\Delta\right)\Bigg|_{\mathrm{T}^i_\ell} = f \,|_{\mathrm{T}^i_\ell} \quad \text{for} \quad \ell = 1, 2, \ldots, N_i - 1 \tag{11}$$

The equations (11) *away* $(2 \leq \ell \leq N_i - 2)$ from the boundary can be written as

$$\sum_{i=1}^{k} \frac{\alpha_i}{h_i^2} \left[U^i_{\ell-1}(\mathrm{n}_i) - 2U^i_\ell(\mathrm{n}_i) + U^i_{\ell+1}(\mathrm{n}_i)\right]$$
$$+ \frac{\gamma}{6}[U^1_{\ell-1}(\mathrm{n}_1) + 4U^1_\ell(\mathrm{n}_1) + U^1_{\ell+1}(\mathrm{n}_1)] = f|_{T^i_\ell} \tag{12}$$

For lines next to the boundary, the equations have similar form with appropriately modified right sides (see [5]) while the full matrix form of these equations is given in Section 3. This discretization has redundant coefficients, there are $\Pi_i(N_i + 1)$ coefficients U^i_ℓ for each i, but 2 are known from equations (10) at the boundaries so there are $\mathcal{K} = \Pi_i(N_i - 1)$ unknowns in each collocation approximation $u^i_\Delta(\mathrm{x})$. This redundancy is handled by requiring that all these approximations agree on the mesh points, that is

$$u^1_\Delta = u^2_\Delta = \ldots = u^k_\Delta \quad \text{on the mesh} \quad \Delta \tag{13}$$

It has been shown ([7], [9], [5]) that the above described method leads to a second order collocation approximation of u.

2.2 The Fourth Order Line Cubic Spline Collocation (LCSC) Method

To derive the fourth order LCSC method we use high order approximations of the derivatives $D^j_{x_i}u$, $j = 1, 2$. These approximations are defined as appropriate linear combinations of S and its derivatives at the mesh points [7]. Specifically, we approximate the second derivatives in the PDE operator by the difference scheme

$$D^2_{x_i} u^i_\Delta \Big|_{\mathbf{T}^i_\ell} = \tag{14}$$
$$\left(U^i_{\ell-2}(\mathbf{n}_i) + 8U^i_{\ell-1}(\mathbf{n}_i) - 18U^i_\ell(\mathbf{n}_i) + 8U^i_{\ell+1}(\mathbf{n}_i) + U^i_{\ell+1}(\mathbf{n}_i)\right)/(12h_i^2).$$

The collocation equations corresponding to the mesh points on a line $\mathcal{L}^i$ *away* ($2 \le \ell \le N_i - 2$) from the boundary are written at the point T^i_ℓ as

$$\sum_{i=1}^{k} \frac{\alpha_i}{12h_i^2}\left[U^i_{\ell-2}(\mathbf{n}_i) + 8U^i_{\ell-1}(\mathbf{n}_i) - 18U^i_\ell(\mathbf{n}_i) + 8U^i_{\ell+1}(\mathbf{n}_i) + U^i_{\ell+2}(\mathbf{n}_i)\right]$$
$$+ \frac{\gamma}{6}[U^1_{\ell-1}(\mathbf{n}_i) + 4U^1_\ell(\mathbf{n}_i) + U^1_{\ell+1}(\mathbf{n}_1)] = f|_{\mathbf{T}^i_\ell} \tag{15}$$

For lines *close* to the boundary we have similar forms with appropriately modified right sides (see [5]). This discretization has redundant coefficients just as the second order LCSC method. The same conditions (13) are used to reduce the number of coefficients to the number of equations. The nature of this discretization is changed from the previous in that the stencils have 5, rather than 3, points along each coordinate direction.

3 Iterative Solution of the Interior Line Cubic Spline Collocation Equations

The LCSC equations can be written in the form

$$\sum_{i=1}^{k} A_i \mathbf{U}^i(\mathbf{n}) = \mathbf{F}^1, \tag{16}$$

where the coefficient matrices are defined by[1]

$$A_i \equiv \left(\prod_{j=1}^{k-i} \otimes I\right) \otimes \mathcal{E}_i \otimes \left(\prod_{j=k-i+2}^{k} \otimes I\right), \quad i = 1, \ldots, k, \tag{17}$$

with

$$\mathcal{E}_1 = \frac{\alpha_1}{h_1^2}\mathcal{H}_1 + \frac{\gamma}{6}\mathcal{T}_4^1 \text{ and } \mathcal{E}_i = \frac{\alpha_i}{h_i^2}\mathcal{H}_i, \quad i = 2, \ldots, k. \tag{18}$$

[1] For the tensor product properties which are to be used in the sequel the reader is referred to [6] and [11].

The matrix $\mathcal{H}_i$ depends both on the discretization scheme applied and the nature (Dirichlet (**D**) or Neumann (**N**)) of the boundary conditions on the *left* and *right* end-points of the line direction with which the matrix $\mathcal{H}_i, i = 1, \ldots, k$, is associated, while I denotes a unit matrix of appropriate order. Specifically, $\mathcal{H}_i = T^i_{\alpha,\beta}$ for the second order discretization and $\mathcal{H}_i = \frac{1}{12} T^i_{10} T^i_{\alpha,\beta}$ for the fourth order one. The matrices $T^i_{\alpha,\beta}, T^i_4$ and T^i_{10} are given by

$$T^i_{\alpha,\beta} := \begin{pmatrix} \alpha & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & \beta \end{pmatrix}_{(N_i-1)\times((N_i-1)} \quad , \tag{19}$$

$$T^i_4 = 6I + T^i_{\alpha,\beta} \text{ and } T^i_{10} = 12I + T^i_{-2,-2},$$

where

$$(\alpha, \beta) = \begin{cases} (-2,-2) & \text{in case of (D) conditions on both ends} \\ (-1,-2) & \text{in case of (N) conditions on the } \textit{left} \text{ and} \\ & \text{(D) ones on the } \textit{right} \text{ end} \\ (-2,-1) & \text{in case of (D) conditions on the } \textit{left} \text{ and} \\ & \text{(N) ones on the } \textit{right} \text{ end} \\ (-1,-1) & \text{in case of (N) conditions on both ends} \end{cases} . \tag{20}$$

NOTE: Since in the analysis which will follow, the existence of A_1^{-1} is a necessary requirement, we assume, without loss of generality, that on at least one of the faces of Ω perpendicular to the x_1-direction (**D**) boundary conditions have been imposed so that the invertibility of A_1 will be guaranteed even if $\gamma = 0$.

In both cases the system (16) is under-determined with $\mathcal{K} = \prod_{j=1}^{k}(N_j - 1)$ equations and $k\mathcal{K}$ unknowns. For its completion we consider the $(k-1)\mathcal{K}$ interpolation equations (13). If we order them according to the ordering of unknowns $\mathbf{U}^1(\mathbf{n})$ we obtain the equations

$$-B_i \mathbf{U}^1(\mathbf{n}) + D_i \mathbf{U}^i(\mathbf{n}) = \mathbf{F}^i \ , \ i = 2, \ldots, k \tag{21}$$

where

$$B_i = \left(\prod_{j=1}^{k-1} \otimes I \right) \otimes T_4^1 \ , \ D_i = \left(\prod_{j=1}^{k-i} \otimes I \right) \otimes T_4^i \otimes \left(\prod_{j=k-i+2}^{k} \otimes I \right) , \ i = 2, \ldots, k \tag{22}$$

where T_4^i is given as in (19) above. Most of the components of $\mathbf{F}^i$ are zero, except those that include the effect of the elimination of the boundary conditions. The collection of all equations (16) and (21) produces the order $k\mathcal{K}$ system of linear equations

$$\begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_k \\ -B_2 & D_2 & & & \\ -B_3 & & D_3 & & \\ \vdots & & & \ddots & \\ -B_k & & & & D_k \end{pmatrix} \begin{pmatrix} \mathbf{U}^1 \\ \mathbf{U}^2 \\ \mathbf{U}^3 \\ \vdots \\ \mathbf{U}^k \end{pmatrix} = \begin{pmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \vdots \\ \mathbf{F}^k \end{pmatrix} \tag{23}$$

for the unknown $\mathbf{U}^i$.

As it becomes obvious, due to the Neumann conditions on some of the faces of Ω the various matrices, which have been obtained after the discretization and which appear in (18) and (22), are not linear or quadratic functions of the matrix $T_{-2,-2}$ as this was the case where *only* Dirichlet boundary conditions were imposed on $\partial\Omega$. Besides, in the case of the fourth order discretization scheme the product $T_{10}^iT_{\alpha,\beta}^i$ is not a symmetric matrix in the case where $(\alpha,\beta) \neq (-2,-2)$. So, the analysis presented in Section 3 of [5] does not always apply. In order to give a unified analysis we should base it on a different background material which is developed in the sequel in the form of five Lemmas.

Lemma 3.1 *If $A \in \mathcal{C}^{m,n}$ and $B \in \mathcal{C}^{n,m}$ then AB and BA have the same, different from zero, eigenvalues. If $m = n$ then AB and BA have precisely the same eigenvalues.*

Proof. See [13] (Thm 1.12, p16). □

Lemma 3.2 *If A is Hermitian positive (nonnegative) definite there exists a unique Hermitian positive (nonnegative) definite matrix, denoted by $A^{1/2}$, such that $(A^{1/2})^2 = A$.*

Proof. See [13] (Thm 2.7, p22). □

Lemma 3.3 *If A and B are Hermitian positive definite matrices of the same order then i) AB (and BA) possesses a complete set of linearly independent eigenvectors and ii) Let $X \in \mathcal{C}^{n,n}$ and $\sigma(X) \subset \mathcal{R}$, where $\sigma(.)$*

denotes the spectrum of a matrix. Let also λ_X, $\lambda_{X,m}$ and $\lambda_{X,M}$ denote any, the smallest and the largest eigenvalue of X respectively. Then for the eigenvalues of the matrix AB in (i) there hold :

$$0 < \lambda_{A,m}\lambda_{B,m} \le \lambda_{AB,m} \le \lambda_{AB,M} \le \lambda_{A,M}\lambda_{B,M}. \tag{24}$$

Proof. Let $A^{1/2}$ and $B^{1/2}$ be the unique square roots of A and B defined in Lemma 3.2. The matrices AB and $A^{1/2}\ B\ A^{1/2} = (B^{1/2}A^{1/2})^H$ $(B^{1/2}A^{1/2})$ are similar, with $(\cdot)^H$ denoting the complex conjugate transpose of a given matrix. Obviously, the matrix $(B^{1/2}A^{1/2})^H\ (B^{1/2}A^{1/2})$ is Hermitian and positive definite. From the similarity property and the fact that $A^{1/2}BA^{1/2}$ is Hermitian the validity of the assertion in (i) follows. Now, it is easily seen that $\lambda_{AB} \le \lambda_{AB,M} = \rho(AB) \le \| AB \|_2 \le \| A \|_2 \| B \|_2 = \rho(A)\rho(B) = \lambda_{A,M}\lambda_{B,M}$ hold, proving the two rightmost inequalities in (24). On the other hand, since $\det(AB) = \det(A)\det(B) > 0$, $(AB)^{-1}$ exists and $\frac{1}{\lambda_{AB}} \in \sigma((AB)^{-1})$ implies $0 < \frac{1}{\lambda_{AB}} \le \frac{1}{\lambda_{AB,m}} = \rho((AB)^{-1}) \le \| B^{-1}A^{-1} \| \le \| B^{-1} \|_2 \| A^{-1} \|_2 = \frac{1}{\lambda_{B,m}} \cdot \frac{1}{\lambda_{A,m}}$ proving that the three leftmost inequalities in (24) hold. □

NOTE: The assertions of Lemma 3.3 hold even if one of A or B is nonnegative definite. The only difference is that the two leftmost inequalities in (24) become equalities, that is $0 = \lambda_{A,m}\lambda_{B,m} = \lambda_{AB,m}$. Indeed, the proof for the assertion in (i) is the same if B is singular, while if A is singular one uses the similarity of the matrices AB and $B^{1/2}AB^{1/2}$. The proof for the rightmost inequalities in (24) is exactly the same as before. For the leftmost ones we simply observe that $\det(AB) = \det(A)\det(B) = 0$.

Lemma 3.4 *Consider the real symmetric tridiagonal matrix $T_{\alpha,\beta}$ of order n given by*

$$T_{\alpha,\beta} := \begin{pmatrix} \alpha & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & \beta \end{pmatrix}, \tag{25}$$

where (α,β) = *(-2,-2), (-2,-1), (-1,-2), (-1,-1). Its eigenvalues* $\lambda := \lambda_{T_{\alpha,\beta}}$ *are given for* $\ell = 0, \ldots, n-1$ *by the expressions*

$$i)\ \text{If } (\alpha,\beta) = (-2,-2) \text{ then } \lambda_{T_{\alpha,\beta}} = -4\sin^2\left(\frac{(\ell+1)\pi}{2(n+1)}\right),$$

$$ii)\ \text{If } (\alpha,\beta) = (-2,-1) \text{ or } (-1,-2) \text{ then } \lambda_{T_{\alpha,\beta}} = -4\sin^2\left(\frac{(2\ell+1)\pi}{2(2n+1)}\right), \tag{26}$$

$$iii)\ \text{If } (\alpha,\beta) = (-1,-1) \text{ then } \lambda_{T_{\alpha,\beta}} = -4\sin^2\left(\frac{\ell\pi}{2n}\right).$$

Proof. Although in at least the cases (26i) and (26iii) the above expressions are well-known in the literature we give in the sequel a unified way of obtaining them all simultaneously. First, we note that it can be proved that all matrices, with the exception of $T_{-1,-1}$ which is non-positive definite, are negative definite. It is $\rho(T_{\alpha,\beta}) \leq \| T_{\alpha,\beta} \|_\infty = 4$ and since it can be checked that $-4 \notin \sigma(T_{\alpha,\beta})$ it is concluded that $\sigma(T_{\alpha,\beta)} \subset (-4,0]$. To determine all the different from zero eigenvalues λ of $T_{\alpha,\beta}$ let $x = [x_1, \ldots, x_n]^T$ denote the associated eigenvectors. From $T_{\alpha,\beta}x = \lambda x$, one obtains

$$\begin{aligned} \alpha x_1 + x_2 &= \lambda x_1, \\ x_{i-1} - 2x_i + x_{i+1} &= \lambda x_i, \quad i = 2, \ldots, n-1, \\ x_{n-1} + \beta x_n &= \lambda x_n. \end{aligned}$$

The set of the above equations can be written as

$$x_{i-1} - (2+\lambda)x_i + x_{i+1} = 0, \quad i = 1, \ldots, n, \tag{27}$$

provided one sets

$$\text{i)} \quad x_0 = (2+\alpha)x_1, \quad \text{ii)} \quad x_{n+1}(2+\beta)x_n. \tag{28}$$

The characteristic equation of (27) is

$$r^2 - (2+\lambda)r + 1 = 0, \tag{29}$$

and since we are looking for $\lambda \neq 0, -4$ the two zeros of the quadratic in (29) will be such that $r_1 \neq r_2$ and will satisfy

$$\text{i)} \quad r_1 + r_2 = 2 + \lambda, \quad \text{ii)} \quad r_1 r_2 = 1. \tag{30}$$

Consequently the solution to (27) will be given by

$$x_i = c_1 r_1^i + c_2 r_2^i, \quad i = 0, \ldots, n+1. \tag{31}$$

If we arbitrarily put $x_1 = 1$ and use the restriction on x_0 from (28i), the coefficients c_1 and c_2 can be determined. Next, using the restrictions on x_{n+1} from (28ii) and (30ii) one arrives at

$$r_1^{2n} = \frac{[(2+\alpha)r_1 - 1][(2+\beta)r_1 - 1]}{[r_1 - (2+\alpha)][r_1 - (2+\beta)]}. \tag{32}$$

So, we distinguish three cases:

i) $(\alpha, \beta) = (-2, -2)$: Then $r_1 = \cos\frac{\ell\pi}{n+1} + i \sin\frac{\ell\pi}{n+1}$, $r_2 = \cos\frac{\ell\pi}{n+1}$ - $i \sin\frac{\ell\pi}{n+1}$ and from (30ii) one obtains the n distinct expressions for $\lambda (\neq 0, -4)$ given in (26i).

ii) $(\alpha, \beta) =$ (-2,-1), (-1,-2): This time $r_1 = \cos\left(\frac{(2\ell+1)\pi}{2n+1}\right) + i \sin\left(\frac{(2\ell+1)\pi}{2n+1}\right)$, $r_2 = \cos\left(\frac{(2\ell+1)\pi}{2n+1}\right)$ - $i \sin\left(\frac{(2\ell+1)\pi}{2n+1}\right)$ and again from (30ii) we have the n distinct values for $\lambda (\neq 0, -4)$ in (26ii).

iii) $(\alpha, \beta) = (-1, -1)$: Working in the same way we obtain the $n-1$ distinct expressions $\lambda = 4 \sin^2 \frac{\ell\pi}{2n}$, $\ell = 1, \ldots, n-1$ $(\lambda \neq 0, -4)$. If we incorporate the eigenvalue $\lambda = 0$ as well we have the expressions in (26iii). □

Lemma 3.5 *Let the $n \times n$ matrices A_i , $i = 1, \ldots, k$, possess complete sets of linearly independent eigenvectors $\mathbf{y}^{(i,j)}$, with corresponding eigenvalues $\lambda_{A_i}^{(j)}$ $(j = 1, \ldots, k$, $i = 1, \ldots, k)$. Then the matrix $A \equiv A_1 \otimes A_2 \otimes \ldots \otimes A_k$ possesses the n^k linearly independent eigenvectors $\mathbf{y}^{(\underline{j})} \equiv \mathbf{y}^{(1,j_1)} \otimes \mathbf{y}^{(2,j_2)} \otimes \ldots \otimes \mathbf{y}^{(k,j_k)}$, where $\underline{j} \equiv (j_1, \ldots, j_k)$ and $j_i = 1, \ldots, n$, $i = 1, \ldots, k$, with corresponding eigenvalues $\lambda^{(\underline{j})} = \lambda_{A_1}^{(j_1)} \lambda_{A_2}^{(j_2)} \ldots \lambda_{A_k}^{(j_k)}$.*

Proof. First, using tensor product properties we can easily verify that $A\mathbf{y}^{(\underline{j})} = \lambda^{(\underline{j})}\mathbf{y}^{(\underline{j})}$ In order to prove the linear independence of the $\mathbf{y}^{(\underline{j})}$'s we construct the $n^k \times n^k$ matrix Y whose columns are the n^k eigenvectors of A in the following order

$$\begin{aligned} Y &= [\mathbf{y}^{(1,1)} \otimes \mathbf{y}^{(2,1)} \otimes \ldots \otimes \mathbf{y}^{(k,1)}, \\ &\quad \mathbf{y}^{(1,1)} \otimes \mathbf{y}^{(2,1)} \otimes \ldots \otimes \mathbf{y}^{(k,2)}, \\ &\quad \ldots, \mathbf{y}^{(1,1)} \otimes \mathbf{y}^{(2,1)} \otimes \ldots \otimes \mathbf{y}^{(k,n)}, \\ &\quad \mathbf{y}^{(1,2)} \otimes \mathbf{y}^{(2,1)} \otimes \ldots \otimes \mathbf{y}^{(k,1)}, \\ &\quad \mathbf{y}^{(1,2)} \otimes \mathbf{y}^{(2,1)} \otimes \ldots \otimes \mathbf{y}^{(k,2)}, \\ &\quad \ldots, \mathbf{y}^{(1,2)} \otimes \mathbf{y}^{(2,2)} \otimes \ldots \otimes \mathbf{y}^{(k,n)}, \\ &\quad \ldots, \mathbf{y}^{(1,n)} \otimes \mathbf{y}^{(2,n)} \otimes \ldots \otimes \mathbf{y}^{(k,n)}] \end{aligned}$$

$$= Y_1 \otimes Y_2 \otimes \ldots \otimes Y_k,$$

where

$$Y_i = \left[\mathbf{y}^{(i,1)}\mathbf{y}^{(i,2)} \ldots \mathbf{y}^{(i,n)}\right], \quad i = 1, \ldots, k.$$

From the assumed linear independence of $\mathbf{y}^{(i,j)}$, $j = 1, \ldots, n$, for each $i = 1, \ldots, k$, we conclude that Y_i^{-1}, $i = 1, \ldots, k$, and $Y^{-1} = Y_1^{-1} \otimes Y_2^{-1} \otimes \ldots \otimes Y_k^{-1}$ exist. This implies the linear independence of the n^k eigenvectors and concludes the proof of the lemma. □

Having developed the background material required we are able to go on with the analysis of the block Jacobi (J),

$$K\mathbf{U}^{(s+1)} = (L + M)\mathbf{U}^{(s)} + \mathbf{F} \ , \ s = 0, 1, 2, \ldots,$$

the block Extrapolated Jacobi (EJ),

$$K\mathbf{U}^{(s+1)} = (1 - \omega)K\mathbf{U}^{(s)} + \omega(L + M)\mathbf{U}^{(s)} + \omega\mathbf{F} \ , \ s = 0, 1, 2, \ldots,$$

and the block Successive Overrelaxation (SOR)

$$K\mathbf{U}^{(s+1)} = (1 - \omega)K\mathbf{U}^{(s)} + \omega L\mathbf{U}^{(s+1)} + \omega M\mathbf{U}^{(s)} + \omega\mathbf{F} \ , \ s = 0, 1, 2, \ldots,$$

methods associated with the linear system (23), where

$$K \equiv diag(A_1, D_2, D_3, \ldots, D_k),$$

and $-L$ and $-M$ are the strictly lower and the strictly upper triangular parts of the matrix coefficient in (23).

The block Jacobi iteration matrix $J \equiv K^{-1}(L + M) = \{J_{i,j}\}$, associated with the matrix coefficient in (23), can be described as

$$J_{1,1}=0,\ \{\{J_{i,j}\}_{j=2}^{k}\}_{i=2}^{k}=0,\ \{H_j := J_{1,j} = -A_1^{-1}A_j\}_{j=2}^{k},$$

$$\{G_i := J_{i,1} = D_i^{-1}B_i\}_{i=2}^{k}. \tag{33}$$

Let H and G denote the matrices

$$H := [H_2, H_3, \ldots, H_k],\ G := \left[G_2^T, G_3^T, \ldots, G_k^T\right]^T \tag{34}$$

of dimensions $\mathcal{K} \times (k-1)\mathcal{K}$ and $(k-1)\mathcal{K} \times \mathcal{K}$ respectively. Consider then the matrix J^2 which, apparently, is a block diagonal matrix with diagonal blocks HG and GH of orders $\mathcal{K}$ and $(k-1)\mathcal{K}$ respectively. By virtue of Lemma 3.1 it is

$$\sigma(J^2) \setminus \{0\} = \sigma(HG) \setminus \{0\} = \sigma(GH) \setminus \{0\}. \tag{35}$$

However,

$$HG = \sum_{j=2}^{k} H_j G_j \tag{36}$$

where each term in the above sum can be found by using (33), (17), (22) and simple tensor product properties. Specifically, it is

$$H_jG_j = -\Pi_{\ell=1}^{k-j}(\otimes I) \otimes (\mathcal{H}_j D_j^{-1}) \otimes (\Pi_{\ell=k-j+2}^{k-1} \otimes I) \otimes (A_1^{-1}T_4^1),\ j = 2, \ldots, k, \tag{37}$$

where for the second order LCSC, it will be

$$S_j := \mathcal{H}_j D_j^{-1} = \frac{\alpha_j}{h_j^2} T_{\alpha,\beta}^j \left(6I + T_{\alpha,\beta}^j\right)^{-1},\ j = 2, \ldots, k, \tag{38}$$

$$S_1 := A_1^{-1}T_4^1 = \left[\frac{\alpha_1}{h_1^2} T_{\alpha,\beta}^1 \left(6I + T_{\alpha,\beta}^1\right)^{-1} + \frac{\gamma}{6} I\right]^{-1} \tag{39}$$

while for the fourth order one we shall have

$$S_j := \mathcal{H}_j D_j^{-1} = \frac{1}{12}\left(12I + T_{-2,-2}^j\right) \frac{\alpha_j}{h_j^2} T_{\alpha,\beta}^j \left(6I + T_{\alpha,\beta}^j\right)^{-1},\ j = 2, \ldots, k, \tag{40}$$

$$S_1 := A_1^{-1}T_4^1 = \left[\left(6I + T^1_{\alpha,\beta}\right)^{-1} \frac{1}{12}\left(12I + T^1_{-2,-2}\right)\frac{\alpha_1}{h_1^2}T^1_{\alpha,\beta} + \frac{\gamma}{6}I\right]^{-1} \quad (41)$$

Due to the presence of the unit matrix factors in the tensor product form (37) all H_jG_j's will possess a linearly independent set of $\mathcal{K}$ common eigenvectors if and only if each $\mathcal{H}_jD_j^{-1}$, $j = 2,\ldots,k$ and $A_1^{-1}T_4^1$ possess complete sets of linearly independent eigenvectors. For the matrices in (39) this is obvious because $T^j_{\alpha,\beta}$, $j = 2,\ldots,k$, is real symmetric negative (or non-positive) definite, $T^1_{\alpha,\beta}$ is real symmetric negative definite and in addition $\sigma(T^j_{\alpha,\beta}) \subset (-4,0]$, $j = 1,\ldots,k$, (see Lemma 3.4), $\alpha_j < 0$, $j = 1,\ldots,k$, and $\gamma \geq 0$. As one can readily see, each of the matrices S_j in (40) is the product of the two real symmetric positive definite matrices $\frac{1}{12}(12I + T^j_{-2,-2})$ and $\frac{\alpha_j}{h_j^2}T^j_{\alpha,\beta}(6I + T_{\alpha,\beta})^{-1}$. (The second matrix factor may be nonnegative definite.) Therefore Lemma 3.2 (or its Note) applies. For the matrix $A_1^{-1}T_4^1$ in (41) one simply observes that the first matrix term in the brackets is similar to $\frac{1}{12}(12I + T^1_{-2,-2})\frac{\alpha_1}{h_1^2}\, T^1_{\alpha,\beta}(6I + T^1_{\alpha,\beta})^{-1}$ which, in turn, is of exactly the same form as the matrices S_j in (40), except that the second matrix factor considered previously is now always positive definite. Consequently, in all possible cases, we are dealing with, all terms $H_j\, G_j$ in (37) possess a linearly independent set of $\mathcal{K}$ common eigenvectors. By virtue of this result and in view of Lemma 3.5, it is implied from (37) that

$$\lambda_{HG} = -\lambda_{S_1}\sum_{j=2}^{k}\lambda_{S_j}, \qquad (42)$$

where λ_X is used to denote any eigenvalue of the matrix X. However, from the previous discussion there follows that $\lambda_{S_i} \geq 0$, $j = 2,\ldots,k$, while $\lambda_{S_1} > 0$. Therefore $\lambda_{HG} \leq 0$. So, from (35) it is implied that J^2 possesses non-positive eigenvalues and hence the block Jacobi matrix J has a purely imaginary spectrum. From the analysis so far it becomes clear that the eigenvalues of J^2 and therefore those of J can be given analytically in the following cases:

α) In all the cases of the second order LCSC we are dealing with when Dirichlet *and/or* Neumann boundary conditions are imposed on the faces of $\partial\Omega$.

β) In the fourth order one when *only* Dirichlet boundary conditions are imposed on $\partial\Omega$.

This is an immediate consequence of the fact that each matrix S_j, $j = 1,\ldots,k$, in (38)-(41) is a simple *real rational* well-defined matrix function of the matrix $T^j_{\alpha,\beta}$, $j = 1,\ldots,k$, in case (α) and of the matrix $T^j_{\alpha,\beta}$, $j = 2,\ldots,k$, and $T^1_{-2,-2}$ in case (β) respectively.

Having in mind the various conclusions we have arrived at in the analysis so far, one can state the following Theorem in which the analytic expressions for the eigenvalues of the block Jacobi matrix J in (33) are given.

Theorem 3.1 *The eigenvalues μ of the block Jacobi iteration matrix $K^{-1}(L+M)$ defined in (33) are $\mu = 0$ of multiplicity $(k-2)K$ and pure imaginary given for the second order collocation scheme by*

$$\mu = \pm i \left\{ \frac{\sum_{j=2}^{k} \frac{\alpha_j}{h_j^2} \lambda^j_{\alpha,\beta} (6 + \lambda^j_{\alpha,\beta})^{-1}}{\frac{\alpha_1}{h_1^2} \lambda^1_{\alpha,\beta} (6 + \lambda^1_{\alpha,\beta})^{-1} + \frac{\gamma}{6}} \right\}^{1/2} \tag{43}$$

while for the fourth order scheme, where (2a) is assumed to be subjected to Dirichlet boundary conditions only, by

$$\mu = \pm i \left\{ \frac{\sum_{j=2}^{k} \frac{1}{12}(12 + \lambda^j_{-2,-2}) \frac{\alpha_j}{h_j^2} \lambda^j_{-2,-2} (6 + \lambda^j_{-2,-2})^{-1}}{\frac{1}{12}(12 + \lambda^1_{-2,-2}) \frac{\alpha_1}{h_1^2} \lambda^1_{-2,-2} (6 + \lambda^1_{-2,-2})^{-1} + \frac{\gamma}{6}} \right\}^{-1/2} . \tag{44}$$

In (43) and (44) $\lambda^j_{\alpha,\beta}$, $j = 1,\ldots,k$, with (α,β) = (-2,-2), (-2,-1), (-1,-2), (-1,-1), are the $n = N_j - 1$ eigenvalues of the matrix $T^j_{\alpha,\beta}$, $j = 1,\ldots,k$, as these are given in Lemma 3.4. □

NOTE: It should be pointed out that analytic expressions for the eigenvalues of J can *not* be derived, in terms of the matrices $T^j_{\alpha,\beta}$ involved, in the case of the fourth order LCSC where on at least one face of $\partial\Omega$ (N) boundary conditions are imposed. This is due to the non-commutativity of the matrices $T^j_{-2,-2}$ and $T^j_{\alpha,\beta}$ for $(\alpha,\beta) \neq (-2,-2)$. However, one can trivially give analytic expressions in terms of the eigenvalues of the matrices $(12I + T^j_{2,-2})\, T^j_{\alpha,\beta}\, (6I + T^j_{\alpha,\beta})^{-1}$, $j = 1,\ldots,k$, and also, by virtue of Lemma 3.3, lower and upper bounds for the eigenvalues of $H_j\, G_j$, $j = 1,\ldots,k$, in terms of the extreme eigenvalues of $T^j_{-2,-2}$, and $T^j_{\alpha,\beta}$, $j = 1, \ldots, k$, of Lemma 3.4.

To derive the spectral radii of the Jacobi matrices of Theorem 3.1 and also upper bounds in the case of the previous Note, we introduce some notations first. Let, then, $\lambda^j_{\alpha,\beta}$ denote the eigenvalues of the matrices $T^j_{\alpha,\beta}$, $j = 1, \ldots, k$, as in Theorem 3.1. Let also

$$c_j := c_j(\alpha, \beta) := \min \lambda^j_{\alpha,\beta},\ j = 1, \ldots, k, \tag{45}$$

$$s_j := s_j(\alpha, \beta) := \max \lambda^j_{\alpha,\beta},\ j = 1, \ldots, k,$$

denote the smallest and the largest eigenvalues $\lambda^j_{\alpha,\beta}$, $j = 1, \ldots, k$, respectively. Finally, let us define the quantities

$$y_j := y_j(\lambda^j_{\alpha,\beta}) := \frac{\alpha_j}{h_j^2} \lambda^j_{\alpha,\beta}(6 + \lambda^j_{\alpha,\beta})^{-1}, \tag{46}$$

$$z_j := z_j(\lambda^j_{-2,-2}, \lambda^j_{\alpha,\beta}) := \frac{1}{12}(12 + \lambda^j_{-2,-2}) y_j \tag{47}$$

in terms of the eigenvalues $\lambda^j_{\alpha,\beta}$ of the matrices $T^j_{\alpha,\beta}$, $j = 1, \ldots, k$. (It is obvious that we have omitted the index j from the pair of subscripts (α, β) to simplify the notation.). Then we have:

Theorem 3.2 *(i) The spectral radii of the block Jacobi matrices corresponding to the second and the fourth order collocation schemes considered in Theorem 3.1 are defined by the following expressions*

$$\rho^2(J) \equiv \sum_{j=2}^{k} y_j(c_j)/(y_1(s_1) + \frac{\gamma}{6}) \tag{48}$$

and

$$\rho^2(J) \equiv \frac{\sum_{j=2}^{k} z_j\left(c_j(-2,-2), c_j(\alpha,\beta)\right)}{z_1\left(s_1(-2,-2), s_1(\alpha,\beta)\right) + \frac{\gamma}{6}}. \tag{49}$$

(ii) Moreover the expression (49) is also a strict upper bound for the square of the spectral radius of the Jacobi matrix in the case of the fourth order scheme corresponding to an elliptic problem (1b) where on at least one of the faces of $\partial\Omega$ (N) boundary conditions are imposed.

Proof: We notice that in (43) and (44) $\alpha_j < 0$, $\lambda^j_{\alpha,\beta} \in [c_j, s_j]$, $j = 1,\ldots,k$, and $\gamma \geq 0$. Hence the expressions $y_j := y_j(\lambda^j_{\alpha,\beta}) := y_j(\lambda)$ in (46) are nonnegative and independent of each other. So are the expressions $z_j := z_j(\lambda^j_{-2,-2}, \lambda^j_{-2,-2}) := z(\lambda,\lambda)$ in (47). Therefore, for the extreme values of $y_j(\lambda)$, $\lambda \in [c_j, s_j]$, $j = 1,\ldots,k$, which will lead from (43) to the determination of the spectral radius of the block Jacobi matrix of the second order collocation scheme we have

$$\frac{\partial y_j}{\partial \lambda} = \frac{6}{(6+\lambda)^2}\frac{\alpha_j}{h_j^2} < 0.$$

Consequently,

$$\max y_j = y_j(c_j), \ \min y_j = y_j(s_j). \tag{50}$$

For the corresponding quantity of the fourth order scheme of Theorem 3.1, working in a similar way we obtain

$$\frac{\partial z_j}{\partial \lambda} = \frac{1}{12(6+\lambda)^2}\frac{\alpha_j}{h_j^2}[(6+\lambda)^2 + 36] < 0,$$

which implies that

$$\max z_j = z_j\left(c_j(-2,-2), c_j(-2,-2)\right), \tag{51}$$

$$\min z_j = z_j\left(s_j(-2,2), s_j(-2,-2)\right).$$

It is obvious now that from (43) - (47) and (50), (51), the results (48) and (49) follow, which conclude the proof for part (i) of the theorem. For part (ii) we should bear in mind the analysis preceding the statement of Theorem 3.1 and referring to the matrices in (40) and (41), especially for those indices $j = 1,\ldots,k$ for which $(\alpha,\beta) \neq (-2,-2)$, and also the Note immediately following Theorem 3.1. As was noticed there, and in view of Lemma 3.3, its Note and Lemma 3.4, the lower and upper bounds for the eigenvalues of the matrices in (40) and (41) depend directly on the extreme eigenvalues of the two matrices $\frac{1}{12}(12I + T^j_{-2,-2})$ and $\frac{\alpha_j}{h_j^2}T^j_{\alpha,\beta}(6I + T^j_{\alpha,\beta})^{-1}$. These are readily seen to be $\frac{1}{12}(12 + c_j(-2,2))$ and $\frac{1}{12}(12 + s_j(-2,-2))$ for the minimum and the maximum eigenvalues for the former matrix and $y_j(c_j)$ and $y_j(s_j)$ for the latter one, where c_j, s_j and y_j are the expressions in (45) and (46). From Lemma 3.3 and its Note using the expressions (47) the bound for the spectral radius of the corresponding block Jacobi matrix is readily shown to be the expression in the right hand side of (49).□

REMARK: The analysis so far has been made on the assumption that the x_1-direction is somehow predetermined. However, since there are k possible choices for the x_1-direction in a particular case, the choice should be made in such a way as to give the smallest possible values in (48) and (49). Consider then the quantities

$$\min_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{k} y_j(c_j)/(y_i(s_i) + \frac{\gamma}{6}) \right) \tag{52}$$

taken over all $i = 1, \ldots, k$ for which they are well-defined and also the quantities

$$\min_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{\kappa} \frac{z_j\left(c_j(-2,2), c_j(\alpha,\beta)\right)}{z_i\left(s_i(-2,-2), s_i(\alpha,\beta)\right) + \frac{\gamma}{6}} \right) \tag{53}$$

in the same way. The indices i, in (52) and (53), for which the corresponding minima take place should be interchanged with the index 1 and therefore the x_i-direction should be taken as the x_1-direction. If in either case (52) or (53), more than one index i gives the same minimum value then anyone will do. It should be noted that after having chosen the x_1-direction in the way described the expressions in (48) and (49) give then the smallest possible values for the spectral radius or for an upper bound of it, as was explained, for the particular problem at hand.□

From this point on, the analysis is almost identical with that in [5] and the interested reader is referred to it. For completeness, we may simply mention some of the theoretical results obtained in [5], which are based on the corresponding theory developed in [12], [13] and in [1], [2], [3], [4], [10].

(i) The block Extrapolated Jacobi (EJ) method and the block SOR method corresponding to the block Jacobi method of this paper converge for values of their parameters varying in some open intervals whose left end is 0 and the right one is a function or $\rho(J)$. However, the optimum SOR is always superior to the optimum EJ and the corresponding optimal parameters for the SOR method are given by

$$\omega_{opt} = \frac{2}{1+\sqrt{1+\rho^2(J)}}, \; \rho(\mathcal{L}_{\omega_{opt}}) = 1 - \omega_{opt}. \tag{54}$$

(ii) For the efficiency of both the serial and the parallel iterative solution of the linear system (23), a cyclic natural ordering of the unknowns U^i, $i = 2, \ldots, k$, is adopted according to which

$$\mathcal{U}^i \equiv \left\{ \cdots \left\{ \left\{ \cdots \left\{ U^i_{j_1,\ldots,j_{i-1},j_i,\ldots,j_k} \right\}_{j_i=1}^{N_i-1} \cdots \right\}_{j_k=1}^{N_k-1} \right\}_{j_1=1}^{N_1-1} \cdots \right\}_{j_{i-1}=1}^{N_{i-1}-1}, \tag{55}$$

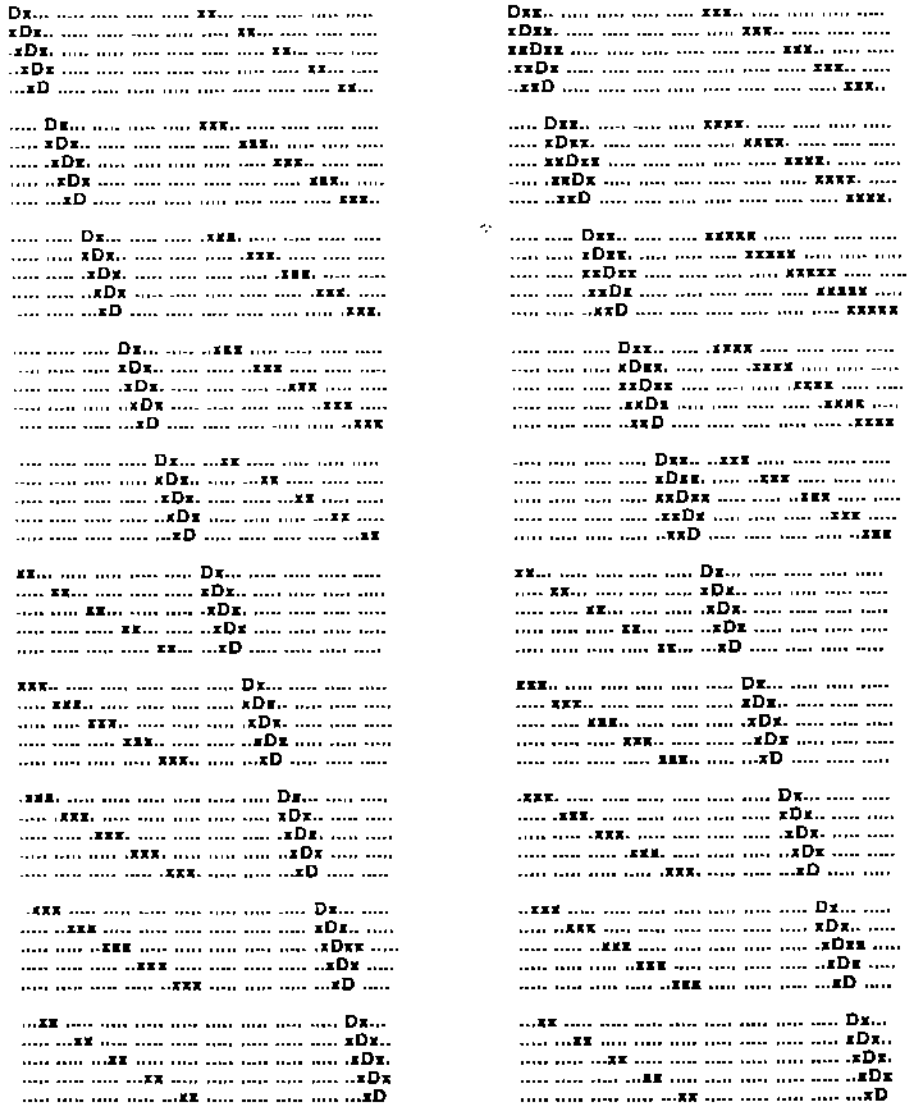
the equations in (16) are reordered according to the ordering of $\mathcal{U}^1$ and each block of the auxiliary conditions (21) according to the ordering of the unknowns $\mathcal{U}^i$. It is then proved that the new coefficient matrix $\mathcal{A}$ is obtained by a permutation similarity transformation of the matrix coefficient A in (23) having the same $k \times k$ block structure and, therefore, a similar block Jacobi matrix $\mathcal{J}$ to the previous one J. Consequently, the convergence results are identically the same so that all the formulas in connection with eigenvalues, spectral radii, etc. of the Jacobi, the Extrapolated Jacobi and the SOR method studied in this section remain unchanged.

(iii) The new structure of the collocation coefficient matrix $\mathcal{A}$, for the second and fourth order scheme in 2-dimensions, is given in Figure 1.

4 Numerical verification of convergence

In this section we present the results of some numerical experiments that verify the convergence properties of the iterative solution methods of Section 3. We should mention that although we present numerical data for only the $O(h^2)$, 3-dimensional case, these are very representing for problems with different dimensionality or $O(h^4)$ discretization schemes. For experimental data on the convergence properties of the line cubic spline collocation method (LCSC) and the iterative solvers in the case of Dirichlet only boundary conditions the reader is referred to [5] and [7]. The parallel implementation details and the performance of the iterative LCSC schemes, for 2-dimensional problems, on several SIMD and MIMD architectures can be found in [8].

Specifically we have applied the LCSC discretization techniques with uniform meshes to approximate the known solution of the following PDE



(a) (b)

Figure 1: Structure of matrices from the second (a) and fourth (b) order LCSC collocation methods for a 2 dimensional problem. We have $N_1 = N_2 = 6$ with the notation D = diagonal non-zero element, x = off diagonal non-zero element and . = zero element.

problems

PDE 1: $D_x^2u + D_y^2u + D_z^2u = -f$
PDE 2: $4D_x^2u + 2D_y^2u + D_z^2u - 10u = -f$
PDE 3: $D_x^2u + \frac{1+x-y+z}{8}D_y^2u + \frac{x+z}{9}D_z^2u + e^{x+y+z}u = f$

All these problems are considered on the unit cube subject to one of the following types of boundary conditions

BC 1: Dirichlet boundary conditions on all faces of Ω.

BC 2: Neumann boundary conditions on the faces $y = 1$ and $z = 0$ of Ω and Dirichlet ones on the rest.

BC 3: Neumann boundary conditions on the face $x = 1$ of Ω and Dirichlet on the rest.

The right hand side f is selected so that the true solution is

$$u(x, y, z) = e^{x+y+z}(x^2 - x)(y^2 - y)(z^2 - z)$$

The linear systems from the LCSC discretization were solved by the iteration of Section 3, the termination criterion of the iteration being that $||U^{(s+1)} - U^{(s)}||_\infty$ is within the interval $(0, 10^{-7})$. All experiments were performed, in double precision, on a SUN4-110 workstation.

In Figures 2 and 3 we present the theoretically estimated and the experimentally computed values of the optimum SOR relaxation parameter ω_{opt} for the two elliptic operators in PDE 1 and PDE 2 respectively, subject to the three different types of boundary conditions mentioned above.

In Figure 4 we present the experimentally computed values of the optimum SOR relaxation parameter ω_{opt} for the elliptic operator PDE 3 subject to the three different boundary conditions.

Table 1 presents the SOR iterations required to solve the discretized equations using a 10^{-7} stopping criterion for the PDE problems considered in Figure 2.

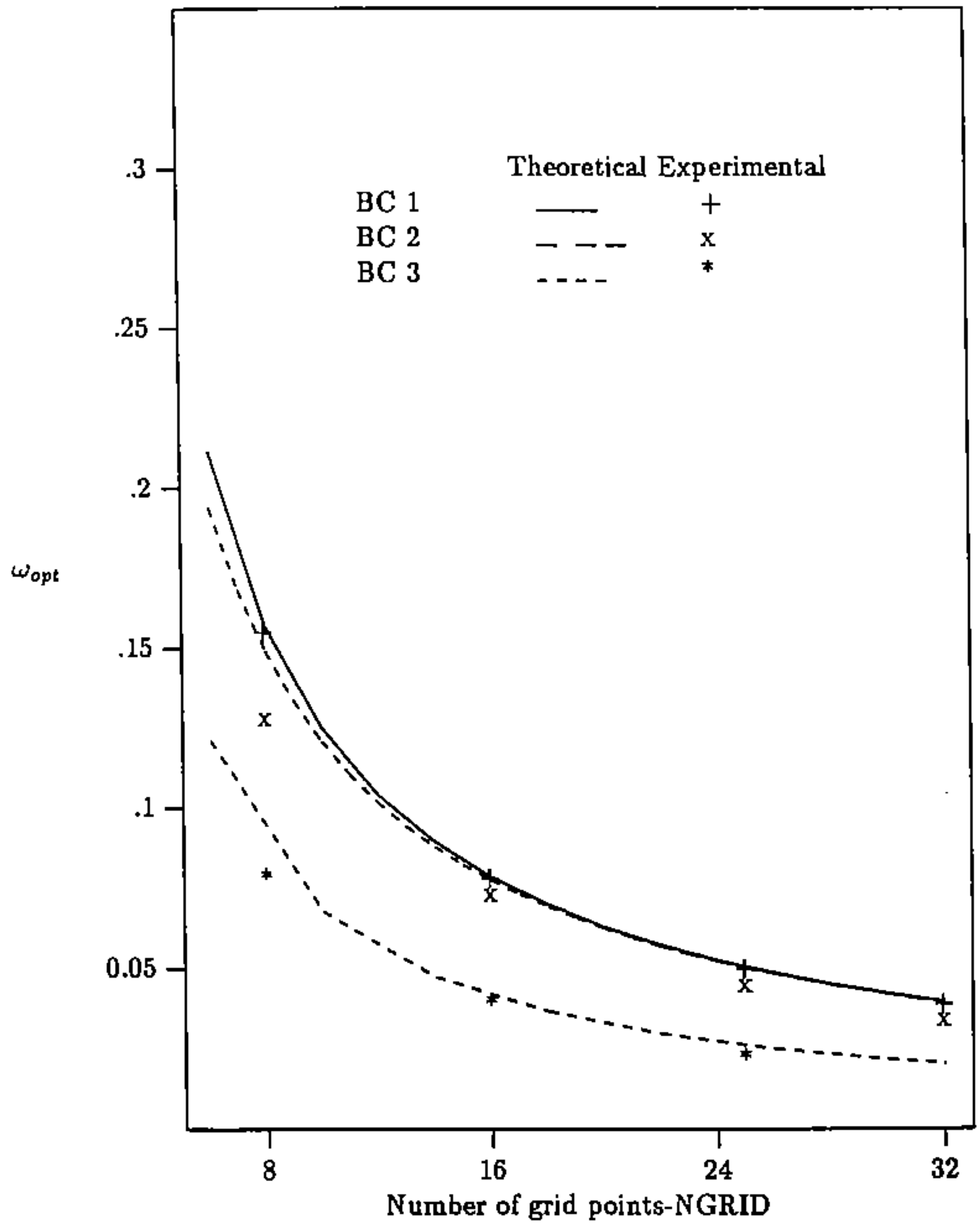


Figure 2: The graph of the ω_{opt} vs. the number of grid points for the second order LCSC method applied to PDE 1, with different types of boundary conditions on faces of $\partial\Omega$.

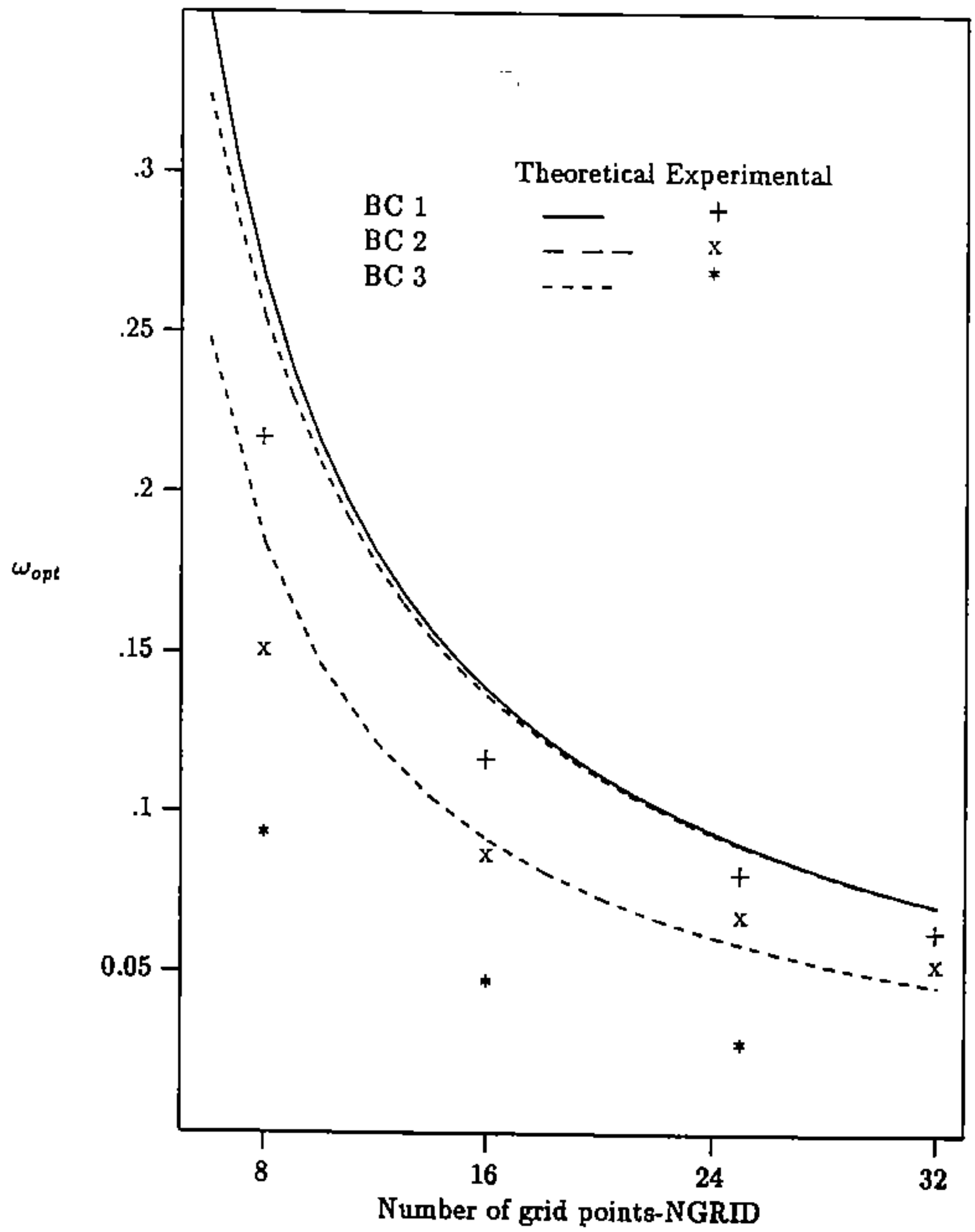


Figure 3: The graph of the ω_{opt} vs. the number of grid points for the second order LCSC method applied to PDE 1, with different types of boundary conditions on faces of $\partial\Omega$.

Table 1: The required number of iterations of the SOR method to solve the second order LCSC equations for the elliptic problems considered in Figure 2.

NGRID	BC 1	BC 2	BC 3
8	42	55	85
16	63	98	172
24	73	125	255
32	105	169	320

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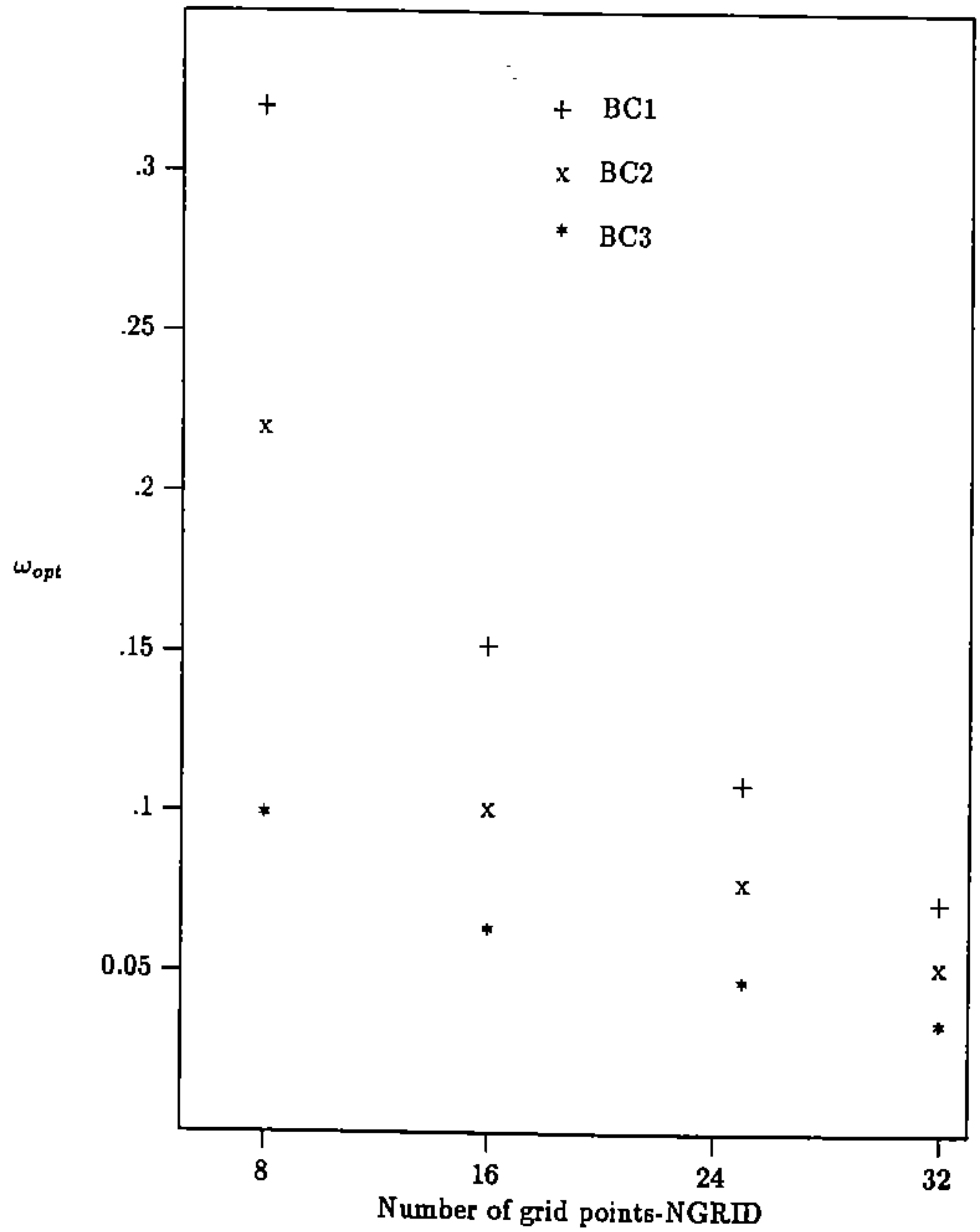

Figure 4: The graph of the ω_{opt} vs. the number of grid points for the second order LCSC method applied to PDE 3, with different types of boundary conditions on faces of $\partial\Omega$.

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